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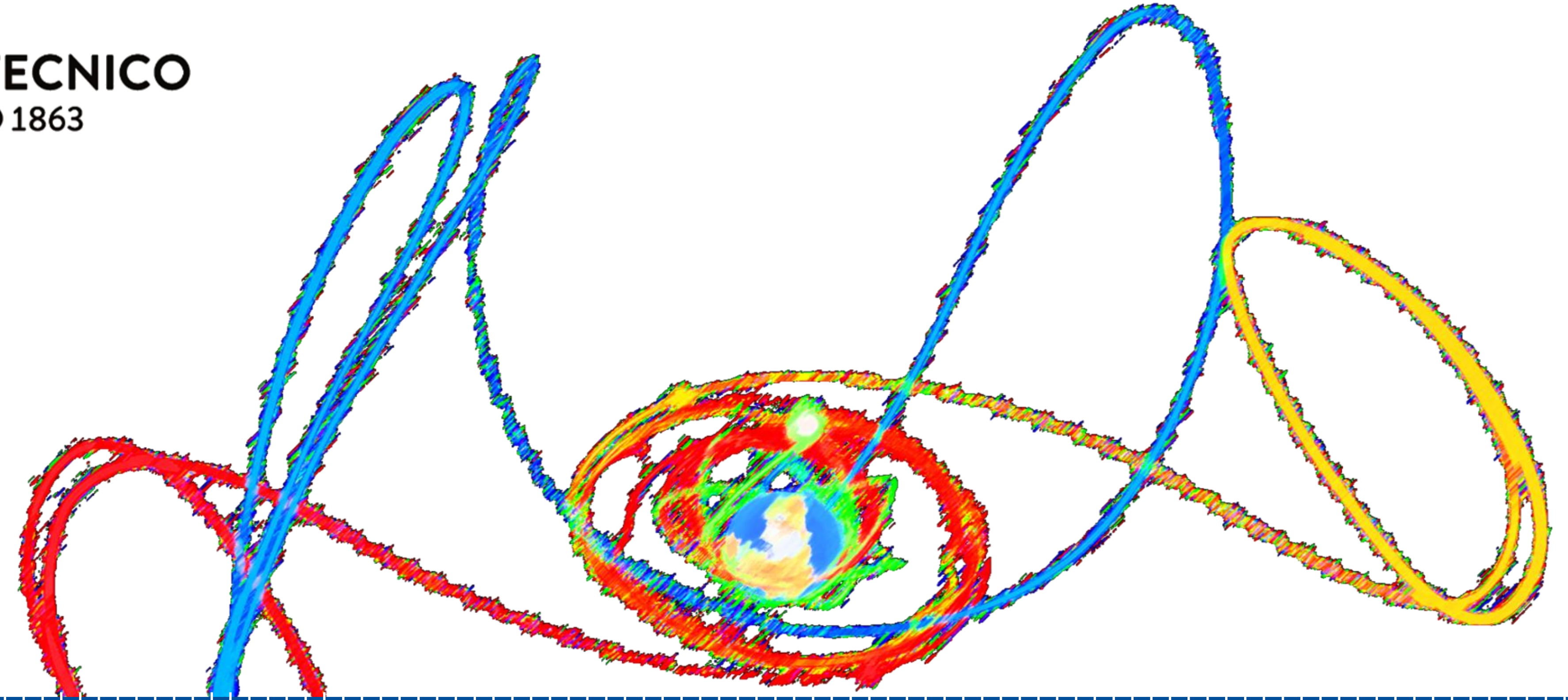

Orbital Mechanics
Module 1: The two body problem

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Module Contents

The two body problem

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NUMERICAL INTEGRATION OF DYNAMICAL SYSTEMS

Ordinary Differential Equations

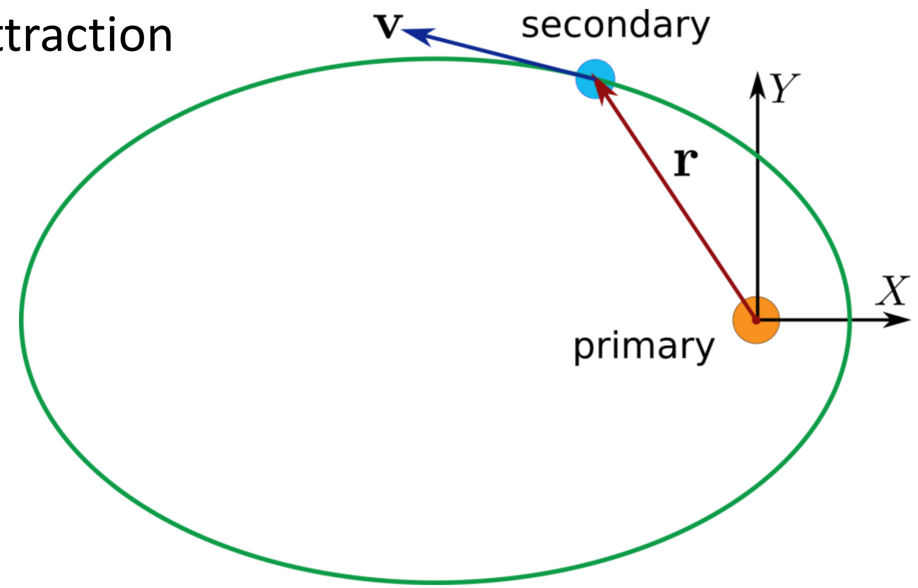
An **ordinary differential equation** (ODE) is an equation F involving one or more functions $\mathbf{x}$ of an independent variable t (usually **time**) and their derivatives:

$$F\left(t; \mathbf{x}(t), \dot{\mathbf{x}}(t), \ddot{\mathbf{x}}(t), \dots, \mathbf{x}^{(n)}(t)\right) = 0$$

Many dynamical systems in nature can be modelled through a system of ODEs

Example: Two body problem

Motion of two bodies subjected only to their mutual gravitational attraction



Equations for the motion of the secondary body around the primary body:

$$\ddot{\mathbf{r}} + \frac{\mu}{r^3}\mathbf{r} = \mathbf{0}$$

where:

$\mathbf{r}$: Position vector of the secondary, in an inertial reference frame centred on the primary
μ: gravitational parameter of primary body

Reduction to a first-order system

The **order** of an ODE is that of the highest-order derivative involved.

Any ODE can be reduced to a **first-order system** by treating the derivatives up to order *n-1* as **independent variables**. In explicit form:

$$\frac{\mathrm{d}\mathbf{y}(t)}{\mathrm{d}t} = \mathbf{f}(t; \mathbf{y}(t))$$

where $\mathbf{y} = \left[\mathbf{x}, \dot{\mathbf{x}}, \ \dots, \mathbf{x}^{(n-1)}\right]$.

Example: Two body problem – reduction to a first-order ODE system

New state vector that includes the first derivative of the position (i.e., the velocity)

$$\mathbf{s} = \begin{bmatrix} \mathbf{r} \\ \mathbf{v} \end{bmatrix} = \begin{bmatrix} r_x \\ r_y \\ r_z \\ v_x \\ v_y \\ v_z \end{bmatrix}$$

Velocity definition:

$$\dot{\mathbf{r}} = \mathbf{v}$$

Equations of motion:

$$\dot{\mathbf{v}} = -\frac{\mu}{r^3}\mathbf{r}$$

$$\dot{\mathbf{s}} = \begin{bmatrix} \dot{\mathbf{r}} \\ \dot{\mathbf{v}} \end{bmatrix} = \begin{bmatrix} \dot{r}_x \\ \dot{r}_y \\ \dot{r}_z \\ \dot{v}_x \\ \dot{v}_y \\ \dot{v}_z \end{bmatrix} = \begin{bmatrix} v_x \\ v_y \\ v_z \\ -\frac{\mu}{r^3} r_x \\ -\frac{\mu}{r^3} r_y \\ -\frac{\mu}{r^3} r_z \end{bmatrix}$$

Resolution of an ODE system

If $\mathbf{f}$ is *regular enough*, the solution of the first order ODE system for given initial conditions $\mathbf{y}(t_0) = \mathbf{y}_0$ exists and is unique (Cauchy-Lipschitz or Picard-Lindelöf theorem)

- In some cases, a closed form (analytic) solution can be found
- In general, we will resort to **numerical integration schemes**

Example: Two body problem – analytic solution

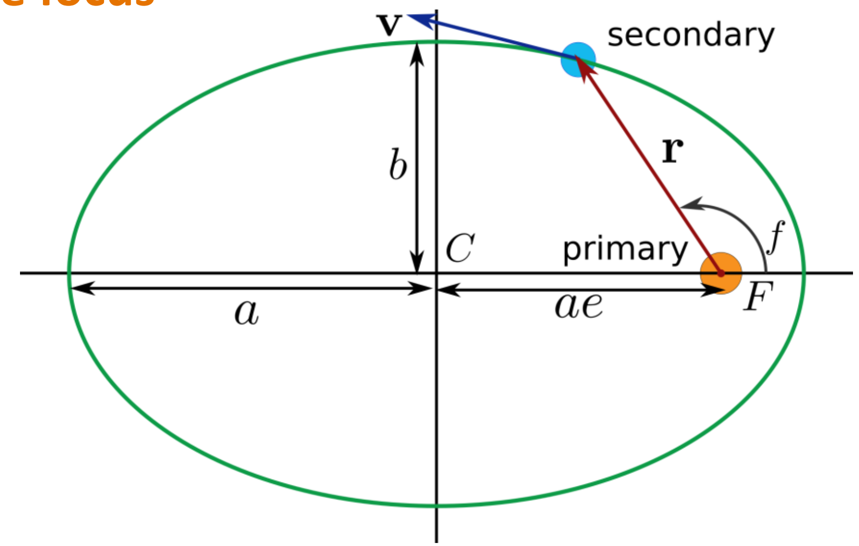
First Kepler's law: In an inertial frame centred on one of the bodies (**the primary**), the **other body** describes a **conic** (ellipse, hyperbola, or parabola) with the **primary in one focus**

Polar equation of the orbit:

$$r = \frac{p}{1 + e\cos f}$$

$\mathbf{r}$: Position vector of the secondary body
μ: gravitational parameter of primary body
p: semilatus rectum of the conic
e: eccentricity of the conic
f: true anomaly

F: focus
C: centre of the conic
a: semimajor axis
b: semiminor axis



Numerical resolution of ODEs

Euler scheme

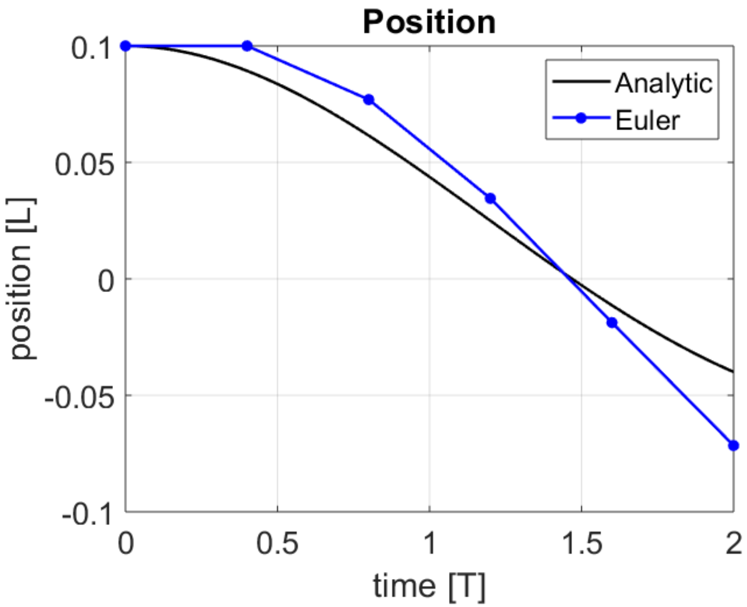
The simplest numerical scheme for solving ODEs is the **Euler scheme**. Consider the Taylor expansion of $\mathbf{y}(t)$ around t_k:

$$\mathbf{y}(t) = \mathbf{y}(t_k) + \dot{\mathbf{y}}(t_k)(t - t_k) + O(t - t_k)^2$$

Retaining only the first order term, **and recalling that $\dot{\mathbf{y}}(t) = \mathbf{f}(t, \mathbf{y})$**:

$$\mathbf{y}(t_{k+1}) = \mathbf{y}_{k+1} \approx \mathbf{y}_k + \mathbf{f}(t_k, \mathbf{y}_k)\Delta t$$

That is, we approximate the value of $\mathbf{y}$ at time step $t_{k+1} = t_k + \Delta t$ from its value at t_k. Geometrically, this is equivalent to treating $\mathbf{y}$ between t_k and t_{k+1} as a straight line with the slope given by $\mathbf{f}_k$.



The values of t_k and $\mathbf{y}_k$ are not known a priori, so $\mathbf{f}_k = \mathbf{f}(t_k, \mathbf{y}_k)$ has to be evaluated as the numerical solver advances

Euler scheme requires very small time steps to achieve high precision (local error is of order $\ddot{\mathbf{y}}_k \Delta t^2$). This leads to large number of steps and computational time.

In practice, we will use more advanced integration schemes already implemented in MATLAB

Integrating ODEs with MATLAB

MATLAB provides several solvers for systems of ODEs

- Based on different integration schemes, with different properties
- **We have to select the most appropriate one for our dynamics**

Solver	Problem Type	Accuracy	When to Use
`ode45`	Nonstiff	Medium	Most of the time. `ode45` should be the first solver you try.
`ode23`		Low	`ode23` can be more efficient than `ode45` at problems with crude tolerances, or in the presence of moderate stiffness.
`ode113`		Low to High	`ode113` can be more efficient than `ode45` at problems with stringent error tolerances, or when the ODE function is expensive to evaluate.
`ode15s`	Stiff	Low to Medium	Try `ode15s` when `ode45` fails or is inefficient and you suspect that the problem is stiff. Also use `ode15s` when solving differential algebraic

Integrating ODEs with MATLAB

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ode45

Explicit Runge-Kutta (4,5) formula (the Dormand-Prince pair)

- Very versatile
- Low performance for stiff problems
- Low performance for high accuracy requirements

Integrating ODEs with MATLAB

ode113

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ode113

Variable-step, variable-order Adams-Bashforth-Moulton Predictor-Corrector solver of orders 1 to 13

- Less function calls than ode45 (less evaluations of the ODE function)

Syntax for ODE solvers

```
[t, y] = odeXX( odefun, tspan, y0, options )
```

Syntax for ODE solvers

Inputs: ODE function

```
[t, y] = odeXX( odefun, tspan, y0, options )
```

`odefun` is a MATLAB function for the **right-hand side of the first order ODE system**, that is, function $\mathbf{f}$ in:

$$\frac{\mathrm{d}\mathbf{y}(t)}{\mathrm{dt}} = \mathbf{f}(t;\mathbf{y}(t))$$

This function has to be of the form

```
function dy = odefun( t, y )
    .....
    .....
end
```

⚠

`t` is a scalar

`y` and `dy` are column vectors of dimension [n x 1]

The order of inputs t and y must be respected

How do we introduce additional parameters (such as μ in the two-body problem)?

Syntax for ODE solvers

Inputs: ODE function

```
[t, y] = odeXX( odefun, tspan, y0, options )
```

Two ways of introducing additional parameters:

1. Additional inputs added at the end of the argument list of `odeXX` (i.e., after `options`) are passed directly to `odefun`:

```
[t, y] = odeXX( odefun, tspan, y0, options, par1, par2, … )

         dy = odefun( t, y, par1, par2, … )
```

2. Use an **anonymous function**:

```
[t, y] = odeXX(@(t,y) odefun(t,y,par1,par2), tspan, y0, options )
```

This creates an unnamed function with inputs `t` and `y`, to be used by `odeXX` as a wrapper for `odefun`.

Syntax for ODE solvers

Inputs: Time span

```
[t, y] = odeXX( odefun, tspan, y0, options )
```

Time span for the integration. There are two possibilities:

1. **`tspan = [ tstart  tend  ]`**
   If `tspan` is a 2-elements array, they represent the initial and final times for the integration, respectively. Output is given at the **internal time steps used by the solver.**

2. **`tspan = [ tstart  t1  t2  …  tend  ]`**
   If `tspan` is a monotonic array of more than 2 elements, the solver returns the value of `y` only at the times in `tspan`. The first and last elements of `tspan` still represent the initial and final time of the integration. **This does not affect the internal time steps automatically decided by the solver (it uses its own time steps to integrate, and afterwards interpolates to get the values for the requested times).**

⚠ The time values in `tspan` must be strictly monotonic!

Syntax for ODE solvers

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Inputs: Initial conditions

```
[t, y] = odeXX( odefun, tspan, y0, options )
```

Column array with the initial conditions for the state (i.e., the value of `y` at the first time given in `tspan`, $\mathbf{y}(t_{\text{start}}) = \mathbf{y}_0$). It must have the same dimensions as `y` and `dy`.

Syntax for ODE solvers

Inputs: Options

```
[t, y] = odeXX( odefun, tspan, y0, options )
```

Object containing optional parameters for the ODE solver. It is created using the `odeset` function (check the documentation center). For example:

```
options = odeset( 'RelTol', 1e-13, 'AbsTol', 1e-14 );
```

For now, we consider only the relative and absolute tolerances (`RelTol` and `AbsTol`, respectively). **The internal time steps used by the solver are automatically chosen to fulfill the tolerances (they do not depend on `tspan`)**

There are many (powerful) options that can be configured. Check the **documentation center page** about `odeset` for more information

Default tolerance values, `RelTol=1e-3` and `AbsTol=1e-6`, are too loose for orbit propagation. Don't forget to set more stringent ones!

Syntax for ODE solvers

Outputs: Two possibilities

```
[t, y] = odeXX( odefun, tspan, y0, options )
```

`t`: m x 1 array with the times at each of the m time steps.

`y`: m x n array with the n states at each of the m time steps. That is, row m corresponds to the state at the m-th time step.

```
sol = odeXX( odefun, tspan, y0, options )
```

`sol`: MATLAB structure containing detailed information about the solution:

- Time and state at the time steps decided by the integrator are stored in `sol.x` and `sol.y`, respectively. **Beware**, they are transposed with respect to `t` and `y`: `sol.x` is a 1 x m array, and `sol.y` is a n x m array with each column corresponding to a different time step.
- The state for other times can be obtained using the function `deval`.

Back to the two-body problem

ODE function

```
function dy = ode_2bp( ~, y, mu )
%ode_2bp ODE system for the two-body problem (Keplerian motion)
%
% PROTOTYPE
%   dy = ode_2bp( t, y, mu )
%
% INPUT:
%   t[1]        Time (can be omitted, as the system is autonomous) [T]
%   y[6x1]      State of the body ( rx, ry, rz, vx, vy, vz ) [ L, L/T ]
%   mu[1]       Gravitational parameter of the primary [L^3/T^2]
%
% OUTPUT:
%   dy[6x1]     Derivative of the state  [ L/T^2, L/T^3 ]
%
% CONTRIBUTORS:
%   Juan Luis Gonzalo Gomez
%
% VERSIONS
%   2018-09-26: First version
%
% ------------------------------------------------------------------------

% Position and velocity
r = y(1:3);
v = y(4:6);

% Distance from the primary
rnorm = norm(r);

% Set the derivatives of the state
dy = [  v
        (-mu/rnorm^3)*r ];

end
```

Back to the two-body problem

ODE function

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end
```

Use ~ to omit unneeded inputs (like time in the two-body problem)

Always document your code!
You can take this example as template.

Back to the two-body problem

Main script

```
% Physical parameters
mu_E = astroConstants(13); % Earth's gravitational parameter [km^3/s^2]

% Initial condition
r0 = [ 26578.137; 0; 0 ]; % [km]
v0 = [ 0; 2.221; 3.173 ]; % [km/s]
y0 = [ r0; v0 ];

% Set time span
a = 1/( 2/norm(r0) - dot(v0,v0)/mu_E ); % Semi-major axis [km]
T = 2*pi*sqrt( a^3/mu_E ); % Orbital period [1/s]
tspan = linspace( 0, 2*T, 1000 );

% Set options for the ODE solver
options = odeset( 'RelTol', 1e-13, 'AbsTol', 1e-14 );
% Perform the integration
[ T, Y ] = ode113( @(t,y) ode_2bp(t,y,mu_E), tspan, y0, options );

% Plot the results
figure()
plot3( Y(:,1), Y(:,2), Y(:,3), '-' )
xlabel('X [km]'); ylabel('Y [km]'); zlabel('Z [km]');
title('Two-body problem orbit');
axis equal;
grid on;
```

Exercise 1: Two-body problem

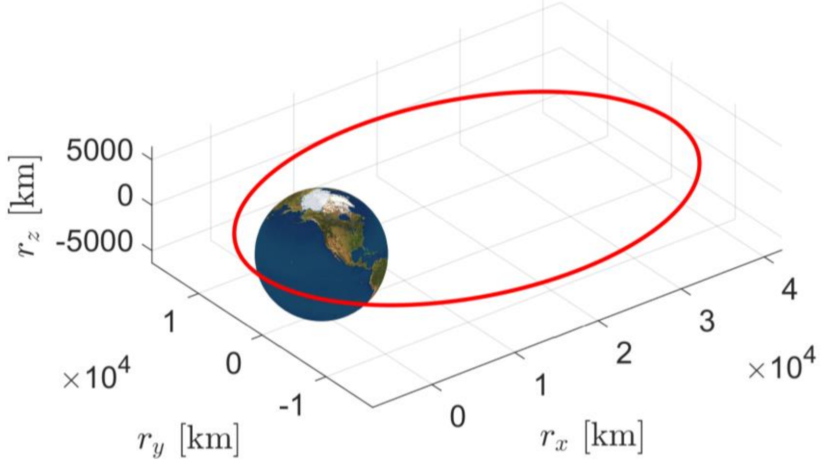
Exercise 1: Integrate numerically a Keplerian orbit (two-body problem)

1. Implement the code to propagate the orbit[1]:
   - Identify the **states** of the system and the **physical parameters** involved
   - Write the *second-order ODE* describing dynamics
   - Reduce the problem to a *first-order ODE system*
   - Implement the `odefun` MATLAB **function** for this ODE system
   - Write a **main script** to integrate numerically the system, **choosing one of MATLAB's solvers and setting its options as needed**.

Equations of motion:

$$\ddot{\mathbf{r}} + \frac{\mu}{r^3}\mathbf{r} = \mathbf{0}$$

μ: gravitational parameter of primary body



[1] **Orbit propagation**: prediction of the orbital characteristics of a body at some future date given the current orbital characteristics.

Exercise 1: Two-body problem

Exercise 1: Integrate numerically a Keplerian orbit (two-body problem)

2. Propagate the orbit for different initial conditions $(\mathbf{r}_0, \mathbf{v}_0)$ and primary attracting body:
   - You can get the gravitational parameter μ of different bodies in the solar system from function `astroConstants.m` in WeBeep.
   - To have a closed orbit (elliptical orbit), your initial conditions should correspond to an energy $\varepsilon < 0$
3. Analyse the results:
   - Plot the orbit over 1 period T
     – Only elliptical (i.e., closed) orbits have a period. Hyperbolic and parabolic (i.e., open) orbits can also be computed, but they will never close.
   - Plot angular momentum vector $\mathbf{h}$ and eccentricity vector $\mathbf{e}$, and check that they remain constant in magnitude and direction
   - Check that $\mathbf{h}$ and $\mathbf{e}$ remain perpendicular (hint: plot the error of a suitable test condition)
   - Plot the specific energy ε, and check if it is constant in time
   - Plot the evolution of the radial and transversal components of the velocity

Exercise 1: Two-body problem

Useful relations

Orbital period

$$T = 2\pi\sqrt{a^3/\mu}$$

Specific energy

$$\varepsilon = \frac{v^2}{2} - \frac{\mu}{r} = -\frac{\mu}{2a}$$

Angular momentum

$$\mathbf{h} = \mathbf{r} \times \mathbf{v}$$

Eccentricity vector

$$\mathbf{e} = \frac{1}{\mu}\mathbf{v} \times \mathbf{h} - \frac{\mathbf{r}}{r}$$

True anomaly

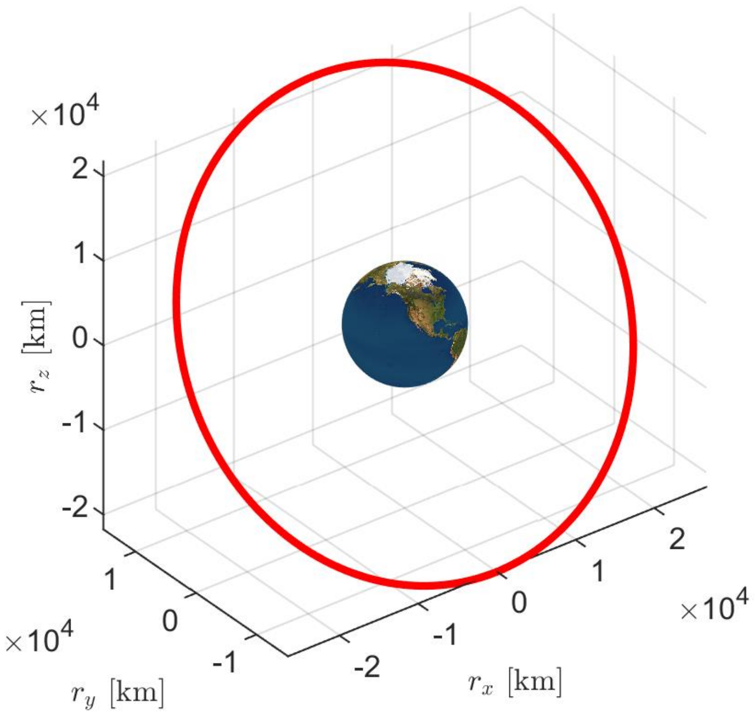
$$\cos f = \frac{\mathbf{r} \cdot \mathbf{e}}{re}$$

Exercise 1: Two-body problem

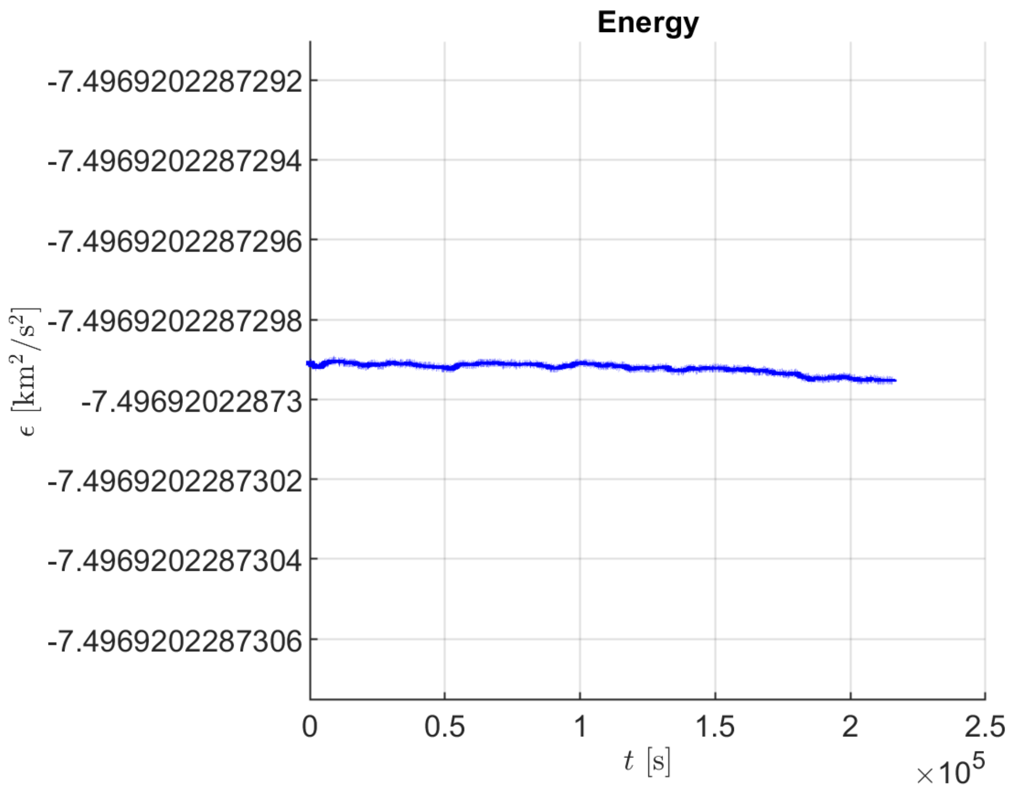
Sample solution: Quasi-circular Medium Earth Orbit

Parameters and initial condition

$$\mathbf{r}_0 = [\,26578.137, 0, 0\,]\ \mathrm{km}$$
$$\mathbf{v}_0 = [\,0, 2.221, 3.173\,]\ \mathrm{km/s}$$



Orbit representation



Specific energy

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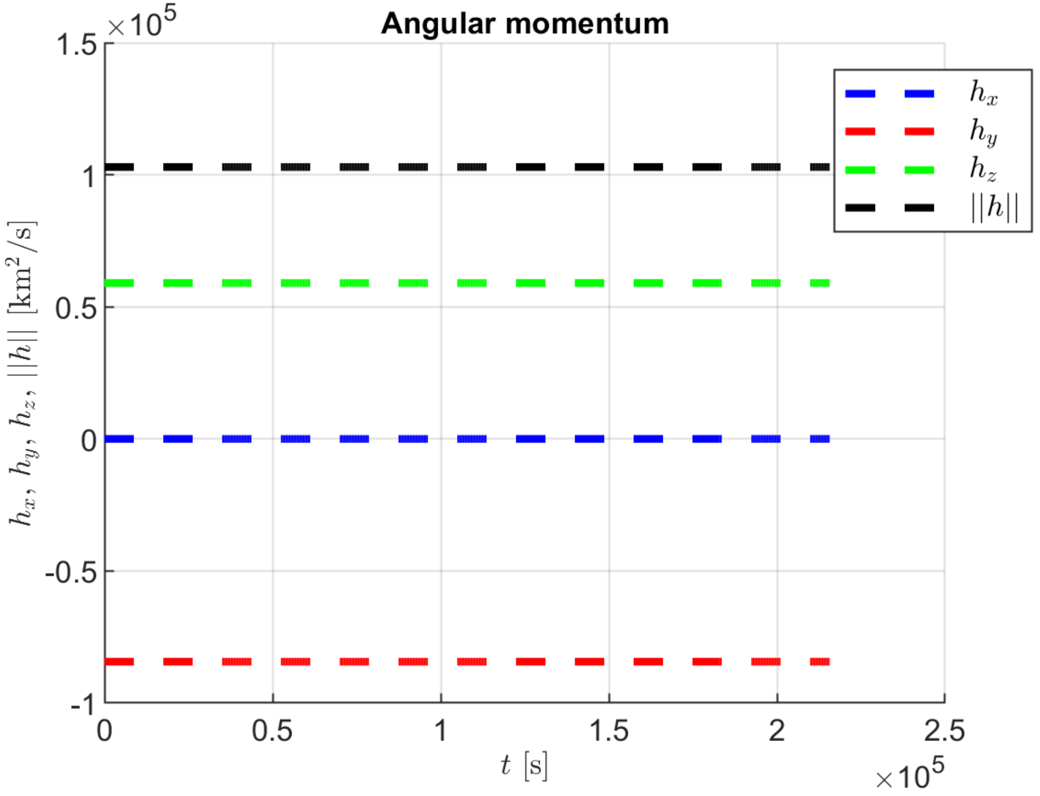
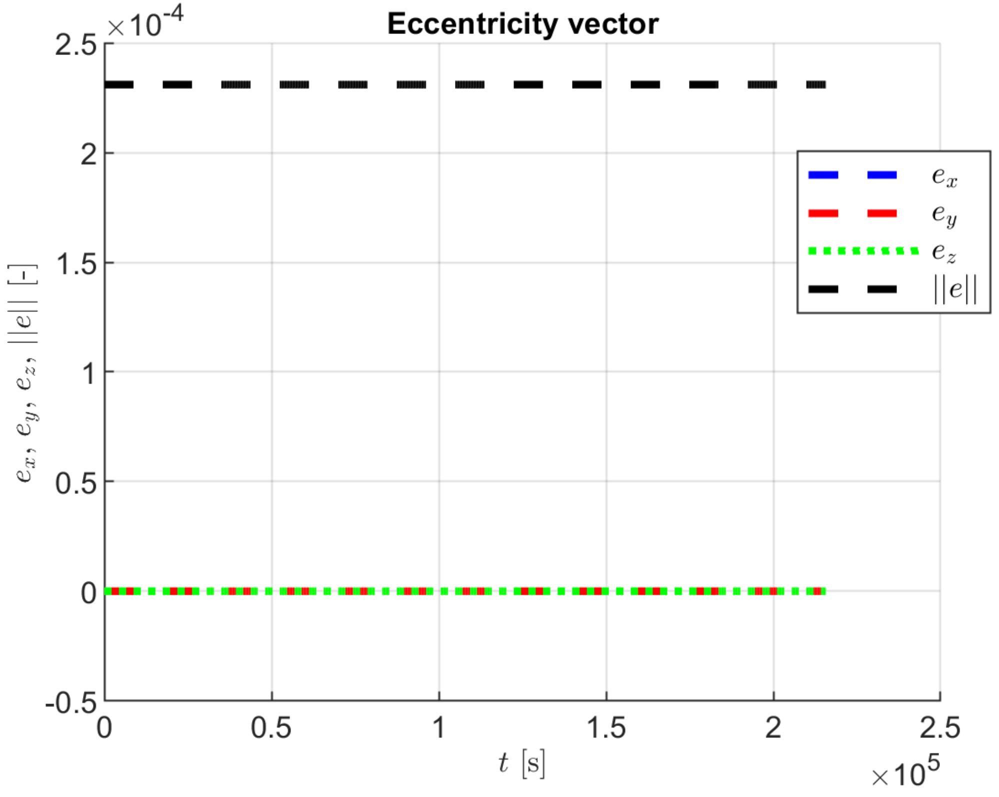


Angular momentum



Eccentricity vector

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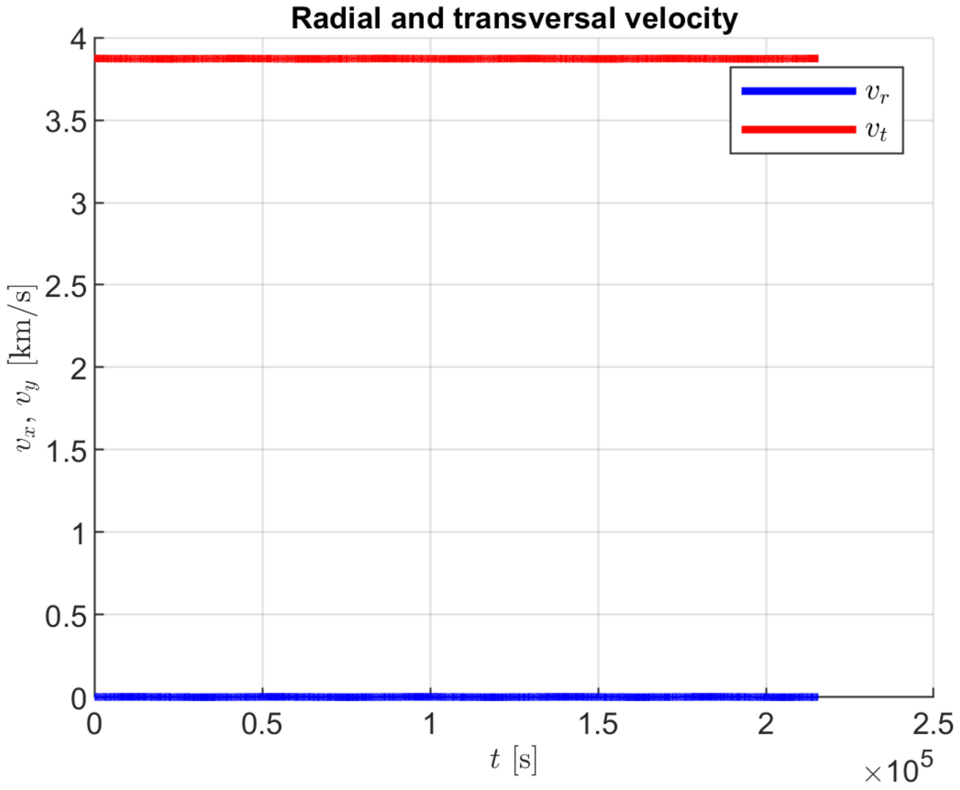
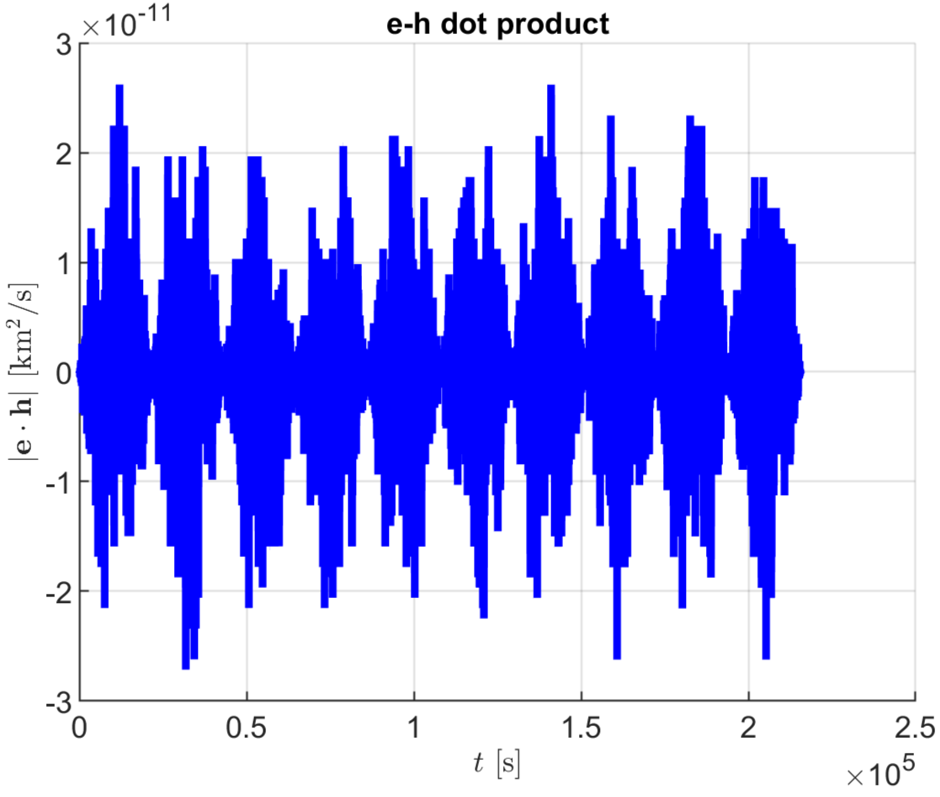


Radial and transversal velocity



Perpendicularity condition for $\mathbf{e}$ and $\mathbf{h}$

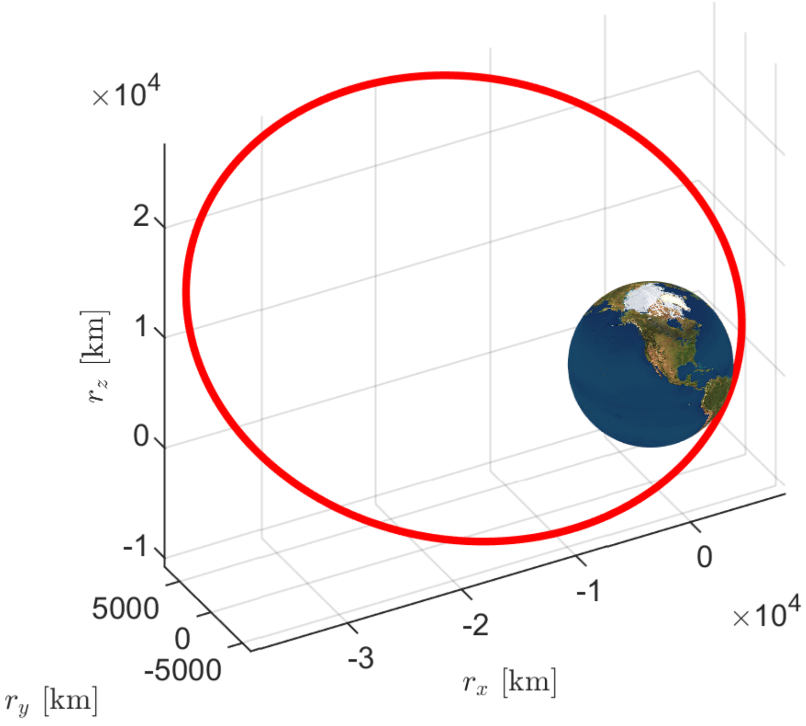
Exercise 1: Two-body problem

Sample solution: Highly eccentric and inclined orbit

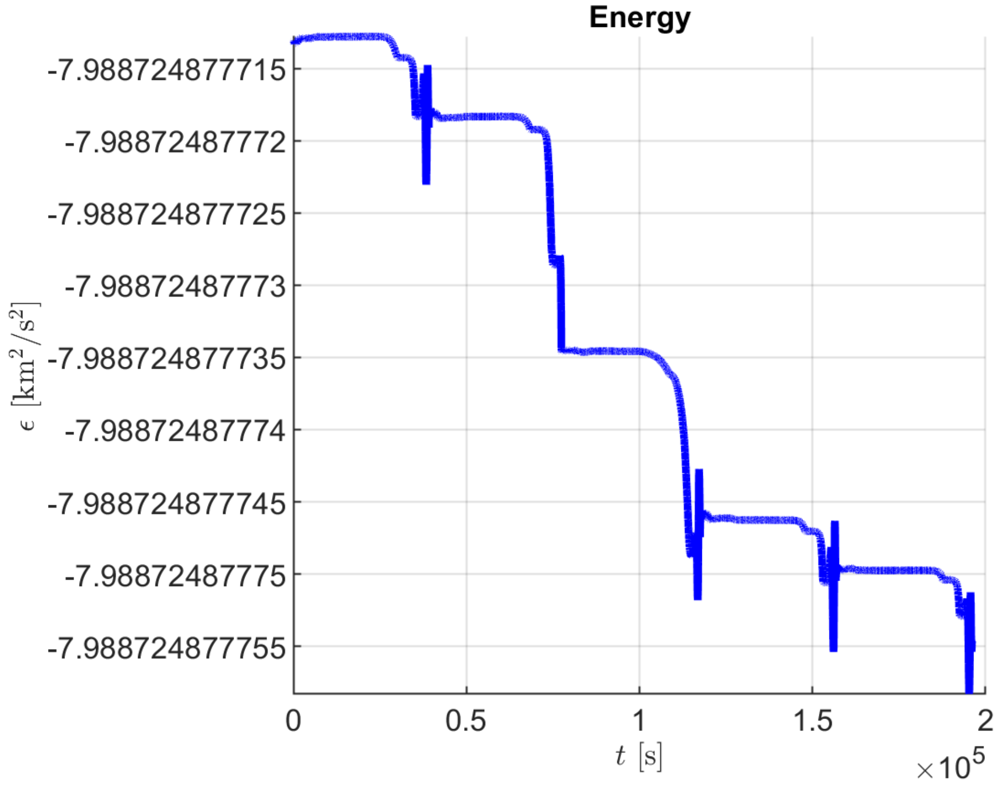
Parameters and initial condition

$$\mathbf{r}_0 = [6495, -970, -3622] \text{ km}$$

$$\mathbf{v}_0 = [4.752, 2.130, 7.950] \text{ km/s}$$



Orbit representation



Specific energy

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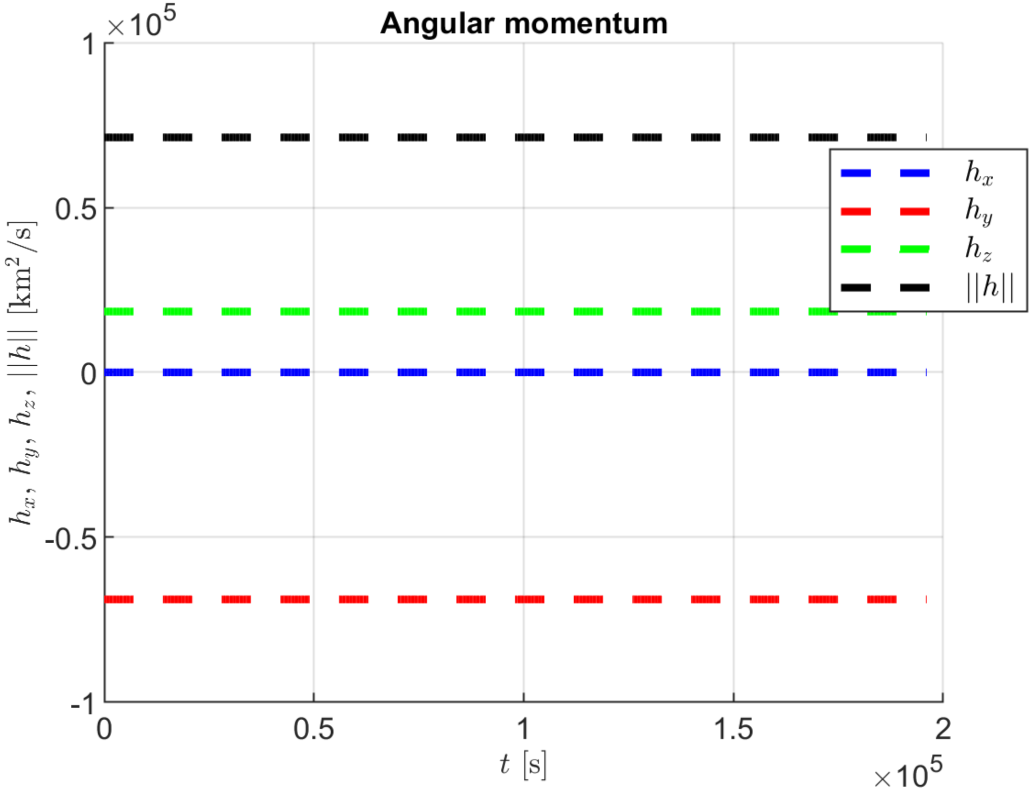
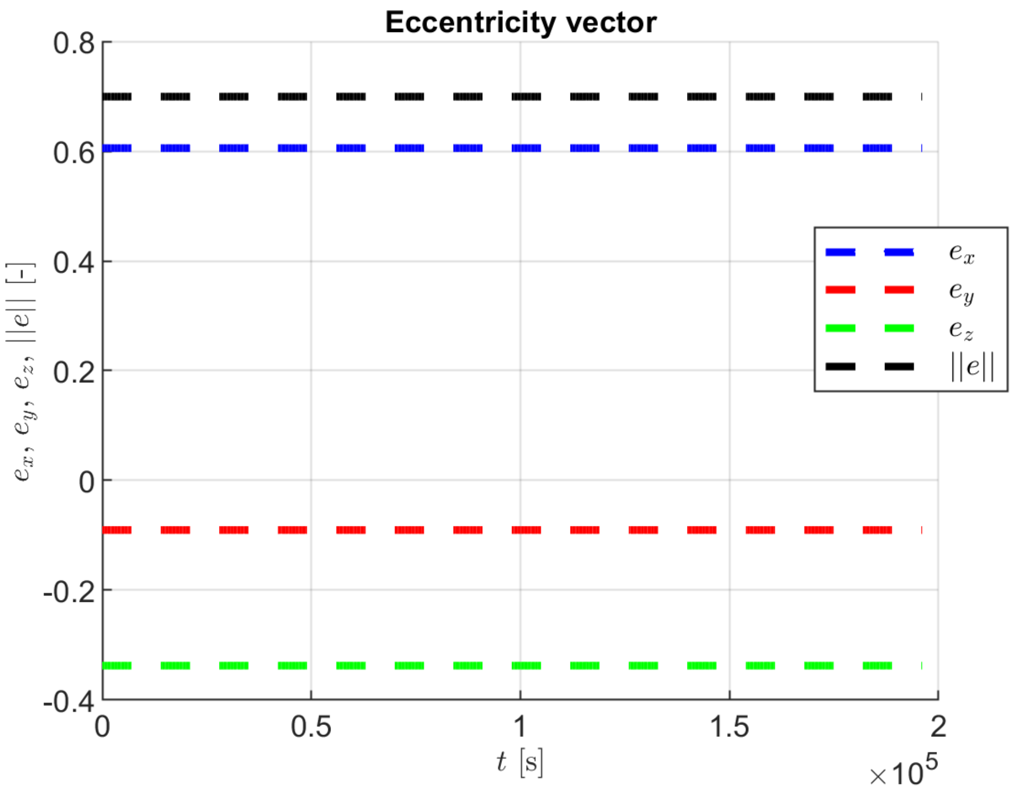


Angular momentum



Eccentricity vector

Exercise 1: Two-body problem

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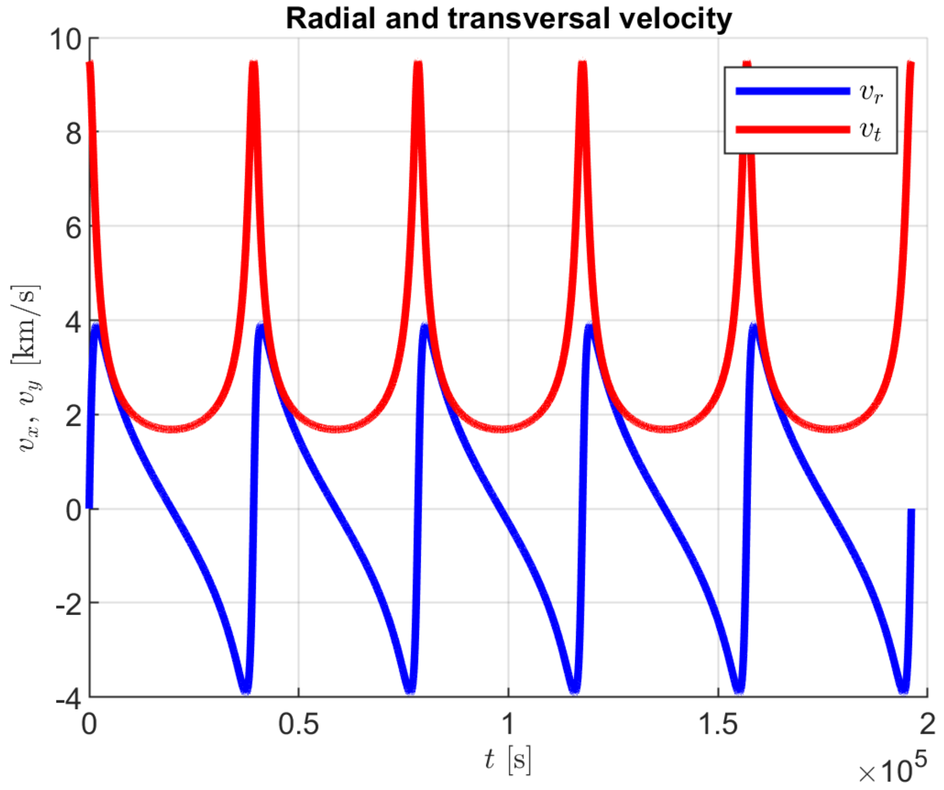
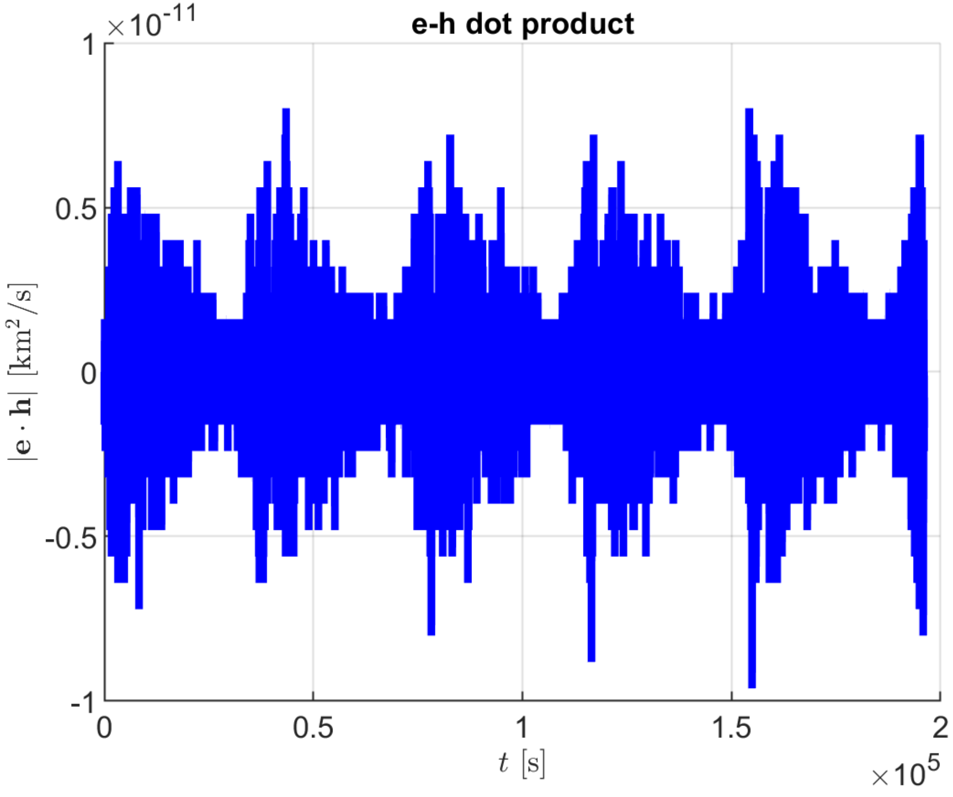


Radial and transversal velocity



Perpendicularity condition for **e** and **h**

Perturbed two-body problem

Non-spherical gravity field and other perturbations

- The two-body problem only accounts for **2 bodies with spherical gravity fields:**
  - In real life, the two bodies **will not be homogeneous spheres**
  - Many other perturbations will be present, depending on the orbital region: atmospheric drag, solar radiation pressure, gravitational attraction from other bodies, etc.
  - Spacecraft may perform manoeuvres using their propulsion systems

- Non-sphericity of Earth:
  - The Earth bulges out at the equator due to centrifugal forces, taking the form of an **oblate spheroid**
  - **Zonal variations:** the oblateness causes the gravitational field to depend not only on distance, but also on **latitude**. These **zonal variations** of the gravitational field can be expressed mathematically as a **series**, being the major contribution the **second zonal harmonic** $\boldsymbol{J_2}$

Perturbed two-body problem

Earth's oblateness – Effect of J_2

In Cartesian formulation ($\mathbf{r}$ and $\mathbf{v}$ in inertial reference frame), the perturbation due to the **second zonal harmonic J_2** can be expressed as:

$$\ddot{\mathbf{r}} = -\frac{\mu}{r^3}\mathbf{r} + \mathbf{a}_{J_2}$$

$$\mathbf{a}_{J_2} = \frac{3}{2}\frac{J_2 \mu R_e^2}{r^4}\left[\frac{x}{r}\left(5\frac{z^2}{r^2}-1\right)\hat{\mathbf{i}} + \frac{y}{r}\left(5\frac{z^2}{r^2}-1\right)\hat{\mathbf{j}} + \frac{z}{r}\left(5\frac{z^2}{r^2}-3\right)\hat{\mathbf{k}}\right]$$

where (values of R_e and J_2 taken from [1]):

- $R_e = 6378.137$ km is Earth's equatorial radius
- $J_2 = 0.00108263$
- $\mathbf{r} = x\,\hat{\mathbf{i}} + y\,\hat{\mathbf{j}} + z\,\hat{\mathbf{k}}$

[1] https://nssdc.gsfc.nasa.gov/planetary/factsheet/earthfact.html

Exercise 2: Perturbed two-body problem

Effect of J_2

Exercise 2: Earth orbit propagation with J_2

1. Modify the function from **Exercise 1** to include also the effect of J_2
   - Add physical parameters J_2 and R_e as inputs to the function
   - Remember that the value of $\mathbf{a}_{J_2}$ has to be recomputed at each time step

2. Repeat the analysis in point **3.** of **Exercise 1**, for initial conditions corresponding to Earth-bound elliptical orbits
   - Plot together and compare the results with and without J_2
   - Are all the conservation principles for the 2BP still valid for the orbit propagation including J_2?
   - Propagate for a 1-year time span and check how the differences between both models accumulate in time

Data

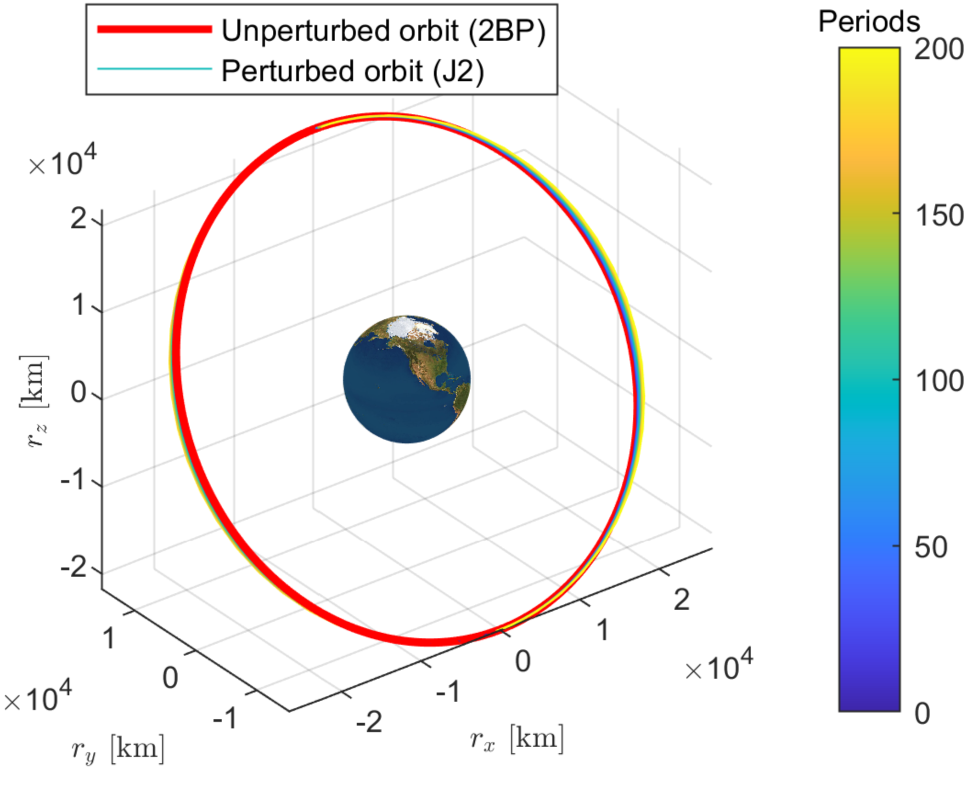
$\mu_\oplus$, R_e, and J_2 from `astroConstants.m`

Exercise 2: Perturbed two-body problem

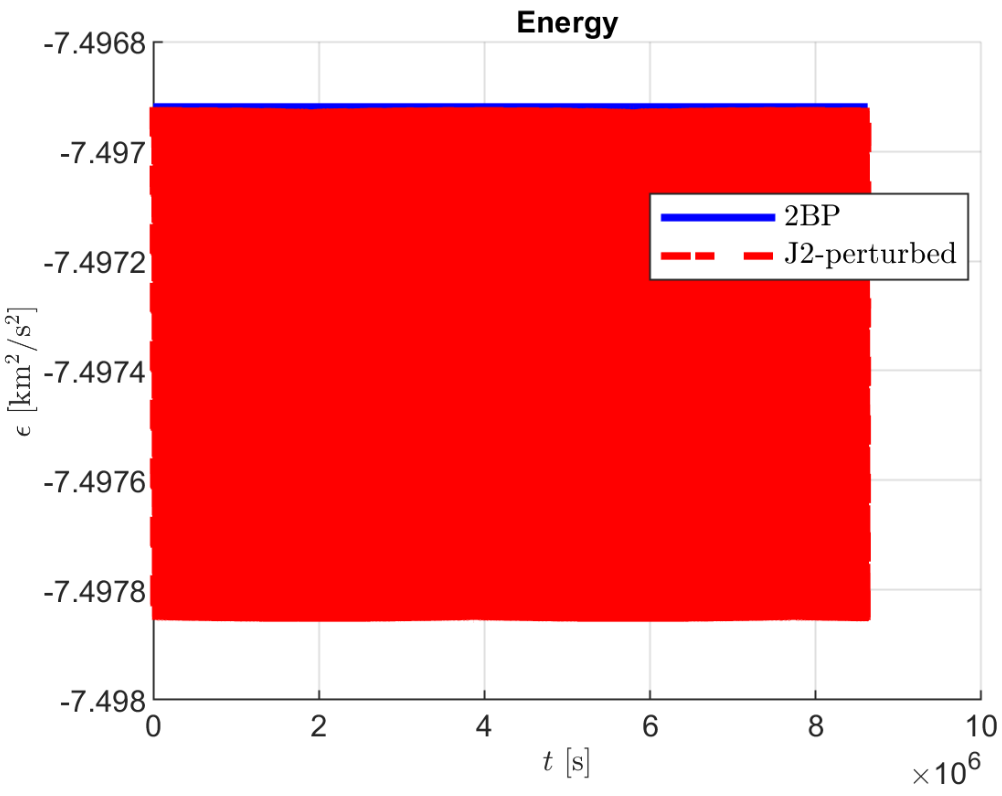
Sample solution: Quasi-circular Medium Earth Orbit

Parameters and initial condition

$$\mathbf{r}_0 = [\,26578.137, 0, 0\,]\ \text{km}$$
$$\mathbf{v}_0 = [\,0, 2.221, 3.173\,]\ \text{km/s}$$



Orbit representation



Specific energy

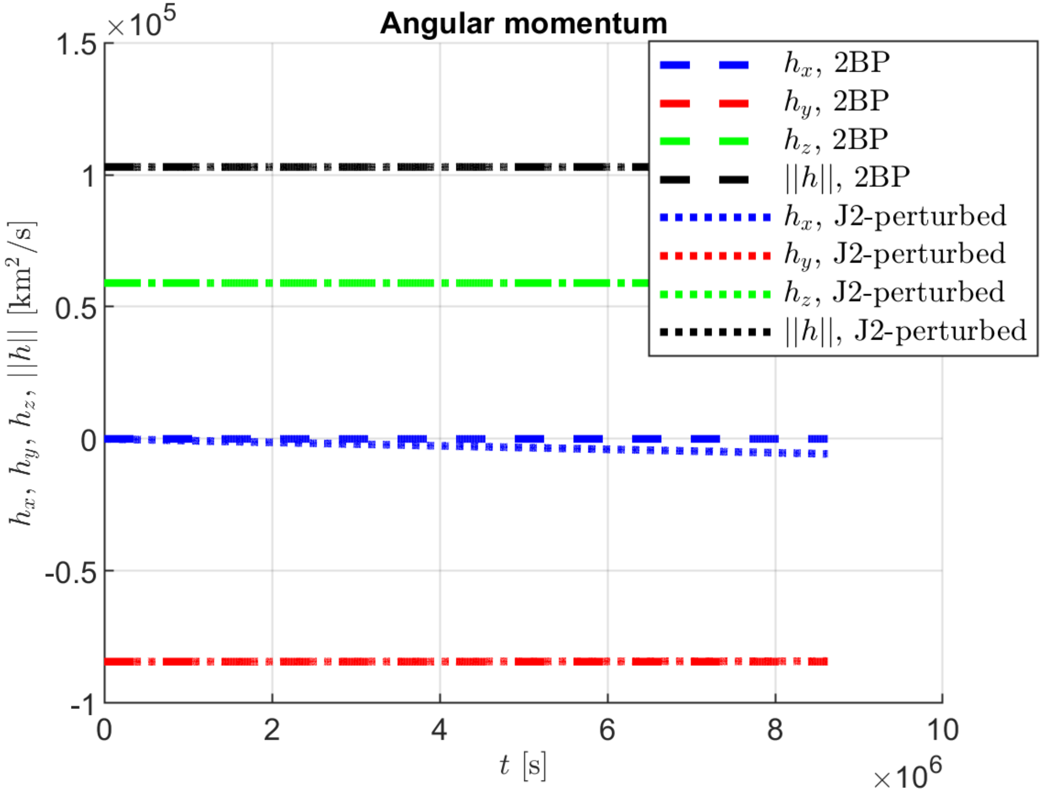
Exercise 2: Perturbed two-body problem

Sample solution: Quasi-circular Medium Earth Orbit

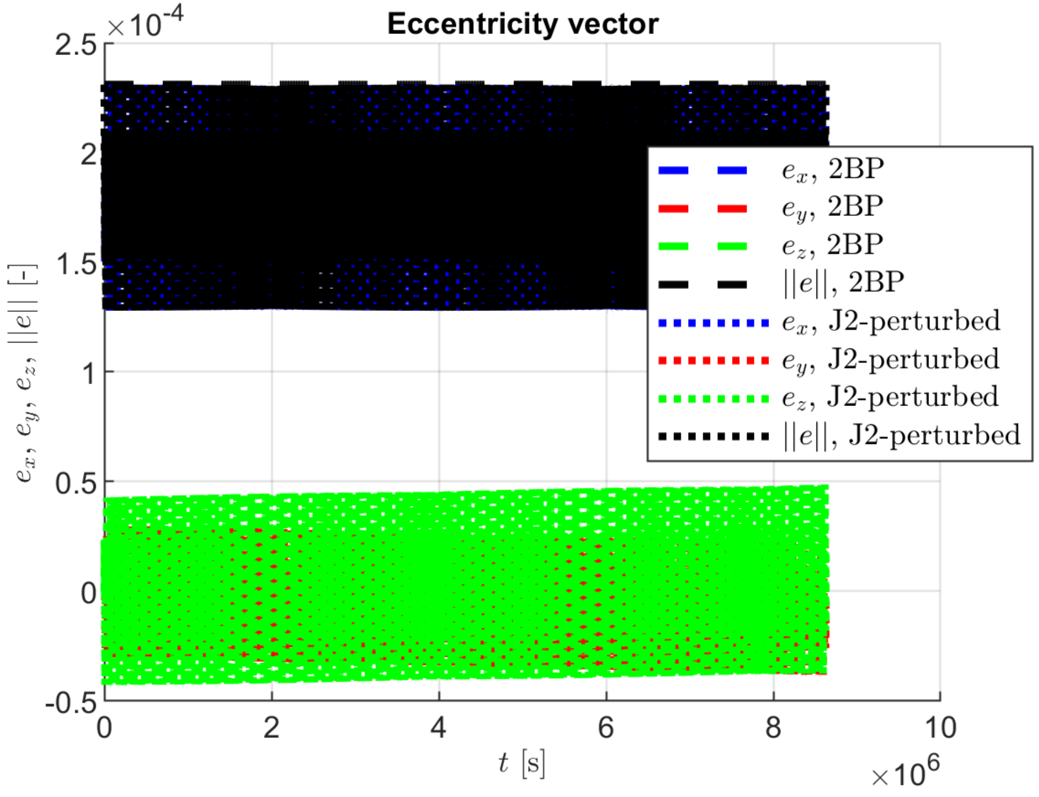
Parameters and initial condition

$$\mathbf{r}_0 = [\,26578.137, 0, 0\,]\ \text{km}$$

$$\mathbf{v}_0 = [\,0, 2.221, 3.173\,]\ \text{km/s}$$



Angular momentum



Eccentricity vector

Exercise 2: Perturbed two-body problem

Sample solution: Quasi-circular Medium Earth Orbit

Parameters and initial condition

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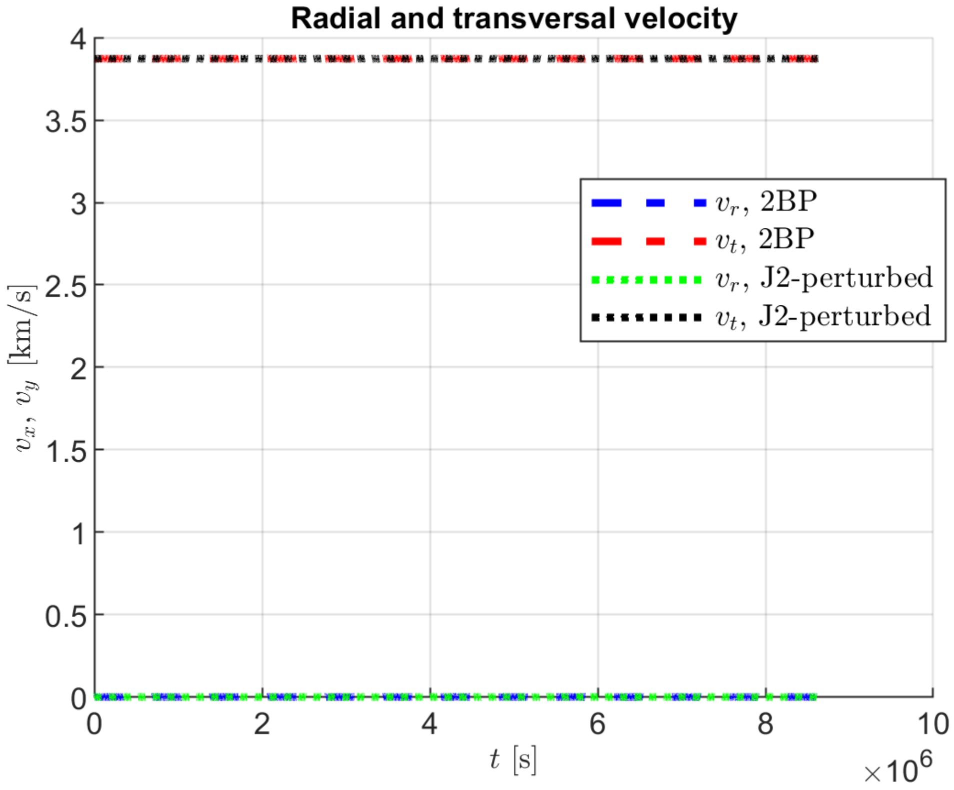
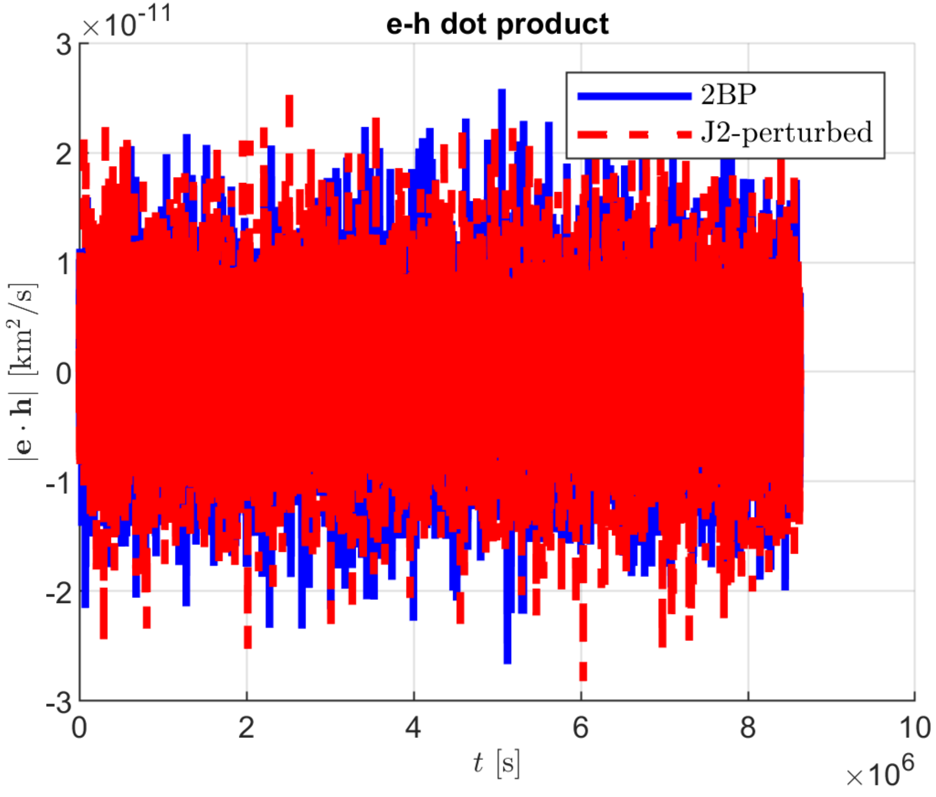
$$\mathbf{v}_0 = [\,0, 2.221, 3.173\,]\ \text{km/s}$$



Radial and transversal velocity



Perpendicularity condition for $\mathbf{e}$ and $\mathbf{h}$

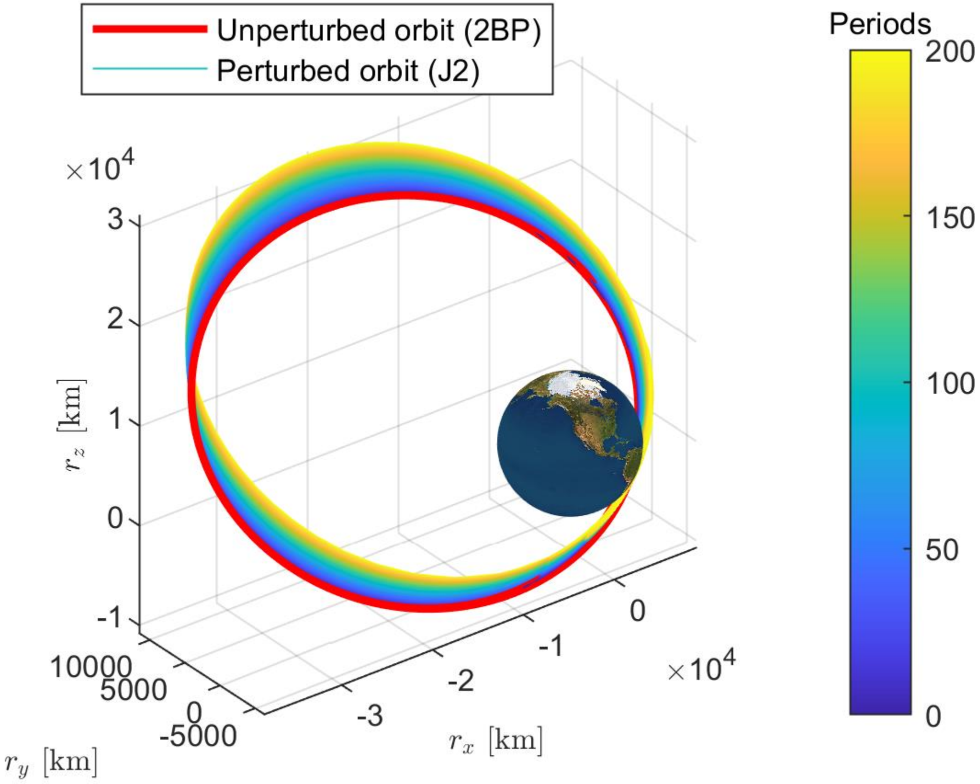
Exercise 2: Perturbed two-body problem

Sample solution: Highly eccentric and inclined orbit

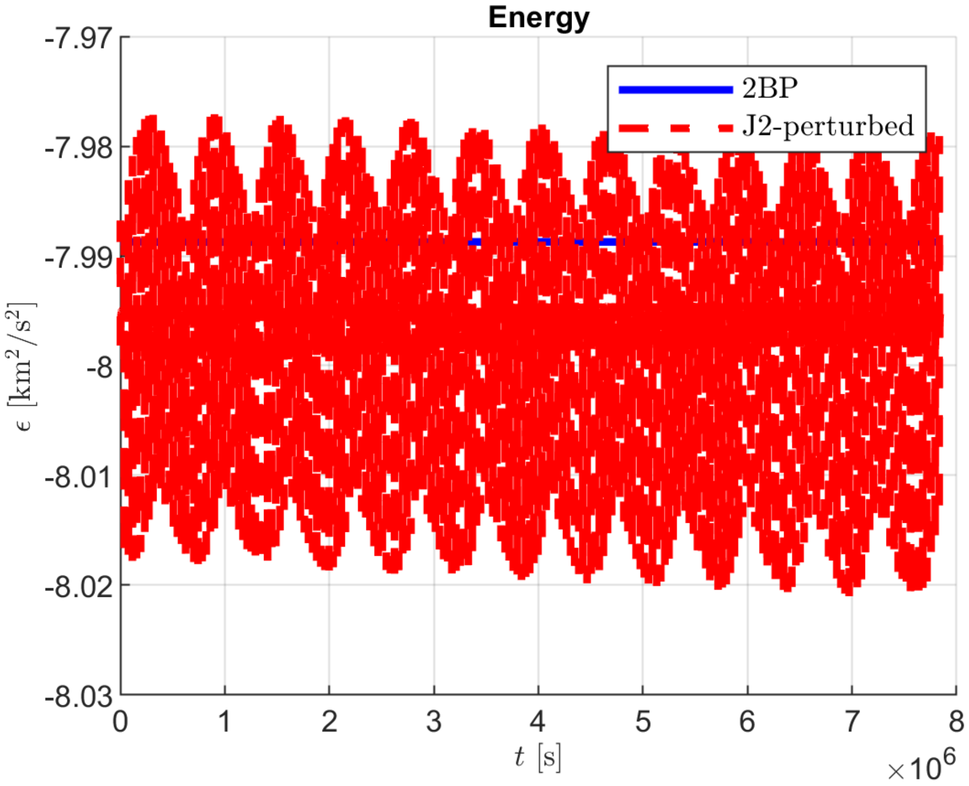
Parameters and initial condition

$$\mathbf{r}_0 = [6495, -970, -3622] \text{ km}$$

$$\mathbf{v}_0 = [4.752, 2.130, 7.950] \text{ km/s}$$



Orbit representation



Specific energy

Exercise 2: Perturbed two-body problem

Sample solution: Highly eccentric and inclined orbit

Parameters and initial condition

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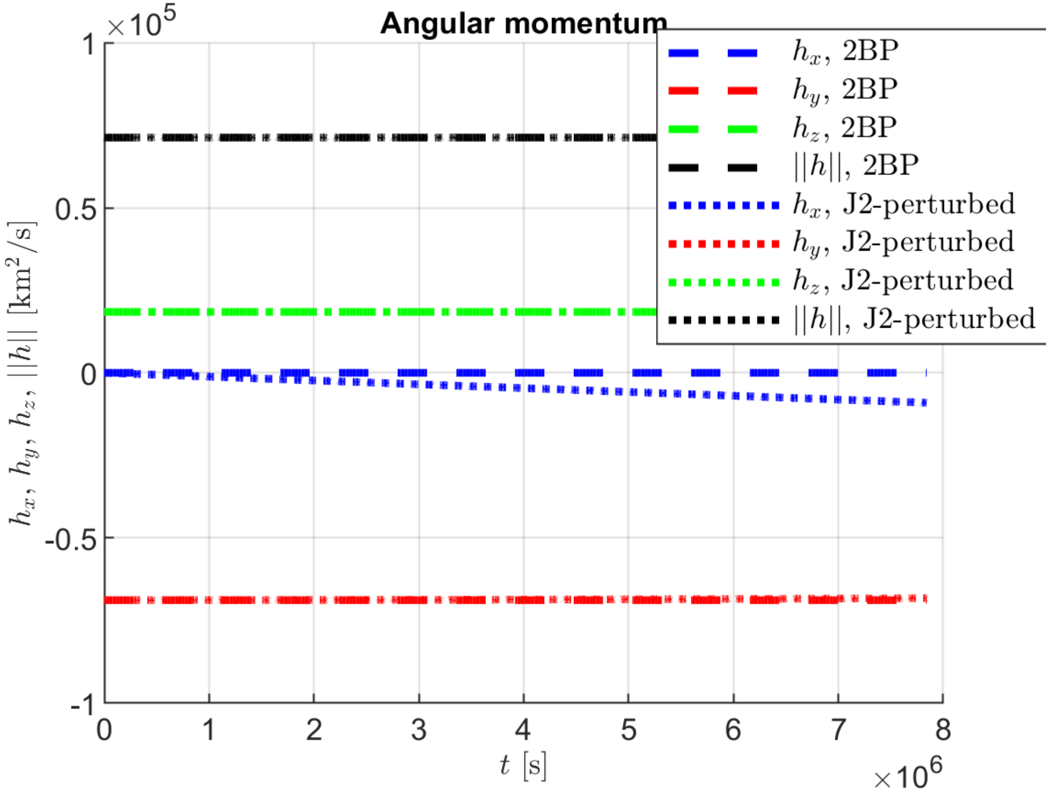
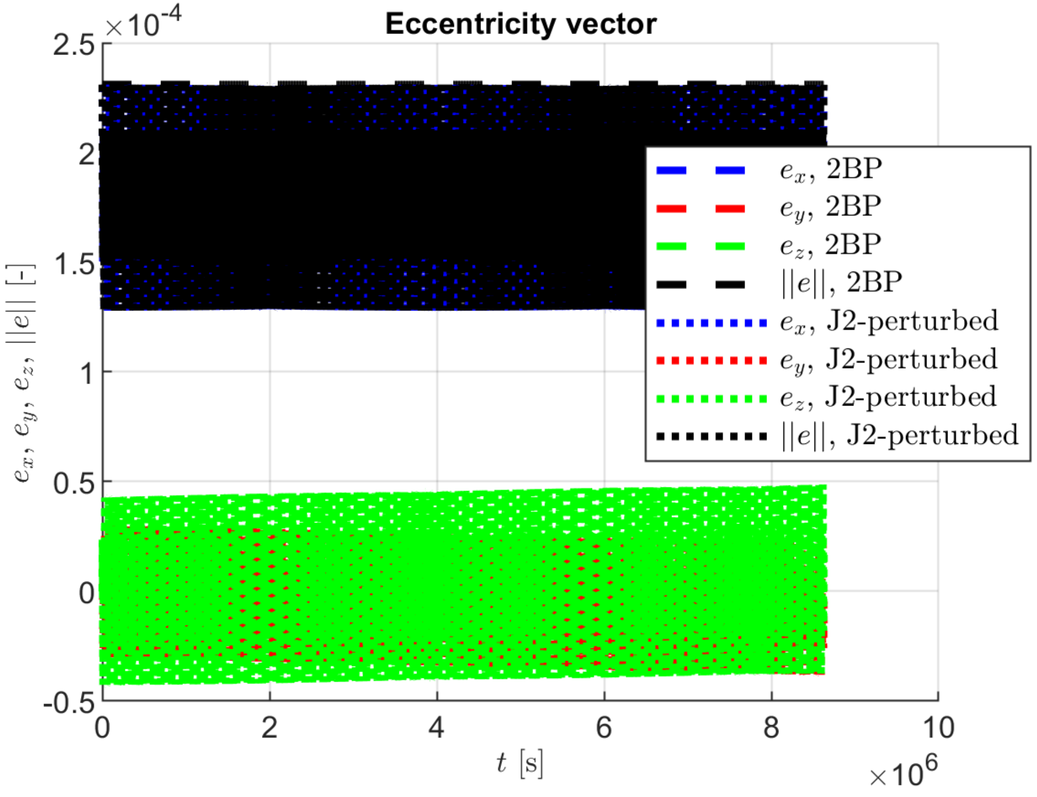


Angular momentum



Eccentricity vector

Exercise 2: Perturbed two-body problem

Sample solution: Highly eccentric and inclined orbit

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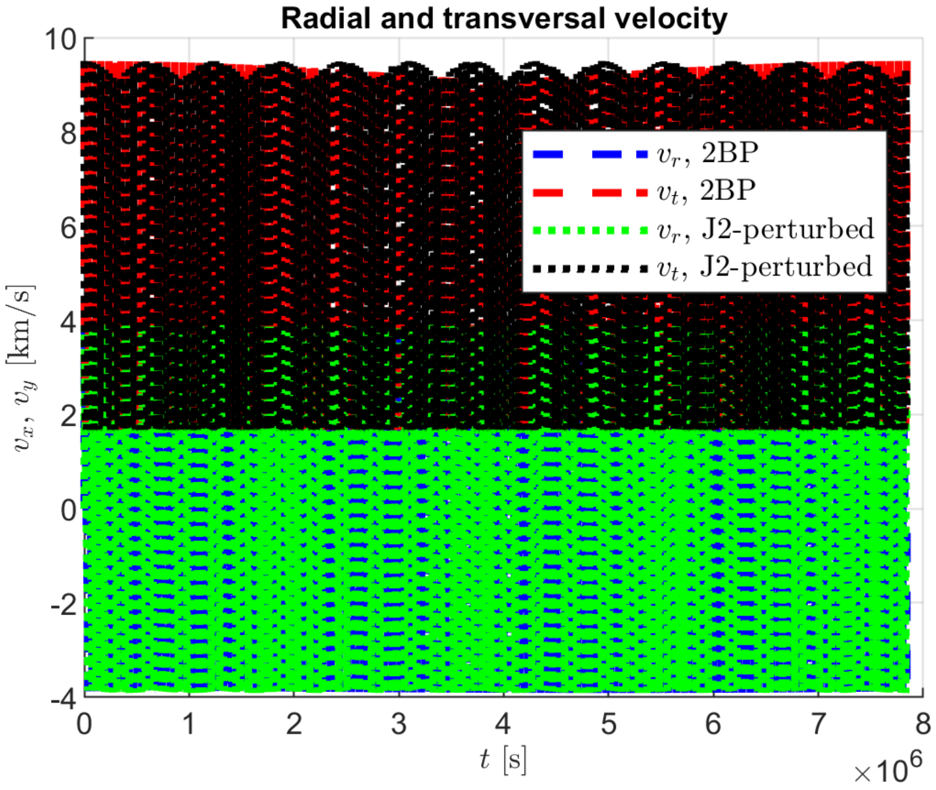
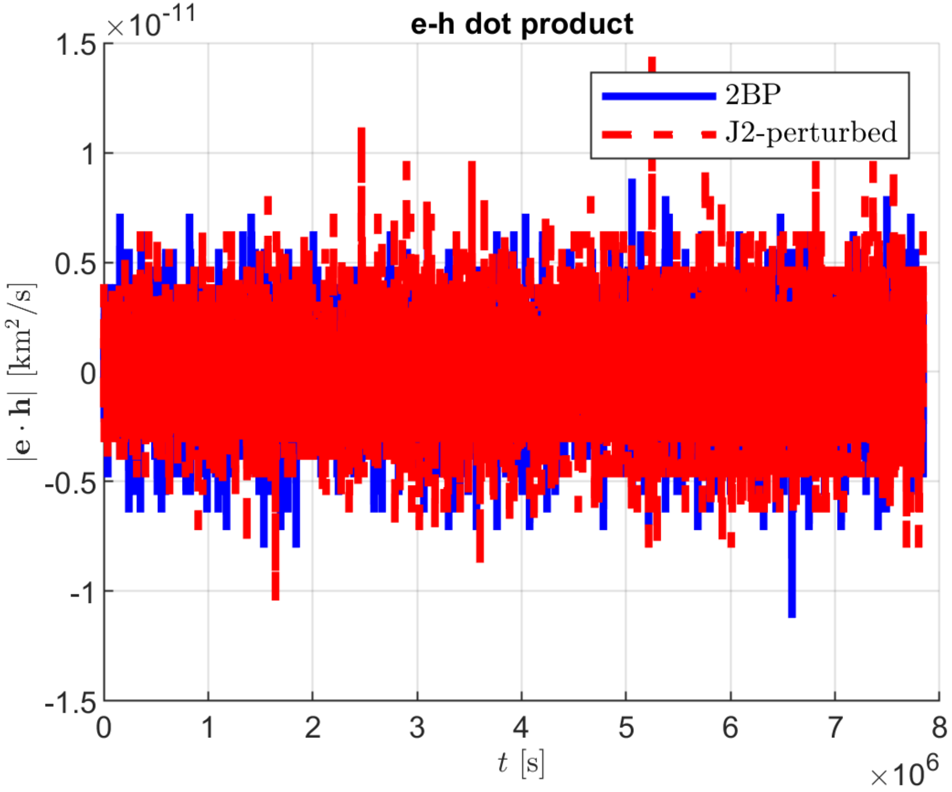


Radial and transversal velocity



Perpendicularity condition for $\mathbf{e}$ and $\mathbf{h}$

ROOT FINDING

Root-finding

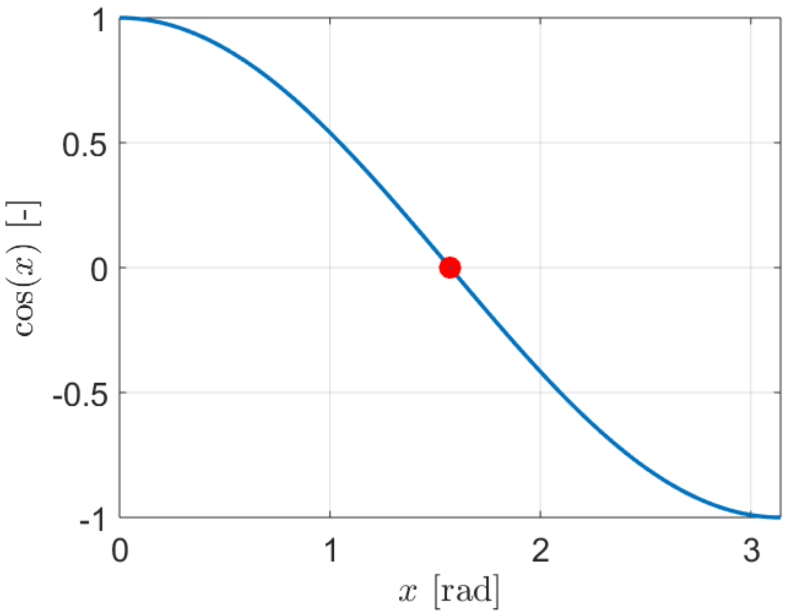
A basic problem in engineering

A root (or zero) of a function $\boldsymbol{F}(\boldsymbol{x})$ is a number $\boldsymbol{x_0}$ such that:

$$\boldsymbol{F}(\boldsymbol{x_0}) = \boldsymbol{0}$$

$\boldsymbol{F}$ and $\boldsymbol{x_0}$ can be scalars or vectors of the same dimension

- Root-finding is a very common problem in engineering
  - A classic Orbital Mechanics example is **solving Kepler's equation**
- There are many root-finding algorithms:
  - Bisection
  - Secant
  - Regula falsi
  - Newton's method
  - etc.



Root-finding

MATLAB's root-finding algorithms

- **`fsolve`**
  - Can solve systems of non-linear equations
  - Includes several algorithms to choose from
  - Very robust and versatile

- **`fzero`**
  - Works only with scalar equations
  - Combines bisection, secant, and inverse quadratic interpolation methods
  - Can be faster than **`fsolve`**, especially if bounds for $\boldsymbol{x_0}$ are provided

When the change of the function value in the last iteration is smaller in magnitude than `TolFun` or `FunctionTolerance` (default `1e-6`)

When the the relative change of $\boldsymbol{x}$ in the last iteration is smaller in magnitude than `TolX` (default `2e-16`)

Termination Condition

Read the documentation pages on **`fsolve`** and **`fzero`** to learn how to use them!

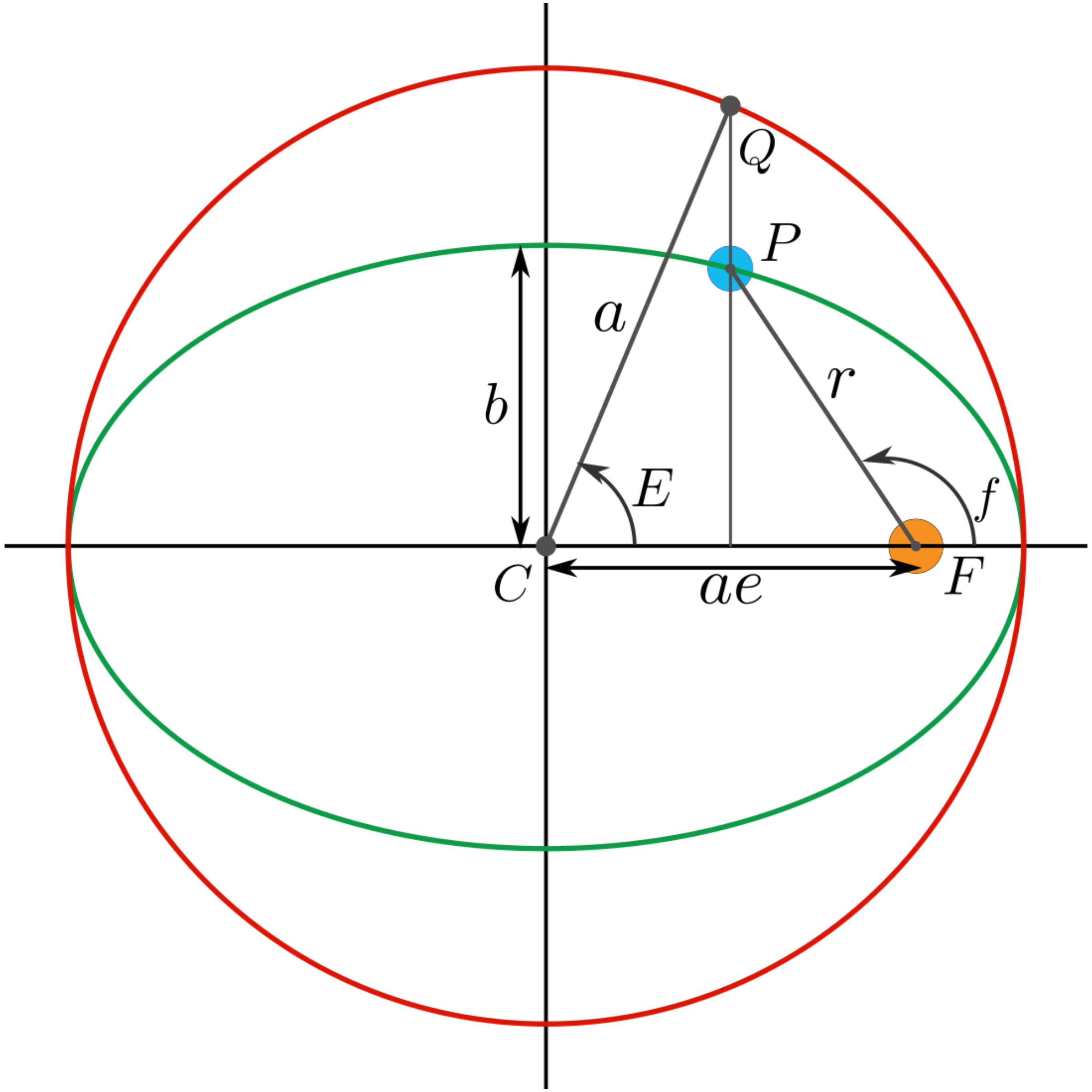
Both require as inputs the function to be solved and an initial guess for the solution

- **Anonymous functions** can be helpful

Exercise 3: Kepler's equation

Angles in the 2BP

- **True anomaly f**: angular position of body P, measured from the direction of periapsis.
- **Eccentric anomaly E**: angular position of point Q, which is the projection (perpendicular to the semimajor axis) of P onto a **circle** with radius equal to the semimajor axis of the ellipse and same centre C. The relation between f and E is purely geometrical.
- **Mean anomaly M** (not in the figure): angular position of a *fictitious body* orbiting the **circular orbit**, with constant angular velocity and the same period as the real orbit.
  M is measured over the same circle as E, but they differ because E does not have constant velocity (it follows P).



Exercise 3: Kepler's equation

Kepler's equation: time law for the 2BP

Kepler's equation provides a relation between E and M, depending on the orbit's eccentricity e. For elliptical orbits (i.e., with $e < 1$):

$$M = E - e \sin E \ .$$

M is related to time t through the mean motion n (M_0 is the mean anomaly at reference time t_0):

$$M = M_0 + n(t - t_0) = M_0 + \sqrt{\frac{\mu}{a^3}}(t - t_0) \ ,$$

E is related to true anomaly f (geometrical relations):

$$\cos f = \frac{\cos E - e}{1 - e \cos E} \ , \quad \tan \frac{f}{2} = \sqrt{\frac{1+e}{1-e}} \tan \frac{E}{2} \ .$$

Therefore, **Kepler's equation gives us the angular position of the orbiting body (true anomaly) as an implicit function of time (time law for the Keplerian orbit).**

Exercise 3: Kepler's equation

Handling multiple revolutions

- If $M < 0$ or $M \geq 2\pi$ radians, we may want to account for the **number of revolutions**
  - This is straightforward, because a complete revolution in M also corresponds to a complete revolution in E and f

- A simple algorithm to do this:
  1. Reduce M to $\overline{M} \in [0,2\pi]$ rad plus a whole number of revolutions $k \in \mathbb{Z}$, so that $M = \overline{M} + k2\pi$
     - **Hint:** Take a look at functions like `floor`, `ceil`, `fix`, `mod`, `wrapTo2Pi`, `wrapToPi`.
  2. Solve Kepler's equation for $\overline{M}$
  3. Compute the corresponding true anomaly $\overline{f} \in [0,2\pi]$ rad
  4. Add the number of complete revolutions k, $f = \overline{f} + k2\pi$

Exercise 3: Kepler's equation

Implement a solver for Kepler's equation

Exercise 3a: **Implement a solver for Kepler's equation**

- **Inputs:**
  - Time t
  - eccentricity e
  - semimajor axis a
  - gravitational parameter of the primary μ
  - reference initial time t_0 and true anomaly f_0
- **Outputs**:
  - true anomaly f
- Try and compare using both `fsolve` and `fzero`
- Be very careful with the ranges of the angles to prevent discontinuities

Exercise 3: Kepler's equation

Hints

- Read the documentation pages for `fsolve` and `fzero`
  - You may need to set the tolerance (error) for the solution
- You can use an anonymous function to pass the function to be solved to `fsolve` and `fzero`
- A good initial guess for the root is [2]: $E_{\text{guess}} = M + \frac{e \sin M}{1 - \sin(M+e) + \sin M}$
- You can add additional inputs as you see fit
  - It is possible to call a function without passing all the inputs. You can check the number of inputs passed using `nargin`. This way, you can program a function to have some inputs as 'optional'. Example:

```
% Seventh input is the tolerance 'tol', treated as optional
if nargin<7
    tol = 1e-6; % Default value used if input not given
end
```

[2] Battin, R., *An Introduction to the Mathematics and Methods of Astrodynamics*, AIAA Education Series, 1999

Exercise 3: Kepler's equation

Plot the evolution of true anomaly with time

Exercise 3b: Plot $f(t)$ for different orbits

1. Create a function that computes the true anomaly for an orbit with fixed e, a, and μ, for an array of times with N-points covering k periods of the orbit
   - This new function can reuse the function you created in **Exercise 3a**
   - You can create arrays of uniformly distributed points in MATLAB using the **:** operator (colon) or the `linspace` function
   - The period of an orbit is $T = 2\pi/n$, where n is the mean motion
2. Compute $f(t)$ for the following values:
   - $a = 7000$ km
   - μ_{Earth} from `astroConstants.m`
   - $f_0(t_0 = 0) = 0$ rad
   - $k = 2, N \geq 100$
   - Six different eccentricities: $e = 0, 0.2, 0.4, 0.6, 0.8, 0.95$

Exercise 3: Kepler's equation

Plot the evolution of true anomaly with time

Exercise 3b: **Plot $f(t)$ for different orbits**

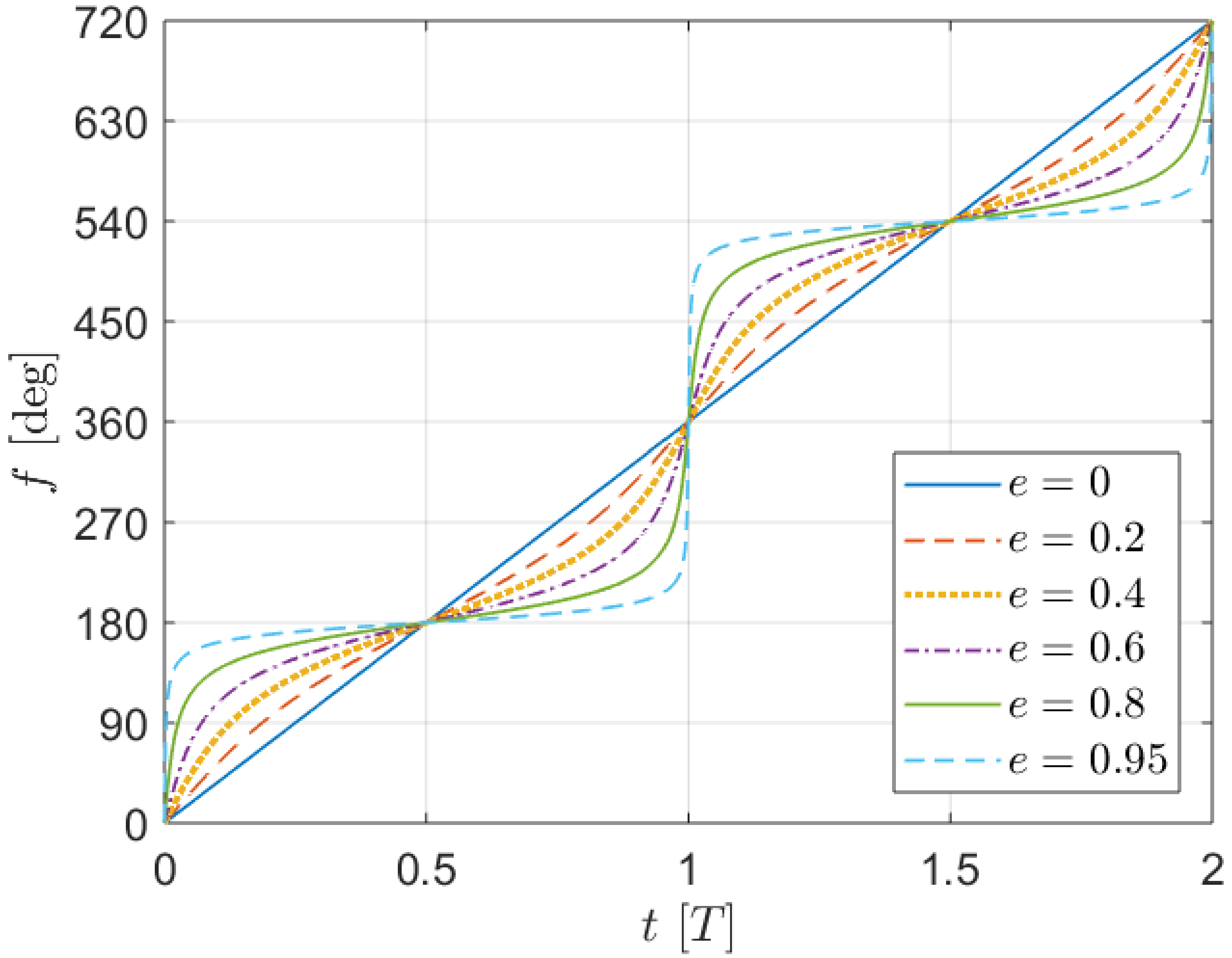
3. Plot $f(t; e)$ for all the cases in **2.** in a single 2D plot, using the `plot` command
   - Remember to set the axes and other properties of the plot
   - Hint: You can plot several lines at the same time with a single `plot` command, or you can use `hold on` to add them one by one

4. Plot $f(t; e)$ as a 3D surface using the `surf` command
   - Represent time in the x axis, eccentricity in the y axis, and true anomaly in the z axis. Use the **documentation center** to check the correct way of passing these inputs to `surf`
   - You can compute $f(t; e)$ for additional values of e to get a smoother plot

5. Compare with the results obtained from the numerical propagation of the 2BP in **Exercise 2**

Exercise 3: Kepler's equation

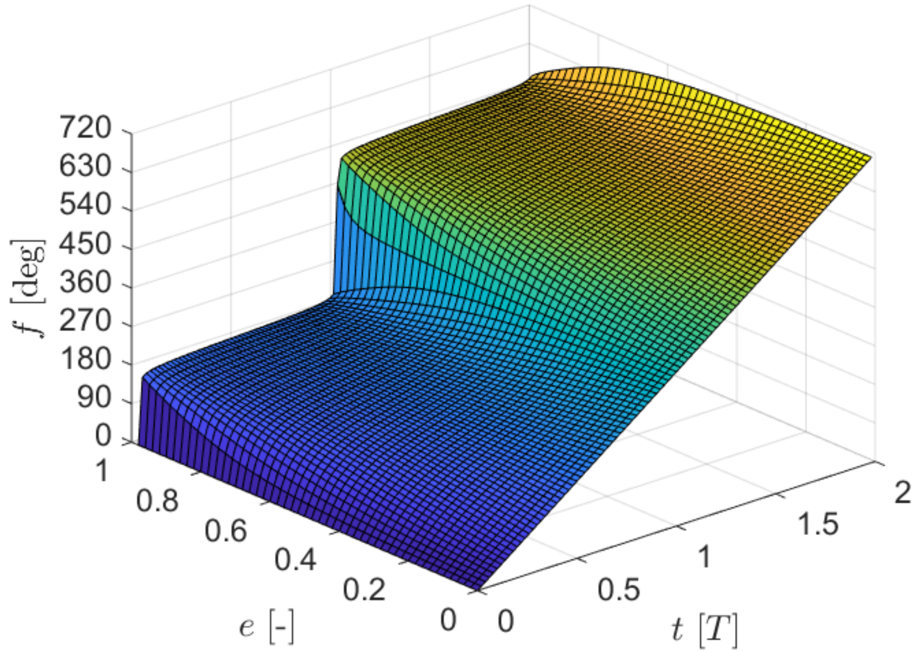
Solution: 2D plot

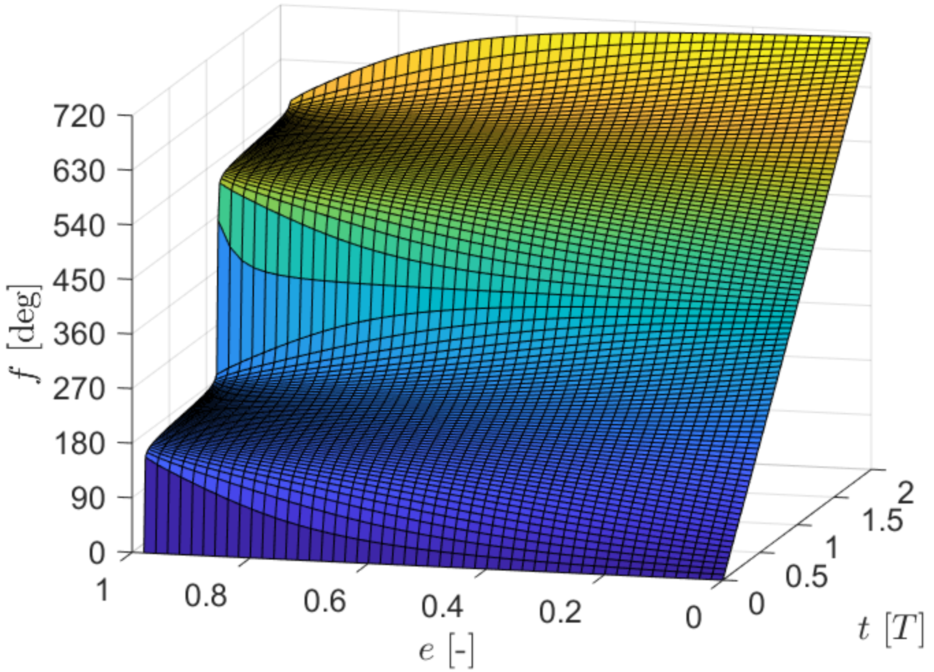


Evolution of true anomaly with time for $a = 7000$ km, Earth as primary body, and different eccentricities

Exercise 3: Kepler’s equation

Solution: surface plot





Evolution of true anomaly with time and eccentricity for
$a = 7000$ km and Earth as primary body