ASEN 6080: Statistical Orbit Determination
Homework 3 Supplemental Materials

Dr. Takahashi*

State Noise Compensation

We will develop a simple state noise compensation algorithm. This algorithm adds process noise (unmodeled acceleration, or stochastic acceleration) to the acceleration equations of the system, which results in an additional term being added to the time update of the estimation error covariance matrix for the Kalman filter.

General Discussion

The process noise is added to the differential equation of motion given by

$$\dot{\mathbf{X}} = \mathbf{F}(\mathbf{X}, \mathbf{u}, t) = \mathbf{F}(\mathbf{X}, t) + \mathbf{B}(t)\mathbf{u}(t) \tag{1}$$

where $\mathbf{u}(t)$ is white Gaussian process noise with

$$E[\mathbf{u}(t)] = \mathbf{0} \tag{2}$$

$$E[\mathbf{u}(t_i)\mathbf{u}^T(t_j)] = E[\Delta\mathbf{u}(t_i)\Delta\mathbf{u}^T(t_j)] = \tilde{\mathbf{Q}}(t_i)\delta_{ij} \tag{3}$$

where δ_{ij} is the Kronecker delta. Note that, in Equation 3, $\tilde{\mathbf{Q}}(t)$ is the *continuous* process noise covariance. The Kronecker delta only activates it at $t_i = t_j$.

The matrix, $\mathbf{B}$, maps the process noise into the state derivatives. Although it is not necessary, we will assume for simplicity that $\tilde{\mathbf{Q}}(t_i)$ is constant.

By expanding Equation 1 in a Taylor series about the reference, we get

$$\begin{aligned}
\dot{\mathbf{X}} + \Delta\dot{\mathbf{X}} &= \mathbf{F}(\mathbf{X} + \Delta\mathbf{X}, \mathbf{u} + \Delta\mathbf{u}, t) \\
&= \mathbf{F}(\mathbf{X}, t) + \left.\frac{\partial \mathbf{F}}{\partial \mathbf{X}}\right|_{\mathbf{X}(t)} \Delta\mathbf{X} + \mathbf{B}(t)(\mathbf{u}(t) + \Delta\mathbf{u}(t))
\end{aligned} \tag{4}$$

Using Equation 1, Equation 4 becomes

$$\Delta\dot{\mathbf{X}} = \mathbf{A}(t)\Delta\mathbf{X} + \mathbf{B}(t)\Delta\mathbf{u}(t) \tag{5}$$

*The original document was prepared by Dr. Born. I edited it for clarity.

where, of course, the $\mathbf{A}(t)$ matrix is defined as $\left.\frac{\partial \mathbf{F}}{\partial \mathbf{X}}\right|_{\mathbf{X}(t)}$. Equation 5 is the state deviation propagation equation including process noise for a linear system. Equation 5 has the general solution below obtained by the variation of parameters:

$$\Delta \mathbf{X}(t) = \mathbf{\Phi}(t, t_0)\Delta \mathbf{X}(t_0) + \int_{t_0}^{t} \mathbf{\Phi}(t, \tau)\mathbf{B}(t)\Delta \mathbf{u}(\tau) d\tau \tag{6}$$

It is evident that Equation 6 contains the deterministic term and the random but bounded stochastic term. From Equation 3 and 6, we can obtain the time update for the covariance:

$$\begin{aligned} \mathbf{P}_{i+1}^{-} &= \mathbf{\Phi}(t_{i+1}, t_i)\mathbf{P}_i^{+}\mathbf{\Phi}(t_{i+1}, t_i)^T + \int_{t_i}^{t_{i+1}} \mathbf{\Phi}(t_{i+1}, \tau)\mathbf{B}(\tau)\tilde{\mathbf{Q}}\mathbf{B}(\tau)^T\mathbf{\Phi}(t_{i+1}, \tau)^T d\tau \\ &= \mathbf{\Phi}(t_{i+1}, t_i)\mathbf{P}_i^{+}\mathbf{\Phi}(t_{i+1}, t_i)^T + \mathbf{Q}(t_{i+1}, t_i) \end{aligned} \tag{7}$$

where we replaced the quadrature in the second term with the *discrete* process noise covariance $\mathbf{Q}(t_{i+1}, t_i)$. Equation 7 states that the deterministic and random components are orthogonal and each uncertainty contribution RSS (root-sum-square) to yield the total uncertainty.

Assuming constant $\mathbf{u}$ between t_{i+1} and t_i in Equation 6, Equation 7 simplifies to yield

$$\mathbf{P}_{i+1}^{-} \approx \mathbf{\Phi}(t_{i+1}, t_i)\mathbf{P}_i^{+}\mathbf{\Phi}(t_{i+1}, t_i)^T + \mathbf{\Gamma}(t_{i+1}, t_i)\tilde{\mathbf{Q}}\mathbf{\Gamma}(t_{i+1}, t_i)^T \tag{8}$$

where $\mathbf{\Gamma}(t_{i+1}, t_i)$ is given by

$$\mathbf{\Gamma}(t_{i+1}, t_i) = \int_{t_i}^{t_{i+1}} \mathbf{\Phi}(t_{i+1}, \tau)\mathbf{B}(\tau) d\tau \tag{9}$$

In the sections below, we will derive the expressions for $\mathbf{Q}(t_{i+1}, t_i)$. To summarize, the relationship between the *discrete* process noise covariance and the *continuous* process noise covariance is

$$\mathbf{Q}(t_{i+1}, t_i) = \int_{t_i}^{t_{i+1}} \mathbf{\Phi}(t_{i+1}, \tau)\mathbf{B}(\tau)\tilde{\mathbf{Q}}\mathbf{B}(\tau)^T\mathbf{\Phi}(t_{i+1}, \tau)^T d\tau \approx \mathbf{\Gamma}(t_{i+1}, t_i)\tilde{\mathbf{Q}}\mathbf{\Gamma}(t_{i+1}, t_i)^T \tag{10}$$

Simplified Dynamics for SNC

We assume that the process noise is only being added to the acceleration components of the state. For simplicity, we only work with the spacecraft state in the state vector. Thus, we have

$$\dot{\mathbf{X}} = \begin{bmatrix} \mathbf{V} \\ \\ \ddot{\mathbf{r}} \end{bmatrix} \tag{11}$$

$$\mathbf{u} = \begin{bmatrix} u_{\ddot{x}} \\ u_{\ddot{y}} \\ u_{\ddot{z}} \end{bmatrix} \tag{12}$$

$$\mathbf{B} = \begin{bmatrix} 0_{3\times 3} \\ 1_{3\times 3} \end{bmatrix} \tag{13}$$

In our case $\mathbf{B}(\tau)$ is constant (Equation 13) and $\mathbf{\Phi}(t_{i+1}, \tau)$ is given as follows:

$$\mathbf{\Phi}(t_{i+1}, \tau) = \begin{bmatrix} \frac{\partial \mathbf{r}(t_{i+1})}{\partial \mathbf{r}(\tau)} & \frac{\partial \mathbf{r}(t_{i+1})}{\partial \dot{\mathbf{r}}(\tau)} \\ \frac{\partial \dot{\mathbf{r}}(t_{i+1})}{\partial \mathbf{r}(\tau)} & \frac{\partial \dot{\mathbf{r}}(t_{i+1})}{\partial \dot{\mathbf{r}}(\tau)} \end{bmatrix} = \begin{bmatrix} \phi_1(\tau) & \phi_2(\tau) \\ \phi_3(\tau) & \phi_4(\tau) \end{bmatrix} \tag{14}$$

Thus, the integrand in Equation 9 becomes

$$\mathbf{\Phi}(t_{i+1}, \tau)\mathbf{B}(\tau) = \begin{bmatrix} \phi_1(\tau) & \phi_2(\tau) \\ \phi_3(\tau) & \phi_4(\tau) \end{bmatrix} \begin{bmatrix} 0_{3\times 3} \\ 1_{3\times 3} \end{bmatrix} = \begin{bmatrix} \phi_2(\tau) \\ \phi_4(\tau) \end{bmatrix} \tag{15}$$

Using Equation 9 and 15, one can carry out the integration as a quadrature. Alternatively one can approximate the values of $\phi_2(\tau)$ and $\phi_4(\tau)$ to carry out the integration analytically. We will do the latter.

We approximate the motion of the spacecraft as the rectilinear motion assuming a small time interval between t_{i+1} and τ. Note that, as shown in Equation 14, the reference epoch for the STM is τ. Thus,

$$\mathbf{X}(t_{i+1}) \approx \mathbf{X}(\tau) + \dot{\mathbf{X}}(\tau)(t_{i+1} - \tau) \tag{16}$$

which yields

$$\frac{\partial \mathbf{X}(t_{i+1})}{\partial \mathbf{X}(\tau)} \approx 1_{6\times 6} \tag{17}$$

$$\frac{\partial \mathbf{X}(t_{i+1})}{\partial \dot{\mathbf{X}}(\tau)} \approx (t_{i+1} - \tau)1_{6\times 6} \tag{18}$$

Thus, we get

$$\phi_4(\tau) = \frac{\partial \mathbf{r}(t_{i+1})}{\partial \dot{\mathbf{r}}(\tau)} \approx 1_{3\times 3} \tag{19}$$

$$\phi_2(\tau) = \frac{\partial \dot{\mathbf{r}}(t_{i+1})}{\partial \dot{\mathbf{r}}(\tau)} \approx (t_{i+1} - \tau)1_{3\times 3} \tag{20}$$

Now, we have two ways to solve for the process noise contribution to the covariance in Equation 7. We can work with Equation 7 directly or solve for $\mathbf{\Gamma}(t_{i+1}, t_i)$ in Equation 9 and substitute it into Equation 8.

SNC Time Update - Direct Integration

Here, we attempt to directly integrate the stochastic contribution in Equation 7. Based on the simplified dynamics/STM above, $\mathbf{Q}(t_{i+1}, t_i)$ becomes

$$\begin{aligned}
\mathbf{Q}(t_{i+1}, t_i) &= \int_{t_i}^{t_{i+1}} \mathbf{\Phi}(t_{i+1}, \tau)\mathbf{B}(\tau)\tilde{\mathbf{Q}}\mathbf{B}(\tau)^T \mathbf{\Phi}(t_{i+1}, \tau)^T d\tau \\
&= \int_{t_i}^{t_{i+1}} \begin{bmatrix} \phi_2(\tau)\tilde{\mathbf{Q}}\phi_2(\tau)^T & \phi_2(\tau)\tilde{\mathbf{Q}}\phi_4(\tau)^T \\ \phi_4(\tau)\tilde{\mathbf{Q}}\phi_2(\tau)^T & \phi_4(\tau)\tilde{\mathbf{Q}}\phi_4(\tau)^T \end{bmatrix} d\tau \\
&\approx \int_{t_i}^{t_{i+1}} \begin{bmatrix} (t_{i+1}-\tau)^2\tilde{\mathbf{Q}} & (t_{i+1}-\tau)\tilde{\mathbf{Q}} \\ (t_{i+1}-\tau)\tilde{\mathbf{Q}} & \tilde{\mathbf{Q}} \end{bmatrix} d\tau \\
&= \begin{bmatrix} -\frac{1}{3}\left[(t_{i+1}-\tau)^3\right]_{t_i}^{t_{i+1}}\tilde{\mathbf{Q}} & -\frac{1}{2}\left[(t_{i+1}-\tau)^2\right]_{t_i}^{t_{i+1}}\tilde{\mathbf{Q}} \\ -\frac{1}{2}\left[(t_{i+1}-\tau)^2\right]_{t_i}^{t_{i+1}}\tilde{\mathbf{Q}} & [\tau]_{t_i}^{t_{i+1}}\tilde{\mathbf{Q}} \end{bmatrix} d\tau \\
&= \begin{bmatrix} \frac{1}{3}(t_{i+1}-t_i)^3\tilde{\mathbf{Q}} & \frac{1}{2}(t_{i+1}-t_i)^2\tilde{\mathbf{Q}} \\ \frac{1}{2}(t_{i+1}-t_i)^2\tilde{\mathbf{Q}} & (t_{i+1}-t_i)\tilde{\mathbf{Q}} \end{bmatrix} \\
&= \Delta t \begin{bmatrix} \frac{1}{3}\Delta t^2\tilde{\mathbf{Q}} & \frac{1}{2}\Delta t\tilde{\mathbf{Q}} \\ \frac{1}{2}\Delta t\tilde{\mathbf{Q}} & \tilde{\mathbf{Q}} \end{bmatrix}
\end{aligned} \tag{21}$$

where $\Delta t = t_{i+1} - t_i$. Note that the correlation coefficient is larger than 1 in Equation 21. This is an unfortunate downside of approximating the dynamics.

SNC Time Update - Approximation

Next, we first solve for $\mathbf{\Gamma}(t_{i+1}, t_i)$ and substitute it into Equation 8. By direct substitution of Equation 19 and 20 into Equation 9, we get

$$
\Gamma(t_{i+1}, t_i) = \int_{t_i}^{t_{i+1}} \begin{bmatrix} \phi_2(\tau) \\ \phi_4(\tau) \end{bmatrix} d\tau = \begin{bmatrix} \frac{\Delta t^2}{2} & 0 & 0 \\ 0 & \frac{\Delta t^2}{2} & 0 \\ 0 & 0 & \frac{\Delta t^2}{2} \\ \Delta t & 0 & 0 \\ 0 & \Delta t & 0 \\ 0 & 0 & \Delta t \end{bmatrix} = \Delta t \begin{bmatrix} \frac{\Delta t}{2} 1_{3\times 3} \\ 1_{3\times 3} \end{bmatrix} \tag{22}
$$

Thus, we get

$$
\begin{aligned}
\mathbf{Q}(t_{i+1}, t_i) &\approx \Gamma(t_{i+1}, t_i)\tilde{\mathbf{Q}}\Gamma(t_{i+1}, t_i)^T \\
&= \Delta t^2 \begin{bmatrix} \frac{\Delta t^2}{4}\tilde{\mathbf{Q}} & \frac{\Delta t}{2}\tilde{\mathbf{Q}} \\ \frac{\Delta t}{2}\tilde{\mathbf{Q}} & \tilde{\mathbf{Q}} \end{bmatrix}
\end{aligned} \tag{23}
$$

The expressions in Equation 21 and 23 are similar but distinctively different. In particular, note that Equation 23 has one extra multiplication by Δt for all terms. Because we do not model or estimate the stochastic acceleration $\mathbf{u}(t)$ in the filter, it is oftentimes desirable to employ a more conservative (i.e., larger) process noise covariance. Therefore, we implement Equation 23 in this class. Of course, **do not add the process noise covariance when the data gap is large**. It will quickly overwhelm the deterministic uncertainty because of Δt. Similarly to Equation 21, Equation 23 has the correlation coefficient of 1 and is not positive definite.

Frames for the Process Noise

In general, we assume the diagonal covariance $\tilde{\mathbf{Q}} = diag[\sigma_{\ddot{x}}^2, \sigma_{\ddot{y}}^2, \sigma_{\ddot{z}}^2]$ because we do not know the correlations between the unmodeled acceleration. The values of each of σ in $\tilde{\mathbf{Q}}$ depends on the magnitude of the unmodeled acceleration, which is usually $5 \times 10^{-11} km/s^2$ or less if you have good dynamical models.

Note that in some situations it is desirable to express the covariance $\tilde{\mathbf{Q}}$ in another frame/set of coordinates (e.g., spacecraft body frame, RTN, spherical coordinates, etc) that is not in the inertial Cartesian frame. In this case, $\tilde{\mathbf{Q}}$ needs to be transformed into the inertial Cartesian frame as follows:

$$
\tilde{\mathbf{Q}}_{XYZ} = \mathbf{R}\tilde{\mathbf{Q}}_{\alpha\beta\gamma}\mathbf{R}^T \tag{24}
$$

where $\mathbf{R} = \dfrac{\partial(XYZ)}{\partial(\alpha\beta\gamma)}$. For example, if we were trying to compensate for atmospheric drag uncertainty, $\tilde{\mathbf{Q}}$ would primarily be along track. In the RTN frame one might assume that $\sigma_R^2 = \sigma_N^2 = 0$ and σ_T^2 corresponds to the uncertainty in the drag acceleration. In this case $\tilde{\mathbf{Q}}_{XYZ}$ may not be diagonal or constant, depending on the orbit geometry. The rotation matrix between the inertial frame and the RTN frame, and therefore Equation 24, needs to be evaluated at each time step t_i (not t_{i+1} because you are mapping $\tilde{\mathbf{Q}}$ from t_i to t_{i+1}) based on the nominal state.

Dynamic Model Compensation

The original document was prepared by D.R. Cruickshank in what was originally offered as ASEN6080: Statistical Orbit Determination II. The author's intent is to add more clarification and streamline their derivation processes in this document.

The fundamental assumption of Dynamic Model Compensation (DMC) is that the state dynamics of the system to be modeled and estimated are subject to stochastic accelerations that are beyond the scope of the explicit (i.e., known) dynamical models. The presence of the unmodeled stochastic acceleration is the same as SNC, but these unknown accelerations are modeled as an Ornstein-Uhlenbeck process, described by a stochastic Langevin equation, in DMC:

$$\dot{\mathbf{w}}(t) \quad = \quad -\mathbf{B_w}\mathbf{w}(t) + \mathbf{u}(t) \tag{25}$$

where $\mathbf{w}(t)$ is the three-vector of stochastic accelerations and $\mathbf{u}(t)$ is a Gaussian process that is uncorrelated in time (i.e., white noise). $\mathbf{u}(t)$ has a mean of zero and a constant covariance below:

$$\tilde{\mathbf{q}}_\mathbf{u} \quad = \quad \begin{bmatrix} \sigma_{ux}^2 & 0 & 0 \\ 0 & \sigma_{uy}^2 & 0 \\ 0 & 0 & \sigma_{uz}^2 \end{bmatrix} \tag{26}$$

$\mathbf{B_w}$ is a constant matrix that consists of the reciprocal of the time constants τ:

$$\mathbf{B_w} \quad = \quad \begin{bmatrix} \beta_x & 0 & 0 \\ 0 & \beta_y & 0 \\ 0 & 0 & \beta_z \end{bmatrix} \quad = \quad \begin{bmatrix} \frac{1}{\tau_x} & 0 & 0 \\ 0 & \frac{1}{\tau_y} & 0 \\ 0 & 0 & \frac{1}{\tau_z} \end{bmatrix} \tag{27}$$

$\mathbf{B}$ and $\tilde{\mathbf{q}}_\mathbf{u}$ are usually assumed to be diagonal with equal values for all three axes for convenience and ease of instruction, but these are not necessary restrictions. Taking these matrices to be diagonal is equivalent to assuming that the acceleration for a given axis is uncorrelated with the accelerations for the other two axes. Certainly, a spatially correlated acceleration model could be used, but this is seldom necessary and introduces unnecessary complication in the present context.

In DMC, we add the stochastic acceleration in Equation 25 to the spacecraft acceleration and estimate it. We construct the state vector as follows:

$$\mathbf{z}(t) \quad = \quad \begin{bmatrix} \mathbf{X}(t) \\ \\ \mathbf{w}(t) \end{bmatrix} \quad = \quad \begin{bmatrix} \mathbf{r}(t) \\ \\ \mathbf{V}(t) \\ \\ \mathbf{w}(t) \end{bmatrix} \tag{28}$$

$\mathbf{X}(t)$ can generally contain any other dynamic/bias parameters (e.g., GM, spherical harmonics, etc) but we ignore those terms because the STM of those parameters w.r.t. the stochastic acceleration $\mathbf{w}$ are zero. By expanding the equation of motion in Taylor series and only retaining the first order terms, we get the general expression for the time derivative of the state deviation:

$$\Delta\dot{\mathbf{z}}(t) = \mathbf{A_z}(t)\Delta\mathbf{z}(t) + \mathbf{G}(t)\Delta\mathbf{u}(t) \tag{29}$$

Equation 29 has the homogeneous solution $\Delta\mathbf{z}(t) = \mathbf{\Phi_z}(t, t_0)\Delta\mathbf{z}(t_0)$. Its general solution can be obtained by variation of parameters as follows:

$$\Delta \mathbf{z}(t) = \mathbf{\Phi}_{\mathbf{z}}(t, t_0)\Delta \mathbf{z}(t_0) + \int_{t_0}^{t} \mathbf{\Phi}_{\mathbf{z}}(t, s)\mathbf{G}(s)\Delta \mathbf{u}(s)ds \tag{30}$$

where s is the dummy variable for time. Equation 30 clearly has both a deterministic component and a random component. The deterministic component relates directly to the state deviation, and the stochastic accelerations further contribute to it in a random, but bounded manner just like SNC.

The form of the mapped covariance for DMC is the same as that for SNC, as shown below:

$$\mathbf{P}^{-}_{\mathbf{z},i+1} = \mathbf{\Phi}_{\mathbf{z}}(t_{i+1}, t_i)\mathbf{P}^{+}_{\mathbf{z},i}\mathbf{\Phi}_{\mathbf{z}}(t_{i+1}, t_i)^T + \int_{t_i}^{t_{i+1}} \mathbf{\Phi}_{\mathbf{z}}(t_{i+1}, s)\mathbf{G}(s)\tilde{\mathbf{q}}_{\mathbf{u}}\mathbf{G}(s)^T\mathbf{\Phi}_{\mathbf{z}}(t_{i+1}, s)^T ds \tag{31}$$

Note that the full covariance with $\mathbf{w}$ in the state vector are mapped in time for DMC. Similarly to SNC, we can assume constant stochastic acceleration for $\mathbf{u}$ in Equation 30 between the two measurement updates. In that case, the total mapped covariance is simply the addition of the covariances from the deterministic and random sources (i.e., the total uncertainties are RSS of the deterministic and random components).

$$\mathbf{P}^{-}_{z,i+1} = \mathbf{\Phi}_{z}(t_{i+1}, t_i)\mathbf{P}^{+}_{z,i}\mathbf{\Phi}_{z}(t_{i+1}, t_i)^T + \mathbf{\Gamma}_{\mathbf{z}}(t_{i+1}, t_i)\tilde{\mathbf{q}}_{\mathbf{u}}\mathbf{\Gamma}_{\mathbf{z}}(t_{i+1}, t_i)^T \tag{32}$$

where $\tilde{\mathbf{q}}_{\mathbf{u}}$ is the same covariance defined in Equation 26 and we used $\mathbf{\Gamma}_{\mathbf{z}}$ for DMC as opposed to $\mathbf{\Gamma}$ for SNC. Of course, $\mathbf{\Gamma}_{\mathbf{z}}$ is defined as follows:

$$\mathbf{\Gamma}_{\mathbf{z}}(t_{i+1}, t_i) = \int_{t_i}^{t_{i+1}} \mathbf{\Phi}_{\mathbf{z}}(t_{i+1}, s)\mathbf{G}(s)ds \tag{33}$$

As with SNC, we can also define the *discrete* process noise covariance for DMC:

$$\mathbf{q}_{\mathbf{u}}(t_{i+1}, t_i) = \int_{t_i}^{t_{i+1}} \mathbf{\Phi}_{\mathbf{z}}(t_{i+1}, s)\mathbf{G}(s)\tilde{\mathbf{q}}_{\mathbf{u}}\mathbf{G}(s)^T\mathbf{\Phi}_{\mathbf{z}}(t_{i+1}, s)^T ds \approx \mathbf{\Gamma}_{\mathbf{z}}(t_{i+1}, t_i)\tilde{\mathbf{q}}_{\mathbf{u}}\mathbf{\Gamma}_{\mathbf{z}}(t_{i+1}, t_i)^T \tag{34}$$

Simplified Dynamics for DMC

Similarly to SNC, we simplify the dynamical model and assume that the motion of the spacecraft is rectilinear with perturbation of the stochastic acceleration. Thus, the equation of motion of the spacecraft becomes

$$\dot{\mathbf{z}}(t) = \begin{bmatrix} \dot{\mathbf{X}}(t) \\ \dot{\mathbf{w}}(t) \end{bmatrix} = \begin{bmatrix} \dot{\mathbf{r}}(t) \\ \dot{\mathbf{V}}(t) \\ \dot{\mathbf{w}}(t) \end{bmatrix} = \begin{bmatrix} \mathbf{V}(t) \\ \mathbf{w}(t) \\ -\mathbf{B}_{\mathbf{w}}\mathbf{w}(t) + \mathbf{u}(t) \end{bmatrix} \tag{35}$$

Realizing that $\mathbf{A}_{\mathbf{z}}(t) = \dfrac{\partial \dot{\mathbf{z}}(t)}{\partial \mathbf{z}(t)}$ and $\mathbf{G}(t) = \dfrac{\partial \dot{\mathbf{z}}(t)}{\partial \mathbf{u}(t)}$, we get

$$\mathbf{A}_{\mathbf{z}}(t) = \begin{bmatrix} 0_{3\times 3} & 1_{3\times 3} & 0_{3\times 3} \\ 0_{3\times 3} & 0_{3\times 3} & 1_{3\times 3} \\ 0_{3\times 3} & 0_{3\times 3} & -\mathbf{B}_{\mathbf{w}} \end{bmatrix} \tag{36}$$

$$\mathbf{G}(t) = \begin{bmatrix} 0_{3\times3} \\ 0_{3\times3} \\ 1_{3\times3} \end{bmatrix} \tag{37}$$

The homogeneous solution for the STM in Equation 30 (i.e., $\mathbf{\Phi_z}(t, t_0)$) can be easily obtained by direct substitution. First, the time derivative of the STM is expressed as follows:

$$\begin{aligned} \dot{\mathbf{\Phi}}_\mathbf{z}(t,t_0) &= \mathbf{A_z}(t)\mathbf{\Phi_z}(t,t_0) = \begin{bmatrix} 0_{3\times3} & 1_{3\times3} & 0_{3\times3} \\ 0_{3\times3} & 0_{3\times3} & 1_{3\times3} \\ 0_{3\times3} & 0_{3\times3} & -\mathbf{B_w} \end{bmatrix} \begin{bmatrix} \phi_\mathbf{rr} & \phi_\mathbf{rV} & \phi_\mathbf{rw} \\ \phi_\mathbf{Vr} & \phi_\mathbf{VV} & \phi_\mathbf{Vw} \\ \phi_\mathbf{wr} & \phi_\mathbf{wV} & \phi_\mathbf{ww} \end{bmatrix} \\ &= \begin{bmatrix} \phi_\mathbf{Vr} & \phi_\mathbf{VV} & \phi_\mathbf{Vw} \\ \phi_\mathbf{wr} & \phi_\mathbf{wV} & \phi_\mathbf{ww} \\ -\mathbf{B_w}\phi_\mathbf{wr} & -\mathbf{B_w}\phi_\mathbf{wV} & -\mathbf{B_w}\phi_\mathbf{ww} \end{bmatrix} \\ &= \begin{bmatrix} \dot{\phi}_\mathbf{rr} & \dot{\phi}_\mathbf{rV} & \dot{\phi}_\mathbf{rw} \\ \dot{\phi}_\mathbf{Vr} & \dot{\phi}_\mathbf{VV} & \dot{\phi}_\mathbf{Vw} \\ \dot{\phi}_\mathbf{wr} & \dot{\phi}_\mathbf{wV} & \dot{\phi}_\mathbf{ww} \end{bmatrix} \end{aligned} \tag{38}$$

with the initial condition

$$\mathbf{\Phi_z}(t,t_0) = \begin{bmatrix} \phi_\mathbf{rr}(t_0,t_0) & \phi_\mathbf{rV}(t_0,t_0) & \phi_\mathbf{rw}(t_0,t_0) \\ \phi_\mathbf{Vr}(t_0,t_0) & \phi_\mathbf{VV}(t_0,t_0) & \phi_\mathbf{Vw}(t_0,t_0) \\ \phi_\mathbf{wr}(t_0,t_0) & \phi_\mathbf{wV}(t_0,t_0) & \phi_\mathbf{ww}(t_0,t_0) \end{bmatrix} = \begin{bmatrix} 1_{3\times3} & 0_{3\times3} & 0_{3\times3} \\ 0_{3\times3} & 1_{3\times3} & 0_{3\times3} \\ 0_{3\times3} & 0_{3\times3} & 1_{3\times3} \end{bmatrix} \tag{39}$$

Firstly, we solve for the STM with respect to the position (i.e., the first column in STM).

- $\dot{\phi}_\mathbf{wr} = -\mathbf{B_w}\phi_\mathbf{wr}$ with $\phi_\mathbf{wr}(t_0,t_0) = 0_{3\times3}$. Thus, $\phi_\mathbf{wr} = 0_{3\times3}$.
- $\dot{\phi}_\mathbf{Vr} = \phi_\mathbf{wr} = 0_{3\times3}$ with $\phi_\mathbf{Vr}(t_0,t_0) = 0_{3\times3}$. Thus, $\phi_\mathbf{Vr} = 0_{3\times3}$.
- $\dot{\phi}_\mathbf{rr} = \phi_\mathbf{Vr} = 0_{3\times3}$ with $\phi_\mathbf{rr}(t_0,t_0) = 1_{3\times3}$. Thus, $\phi_\mathbf{rr} = 1_{3\times3}$.

Secondly, we solve for the STM with respect to the velocity (i.e., the second column in STM).

- $\dot{\phi}_\mathbf{wV} = -\mathbf{B_w}\phi_\mathbf{wV}$ with $\phi_\mathbf{wV}(t_0,t_0) = 0_{3\times3}$. Thus, $\phi_\mathbf{wV} = 0_{3\times3}$.
- $\dot{\phi}_\mathbf{VV} = \phi_\mathbf{wV} = 0_{3\times3}$ with $\phi_\mathbf{VV}(t_0,t_0) = 1_{3\times3}$. Thus, $\phi_\mathbf{VV} = 1_{3\times3}$.
- $\dot{\phi}_\mathbf{rV} = \phi_\mathbf{VV} = 1_{3\times3}$ with $\phi_\mathbf{rV}(t_0,t_0) = 0_{3\times3}$. Thus, $\phi_\mathbf{rV} = (t-t_0)1_{3\times3}$.

Thirdly, we solve for the STM with respect to the stochastic acceleration (i.e., the third column in STM).

- $\dot{\phi}_{\mathbf{ww}} = -\mathbf{B_w}\phi_{\mathbf{ww}}$ with $\phi_{\mathbf{ww}}(t_0, t_0) = 1_{3\times3}$.
  - This is in the form of the differential equation for an exponential function.
  - $\phi_{\mathbf{ww}} = C_0 e^{-\mathbf{B_w}(t-t_0)}$. Because of the initial condition, $C_0 = 1_{3\times3}$
  - Thus, $\phi_{\mathbf{ww}} = e^{-\mathbf{B_w}(t-t_0)}$.
- $\dot{\phi}_{\mathbf{Vw}} = \phi_{\mathbf{ww}} = e^{-\mathbf{B_w}(t-t_0)}$ with $\phi_{\mathbf{Vw}}(t_0, t_0) = 0_{3\times3}$.
  - $\phi_{\mathbf{Vw}} = -[\mathbf{B_w}]^{-1} e^{-\mathbf{B_w}(t-t_0)} + C_1$. Because of the initial condition, $C_1 = [B_w]^{-1}$
  - Thus, $\phi_{\mathbf{Vw}} = [\mathbf{B_w}]^{-1}\left[1_{3\times3} - e^{-\mathbf{B_w}(t-t_0)}\right]$.
- $\dot{\phi}_{\mathbf{rw}} = \phi_{\mathbf{Vw}} = [\mathbf{B_w}]^{-1}\left[1_{3\times3} - e^{-\mathbf{B_w}(t-t_0)}\right]$ with $\phi_{\mathbf{rw}}(t_0, t_0) = 0_{3\times3}$.
  - $\phi_{\mathbf{rw}} = [\mathbf{B_w}]^{-1}\left[(t-t_0)1_{3\times3} + [\mathbf{B_w}]^{-1} e^{-\mathbf{B_w}(t-t_0)}\right] + C_2$. Because of the initial condition, $C_2 = -[\mathbf{B_w}]^{-2}$.
  - Thus, $\phi_{\mathbf{rw}} = [\mathbf{B_w}]^{-1}(t-t_0) - [\mathbf{B_w}]^{-2}\left(1_{3\times3} - e^{-\mathbf{B_w}(t-t_0)}\right)$.

Therefore, the STM ($\mathbf{\Phi_z}(t, t_0)$) is obtained as

$$
\begin{aligned}
\mathbf{\Phi_z}(t, t_0) &= \begin{bmatrix} \phi_{\mathbf{rr}}(t,t_0) & \phi_{\mathbf{rV}}(t,t_0) & \phi_{\mathbf{rw}}(t,t_0) \\ \phi_{\mathbf{Vr}}(t,t_0) & \phi_{\mathbf{VV}}(t,t_0) & \phi_{\mathbf{Vw}}(t,t_0) \\ \phi_{\mathbf{wr}}(t,t_0) & \phi_{\mathbf{wV}}(t,t_0) & \phi_{\mathbf{ww}}(t,t_0) \end{bmatrix} \\
&= \begin{bmatrix} 1_{3\times3} & (t-t_0)1_{3\times3} & [\mathbf{B_w}]^{-1}(t-t_0) - [\mathbf{B_w}]^{-2}\left(1_{3\times3} - e^{-\mathbf{B_w}(t-t_0)}\right) \\ 0_{3\times3} & 1_{3\times3} & [\mathbf{B_w}]^{-1}\left[1_{3\times3} - e^{-\mathbf{B_w}(t-t_0)}\right] \\ 0_{3\times3} & 0_{3\times3} & e^{-\mathbf{B_w}(t-t_0)} \end{bmatrix}
\end{aligned} \tag{40}
$$

where $t - t_0$ is the time interval between measurement updates (i.e., in the CKF notation, this is $t_{i+1} - t_i$). The following two sections will detail the derivation of the *discrete* process noise covariance ($\mathbf{q_u}(t_{i+1}, t_i)$) for DMC.

DMC Time Update - Direct Integration

By direct substitution of Equation 37 and 40 into Equation 34, we get

$$
\begin{aligned}
\mathbf{q_u}(t_{i+1}, t_i) &= \begin{bmatrix} \mathbf{q}_{\mathbf{u},\mathbf{rr}}(t_{i+1},t_i) & \mathbf{q}_{\mathbf{u},\mathbf{rV}}(t_{i+1},t_i) & \mathbf{q}_{\mathbf{u},\mathbf{rw}}(t_{i+1},t_i) \\ \mathbf{q}_{\mathbf{u},\mathbf{Vr}}(t_{i+1},t_i) & \mathbf{q}_{\mathbf{u},\mathbf{VV}}(t_{i+1},t_i) & \mathbf{q}_{\mathbf{u},\mathbf{Vw}}(t_{i+1},t_i) \\ \mathbf{q}_{\mathbf{u},\mathbf{wr}}(t_{i+1},t_i) & \mathbf{q}_{\mathbf{u},\mathbf{wV}}(t_{i+1},t_i) & \mathbf{q}_{\mathbf{u},\mathbf{ww}}(t_{i+1},t_i) \end{bmatrix} \\
&= \int_{t_i}^{t_{i+1}} \mathbf{\Phi_z}(t_{i+1}, s)\mathbf{G}(s)\tilde{\mathbf{q}}_\mathbf{u}\mathbf{G}(s)^T\mathbf{\Phi_z}(t_{i+1}, s)^T ds \\
&= \int_{t_i}^{t_{i+1}} \begin{bmatrix} \phi_{\mathbf{rw}}(t_{i+1},s)\tilde{\mathbf{q}}_\mathbf{u}\phi^T_{\mathbf{rw}}(t_{i+1},s) & \phi_{\mathbf{rw}}(t_{i+1},s)\tilde{\mathbf{q}}_\mathbf{u}\phi^T_{\mathbf{Vw}}(t_{i+1},s) & \phi_{\mathbf{rw}}(t_{i+1},s)\tilde{\mathbf{q}}_\mathbf{u}\phi^T_{\mathbf{ww}}(t_{i+1},s) \\ \phi_{\mathbf{Vw}}(t_{i+1},s)\tilde{\mathbf{q}}_\mathbf{u}\phi^T_{\mathbf{rw}}(t_{i+1},s) & \phi_{\mathbf{Vw}}(t_{i+1},s)\tilde{\mathbf{q}}_\mathbf{u}\phi^T_{\mathbf{Vw}}(t_{i+1},s) & \phi_{\mathbf{Vw}}(t_{i+1},s)\tilde{\mathbf{q}}_\mathbf{u}\phi^T_{\mathbf{ww}}(t_{i+1},s) \\ \phi_{\mathbf{ww}}(t_{i+1},s)\tilde{\mathbf{q}}_\mathbf{u}\phi^T_{\mathbf{rw}}(t_{i+1},s) & \phi_{\mathbf{ww}}(t_{i+1},s)\tilde{\mathbf{q}}_\mathbf{u}\phi^T_{\mathbf{Vw}}(t_{i+1},s) & \phi_{\mathbf{ww}}(t_{i+1},s)\tilde{\mathbf{q}}_\mathbf{u}\phi^T_{\mathbf{ww}}(t_{i+1},s) \end{bmatrix} ds
\end{aligned} \tag{41}
$$

Because we have the diagonal $\tilde{\mathbf{q}}_{\mathbf{u}}$ and the diagonal $\mathbf{B}_{\mathbf{w}}$, Equation 41 simplifies significantly. There are only six independent components. We individually solve for them below using the simplified STM obtained in Equation 40.

First, we solve for the integrals that commonly appear during the derivation.

$$\begin{aligned}
\int_{t_i}^{t_{i+1}} e^{-\mathbf{B}_{\mathbf{w}}(t_{i+1}-s)} ds &= [\mathbf{B}_{\mathbf{w}}]^{-1} \left[e^{-\mathbf{B}_{\mathbf{w}}(t_{i+1}-s)} \right]_{t_i}^{t_{i+1}} \\
&= [\mathbf{B}_{\mathbf{w}}]^{-1} \left[1_{3\times 3} - e^{-\mathbf{B}_{\mathbf{w}}(t_{i+1}-t_i)} \right] \\
&= [\mathbf{B}_{\mathbf{w}}]^{-1} \left[1_{3\times 3} - e^{-\mathbf{B}_{\mathbf{w}}\Delta t} \right]
\end{aligned} \tag{42}$$

$$\begin{aligned}
\int_{t_i}^{t_{i+1}} e^{-2\mathbf{B}_{\mathbf{w}}(t_{i+1}-s)} ds &= \left(\frac{1}{2} [\mathbf{B}_{\mathbf{w}}]^{-1} \right) \left[e^{-2\mathbf{B}_{\mathbf{w}}(t_{i+1}-s)} \right]_{t_i}^{t_{i+1}} \\
&= \frac{1}{2} [\mathbf{B}_{\mathbf{w}}]^{-1} \left[1_{3\times 3} - e^{-2\mathbf{B}_{\mathbf{w}}(t_{i+1}-t_i)} \right] \\
&= \frac{1}{2} [\mathbf{B}_{\mathbf{w}}]^{-1} \left[1_{3\times 3} - e^{-2\mathbf{B}_{\mathbf{w}}\Delta t} \right]
\end{aligned} \tag{43}$$

$$\begin{aligned}
\int_{t_i}^{t_{i+1}} (t_{i+1}-s) e^{-\mathbf{B}_{\mathbf{w}}(t_{i+1}-s)} ds &= \int_{t_i}^{t_{i+1}} (t_{i+1}-s) \frac{\partial \left([\mathbf{B}_{\mathbf{w}}]^{-1} e^{-\mathbf{B}_{\mathbf{w}}(t_{i+1}-s)} \right)}{\partial s} \\
&= \left[(t_{i+1}-s) \left([\mathbf{B}_{\mathbf{w}}]^{-1} e^{-\mathbf{B}_{\mathbf{w}}(t_{i+1}-s)} \right) \right]_{t_i}^{t_{i+1}} - \int_{t_i}^{t_{i+1}} \frac{\partial (t_{i+1}-s)}{\partial s} \left([\mathbf{B}_{\mathbf{w}}]^{-1} e^{-\mathbf{B}_{\mathbf{w}}(t_{i+1}-s)} \right) ds \\
&= -(t_{i+1}-t_i) [\mathbf{B}_{\mathbf{w}}]^{-1} e^{-\mathbf{B}_{\mathbf{w}}(t_{i+1}-t_i)} + [\mathbf{B}_{\mathbf{w}}]^{-1} \int_{t_i}^{t_{i+1}} \left(e^{-\mathbf{B}_{\mathbf{w}}(t_{i+1}-s)} \right) ds \\
&= -(t_{i+1}-t_i) [\mathbf{B}_{\mathbf{w}}]^{-1} e^{-\mathbf{B}_{\mathbf{w}}(t_{i+1}-t_i)} + [\mathbf{B}_{\mathbf{w}}]^{-2} \left[1_{3\times 3} - e^{-\mathbf{B}_{\mathbf{w}}(t_{i+1}-t_i)} \right] \\
&= -\Delta t [\mathbf{B}_{\mathbf{w}}]^{-1} e^{-\mathbf{B}_{\mathbf{w}}\Delta t} + [\mathbf{B}_{\mathbf{w}}]^{-2} \left[1_{3\times 3} - e^{-\mathbf{B}_{\mathbf{w}}\Delta t} \right]
\end{aligned} \tag{44}$$

Note that Equation 44 is simply integration by parts. Then, we have

$$
\begin{aligned}
\mathbf{q}_{\mathbf{u},\mathbf{rr}}(t_{i+1},t_i) &= \int_{t_i}^{t_{i+1}} \phi_{\mathbf{rw}}(t_{i+1},s)\tilde{\mathbf{q}}_{\mathbf{u}}\phi_{\mathbf{rw}}^{T}(t_{i+1},s)ds \qquad (45)\\
&\approx \tilde{\mathbf{q}}_{\mathbf{u}}\left[\mathbf{B_w}\right]^{-2}\int_{t_i}^{t_{i+1}}\Big[(t_{i+1}-s)^2 1_{3\times3}+\left[\mathbf{B_w}\right]^{-2}+\left[\mathbf{B_w}\right]^{-2}e^{-2\mathbf{B_w}(t_{i+1}-s)}\\
&\quad -2\left[\mathbf{B_w}\right]^{-1}(t_{i+1}-s)+2(t_{i+1}-s)\left[\mathbf{B_w}\right]^{-1}e^{-\mathbf{B_w}(t_{i+1}-s)}-2\left[\mathbf{B_w}\right]^{-2}e^{-\mathbf{B_w}(t_{i+1}-s)}\Big]ds\\
&= \tilde{\mathbf{q}}_{\mathbf{u}}\left[\mathbf{B_w}\right]^{-2}\Big[\frac{\Delta t^3}{3}\,1_{3\times3}+\left[\mathbf{B_w}\right]^{-2}\Delta t+\frac{1}{2}\left[\mathbf{B_w}\right]^{-3}\left(1_{3\times3}-e^{-2\mathbf{B_w}\Delta t}\right)-\left[\mathbf{B_w}\right]^{-1}\Delta t^2\\
&\quad -2\Delta t\left[\mathbf{B_w}\right]^{-2}e^{-\mathbf{B_w}\Delta t}+2\left[\mathbf{B_w}\right]^{-3}\left(1_{3\times3}-e^{-\mathbf{B_w}\Delta t}\right)-2\left[\mathbf{B_w}\right]^{-3}\left(1_{3\times3}-e^{-\mathbf{B_w}\Delta t}\right)\Big]\\
&= \tilde{\mathbf{q}}_{\mathbf{u}}\left[\mathbf{B_w}\right]^{-2}\Big[\frac{\Delta t^3}{3}\,1_{3\times3}-\Delta t^2\left[\mathbf{B_w}\right]^{-1}+\Delta t\left[\mathbf{B_w}\right]^{-2}+\frac{1}{2}\left[\mathbf{B_w}\right]^{-3}\left(1_{3\times3}-e^{-2\mathbf{B_w}\Delta t}\right)\\
&\quad -2\Delta t\left[\mathbf{B_w}\right]^{-2}e^{-\mathbf{B_w}\Delta t}\Big]
\end{aligned}
$$

$$
\begin{aligned}
\mathbf{q}_{\mathbf{u},\mathbf{rV}}(t_{i+1},t_i) &= \int_{t_i}^{t_{i+1}} \phi_{\mathbf{rw}}(t_{i+1},s)\tilde{\mathbf{q}}_{\mathbf{u}}\phi_{\mathbf{Vw}}^{T}(t_{i+1},s)ds \qquad (46)\\
&\approx \tilde{\mathbf{q}}_{\mathbf{u}}\left[\mathbf{B_w}\right]^{-2}\int_{t_i}^{t_{i+1}}\Big[(t_{i+1}-s)\Big(1_{3\times3}-e^{-\mathbf{B_w}(t_{i+1}-s)}\Big)\\
&\quad -\left[\mathbf{B_w}\right]^{-1}\Big(1_{3\times3}-2e^{-\mathbf{B_w}(t_{i+1}-s)}+e^{-2\mathbf{B_w}(t_{i+1}-s)}\Big)\Big]ds\\
&= \tilde{\mathbf{q}}_{\mathbf{u}}\left[\mathbf{B_w}\right]^{-2}\Big[\frac{\Delta t^2}{2}1_{3\times3}+\Delta t\left[\mathbf{B_w}\right]^{-1}e^{-\mathbf{B_w}\Delta t}-\left[\mathbf{B_w}\right]^{-2}\left(1_{3\times3}-e^{-\mathbf{B_w}\Delta t}\right)\\
&\quad -\left[\mathbf{B_w}\right]^{-1}\Big\{\Delta t\;1_{3\times3}-2\left[\mathbf{B_w}\right]^{-1}\left(1_{3\times3}-e^{-\mathbf{B_w}\Delta t}\right)+\frac{1}{2}\left[\mathbf{B_w}\right]^{-1}\left(1_{3\times3}-e^{-2\mathbf{B_w}\Delta t}\right)\Big\}\Big]\\
&= \tilde{\mathbf{q}}_{\mathbf{u}}\left[\mathbf{B_w}\right]^{-2}\Big[\frac{\Delta t^2}{2}1_{3\times3}-\Delta t\left[\mathbf{B_w}\right]^{-1}+\Delta t\left[\mathbf{B_w}\right]^{-1}e^{-\mathbf{B_w}\Delta t}+\left[\mathbf{B_w}\right]^{-2}\left(1_{3\times3}-e^{-\mathbf{B_w}\Delta t}\right)\\
&\quad -\frac{1}{2}\left[\mathbf{B_w}\right]^{-2}\left(1_{3\times3}-e^{-2\mathbf{B_w}\Delta t}\right)\Big]
\end{aligned}
$$

$$
\begin{aligned}
\mathbf{q}_{\mathbf{u},\mathbf{VV}}(t_{i+1},t_i) &= \int_{t_i}^{t_{i+1}} \phi_{\mathbf{Vw}}(t_{i+1},s)\tilde{\mathbf{q}}_{\mathbf{u}}\phi_{\mathbf{Vw}}^{T}(t_{i+1},s)ds \qquad (47)\\
&\approx \tilde{\mathbf{q}}_{\mathbf{u}}\int_{t_i}^{t_{i+1}}\left[\mathbf{B_w}\right]^{-2}\Big[1_{3\times3}-2e^{-\mathbf{B_w}(t_{i+1}-s)}+e^{-2\mathbf{B_w}(t_{i+1}-s)}\Big]ds\\
&= \tilde{\mathbf{q}}_{\mathbf{u}}\left[\mathbf{B_w}\right]^{-2}\Big[\Delta t\;1_{3\times3}-2\left[\mathbf{B_w}\right]^{-1}\left(1_{3\times3}-e^{-\mathbf{B_w}\Delta t}\right)\\
&\quad +\frac{1}{2}\left[\mathbf{B_w}\right]^{-1}\left(1_{3\times3}-e^{-2\mathbf{B_w}\Delta t}\right)\Big]
\end{aligned}
$$

$$\begin{aligned}
\mathbf{q}_{\mathbf{u},\mathbf{rw}}(t_{i+1},t_i) &= \int_{t_i}^{t_{i+1}} \phi_{\mathbf{rw}}(t_{i+1},s)\tilde{\mathbf{q}}_{\mathbf{u}}\phi^T_{\mathbf{ww}}(t_{i+1},s)ds \\
&\approx \tilde{\mathbf{q}}_{\mathbf{u}} \int_{t_i}^{t_{i+1}} \left[[\mathbf{B}_{\mathbf{w}}]^{-1}(t_{i+1}-s)e^{-\mathbf{B}_{\mathbf{w}}(t_{i+1}-s)} - [\mathbf{B}_{\mathbf{w}}]^{-2}\left(e^{-\mathbf{B}_{\mathbf{w}}(t_{i+1}-s)} - e^{-2\mathbf{B}_{\mathbf{w}}(t_{i+1}-s)}\right)\right] ds \\
&= \tilde{\mathbf{q}}_{\mathbf{u}}\left[\mathbf{B}_{\mathbf{w}}\right]^{-1}\left[-\Delta t\left[\mathbf{B}_{\mathbf{w}}\right]^{-1}e^{-\mathbf{B}_{\mathbf{w}}\Delta t} + \left[\mathbf{B}_{\mathbf{w}}\right]^{-2}\left[1_{3\times 3} - e^{-\mathbf{B}_{\mathbf{w}}\Delta t}\right]\right] \\
&\quad -\tilde{\mathbf{q}}_{\mathbf{u}}\left[\mathbf{B}_{\mathbf{w}}\right]^{-2}\left[\left[\mathbf{B}_{\mathbf{w}}\right]^{-1}\left(1_{3\times 3} - e^{-\mathbf{B}_{\mathbf{w}}\Delta t}\right) - \frac{1}{2}\left[\mathbf{B}_{\mathbf{w}}\right]^{-1}\left(1_{3\times 3} - e^{-2\mathbf{B}_{\mathbf{w}}\Delta t}\right)\right] \\
&= \tilde{\mathbf{q}}_{\mathbf{u}}\left[\mathbf{B}_{\mathbf{w}}\right]^{-2}\left[-\Delta t e^{-\mathbf{B}_{\mathbf{w}}\Delta t} + \frac{1}{2}\left[\mathbf{B}_{\mathbf{w}}\right]^{-1}\left(1_{3\times 3} - e^{-2\mathbf{B}_{\mathbf{w}}\Delta t}\right)\right]
\end{aligned} \tag{48}$$

$$\begin{aligned}
\mathbf{q}_{\mathbf{u},\mathbf{Vw}}(t_{i+1},t_i) &= \int_{t_i}^{t_{i+1}} \phi_{\mathbf{Vw}}(t_{i+1},s)\tilde{\mathbf{q}}_{\mathbf{u}}\phi^T_{\mathbf{ww}}(t_{i+1},s)ds \\
&\approx \tilde{\mathbf{q}}_{\mathbf{u}} \int_{t_i}^{t_{i+1}} [\mathbf{B}_{\mathbf{w}}]^{-1}\left[e^{-\mathbf{B}_{\mathbf{w}}(t-t_0)} - e^{-2\mathbf{B}_{\mathbf{w}}(t-t_0)}\right] ds \\
&= \tilde{\mathbf{q}}_{\mathbf{u}}\left[\mathbf{B}_{\mathbf{w}}\right]^{-1}\left[\left[\mathbf{B}_{\mathbf{w}}\right]^{-1}\left(1_{3\times 3} - e^{-\mathbf{B}_{\mathbf{w}}(t_{i+1}-t_i)}\right) - \frac{1}{2}\left[\mathbf{B}_{\mathbf{w}}\right]^{-1}\left(1_{3\times 3} - e^{-2\mathbf{B}_{\mathbf{w}}(t_{i+1}-t_i)}\right)\right] \\
&= \tilde{\mathbf{q}}_{\mathbf{u}}\left[\mathbf{B}_{\mathbf{w}}\right]^{-2}\left[-e^{-\mathbf{B}_{\mathbf{w}}\Delta t} + \frac{1}{2}\left(1_{3\times 3} + e^{-2\mathbf{B}_{\mathbf{w}}\Delta t}\right)\right]
\end{aligned} \tag{49}$$

$$\begin{aligned}
\mathbf{q}_{\mathbf{u},\mathbf{ww}}(t_{i+1},t_i) &= \int_{t_i}^{t_{i+1}} \phi_{\mathbf{ww}}(t_{i+1},s)\tilde{\mathbf{q}}_{\mathbf{u}}\phi^T_{\mathbf{ww}}(t_{i+1},s)ds \\
&\approx \tilde{\mathbf{q}}_{\mathbf{u}} \int_{t_i}^{t_{i+1}} e^{-2\mathbf{B}_{\mathbf{w}}(t_{i+1}-s)}ds \\
&= \frac{1}{2}\tilde{\mathbf{q}}_{\mathbf{u}}\left[\mathbf{B}_{\mathbf{w}}\right]^{-1}\left[1_{3\times 3} - e^{-2\mathbf{B}_{\mathbf{w}}\Delta t}\right]
\end{aligned} \tag{50}$$

DMC Time Update - Approximation

$\mathbf{\Gamma}_{\mathbf{z}}$ is defined as follows:

$$
\begin{aligned}
\mathbf{\Gamma_z}(t_{i+1},t_i) &= \int_{t_i}^{t_{i+1}} \mathbf{\Phi_z}(t_{i+1},s)\mathbf{G}(s)ds \\
&= \int_{t_i}^{t_{i+1}} \begin{bmatrix} [\mathbf{B_w}]^{-1}(t_{i+1}-s) - [\mathbf{B_w}]^{-2}\left(1_{3\times3} - e^{-\mathbf{B_w}(t_{i+1}-s)}\right) \\ [\mathbf{B_w}]^{-1}\left[1_{3\times3} - e^{-\mathbf{B_w}(t_{i+1}-s)}\right] \\ e^{-\mathbf{B_w}(t_{i+1}-s)} \end{bmatrix} ds
\end{aligned} \tag{51}
$$

$$
\begin{aligned}
&= \begin{bmatrix} [\mathbf{B_w}]^{-1}\dfrac{(t_{i+1}-t_i)^2}{2} - [\mathbf{B_w}]^{-2}\left[(t_{i+1}-t_i)1_{3\times3} - [\mathbf{B_w}]^{-1}\left(1_{3\times3} - e^{-\mathbf{B_w}(t_{i+1}-t_i)}\right)\right] \\ [\mathbf{B_w}]^{-1}\left[(t_{i+1}-t_i)1_{3\times3} - [\mathbf{B_w}]^{-1}\left(1_{3\times3} - e^{-\mathbf{B_w}(t_{i+1}-t_i)}\right)\right] \\ [\mathbf{B_w}]^{-1}\left(1_{3\times3} - e^{-\mathbf{B_w}(t_{i+1}-t_i)}\right) \end{bmatrix} \\
&= \begin{bmatrix} [\mathbf{B_w}]^{-1}\dfrac{(t_{i+1}-t_i)^2}{2} - [\mathbf{B_w}]^{-2}(t_{i+1}-t_i) + [\mathbf{B_w}]^{-3}\left(1_{3\times3} - e^{-\mathbf{B_w}(t_{i+1}-t_i)}\right) \\ [\mathbf{B_w}]^{-1}(t_{i+1}-t_i) - [\mathbf{B_w}]^{-2}\left(1_{3\times3} - e^{-\mathbf{B_w}(t_{i+1}-t_i)}\right) \\ [\mathbf{B_w}]^{-1}\left(1_{3\times3} - e^{-\mathbf{B_w}(t_{i+1}-t_i)}\right) \end{bmatrix} \\
&= \begin{bmatrix} \dfrac{\Delta t^2}{2}[\mathbf{B_w}]^{-1} - \Delta t[\mathbf{B_w}]^{-2} + [\mathbf{B_w}]^{-3}\left(1_{3\times3} - e^{-\mathbf{B_w}\Delta t}\right) \\ \Delta t[\mathbf{B_w}]^{-1} - [\mathbf{B_w}]^{-2}\left(1_{3\times3} - e^{-\mathbf{B_w}\Delta t}\right) \\ [\mathbf{B_w}]^{-1}\left(1_{3\times3} - e^{-\mathbf{B_w}\Delta t}\right) \end{bmatrix}
\end{aligned} \tag{52}
$$

Equation 51 can be used to compute Equation 34. As with SNC, the DMC process noise covariance is larger for the approximation by $\mathbf{\Gamma_z}$. Because we are estimating the mean value of the process noise by $\mathbf{w}$, we do not want to inflate the covariance more than necessary. Thus, we use the process noise covariance obtained in Equation 45 through 50.