Statistical Orbit Determination

Lecture 5

January 31, 2018

Dynamic Model Compensation

Estimate deterministic part of stochastic accelerations

DYNAMIC MODEL COMPENSATION

DMC: Highlights

For SNC, we assumed the unmodeled accel. is white noise

$$\dot{\vec{x}}(t) = A(t)\vec{x}(t) + B(t)\vec{u}(t)$$

$$\boxed{E[\vec{u}(t)] = 0, E[\vec{u}(t_i)\vec{u}(t_j)^T] = Q(t_i)\delta_{ij}}$$

What if part of unmodeled accel. is deterministic (i.e., *not* entirely white noise)?

Assume accel. is a Gauss-Markov process which is governed by:

$$\dot{\vec{w}}(t) = -B\vec{w}(t) + \vec{u}(t)$$

vector of deterministic accelerations

stationary, Gaussian process uncorrelated in time (white)

$$E[\vec{u}(t)] = 0, E[\vec{u}\vec{u}^T] = \vec{q}_u = \begin{bmatrix} \sigma_x^2 & 0 & 0 \\ 0 & \sigma_y^2 & 0 \\ 0 & 0 & \sigma_x^2 \end{bmatrix}$$

DMC: Highlights

$$\dot{\vec{w}}(t) = -B\vec{w}(t) + \vec{u}(t)$$

$$B = \begin{bmatrix} \beta_x & 0 & 0 \\ 0 & \beta_y & 0 \\ 0 & 0 & \beta_z \end{bmatrix} = \begin{bmatrix} \tau_x^{-1} & 0 & 0 \\ 0 & \tau_y^{-1} & 0 \\ 0 & 0 & \tau_z^{-1} \end{bmatrix}$$

τ = process time constant

For simplicity assumed q_u and B are band-diagonal

Neglect Gaussian term for now:

$$\dot{\vec{w}}(t) = -B\vec{w}(t) + \cancel{\vec{u}(t)} \quad \Rightarrow \quad \dot{\vec{w}}(t) = -B\vec{w}(t)$$

$$\vec{w}(t) = \vec{w}_0 e^{-B(t-t_0)}$$

w_0 = stochastic initial value

DMC: Highlights

Add deterministic accel. terms to filter state so we can estimate their values:

$$\vec{x} = \begin{bmatrix} \vec{r} \\ \vec{v} \\ \vec{w} \end{bmatrix}$$

$$\dot{\vec{x}} = \begin{bmatrix} \vec{v} \\ \dot{\vec{v}} + \vec{w} \\ \dot{\vec{w}} \end{bmatrix}^T$$

Update dynamic model & meas. sensitivity to include deterministic accel.:

$$\dot{\vec{x}} = A'\vec{x}(t) + C\vec{u}(t) \qquad C = \begin{bmatrix} 0_{6x3} \\ I_{3x3} \end{bmatrix}$$

$$A' = \begin{bmatrix} A_{6x6} & D_{6x3} \\ 0_{3x6} & -B_{3x3} \end{bmatrix} \quad D = \begin{bmatrix} 0_{3x3} \\ I_{3x3} \end{bmatrix} \quad \tilde{H} = \begin{bmatrix} \tilde{H} & 0_{mx3} \end{bmatrix}$$

Include white noise in state model

DMC: Highlights

Similar to the SNC algorithm, the DMC process noise is included in the filter time update:

$$\bar{P}(t_i) = \Phi(t_i, t_{i-1}) P(t_{i-1}) \Phi(t_i, t_{i-1})^T + Q(t)$$

Q comes from the stochastic part of the unmodeled accelerations (which we neglected earlier)

The stochastic portion of accels. don't contribute to deterministic state dynamics, but contribute to process noise:

$$Q(t_i) = \int_{t_0}^{t_i} \Phi(t_i, T) C E[\vec{u}(T)\vec{u}(T)^T] C^T \Phi(t_i, T)^T dT$$

Similar to SNC derivation

Recall: $$E[\vec{u}\vec{u}^T] = \begin{bmatrix} \sigma_x^2 & 0 & 0 \\ 0 & \sigma_y^2 & 0 \\ 0 & 0 & \sigma_x^2 \end{bmatrix}$$

DMC: Highlights

Because the stochastic part of Q is a constant matrix, the integral is deterministic:

$$Q(t_i) = \int_{t_0}^{t_i} \Phi(t_i, T) C E[\vec{u}(T)\vec{u}(T)^T] C^T \Phi(t_i, T)^T dT$$

A simplified state transition matrix is used to evaluate integral

Assume velocity time derivative = deterministic accelerations
DO NOT compute simplified STM in your filters!
(numerically integrate given full A′)

$$A' = \begin{bmatrix} 0_{3x3} & I_{3x3} & 0_{3x3} \\ 0_{3x3} & 0_{3x3} & I_{3x3} \\ 0_{3x3} & 0_{3x3} & -B_{3x3} \end{bmatrix}, \quad \dot{\Phi} = A'\Phi$$

STM for any **one** axis only

$$\Rightarrow \Phi(t, t_0) = \begin{bmatrix} 1 & t - t_0 & \beta^{-1}(t - t_0) + \beta^{-2}(e^{-\beta(t-t_0)} - 1) \\ 0 & 1 & \beta^{-1}(1 - e^{-\beta(t-t_0)}) \\ 0 & 0 & e^{-\beta(t-t_0)} \end{bmatrix}$$

DMC: Highlights

- Plug STM back into integral
- Skip over the integration (available on handout)

$$Q = Q_w = \begin{bmatrix} Q_w(r,r) & Q_w(r,v) & Q_w(r,w) \\ Q_w(r,v) & Q_w(v,v) & Q_w(v,w) \\ Q_w(r,w) & Q_w(v,w) & Q_w(w,w) \end{bmatrix}$$

$$\mathbf{Q}_w(r,r) = \sigma_{u_i}^2 \left(\frac{1}{3\beta_i^2}(t-t_0)^3 - \frac{1}{\beta_i^3}(t-t_0)^2 + \frac{1}{\beta_i^4}(t-t_0) - \frac{2}{\beta_i^4} e^{-\beta_i(t-t_0)}(t-t_0) + \frac{1}{2\beta_i^5}\left(1-e^{-2\beta_i(t-t_0)}\right) \right)$$

$$\mathbf{Q}_w(v,r) = \mathbf{Q}_w(r,v) = \sigma_{u_i}^2 \left(\frac{1}{2\beta_i^2}(t-t_0)^2 - \frac{1}{\beta_i^3}(t-t_0) + \frac{1}{\beta_i^3} e^{-\beta_i(t-t_0)}(t-t_0) + \frac{1}{\beta_i^4}\left(1-e^{-\beta_i(t-t_0)}\right) - \frac{1}{2\beta_i^4}\left(1-e^{-2\beta_i(t-t_0)}\right) \right)$$

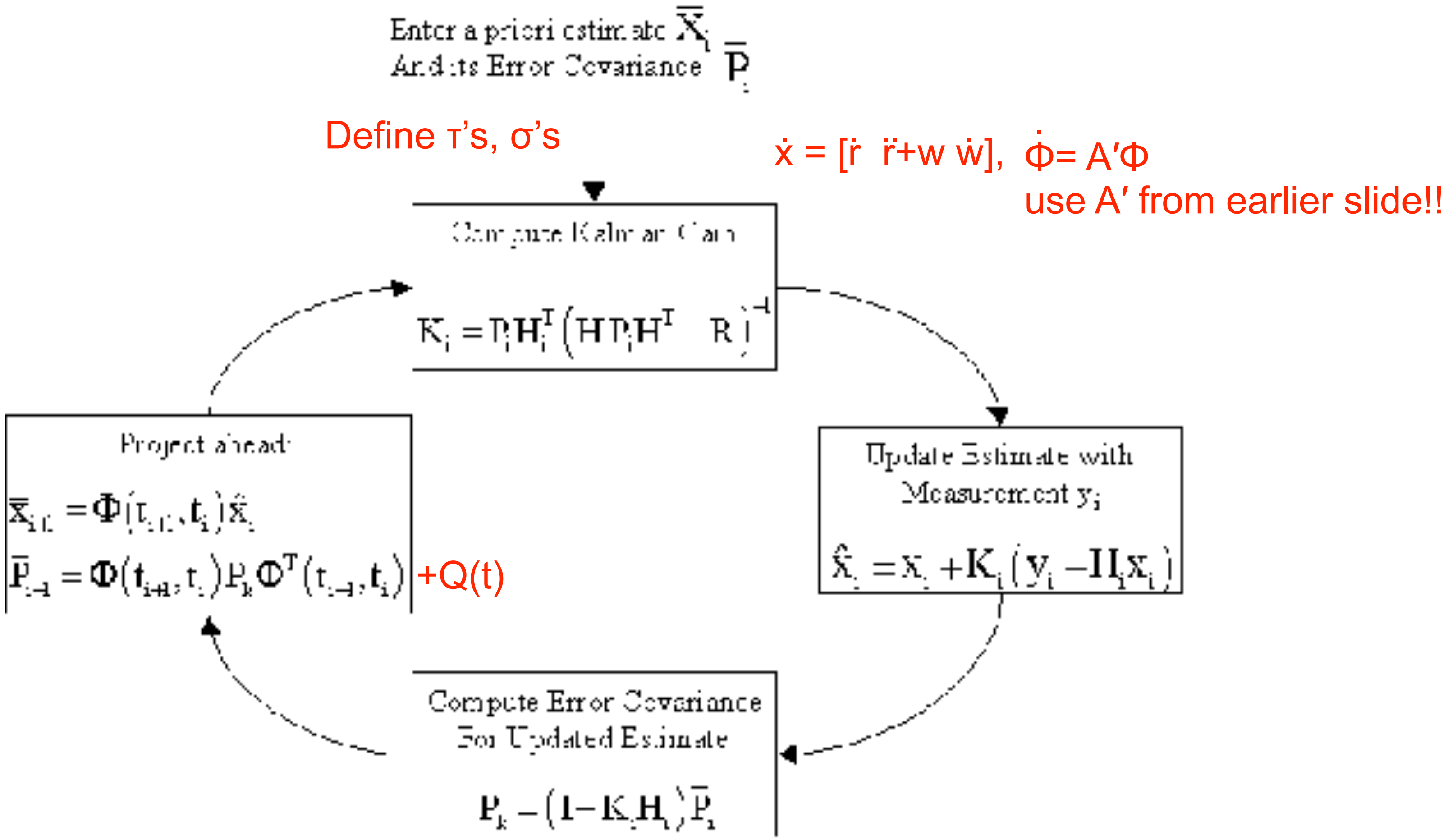
$$\mathbf{Q}_w(w,r) = \mathbf{Q}_w(r,w) = \sigma_{u_i}^2 \left(\frac{1}{2\beta_i^3}\left(1-e^{-2\beta_i(t-t_0)}\right) - \frac{1}{\beta_i^2} e^{-\beta_i(t-t_0)}(t-t_0) \right)$$

$$\mathbf{Q}_w(v,v) = \sigma_{u_i}^2 \left(\frac{1}{\beta_i^2}(t-t_0) - \frac{2}{\beta_i^3}\left(1-e^{-\beta_i(t-t_0)}\right) + \frac{1}{2\beta_i^3}\left(1-e^{-2\beta_i(t-t_0)}\right) \right)$$

$$\mathbf{Q}_w(w,v) = \mathbf{Q}_w(v,w) = \sigma_{u_i}^2 \left(\frac{1}{2\beta_i^2}\left(1+e^{-2\beta_i(t-t_0)}\right) - \frac{1}{\beta_i^2} e^{-\beta_i(t-t_0)} \right)$$

$$\mathbf{Q}_w(w,w) = \frac{\sigma_{u_i}^2}{2\beta_i}\left(1-e^{-2\beta_i(t-t_0)}\right)$$

DMC Flow Chart



DMC: Highlights

- How to select β and q values?
  - Test a range of values & check filter stability at various value combinations
  - Reasonable initial guess of τ ~ orbit period
  - Can use any a priori knowledge about un-modeled accelerations