Pergamon

Computers & Geosciences Vol. 23, No. 10, pp. 1071–1077, 1997

Printed in Great Britain
PII: S0098-3004(97)00110-6
0098-3004/98 $19.00 + 0.00

SPHERICAL HARMONIC COEFFICIENTS FOR THE POTENTIAL OF A CONSTANT-DENSITY POLYHEDRON

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(Received 24 August 1996; accepted 25 August 1997)

Abstract—Recurrent relationships for the definitions of fully normalized spherical harmonic coefficients $\bar{C}_{n,m}$ and $\bar{S}_{n,m}$ are derived and integrated analytically to yield the gravitational potential of a constant-density polyhedron. The algorithm is expressed in a C language computer program. © 1998 Published by Elsevier Science Ltd. All rights reserved

Key Words: Gravitational potential, Polyhedron, Solid spherical harmonics.

INTRODUCTION

Polyhedra have long been used to represent subterranean bodies in geophysical studies. Recently polyhedra have been used to represent entire celestial bodies such as asteroids and small planetary satellites (Simonelli and others, 1993; Hudson and Ostro, 1994; Thomas and others, 1994). The gravitation of such bodies, ore or orb, needs to be evaluated during surveying, prospecting, orbit determination, and so on.

There are three main ways of expressing and calculating an extended body's gravitation—direct integration, approximation via point masses, and approximation using a harmonic expansion.

Pinto and Casas (1996) summarizes papers which directly integrate a polyhedron's gravitation. An additional derivation appears in Werner and Scheeres (1996) work. The advantages of a direct approach are that results are analytic and valid everywhere in free space. The disadvantage might be the amount of computation required to evaluate the formulae.

Some authors approximate a body by filling it with point masses. While simple to derive and program, a point mass approach can yield inaccurate results unless a large number of points are used, increasing the computational burden (Werner and Scheeres, 1996).

For satellite work, potential expansion in solid spherical harmonics has been preferred. Coefficient values have been estimated from satellite orbit perturbations, and have been calculated using the point-mass scheme. Recurrent relationships in the formulae can take advantage of supercomputer architecture and result in fast evaluation.

Some authors calculate harmonic coefficients of a body's gravitational potential by first representing the body's surface radius as an expansion in surface spherical harmonics, then introducing this radius function into the equations which define the solid spherical harmonics for potential, and using orthogonality relationships to simplify. If the density is constant or a simple function of radius, analytic integration in the radial direction is possible. The remaining two integrations over the surface have been achieved numerically (Miller, 1988; Chao and Rubincam, 1989) and analytically (Martinec, Pec and Bursa, 1989; Balmino, 1994).

We have not seen any work which determines harmonic coefficients of the potential of a constant-density polyhedron by direct, analytic integration. This paper is such.

Two main tasks are addressed. One is to determine suitable expressions for the integrands which define $C_{n,m}$, and $S_{n,m}$, the dimensionless, unnormalized parameters describing gravitational potential, and their fully normalized counterparts $\bar{C}_{n,m}$ and $\bar{S}_{n,m}$. This task is solved by deriving recurrent relationships for the integrands, based on well-known, stable recurrences for the Legendre polynomials and associated Legendre functions. It is advantageous to express these integrands as polynomials in Cartesian coordinates instead of the usual map coordinates.

The solution to the other task, integrating Cartesian-coordinate polynomials on a polyhedral domain, is taken from a paper by Lien and Kajiya (1984). The polyhedron is partitioned into a collection of simplices (tetrahedra), each having one vertex at the origin and the opposite face taken from one of the polyhedral faces. A change of variable maps each simplex to one of standard dimensions and orientation. On this standard domain, polynomials of Cartesian coordinates can be integrated analytically.

The ultimate results are straightforward expressions for fully normalized coefficients $\bar{C}_{n,m}$ and

$\bar{S}_{n,m}$ of the original simplex. Summation over all simplices yields the coefficients for the entire polyhedron.

The polyhedron need not be convex or star shaped. The origin does not have to be in the interior. If exterior, contributions from some simplices will be subtracted from the total instead of added.

Other polynomials besides those defining potential harmonic coefficients can be integrated. Lien and Kajiya (1984) derive expressions for the volume of a polyhedron, and the first and second moments necessary for centroid and inertia matrix calculations.

Potential expressed in spherical harmonics

The exterior gravitational potential U of an extended body can be expressed as an infinite series expansion in solid spherical harmonics. One form of the expansion is

$$U = \frac{GM}{r}\left\{1 + \sum_{n=1}^{\infty}\left(\frac{a}{r}\right)^n \sum_{m=0}^{n} P_{n,m}(\sin\phi)[C_{n,m} \times \cos(m\lambda) + S_{n,m}\sin(m\lambda)]\right\}, \quad (1)$$

where G is the gravitational constant and M is the total mass of the body. (ϕ, λ, r) are the latitude, longitude and radius of the field point relative to the coordinate origin. Functions $P_{n,m}(.)$ are the Legendre polynomials $(m = 0)$ or associated Legendre functions $(m > 0)$ of degree n and order m.

Dimensionless parameters $C_{n,m}$ and $S_{n,m}$ encode information about the mass distribution of the extended body. Their definitions are stacked as a two-element column vector and written

$$\begin{bmatrix} C_{n,m} \\ S_{n,m} \end{bmatrix}_{2\times 1} \equiv \frac{1}{M}(2-\delta_{0,m}) \times \frac{(n-m)!}{(n+m)!}\iiint_{\text{extended body}} \times \left(\frac{r'}{a}\right)^n P_{n,m}(\sin\phi')\begin{bmatrix} \cos m\lambda' \\ \sin m\lambda' \end{bmatrix}_{2\times 1} \mathrm{d}m = \iiint_{\text{extended body}} \begin{bmatrix} c_{n,m} \\ s_{n,m} \end{bmatrix}_{2\times 1} \mathrm{d}m, \quad (2)$$

where the integrand is defined as

$$\begin{bmatrix} c_{n,m} \\ s_{n,m} \end{bmatrix}_{2\times 1} \equiv \frac{1}{M}(2-\delta_{0,m})\frac{(n-m)!}{(n+m)!}\left(\frac{r'}{a}\right)^n \times P_{n,m}(\sin\phi')\begin{bmatrix} \cos m\lambda' \\ \sin m\lambda' \end{bmatrix}_{2\times 1}. \quad (3)$$

$\delta_{i,j}$ is the Kronecker delta function. Mass M and distance a are inserted to make parameters $C_{n,m}$ and $S_{n,m}$ dimensionless in Equations (1) and (2). The value of a is often chosen to be the radius of the smallest sphere centered at the origin which circumscribes the extended body. Primed symbols (ϕ', λ', r') represent the latitude, longitude and radius of the differential mass $\mathrm{d}m$ in the extended body, and (x', y', z') will represent its Cartesian coordinates. The distance r' from the origin to the differential mass is $(r')^2 = (x')^2 + (y')^2 + (z')^2$. The distance ρ' in the $x'y'$ plane is $(\rho')^2 = (x')^2 + (y')^2$. The sine and cosine of latitude and longitude of $\mathrm{d}m$ are

$$\sin\phi' = \frac{z'}{r'};\ \cos\phi' = \frac{\rho'}{r'};\ \sin\lambda' = \frac{y'}{\rho'};\ \cos\lambda' = \frac{x'}{\rho'}.$$

Functions $P_{n,m}(.)$ vanish for $m > n$. Tables of $P_{n,m}(.)$ (and by implication $C_{n,m}$ and $S_{n,m}$) are lower triangular where n indexes which row and m which column. Furthermore, since $\sin m\lambda'$ vanishes when $m = 0$, the $S_{n,0}$ coefficients implied by the notation are arbitrary and can be discarded.

INTEGRANDS

Heiskanen and Moritz (1967, chapter 2, section 6) indicate that the integrands $c_{n,m}$ and $s_{n,m}$ are homogeneous polynomials of degree n when expressed in Cartesian coordinates x', y', z' instead of map coordinates ϕ', λ', r'. The tasks addressed in this paper are to determine these polynomials and to integrate them for the case where the extended body is a constant-density polyhedron.

Recurrences for $P_{n,m}$

The following recurrent relationships are known to be stable for the Legendre polynomials $(m = 0)$ and associated Legendre functions $(m > 0)$:

sectorial (diagonal)

$$P_{n,n}(\sin\theta) = (2n-1)\cos\theta P_{n-1,n-1}(\sin\theta),$$

vertical

$$(n-m)P_{n,m}(\sin\theta) = (2n-1)\sin\theta P_{n-1,m}(\sin\theta) - (n+m-1)P_{n-2,m}(\sin\theta).$$

These recurrences for $P_{n,m}$ will now be used to develop recurrences for integrands $c_{n,m}$ and $s_{n,m}$.

Sectorial recurrence

The definition of integrands $c_{n,m}$ and $s_{n,m}$ (Eq. (3) for the sectorial (diagonal) term $m = n$ can be manipulated into a recurrent relationship involving a 2×2 matrix:

$$\begin{bmatrix} c_{n,n} \\ s_{n,n} \end{bmatrix}_{2\times 1} = \frac{1}{M}(2-\delta_{0,n})\frac{(0)!}{(2n)!}\left(\frac{r'}{a}\right)^n \times P_{n,n}(\sin\phi')\begin{bmatrix} \cos n\lambda' \\ \sin n\lambda' \end{bmatrix}_{2\times 1} = \frac{1}{M}\frac{2-\delta_{0,n}}{(2n)!}\left(\frac{r'}{a}\right)^n$$

$$\times\{(2n-1)\cos\phi' P_{n-1,n-1}(\sin\phi')\}$$

$$\times\begin{bmatrix}\cos\lambda' & -\sin\lambda'\\ \sin\lambda' & \cos\lambda'\end{bmatrix}_{2\times2}\times\begin{bmatrix}\cos(n-1)\lambda'\\ \sin(n-1)\lambda'\end{bmatrix}_{2\times1}$$

$$=\frac{2-\delta_{0,n}}{(2n)!}\frac{r'}{a}(2n-1)\cos\phi'\begin{bmatrix}\cos\lambda' & -\sin\lambda'\\ \sin\lambda' & \cos\lambda'\end{bmatrix}$$

$$\times\left\{\left(\frac{1}{M}(2-\delta_{0,n-1})\frac{(0)!}{(2n-2)!}\left(\frac{r'}{a}\right)^{n-1}\right.\right.$$

$$\times P_{n-1,n-1}(\sin\phi')$$

$$\left.\left.\times\begin{bmatrix}\cos(n-1)\lambda'\\ \sin(n-1)\lambda'\end{bmatrix}\right)\Big/\left((2-\delta_{0,n-1})\frac{(0)!}{(2n-2)!}\right)\right\}$$

$$=\frac{2-\delta_{0,n}}{2-\delta_{0,n-1}}\frac{(2n-2)!}{(2n)!}\frac{r'}{a}(2n-1)\frac{\rho'}{r'}$$

$$\times\begin{bmatrix}x'/\rho & -y/\rho'\\ y'/\rho' & x'/\rho'\end{bmatrix}\begin{bmatrix}c_{n-1,n-1}\\ s_{n-1,n-1}\end{bmatrix}$$

$$=(1+\delta_{1,n})\frac{1}{2n}\begin{bmatrix}x'/a & -y'/a\\ y'/a & x'/a\end{bmatrix}\begin{bmatrix}c_{n-1,n-1}\\ s_{n-1,n-1}\end{bmatrix}.$$

The Kronecker delta can be eliminated by enumerating the $n = 1$ case explicitly as well as the $n = 0$ case which anchors the recurrence:

$$n=0:\begin{bmatrix}c_{0,0}\\ s_{0,0}\end{bmatrix}=\frac{1}{M}\begin{bmatrix}1\\ 0\end{bmatrix},\tag{4a}$$

$$n=1:\begin{bmatrix}c_{1,1}\\ s_{1,1}\end{bmatrix}=\frac{1}{M}\begin{bmatrix}x'/a\\ y'/a\end{bmatrix},\tag{4b}$$

$$n>1:\begin{bmatrix}c_{n,n}\\ s_{n,n}\end{bmatrix}=\frac{1}{2n}\begin{bmatrix}x'/a & -y'/a\\ y'/a & x'/a\end{bmatrix}\begin{bmatrix}c_{n-1,n-1}\\ s_{n-1,n-1}\end{bmatrix}.\tag{4c}$$

Vertical recurrence

The vertical recurrence for Legendre functions $P_{n,m}$ involves the preceding two functions $P_{n-1,m}$ and $P_{n-2,m}$. By analogy, coefficients α and β are sought for the integrands such that

$$\begin{bmatrix}c_{n,m}\\ s_{n,m}\end{bmatrix}=\alpha\begin{bmatrix}c_{n-1,m}\\ s_{n-1,m}\end{bmatrix}+\beta\begin{bmatrix}c_{n-2,m}\\ s_{n-2,m}\end{bmatrix},$$

i.e.

$$\frac{1}{M}(2-\delta_{0,m})\frac{(n-m)!}{(n-m)!}\left(\frac{r'}{a}\right)^{n}P_{n,m}(\sin\phi')\begin{bmatrix}\cos m\lambda'\\ \sin m\lambda'\end{bmatrix}$$

$$=\alpha\frac{1}{M}(2-\delta_{0,m})\frac{(n-1-m)!}{(n-1+m)!}\left(\frac{r'}{a}\right)^{n-1}$$

$$\times P_{n-1,m}(\sin\phi')\begin{bmatrix}\cos m\lambda'\\ \sin m\lambda'\end{bmatrix}+\beta\frac{1}{M}(2-\delta_{0,m})$$

$$\times\frac{(n-2-m)!}{(n-2+m)!}\left(\frac{r'}{a}\right)^{n-2}P_{n-2,m}(\sin\phi')\begin{bmatrix}\cos m\lambda'\\ \sin m\lambda'\end{bmatrix}.$$

After the vertical recurrence for $P_{n,m}$ is substituted on the left-hand-side and much cancellation, there results

$$\frac{(n-m-1)}{(n+m)(n+m-1)}\left(\frac{r'}{a}\right)^{2}[(2n-1)$$

$$\times\sin\phi' P_{n-1,m}(\sin\phi')-(n+m-1)P_{n-2,m}(\sin\phi')]$$

$$=\alpha\frac{n-m-1}{n+m-1}\frac{r'}{a}P_{n-1,m}(\sin\phi')+\beta P_{n-2,m}(\sin\phi').$$

Upon collection on $P_{n-1,m}$ and $P_{n-2,m}$ and further simplification, the coefficients are found to be

$$\alpha=\frac{2n-1}{n+m}\frac{z'}{a};\quad \beta=-\frac{n-m-1}{n+m}\left(\frac{r'}{a}\right)^{2}.$$

Thus, the vertical recurrence for integrands $c_{n,m}$ and $s_{n,m}$ is

$$(n+m)\begin{bmatrix}c_{n,m}\\ s_{n,m}\end{bmatrix}=(2n-1)\frac{z'}{a}$$

$$\times\begin{bmatrix}c_{n-1,m}\\ s_{n-1,m}\end{bmatrix}-(n-m-1)\left(\frac{r'}{a}\right)^{2}\begin{bmatrix}c_{n-2,m}\\ s_{n-2,m}\end{bmatrix}.\tag{5}$$

Subdiagonal recurrence

In the special case of the subdiagonal element $m = n-1$, the vertical recurrence simplifies to only

$$\begin{bmatrix}c_{n,n-1}\\ s_{n,n-1}\end{bmatrix}=\frac{z'}{a}\begin{bmatrix}c_{n-1,n-1}\\ s_{n-1,n-1}\end{bmatrix}.\tag{6}$$

Fully normalized integrands

With increasing n,m the associated Legendre functions $P_{n,m}$ becomes huge and the coefficients $C_{n,m}$ and $S_{n,m}$ become small. Computer overflow and underflow might occur when these are calculated separately, even though products $C_{n,m}\ P_{n,m}$ and $S_{n,m}\ P_{n,m}$ in Equation (1) remain of manageable size. One way of addressing this problem (Heiskanen and Moritz, 1967, chapter 1, section 14) is to introduce a normalizing factor $N_{n,m}=\sqrt{(2-\delta_{0,m})(2n+1)(n-m)!/(n+m)!}$ and use overbars to distinguish fully normalized quantities from their unnormalized counterparts:

$$\begin{bmatrix}C_{n,m}\\ S_{n,m}\end{bmatrix}P_{n,m}=\begin{bmatrix}C_{n,m}/N_{n,m}\\ S_{n,m}/N_{n,m}\end{bmatrix}N_{n,m}P_{n,m}\equiv\begin{bmatrix}\bar{C}_{n,m}\\ \bar{S}_{n,m}\end{bmatrix}\bar{P}_{n,m}.$$

Recurrent relationships for fully normalized integrands $\bar{c}_{n,m}$ and $\bar{s}_{n,m}$ resemble those of $c_{n,m}$ and $s_{n,m}$:

sectorial

$$n = 0 : \begin{bmatrix} \bar{c}_{0,0} \\ \bar{s}_{0,0} \end{bmatrix} = \frac{1}{M} \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \tag{7a}$$

$$n = 1 : \begin{bmatrix} \bar{c}_{1,1} \\ \bar{s}_{1,1} \end{bmatrix} = \frac{1}{\sqrt{3}M} \begin{bmatrix} x'/a \\ y'/a \end{bmatrix}, \tag{7b}$$

$$n > 1 : \begin{bmatrix} \bar{c}_{n,n} \\ \bar{s}_{n,n} \end{bmatrix} = \frac{2n-1}{\sqrt{2n(2n+1)}} \times \begin{bmatrix} x'/a & -y'/a \\ y'/a & x'/a \end{bmatrix} \begin{bmatrix} \bar{c}_{n-1,n-1} \\ \bar{s}_{n-1,n-1} \end{bmatrix}, \tag{7c}$$

vertical

$$\begin{bmatrix} \bar{c}_{n,m} \\ \bar{s}_{n,m} \end{bmatrix} = (2n-1)\sqrt{\frac{(2n-1)}{(2n+1)(n+m)(n-m)}}\, \frac{z'}{a} \begin{bmatrix} \bar{c}_{n-1,m} \\ \bar{s}_{n-1,m} \end{bmatrix} - \sqrt{\frac{(2n-3)(n+m-1)(n-m-1)}{(2n+1)(n+m)(n-m)}} \left(\frac{r'}{a}\right)^2 \times \begin{bmatrix} \bar{c}_{n-2,m} \\ \bar{s}_{n-2,m} \end{bmatrix}, \tag{8}$$

subdiagonal

$$\begin{bmatrix} \bar{c}_{n,n-1} \\ \bar{s}_{n,n-1} \end{bmatrix} = \frac{2n-1}{\sqrt{2n+1}} \frac{z'}{a} \begin{bmatrix} \bar{c}_{n-1,n-1} \\ \bar{s}_{n-1,n-1} \end{bmatrix}. \tag{9}$$

INTEGRATION

Thus far the integration over extended body has not been accomplished. All that has been done is to derive recurrent relationships for the integrands $\bar{c}_{n,m}$ and $\bar{s}_{n,m}$. (From the simplicity of their derivation, we suspect these integrand recurrences are already known, but we have not found them in the literature examined to date.) Such recurrent integrands are general and could be applied to bodies other than a constant-density polyhedron.

If the extended body is being approximated by a swarm of point masses, nothing further need be derived. Integration to yield the swarm's harmonic coefficients $\bar{C}_{n,m}$ and $\bar{S}_{n,m}$ involves substituting each point's Cartesian coordinates in the recurrences to calculate that point's various $\bar{c}_{n,m}$ and $\bar{s}_{n,m}$, scaling by the point's mass M_i, and summing over all point masses:

$$\begin{bmatrix} \bar{C}_{n,m} \\ \bar{S}_{n,m} \end{bmatrix} = \sum_{i \in \text{point masses}} \begin{bmatrix} \bar{c}_{n,m}(x_i', y_i', z_i') \\ \bar{s}_{n,m}(x_i', y_i', z_i') \end{bmatrix} M_i.$$

Polynomial integration over a polyhedral domain

If the body has a continuous mass distribution, $\bar{c}_{n,m}$ and $\bar{s}_{n,m}$ must be integrated over the domain of the extended body. At this time two simplifications are made. One is to assume constant density σ. The integral (Eq. 2) becomes

$$\begin{bmatrix} \bar{C}_{n,m} \\ \bar{S}_{n,m} \end{bmatrix} = \sigma \iiint_{\text{extended body}} \begin{bmatrix} \bar{c}_{n,m} \\ \bar{s}_{n,m} \end{bmatrix} \mathrm{d}x'\, \mathrm{d}y'\, \mathrm{d}z'.$$

The other simplification is to assume that the extended body is a polyhedron. The remainder of this section uses the integration technique presented by Lien and Kajiya (1984). The integrand is a polynomial in the Cartesian coordinates of the differential element, and the integration domain is a polyhedron. There are no requirements that the polyhedron be convex or star-shaped, or that the origin be in the interior.

The polyhedron is partitioned into a collection of simplices (tetrahedra). One vertex of each simplex is placed at the origin, and the opposite face of the simplex is one of the polyhedron's faces. The vertex coordinates of a given simplex are denoted (0, 0, 0), (x_1, y_1, z_1), (x_2, y_2, z_2) and (x_3, y_3, z_3), and this simplex is the integration domain. Results from each simplex are accumulated to produce results for the entire polyhedron.

The 1, 2, 3 order of polyhedron vertices must proceed counterclockwise when viewed from the exterior. By following this convention, a determinant calculated below can be either positive or negative. A positive determinant corresponds to a simplex to be added to the total, and a negative determinant corresponds to one to be subtracted.

The next step is to change variables. Coordinates x', y', z' in a given simplex are expressed as functions of new variables X, Y, Z with the mapping

$$\begin{aligned} x'(X, Y, Z) &= x_1 X + x_2 Y + x_3 Z, \\ y'(X, Y, Z) &= y_1 X + y_2 Y + y_3 Z, \\ z'(X, Y, Z) &= z_1 X + z_2 Y + z_3 Z. \end{aligned} \tag{10}$$

Each of these is termed a trinomial. The integration domain in these variables is a "standard" simplex with vertices $(X, Y, Z) = (0, 0, 0)$, (1, 0, 0), (0, 1, 0) and (0, 0, 1). It is particularly easy to deal with.

Before the change of variables, the integrands $\bar{c}_{n,m}$ and $\bar{s}_{n,m}$ are homogeneous polynomials of degree n in x', y', z'. After the change, they are still homogeneous polynomials of the same degree, but are expressed in terms of X, Y, Z:

$$\begin{bmatrix} \bar{c}_{n,m}(x', y', z') \\ \bar{s}_{n,m}(x', y', z') \end{bmatrix} \rightarrow \begin{bmatrix} \bar{c}_{n,m}(X, Y, Z) \\ \bar{s}_{n,m}(X, Y, Z) \end{bmatrix} = \sum_{i+j+k=n} \begin{bmatrix} \bar{\alpha}_{i,j,k} \\ \bar{\beta}_{i,j,k} \end{bmatrix} X^i Y^j Z^k.$$

The summation is taken over all combinations of nonnegative exponents i, j, k which sum to n, the degree of the polynomial (and of the harmonic coefficient). Symbols $\bar{\alpha}_{i,j,k}$ and $\bar{\beta}_{i,j,k}$ represent the X, Y, Z trinomial coefficients.

It might seem that variables x', y', z' must be used in the recurrences to calculate the integrand polynomials, and the X, Y, Z variables are substituted afterwards to calculate $\bar{c}_{n,m}$ and $\bar{s}_{n,m}$. This is not true. The substitutions can be handled automatically by the recurrence machinery. Instead of supplying the scalar x' to the recurrences, the degree-1 trinomial $x_1X + x_2Y + x_3Z$ is supplied, and likewise for y' and z'. In this way, the integrands $\bar{c}_{n,m}$ and $\bar{s}_{n,m}$ are calculated directly in the X, Y, Z variables

With the change of variables, the integrands must be multiplied by the determinant of the Jacobian matrix J of the transformation (Eq. 10). J is constant with respect to the integration variables X, Y, Z.

$$J \equiv \frac{\partial(x', y', z')}{\partial(X, Y, Z)} = \begin{bmatrix} x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \\ z_1 & z_2 & z_3 \end{bmatrix}. \qquad (11)$$

(For reasons given previously, the absolute value of the Jacobian determinant is not taken.)

Final expressions for the integrated harmonic coefficients $\bar{C}_{n,m}$ and $\bar{S}_{n,m}$ are derived in the following way:

$$\begin{bmatrix} \bar{C}_{n,m} \\ \bar{S}_{n,m} \end{bmatrix} = \iiint_{\text{extended body}} \begin{bmatrix} \bar{c}_{n,m} \\ \bar{s}_{n,m} \end{bmatrix} \mathrm{d}m$$

$$= \sigma \sum_{\text{simplices}} \left(\iiint_{\text{simplex}} \begin{bmatrix} \bar{c}_{n,m}(x', y', z') \\ \bar{s}_{n,m}(x', y', z') \end{bmatrix} \mathrm{d}x'\,\mathrm{d}y'\,\mathrm{d}z' \right)$$

$$= \sigma \sum_{\text{simplices}} \left(\iiint_{\text{standard simplex}} \begin{bmatrix} \bar{c}_{n,m}(X, Y, Z) \\ \bar{s}_{n,m}(X, Y, Z) \end{bmatrix} \times \det J\,\mathrm{d}X\,\mathrm{d}Y\,\mathrm{d}Z \right)$$

$$= \sigma \sum_{\text{simplices}} \left(\det J \iiint_{\text{standard simplex}} \times \left(\sum_{i+j+k=n} \begin{bmatrix} \bar{\alpha}_{i,j,k} \\ \bar{\beta}_{i,j,k} \end{bmatrix} X^i Y^j Z^k \right) \mathrm{d}X\,\mathrm{d}Y\,\mathrm{d}Z \right)$$

$$= \sigma \sum_{\text{simplices}} \left(\det J \sum_{i+j+k=n} \begin{bmatrix} \bar{\alpha}_{i,j,k} \\ \bar{\beta}_{i,j,k} \end{bmatrix} \times \left(\iiint_{\text{standard simplex}} X^i Y^j Z^k \,\mathrm{d}X\,\mathrm{d}Y\,\mathrm{d}Z \right) \right).$$

Lien and Kajiya (1984) show the definite integral to equal $i!\,j!\,k!/(i+j+k+3)!$ An equivalent result appears in Section 50 of MacMillan (1930), where it is called the Dirichlet integral. Since $i + j + k = n$, we have the main theoretical result of this paper:

$$\begin{bmatrix} \bar{C}_{n,m} \\ \bar{S}_{n,m} \end{bmatrix} = \sigma \sum_{\text{simplices}} \left(\frac{\det J}{(n+3)!} \sum_{i+j+k=n} i!j!k! \begin{bmatrix} \bar{\alpha}_{i,j,k} \\ \bar{\beta}_{i,j,k} \end{bmatrix} \right). \qquad (12)$$

Harmonic coefficients $\bar{C}_{n,m}$ and $\bar{S}_{n,m}$ for the potential of an arbitrary-shaped, constant-density polyhedron have been calculated with no numerical approximations. Coefficients $\bar{\alpha}_{i,j,k}$, $\bar{\beta}_{i,j,\mathrm{k}}$ for trinomials in X, Y, Z are themselves polynomial functions of simplex coordinates x_1, …, z_3, and are calculated via recurrent relationships given earlier (Eqs. 7–9). The determinant det J is also a function of these coordinates.

IMPLEMENTATION

The ANSI C language program which accompanies this paper is based on a data structure called a trinomial which represents $\Sigma_{i+j+k=n} d_{i,j,k} X^i Y^j Z^k$. The structure contains the trinomial degree n and an array of $(n + 1)(n + 2)]/2$ coefficients $d_{i,j,k}$ corresponding to combinations of exponents i, j, k. Variables X, Y, Z are not stored, only implied. Subroutines implement operations such as addition, multiplication by a scalar, and general multiplication of two trinomials.

Here are two algorithms required to implement the formulae derived above.

Algorithm—trinomial times trinomial

The product of two trinomials involves the Cartesian product of the two sets of coefficients. A typical term is taken from each trinomial, for example, $d_{i,j,k}X^iY^jZ^k$ and $e_{r,s,t}X^rY^sZ^t$. Their symbolic product is $(d_{i,j,k}\ e_{r,s,t})X^{i+r}Y^{j+s}Z^{k+t}$. This indicates the following algorithm: take every pair of numeric coefficients $d_{i,j,k}$ and $e_{r,s,t}$, one from each trinomial, and add their product to the result trinomial coefficient having indices $i + r$, $j + s$, $k + t$. This algorithm is implemented in subroutine TriMult.

Algorithm—integrated trinomial

The multiplication algorithm just given deals with a representation of a trinomial integrand expressed through coefficients $d_{i,j,k}$. However, for this application we desire the definite integral

$$\iiint_{\text{simplex}} (x')^i (y')^j (z')^k \,\mathrm{d}x'\,\mathrm{d}y'\,\mathrm{d}z'$$

$$= \frac{\det J}{(n+3)!} \sum_{i+j+k=n} d_{i,j,k} \cdot i!j!k!.$$

The algorithm is: first calculate a three-dimensional table of mixing factors containing $i!j!k!/(n + 3)!$. Next, determine subscripts i, j, k of each trinomial coefficient $d_{i,j,k}$, multiply each coefficient by the corresponding mixing factor, and accumulate

Table 1. Program output. The first two columns are the degree n and order m. The third and fourth columns are fully normalized coefficients $\bar{C}_{n,m}$ and $\bar{S}_{n,m}$. Coefficients $\bar{S}_{2,1}$ and $\bar{C}_{3,0}$ should be interpreted as zero

```
Potential
coefficients of
polyhedron

                              reference distance    polyhedron density
total              mass 2.2          2.54                 5.52
0                     0:           1.67273
1                     0:          0.285162
1                     1:         -0.0950541                      0
2                     0:          0.0463802
2                     1:         -0.0401664           1.69407e-21
2                     2:          0.0200832            0.0200832
3                     0:         1.05879e-21
3                     1:         -0.00866287           0.0023626
3                     2:           0.012452            0.012452
3                     3:         -0.00305011          -0.00915032
4                     0:         -0.00339679
4                     1:          0.00211806           0.00272322
4                     2:          0.00427913           0.00406518
4                     3:         -0.00240166          -0.00720498
4                     4:        -0.000283038           0.00396253
```

such products. When finished, multiply the scalar accumulation by det J. This algorithm is implemented in function IntegrateOneSimplex.

Printout

A simple main program sets up coefficient tables, then incorporates a single simplex, and prints the resulting coefficient tables. Table 1 exhibits the output produced when the program is run on an Apple Macintosh using Symantec C 7.0.

Other comments about the program

It wastes computer memory to store trinomial coefficients in a three-dimensional array indexed by i, j, k. The known relationship $n = i + j + k$ among the exponents means only one plane through that array would be used. Instead, coefficients are packed in a one-dimensional array and a function of indices i, j, k and degree n is developed to select the proper element. Two suitable (and exactly equivalent) functions are index $(i, j, k; n) = (j + k)(j + k + 1)/2 + k = (n-1)(n-\mathrm{i} + 1)/2 + k$. Either returns a value in the range $[0, (n + 1)(n + 2)/2]$.

Another efficiency occurs when DO loops are set up as shown in Table 2. With these DO loops and the index function just given, the values of index (i, j, k; n) form the sequence 0, 1, 2 ... The C language's pointer arithmetic can be put to very good use.

Table 2. Efficient DO loops

```
do i from n to 0 step -1
do k from 0 to n-i step +1
j = n - i - k
 ... index (i, j, k; n) ...
enddo k
enddo i
```

In Equation (12) the density σ is constant for the entire polyhedron, and determinant det J is constant for each simplex. These constants can be incorporated in the expressions which anchor the recurrence (Eqs. 7a, b):

$$n = 0 \begin{cases} \bar{c}_{0,0} = 1/M \\ \bar{s}_{0,0} = 0 \end{cases} \rightarrow \begin{cases} \bar{c}_{0,0} = \sigma \cdot \det J/M \\ \bar{s}_{0,0} = 0 \end{cases} \tag{13a}$$

$$n = 1 \begin{cases} \bar{c}_{1,1} = x'/\sqrt{3}Ma \\ \bar{s}_{1,1} = y'/\sqrt{3}Ma \end{cases} \rightarrow \begin{cases} \bar{c}_{1,1} = \sigma \cdot \det J \cdot x'/\sqrt{3}Ma \\ \bar{s}_{1,1} = \sigma \cdot \det J \cdot y'/\sqrt{3}Ma \end{cases} \tag{13b}$$

The program has a few inefficiencies. (1) Square arrays "Cnm", "Snm", "vertical1Factors", and "vertical2Factors" have empty upper triangles. Almost half of cubic array "mixingFactors" is unused. (2) The program goes through the motions of calculating entries for the $m = 0$ column of "Snm" even though all are ignored. Such an inefficiency is tolerated, as coefficients are calculated for a given polyhedron only once. (3) Each trinomial is dimensioned to store coefficients of the maximum degree allowed by the compilation (parameter "nmMax"). Most trinomials are of lower degree and do not require so many coefficients. It is felt that using dynamic allocation to eliminate such overallocation would make the program harder to understand and run more slowly.

SUMMARY

A computer program has been developed for calculating the fully normalized harmonic coefficients $\bar{C}_{n,m}$ and $\bar{S}_{n,m}$ for the potential of a constant-density

polyhedron. Nonconvex, nonstar-shaped polyhedra are allowed. The algorithm is based on recurrent relationships derived here for the defining integrands, and an analytic integration technique described elsewhere.

The technique might be used to add small-scale details such as seamounts and ocean trenches to conventional spherical harmonic representations of Earth's gravitational potential. It might also be used for studying orbit evolution and perturbations in the vicinity of asteroids and comets.

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