MASTER’S DEGREE IN COMPUTER ENGINEERING

QUANTUM COMPUTING AND QUANTUM INTERNET

PROJECT DISCUSSION

E91 PROTOCOL

PROFESSORS

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INTRODUCTION

BACKGROUND

Securely sharing cryptographic keys over insecure channels is a fundamental challenge in modern cryptography.

THE PROBLEM

Classical key distribution methods rely on computational hardness, which can be broken by advances in computing.

The E91 protocol leverages quantum mechanics (*entanglement* and the *no-cloning theorem*) to enable infinitely secure key distribution, immune to eavesdropping.

SOLUTION

SCENARIOS TO ANALYZE

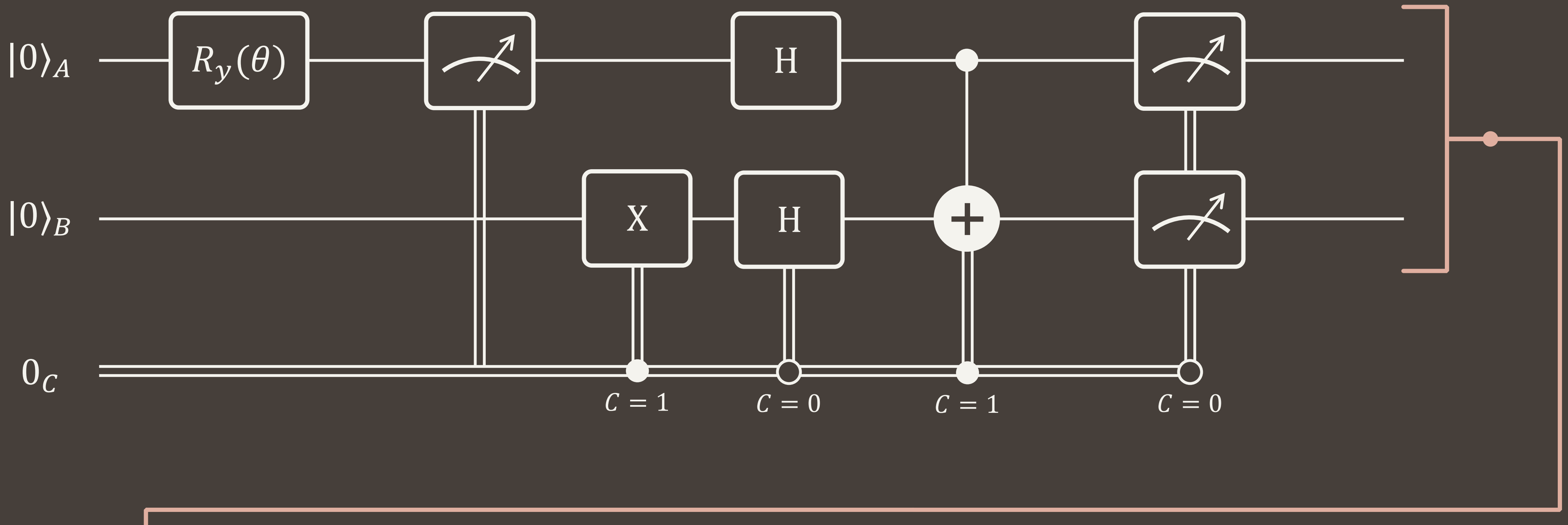
1. IDEAL CONDITIONS

2. CHANNEL ERRORS

3. EAVESDROPPING

THE **SPECIALIZED CIRCUIT** FOR GENERATING **WERNER STATES**

CONTROLLED BY **CHARLIE**



$$\rho_{AB} = \sin^2\left(\frac{\theta}{2}\right) \cdot |\psi^-\rangle\langle\psi^-| + \frac{\cos^2\left(\frac{\theta}{2}\right)}{4} \cdot I$$

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THIS IS A **WERNER STATE** WHEN THE **WERNER PARAMETER** (w) IS SET TO

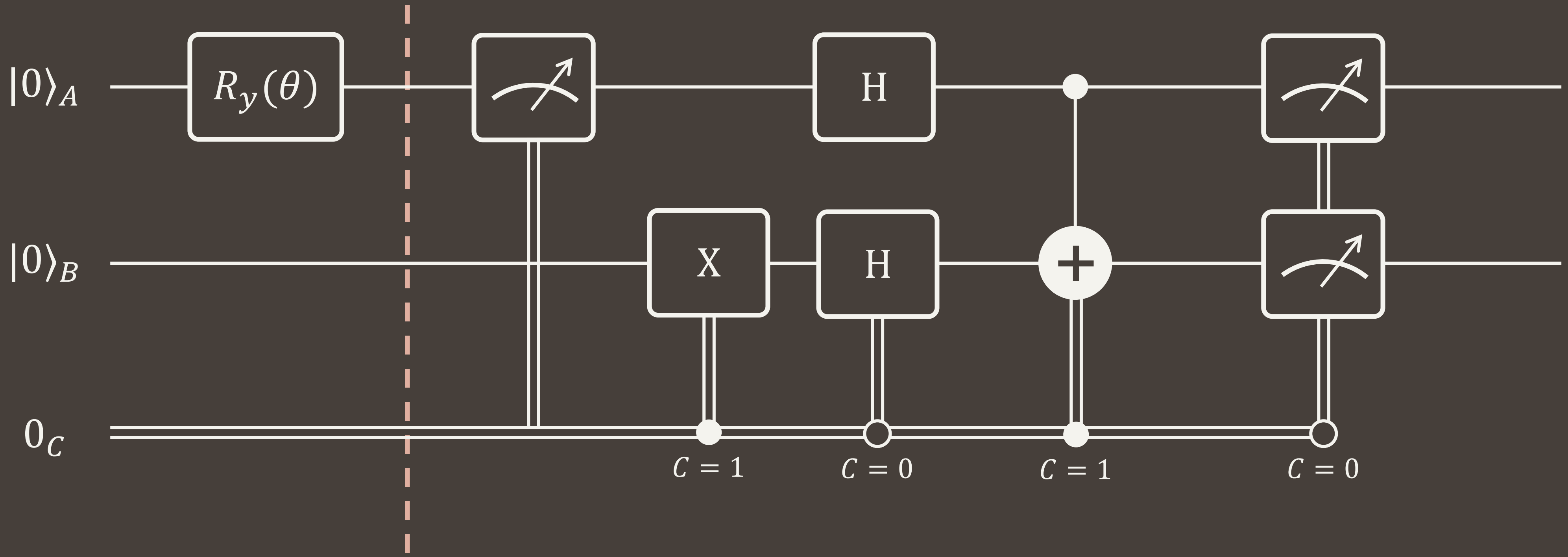
$$w = \sin^2\left(\frac{\theta}{2}\right) \quad \rightarrow \quad \rho_{AB} = w \cdot |\psi^-\rangle\langle\psi^-| + \frac{1-w}{4} \cdot I$$

THE **FIDELITY** CAN BE OBTAINED FROM THE WERNER PARAMETER USING THE FOLLOWING FORMULA:

$$F = \frac{3w+1}{4}$$

1. IDEAL CONDITIONS

CIRCUIT **ANALYSIS**



$$\cos\left(\frac{\theta}{2}\right)|0_B 0_A\rangle + \sin\left(\frac{\theta}{2}\right)|0_B 1_A\rangle$$

MEASUREMENT PROBABILITIES FOR ALICE'S QUBIT

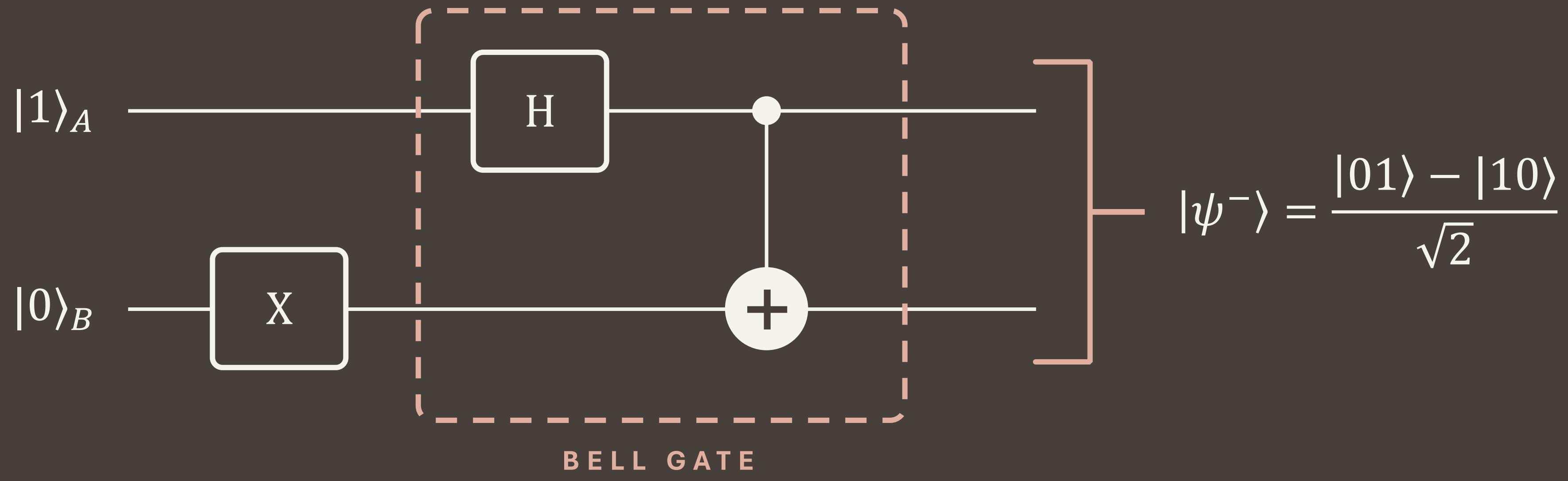
$$P\{0_A\} = \cos^2\left(\frac{\theta}{2}\right)$$

$$P\{1_A\} = \sin^2\left(\frac{\theta}{2}\right)$$

IN THE **IDEAL CASE**, SETTING $\boldsymbol{\theta} = \boldsymbol{\pi}$ SIMPLIFIES THE CIRCUIT AS FOLLOWS

(AFTER MEASURING ALICE'S QUBIT)

$$P\{1_A\} = \sin^2\left(\frac{\pi}{2}\right) = 1 \qquad P\{0_A\} = \cos^2\left(\frac{\pi}{2}\right) = 0$$



2. CHANNEL ERRORS

IN THE PRESENCE OF **CHANNEL ERRORS**, THEY ARE SIMULATED USING THE GENERAL CIRCUIT PREVIOUSLY ILLUSTRATED, WITH THE FOLLOWING θ VALUES:

$$0 \leq \theta < \pi$$

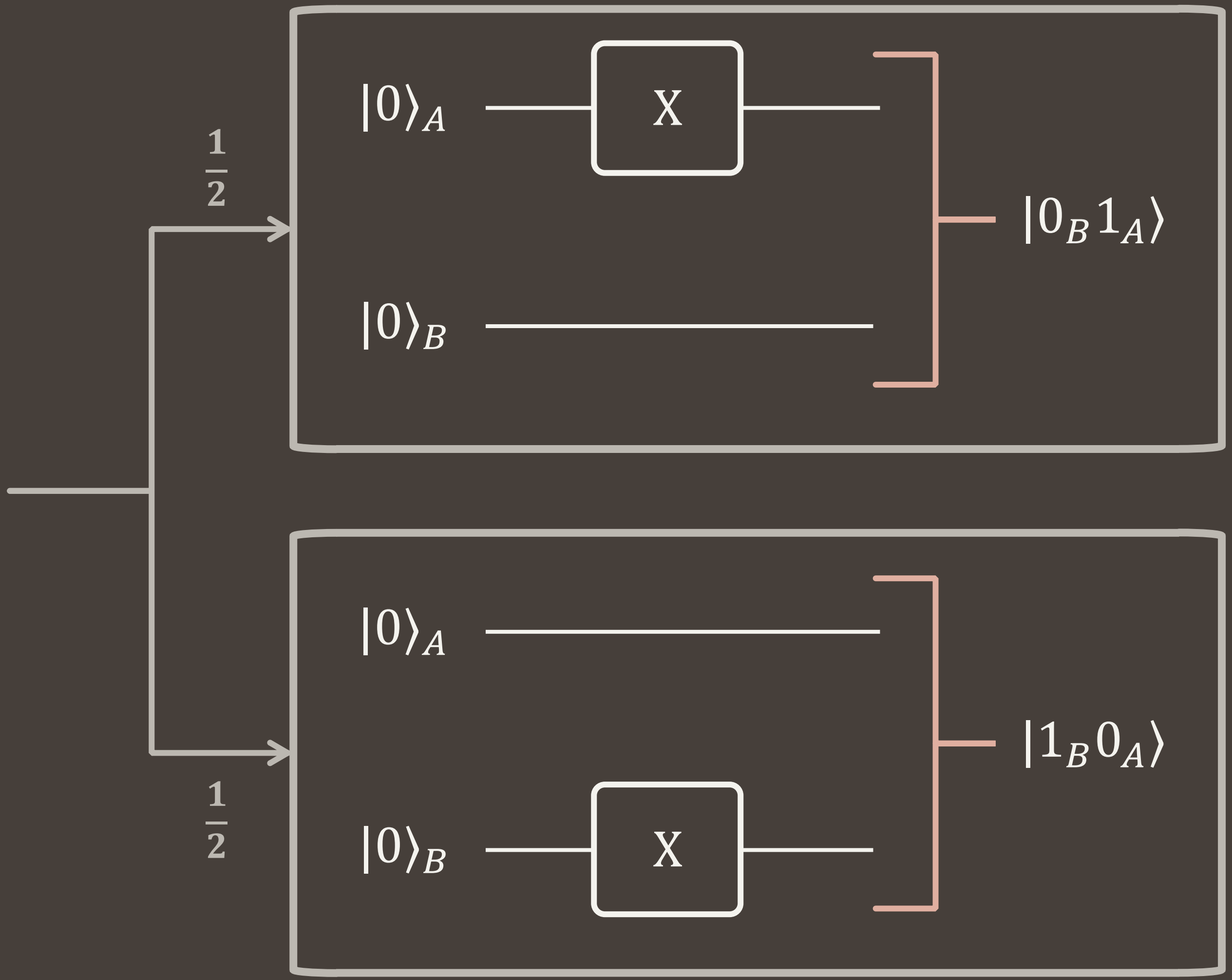
$$\rho_{AB} = \sin^2\left(\frac{\theta}{2}\right) \cdot |\psi^-\rangle\langle\psi^-| + \frac{\cos^2\left(\frac{\theta}{2}\right)}{4} \cdot I$$

NOTE THAT WHEN $\theta = 0$, THE OUTPUT OF THE CIRCUIT WILL BE A COMPLETELY DEPOLARIZED STATE.

3. EAVESDROPPING

IN THE **EAVESDROPPING** SCENARIO **EVE** TAKES CONTROL OF THE SITE-GENERATING ENTANGLED PHOTON PAIRS

EVE SENDS ONE OF THE FOLLOWING TWO STATES TO ALICE AND BOB WITH EQUAL PROBABILITY (1/2), INSTEAD OF AN ENTANGLED PAIR OF PHOTONS.



KEY
DETAILS

PROTOCOL **DETAILS**

OBSERVABLES

ALICE

$$A_0 = Z$$

$$A_1 = X$$

$$A_2 = \frac{Z + X}{\sqrt{2}}$$

BOB

$$B_0 = Z$$

$$B_1 = \frac{Z - X}{\sqrt{2}}$$

$$B_2 = \frac{Z + X}{\sqrt{2}}$$

CHSH CORRELATION VALUE

$$S = |\langle A_0 B_2 \rangle + \langle A_0 B_1 \rangle + \langle A_1 B_2 \rangle - \langle A_1 B_1 \rangle|$$

The **0-indexed** notation is used, as in the Python code, to avoid confusion between slides and implementation.

THEORY

OBSERVABLE

↓

HERMITIAN MATRIX

↓

REAL EIGENVALUES

↓

CAN BE **DISPLAYED** ON THE MEASURING DEVICE

↓

MEASUREMENT CAUSES **COLLAPSE** INTO ONE OF THE **EIGENSPACES** ASSOCIATED WITH THE RELATIVE EIGENVALUE

NOTEWORTHY IMPLEMENTATION STEPS

1. DEFINE THE **OBSERVABLE O**
2. FIND ITS **EIGENVALUES** AND **EIGENVECTORS**

$$\lambda_i \rightarrow |\lambda\rangle_i = \begin{bmatrix} \lambda_i^{(1)} \\ \lambda_i^{(2)} \end{bmatrix} \quad i = 1,2$$

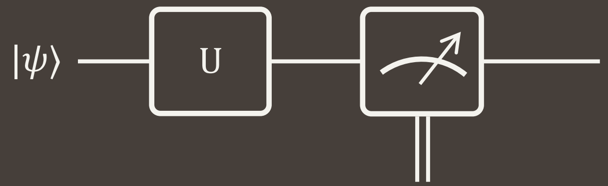
3. DETERMINE THE **UNITARY TRANSFORMATION** MAPPING EIGENVECTORS TO STANDARD BASIS STATES

$$\begin{bmatrix} \lambda_1^{(1)} & \lambda_1^{(2)} & 0 & 0 \\ 0 & 0 & \lambda_1^{(1)} & \lambda_1^{(2)} \\ \lambda_2^{(1)} & \lambda_2^{(2)} & 0 & 0 \\ 0 & 0 & \lambda_2^{(1)} & \lambda_2^{(2)} \end{bmatrix} \cdot \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix} \begin{matrix} \left.\begin{matrix} \\ \end{matrix}\right\} |0\rangle \\ \left.\begin{matrix} \\ \end{matrix}\right\} |1\rangle \end{matrix}$$

4. VERIFY THE FOUND MATRIX IS **UNITARY**

$$U = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \rightarrow UU^\dagger = U^\dagger U = I$$

5. **APPLY** THE UNITARY TRANSFORMATION TO THE QUBIT AND **MEASURE** IN THE STANDARD BASIS



→ THIS ENTIRE PROCESS ENABLES **MEASUREMENT AS IF USING THE ORIGINAL OBSERVABLE O**

METRICS AND PARAMETERS

METRICS TO EVALUATE

CHSH
CORRELATION VALUE

$$S = |\langle A_0B_2 \rangle + \langle A_0B_1 \rangle + \langle A_1B_2 \rangle - \langle A_1B_1 \rangle|$$

MISMATCH RATIO

$$R_{mis} = \frac{m}{l}$$

NUMBER OF **MISMATCHED BITS** BETWEEN ALICE'S AMD BOB'S SECRET KEYS

TOTAL KEY LENGTH

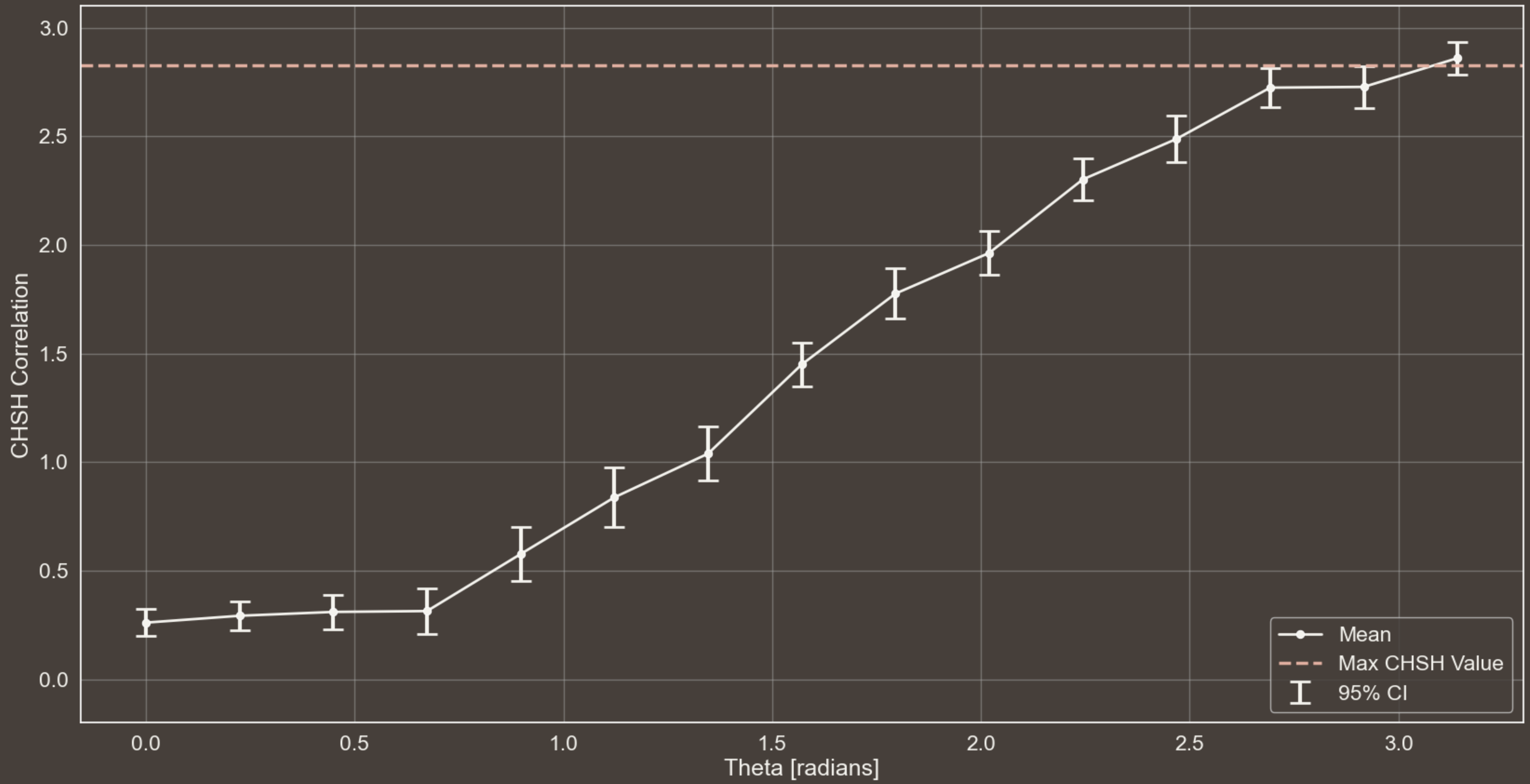
PLOTTED AS A FUNCTION OF $\boldsymbol{\theta}$ OR OF THE **WERNER PARAMETER**

EXPERIMENT **PARAMETERS**

- **n** → NUMBER OF EXECUTIONS PER SETTING
- **θ** → THETA VALUES USED FOR GENERATING THE WERNER STATES
- **Eavesdropping** → WHETHER EVE IS EAVESDROPPING OR NOT
- **#EPR Pairs** → NUMBER OF EPR PAIRS GENERATED IN ONE PROTOCOL EXECUTION

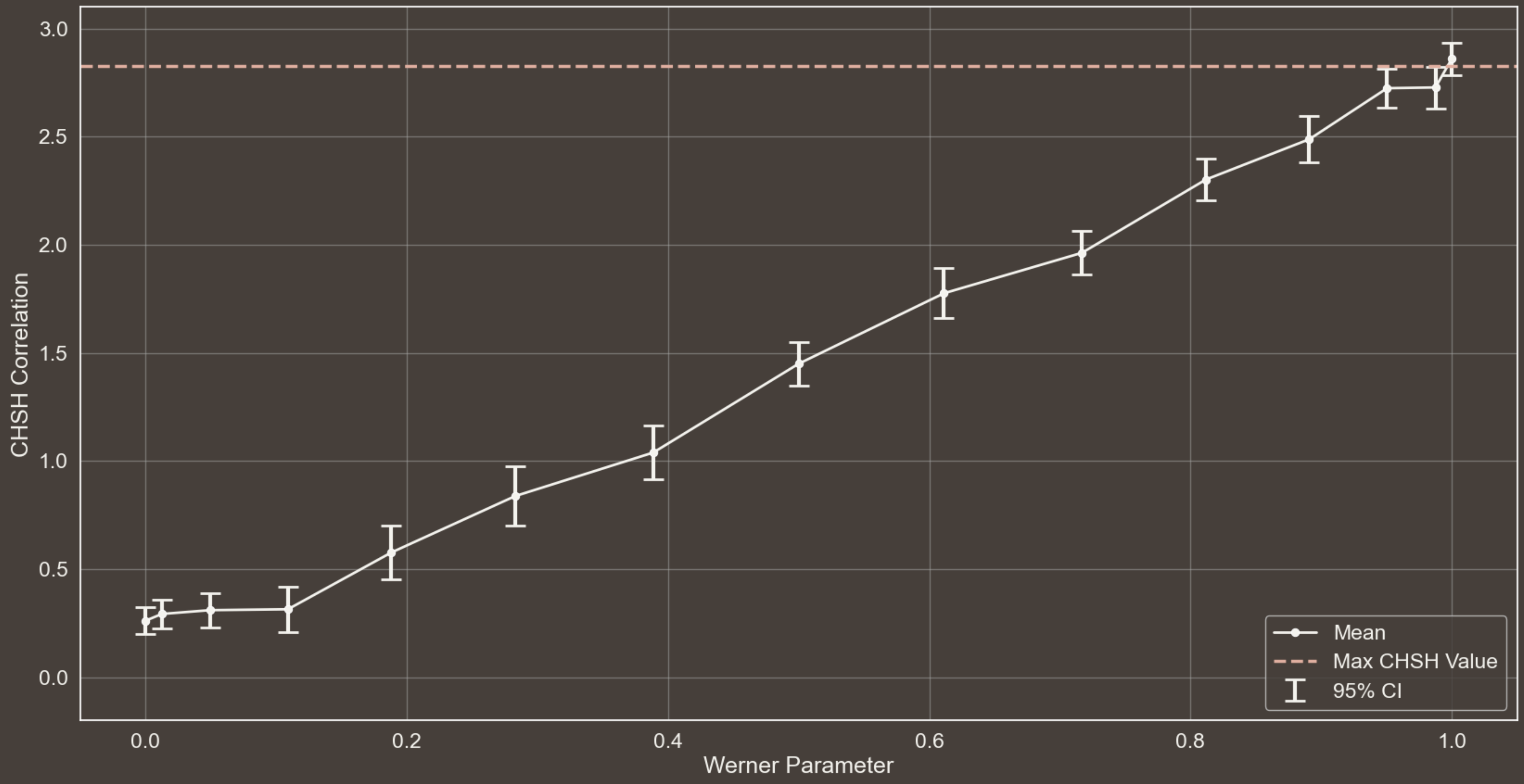
RESULTS
ANALYSIS

CHSH CORRELATION WITH 95% CI **vs THETA**



n → 30 **θ** → $\left[\frac{i}{14}\pi \mid i = 0,1,\ldots,14\right]$ **Eavesdropping** → False **#EPR Pairs** → 300

CHSH CORRELATION WITH 95% CI **vs WERNER PARAMETER**



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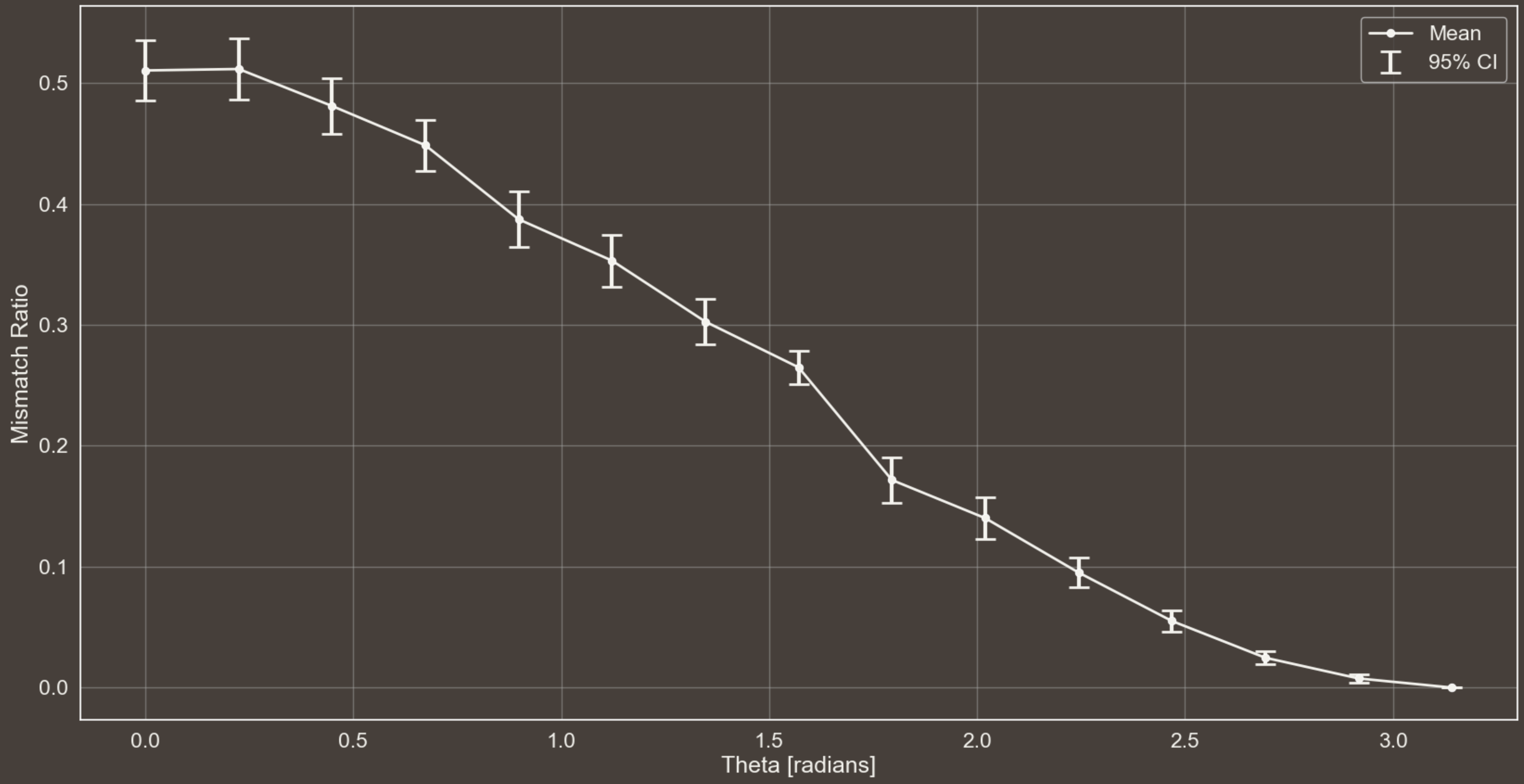
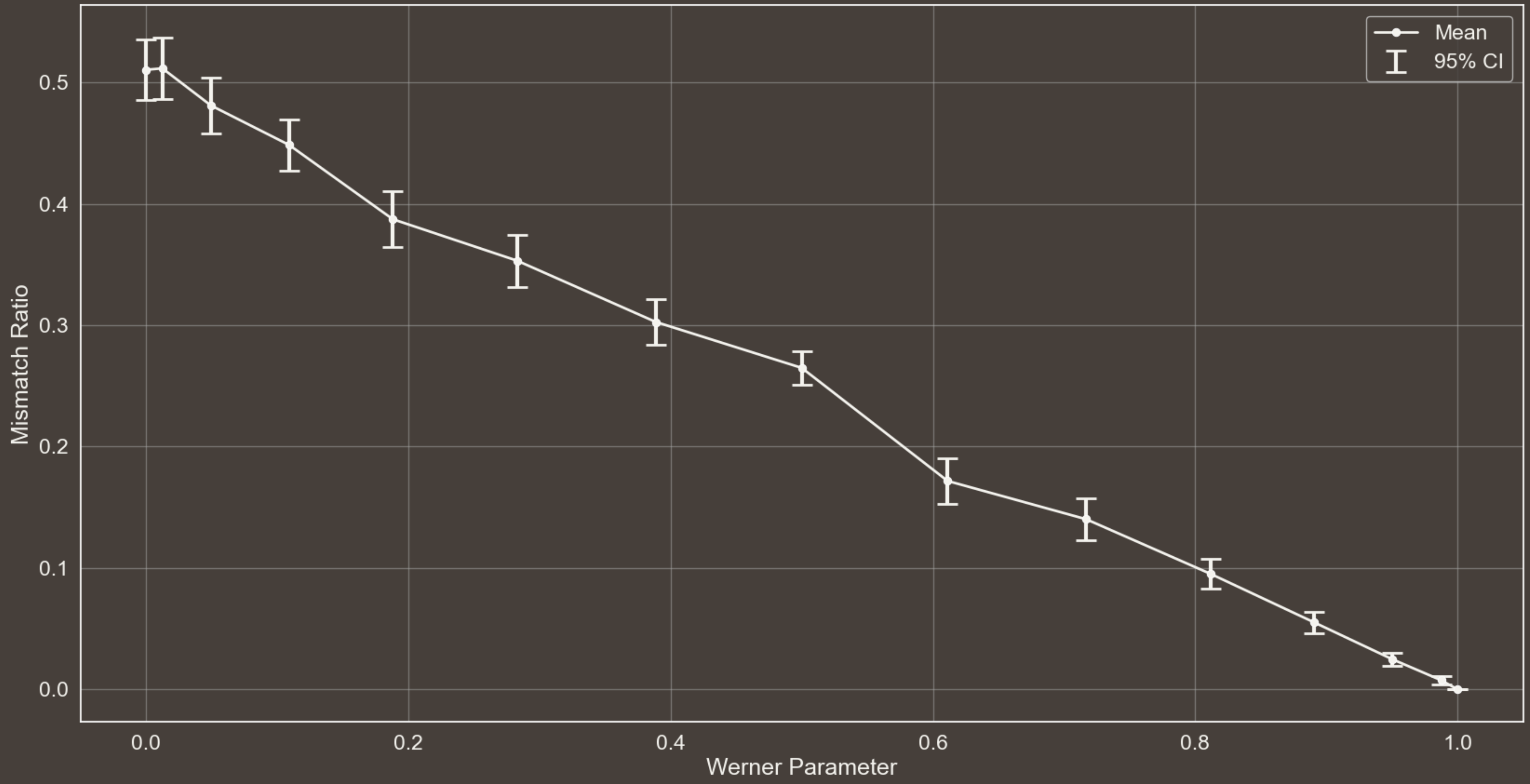
MISMATCH RATIO WITH 95% CI **vs THETA**



n → 30 **θ** → $\left[\frac{i}{14}\pi \mid i = 0,1,\ldots,14\right]$ **Eavesdropping** → False **#EPR Pairs** → 300

MISMATCH RATIO WITH 95% CI vs WERNER PARAMETER



n → 30 **θ** → $\left[\frac{i}{14}\pi \mid i = 0,1,\ldots,14\right]$ **Eavesdropping** → False **#EPR Pairs** → 300

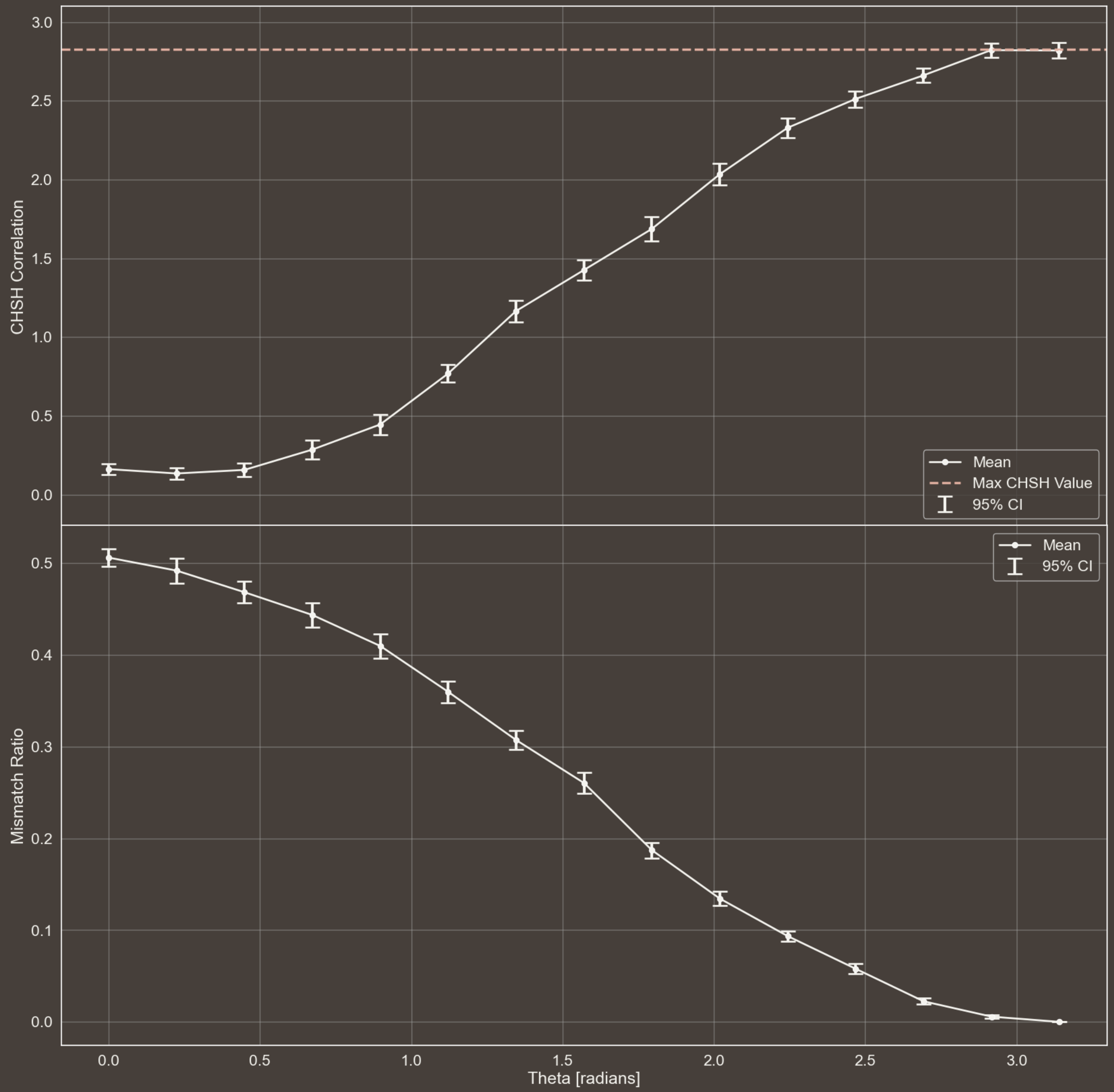
EFFECT OF VARYING **#EPR PAIRS**

n → 30

θ → $\left[\frac{i}{14}\pi \mid i = 0,1,\ldots,14\right]$

Eavesdropping → False

#EPR Pairs → 1000



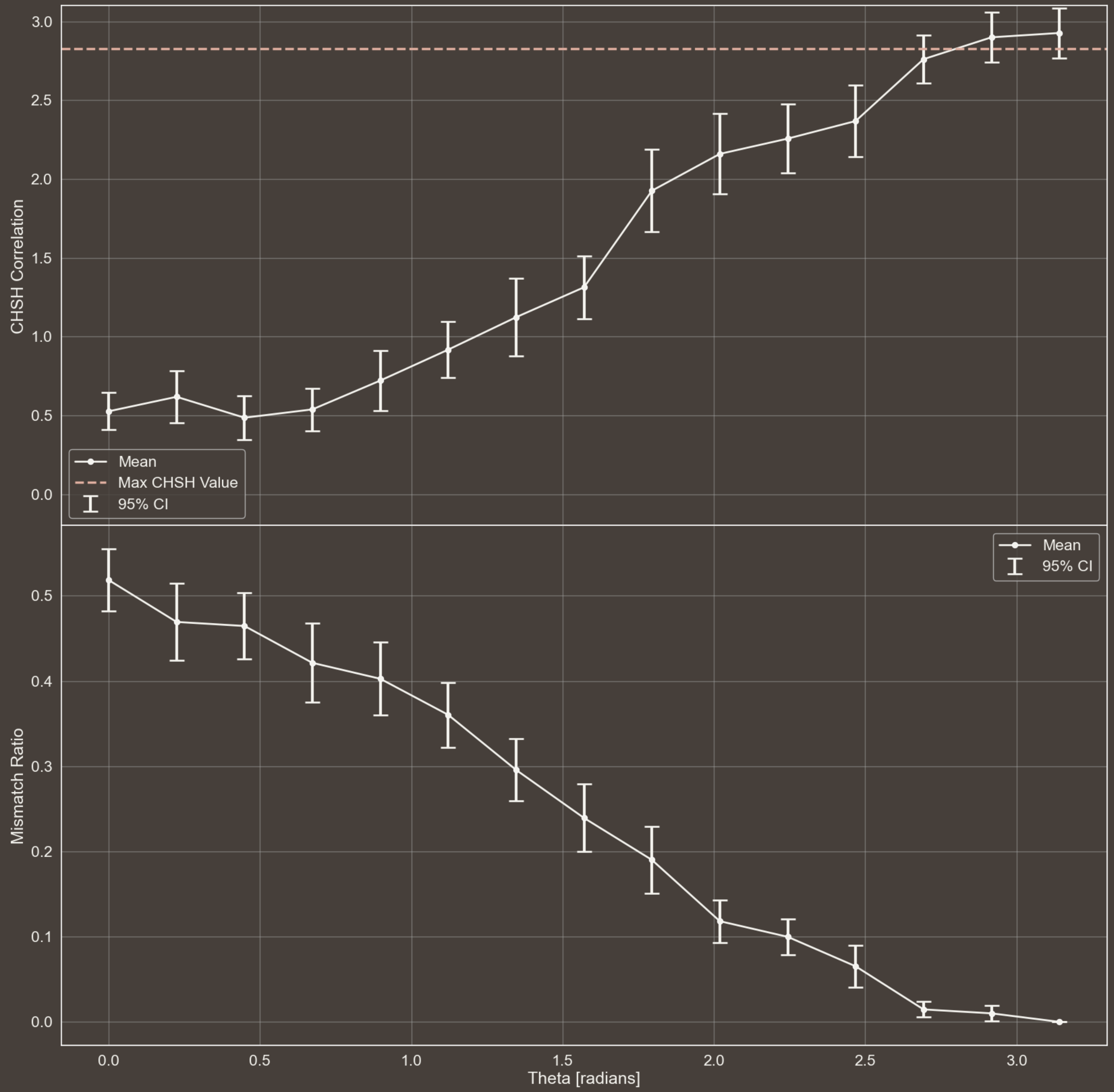
EFFECT OF VARYING **#EPR PAIRS**

n → 30

θ → $\left[\frac{i}{14}\pi \mid i = 0,1,\ldots,14\right]$

Eavesdropping → False

#EPR Pairs → 80



WHAT'S THE EFFECT

OF VARYING THE NUMBER OF ENTANGLED PAIRS PER PROTOCOL EXECUTION ON THE EVALUATED METRICS?

THE PRIMARY EFFECT IS INCREASED **INSTABILITY** IN THE MEASURED QUANTITIES, REFLECTED IN THE **CONFIDENCE INTERVALS**. THIS OCCURS BECAUSE USING FEWER EPR PAIRS PER PROTOCOL EXECUTION AMPLIFIES **STATISTICAL NOISE**.

CONCRETE EXAMPLE

THE NUMBER OF BITS IN THE RESULTING KEY TENDS TOWARD $\frac{2}{9}$, AS ONLY TWO OUT OF NINE POSSIBLE CHOICES OF OBSERVABLES BY ALICE AND BOB CONTRIBUTE TO KEY GENERATION. WITH MORE EPR PAIRS, THE KEY LENGTH RATIO APPROACHES $\frac{2}{9}$, WHEREAS WITH FEWER PAIRS, NOISE BECOMES MORE PRONOUNCED.

THE SAME REASONING APPLIES TO THE EVALUATED METRICS.

WHAT WILL BE THE **RESULTS** IN THE SCENARIO WHERE THERE IS **EVE** THE EAVESDROPPER?

IN THIS SCENARIO, EVE SENDS ALICE AND BOB ONLY QUBIT PAIRS IN THE STATES $|01\rangle$ AND $|10\rangle$.

SINCE THESE STATES ARE NOT ENTANGLED, THE **CHSH CORRELATION** SHOULD REFLECT THIS. INSTEAD OF REACHING THE THEORETICAL MAXIMUM OF $2\sqrt{2}$, THE VALUE WILL BE SIGNIFICANTLY LOWER. THIS ALLOWS ALICE AND BOB TO DETECT THE ANOMALY AND ABORT THE PROTOCOL.

REGARDING THE **MISMATCH RATIO**, WHEN BOTH ALICE AND BOB MEASURE USING THE Z OBSERVABLE, THEY WILL OBTAIN ANTI-CORRELATED RESULTS, PRODUCING A CORRECT MATCHING BIT FOR KEY GENERATION. HOWEVER, WHEN THEY BOTH MEASURE USING $\frac{Z+X}{\sqrt{2}}$, THE OUTCOME IS LESS STRAIGHTFORWARD, REQUIRING FURTHER CALCULATIONS.

MEASURING IN THIS OBSERVABLE REQUIRES A UNITARY TRANSFORMATION THAT MAPS ITS EIGENVECTORS TO THE STANDARD BASIS, FOLLOWED BY A MEASUREMENT IN THE Z BASIS. THE REQUIRED UNITARY U IS GIVEN BELOW AND MUST BE APPLIED TO BOTH QUBITS BEFORE MEASUREMENT.

$$U = \begin{bmatrix} 0.924 & 0.383 \\ -0.383 & 0.924 \end{bmatrix}$$

Results are **rounded to three decimal places** in the slides for presentation purposes. In Python, more decimals are considered for higher accuracy.

$$U = \begin{bmatrix} 0.924 & 0.383 \\ -0.383 & 0.924 \end{bmatrix}$$

CONSIDER THE CASE WHERE EVE SENDS $|10\rangle$ TO ALICE AND BOB (THE ANALYSIS FOR THE OTHER STATE IS ANALOGOUS). THE STATE OF THE QUBITS AFTER APPLYING U TO BOTH CAN BE COMPUTED.

$$|10\rangle \cdot (U \otimes U) = (0.383|0\rangle + 0.924|1\rangle) \otimes (0.924|0\rangle - 0.383|1\rangle) =$$

$$= 0.354|00\rangle - 0.147|01\rangle + 0.854|10\rangle - 0.354|11\rangle$$

ALL FOUR MEASUREMENT OUTCOMES IN THE Z BASIS ARE POSSIBLE. WHEN THE OUTCOMES ARE 00 OR 11, THE RESULTS ARE CORRELATED, CAUSING BOB, UPON FLIPPING HIS VALUE, TO INTRODUCE A MISMATCH BETWEEN HIS KEY AND ALICE'S. THE PROBABILITY OF OBTAINING CORRELATED RESULTS IS GIVEN BY:

$$0.354^2 + 0.354^2 = 0.251$$

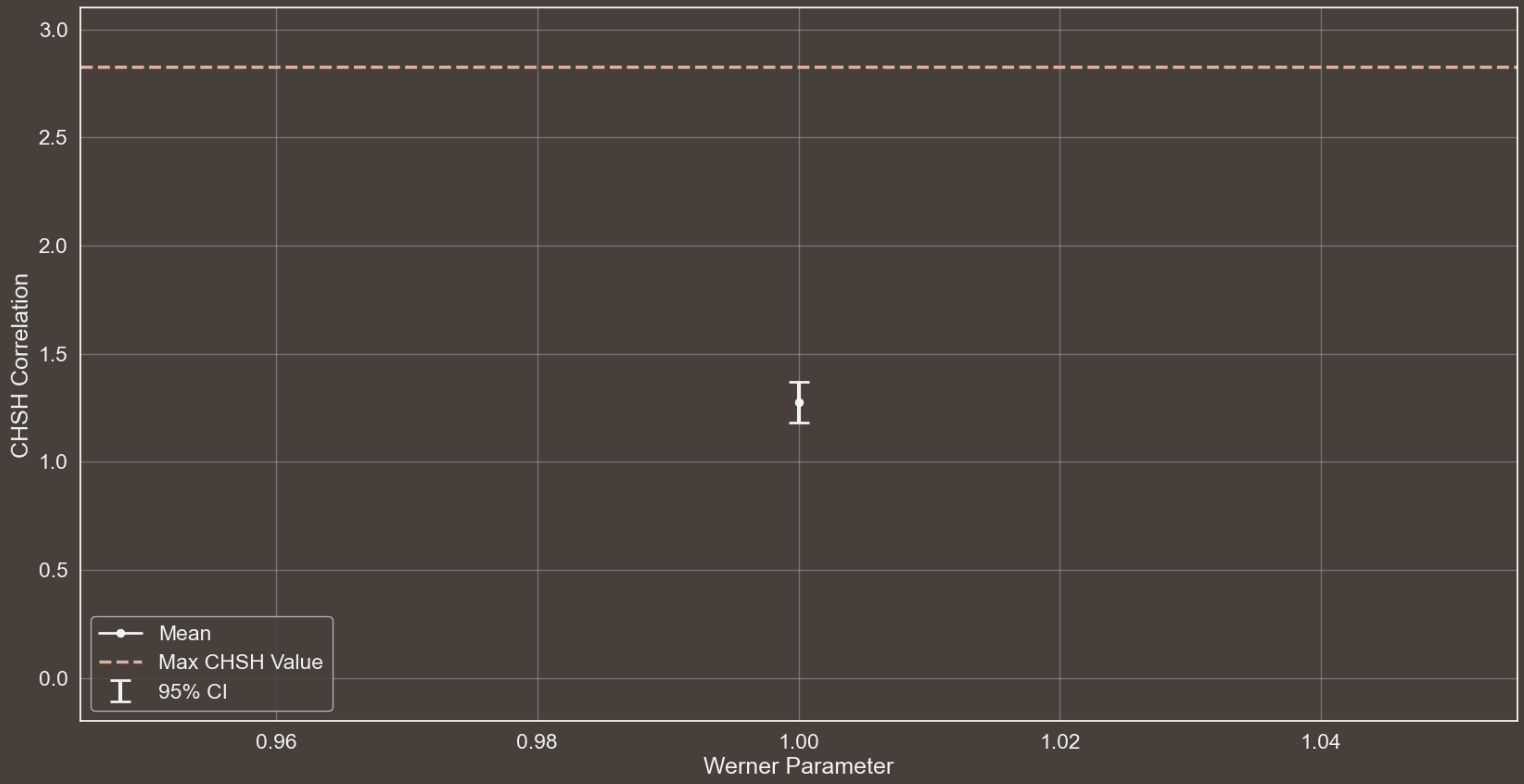
SINCE THIS SCENARIO REPRESENTS ONLY ONE OF THE TWO CASES CONTRIBUTING TO KEY GENERATION (THE OTHER BEING WHEN BOTH ALICE AND BOB CHOOSE Z AS THE OBSERVABLE, WHICH DOES NOT INTRODUCE MISMATCHED BITS), MULTIPLYING BY $\frac{1}{2}$ GIVES THE TOTAL PROBABILITY OF KEY ERRORS.

$$0.251 \cdot \frac{1}{2} = 0.125$$

IN THE MISMATCH RATIO PLOT, THE CONFIDENCE INTERVAL SHOULD ENCOMPASS THIS VALUE

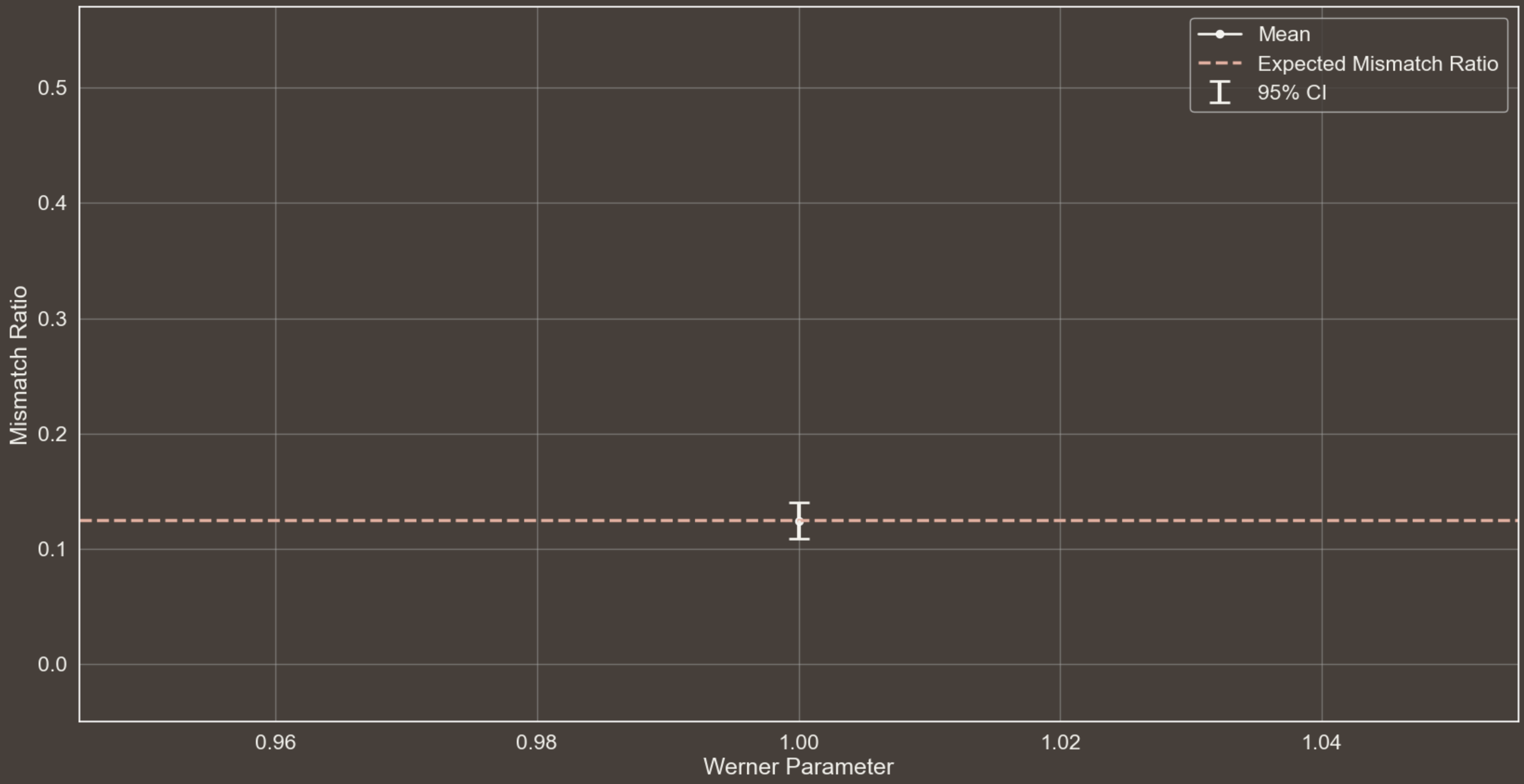
Results are **rounded to three decimal places** in the slides for presentation purposes. In Python, more decimals are considered for higher accuracy.

CHSH CORRELATION WITH 95% CI **vs WERNER PARAMETER**



n → 30 **θ** → $[\pi]$ **Eavesdropping** → True **#EPR Pairs** → 300

MISMATCH RATIO WITH 95% CI vs WERNER PARAMETER



n → 30 **θ** → $[\pi]$ **Eavesdropping** → True **#EPR Pairs** → 300

REFERENCES

[1] CHSH INEQUALITY
https://en.wikipedia.org/wiki/CHSH_inequality

[2] QUANTUM CORRELATION
https://en.wikipedia.org/wiki/Quantum_correlation

[3] TSIRELSON'S BOUND
https://en.wikipedia.org/wiki/Tsirelson's_bound

PALETTE

coolors.co/palette/463f3a-8a817c-bcb8b1-f4f3ee-e0afa0

ICONS

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THANKS

FOR YOUR ATTENTION