UNIVERSITY OF PISA

COMPUTER ENGINEERING MASTER DEGREE

Performance Evaluation of Computer Systems and Networks

Quick Checkout

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Contents

1 Description

This project aims to conduct an in-depth study of the supermarket checkout system, focusing on the behaviour of both quick and normal checkout tills.

2 Objectives

Analyze the Mean Waiting Time and the Mean Response Time of the system with different settings of quick and normal checkout tills.

3 Key Performance Indices (KPIs)

The specific areas of interest include:

$$E[W] \qquad E[N_q] \qquad E[t_S]$$

4 Model

The model chosen to represent the system is the following (Figure 1):

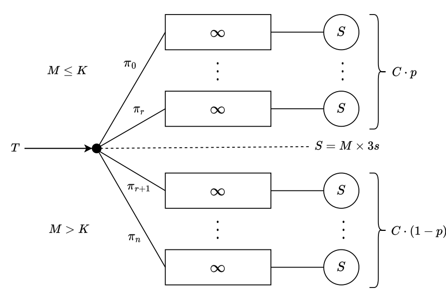


Figure 1: Model of the System.

The symbols utilized are:

- M: RV that represents **the number of items in a customer's cart**.
- T: RV that represents **the inter-arrival time of customers**.
- C: **total number of tills**.
- p: **percentage of quick checkout tills**.
- K: **threshold** for a costumer to be routed in the quick subsystem.

- $\begin{cases} \pi_i = \dfrac{\alpha(K)}{C \cdot p} & 0 \leq i \leq r \\ \pi_j = \dfrac{(1-\alpha(K))}{C \cdot (1-p)} & r+1 \leq j \leq n \end{cases}$ with $\alpha(K) = P\{M \leq K\}$.

The system architecture is represented by **parallel M/G/1 subsystems**. This is due to the **probabilistic node** which truncates the distributions of M (i.e. the truncation involves also the service times, since $t_S = M \times 3\text{s}$).

5 Assumptions and Validation

In the context of the assumptions, it is necessary to mention the following:

1. The size of the queues is **infinite**, and individuals **cannot surpass** others ahead (i.e. FIFO).
2. Once in line, **it is not possible to leave the queue**.
3. A customer with a small number of items ($M \leq K$) can **only** join quick tills.
4. During a simulation, **it is not possible to modify the ratio between normal and quick tills**, and it's not possible to add more of them in situations of high load.
5. **The idle time in case of at least one costumer in queue is zero**.
6. **A customer joins the queue with fewest people**. However, **no distinction** is made if a queue is empty and someone is being served or empty with no one being served.

Taking any supermarket into consideration, the formulated assumptions are not so extreme:

1. Even in a real system, the size of queues in a supermarket does not have a predefined limit. Additionally, customers follow a FIFO policy, as the customer who joins the queue first is unlikely to let others pass ahead of them.
2. Once a person is in line, they will never leave the cart or the supermarket.
3. If a customer has few items, they will always prefer to go to quick tills.
4. It's only considered the steady-state case of a supermarket where all the checkout tills are already open, and it is not possible to open new ones (nor close them).
5. If the cashier has customers in line, they do not stop to take a break.

These observations are all guided by **common sense**.

6 Factors and Parameters

The factors involved are:

- **The percentage of quick checkout tills**: p.
- **The threshold of items**: K.

The parameters are:

- **The total number of tills:** C.
- **Distribution of the inter-arrival time of customers**: $exp(\lambda_T)$.
- **Distribution of the number of items in a customer's cart**: $exp(\lambda_M)$ and $lognormal(\mu, \sigma^2)$.

7 Simulation Tool and Implementation

The simulation tool utilized for this project is **OMNeT++**, an open-source, modular framework designed for discrete-event system modeling and simulation.

To implement the theoretical model, the following modules were taken from the **queueinglib** package found in OMNeT++:

- **Source**: generates customers based on the arrival time distribution T.
- **Classifier**: represents the load balancer that routes customers to the two subsystems.
- **Queue**: represents the checkout till queue.
- **Local-Sink**: represents a cashier.
- **Global-Sink**: collects statistics on the simulation execution.

Apart from minor adjustments, the module that received a significant modification is the **Classifier**. Two policies were introduced: *"equally likely"* and *"join the shortest queue"*, both of which can be configured before the simulation execution.

The difference between **Local-Sink** and **Global-Sink** lies in the fact that the latter is in charge of calculating statistics related to the whole system, while the former is only in charge of the queue to which it is connected to.

These modules can be visualized in Figure 2.

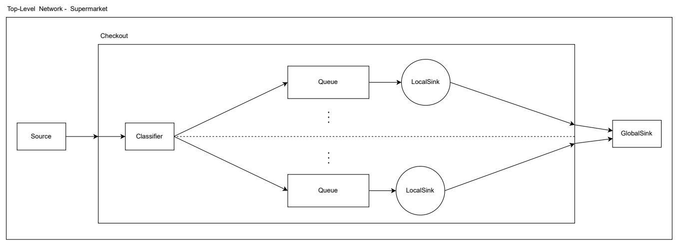


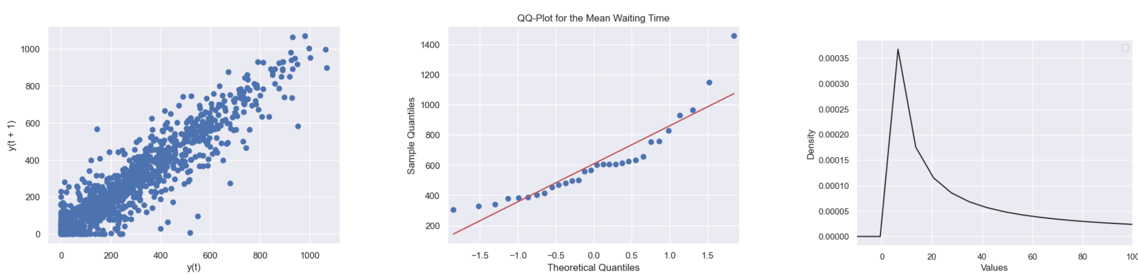
Figure 2: Model of the System.

8 Verification of the Implemented Simulation Model

8.1 Correlation and Lognormal behaviour of the Waiting Time

All the results showed correlation (Figure 3a) and a lognormal behaviour of the distribution of the Mean Waiting Time (Figure 3b) because the Waiting Times inside a simulation are themselves distributed like a lognormal (Figure 3c) and hence the hypotheses of the CLT are not satisfied. Since these observations, sub-sampling was applied until the plot showed low correlation between data, and the Cox Method [1] for the CI was used (95%):

$$\bar{Y} + \frac{S^2}{2} \pm z\sqrt{\frac{S^2}{n} + \frac{S^4}{2(n-1)}}$$



(a) Lag Plot of the Waiting Times of a till.

(b) QQ Plot of the Mean Waiting Time of a till.

(c) Lognormal distribution of the Waiting Times of a till.

Figure 3: Correlation and Lognormal behaviour.

8.2 Consistency Test

In the **Consistency Test** is expected to see that when **halving both the Mean inter-arrival time** ($E[T]$) **and the Mean number of items in a customer's cart** ($E[M]$), **the mean waiting time in each queue halves accordingly (Appendix A)**.

The configuration used are (Table 1):

Table 1: Consistency Test configurations.

	E[T] [s]	**E[M] [items]**	**K [items]**
Slow Behaviour Experiment	60	40	16
Balanced Behaviour Experiment	30	20	8
Fast Behaviour Experiment	15	10	4

K was determined such that $\alpha(K) = 0.33$ considering the exponential distribution. The results can be shown in Figure 4.

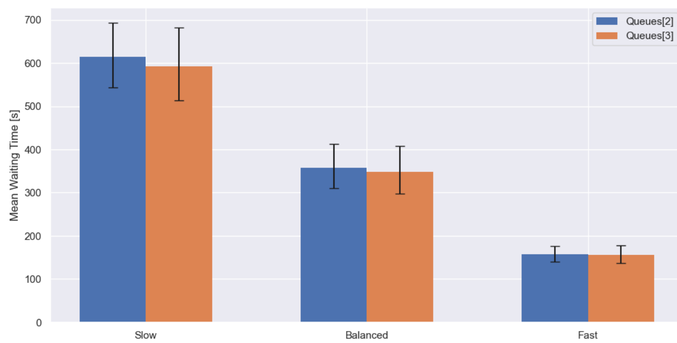


Figure 4: Queues[2] and Queues[3] (Normal Tills) for the Slow, Balanced and Fast configurations.

8.3 Degeneracy Test

In Degeneracy Tests extreme scenarios are tested, in particular:

- $K = 0$ and $p = 0.5$ (counterpart $K = 1000$ and $p = 0.5$)
- $p = 0$ (counterpart $p = 1$), $K = N/A$

In the first case, it's expected **not to receive** customers in the queues which are impossible to reach due to K. In the second case, given that all tills are of the same type, it is expected a **lower mean number of customers** in the queues. The configuration used is:

$$E[T] = 30 \qquad E[M] = 20 \qquad C = 4$$

8.3.1 Degeneracy Tests results

It's worth noticing that test 3 and 4 represent basically the **same scenario**.

Table 2: Mean and Max Number of Customers in the Queues with $K = 0$ and $p = 0.5$.

	Q[0]	Q[1]	Q[2]	Q[3]
Mean	0.0	0.0	18.2632	17.7777
Max	0.0	0.0	36.9281	36.3864

Table 3: Mean and Max Number of Customers in the Queues with $K = 1000$ and $p = 0.5$.

	Q[0]	Q[1]	Q[2]	Q[3]
Mean	18.2632	17.7777	0.0	0.0
Max	36.9281	36.3864	0.0	0.0

Table 4: Mean Number of Customers in the Queues with $p = 0$.

	Q[0]	Q[1]	Q[2]	Q[3]
Mean	0.6005	0.365484	0.1636	0.0498

Table 5: Mean Number of Customers in the Queues with $p = 1$.

	Q[0]	Q[1]	Q[2]	Q[3]
Mean	0.6005	0.3655	0.1636	0.0498

8.4 Continuity Test

In the Continuity Test, with configuration:

$$E[T] = 30 \qquad E[M] = 20 \qquad K = 8 \qquad C = 4$$

what is expected is that, by slightly increasing $E[T]$, **the Global Mean Waiting Time, as seen by the Global-Sink, decreases accordingly** (Figure 5).

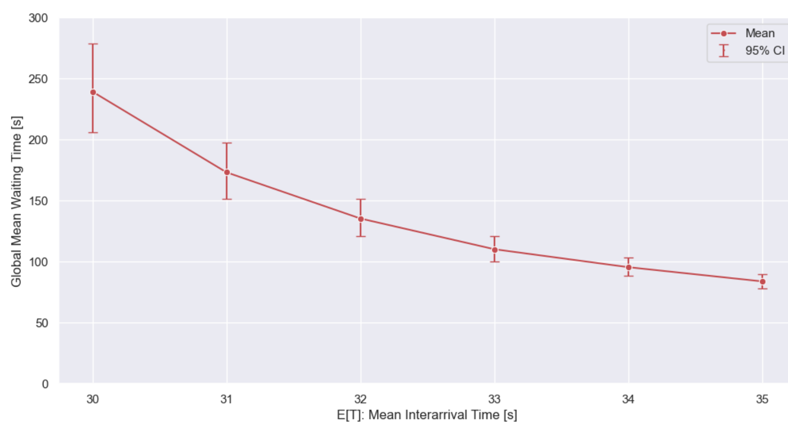


Figure 5: Continuity Test: Global Mean Waiting Time for $E[T]$ from 30 to 35.

8.5 Verification against the Theoretical Model

Thanks to a correct implementation, the results (Appendix B for tables) obtained in the simulation are **coherent** with the theoretical model in the discrete case (Appendix C) and *"Equally likely"* configuration. Indeed, in the proposed theoretical model, **the load balancing is static and independent of the actual state of the system**.

8.5.1 Indexes Formula

$$E[t_{S_i}] = E[M_i] \cdot 3$$

$$\rho_i = \lambda_{T_i} \cdot E[t_{S_i}] \qquad E[N_i] = \rho_i + \frac{\rho_i^2 + \lambda_{T_i}^2 \cdot Var(t_{S_i})}{2 \cdot (1 - \rho_i)}$$

$$E[Nq_i] = E[N_i] - \rho_i \qquad E[W_i] = \frac{E[Nq_i]}{\lambda_{T_i}}$$

The same goes for j (where i is a till in the quick checkout subsystem, j in the normal one). The *pdf*s utilized for the evaluation of $E[M_i]$ and $E[M_j]$ are $\exp(\lambda_M)$ and $lognormal(\mu, \sigma^2)$, with

$$\mu = \ln\left(\frac{E[M]^2}{\sqrt{Var(M) + E[M]^2}}\right) \qquad \sigma = \sqrt{\ln\left(1 + \frac{Var(M)}{E[M]^2}\right)}$$

9 Factors calibration

Tuning the simulation parameters to **ensure realism** in our scenarios is needed. Drawing from the information presented in the article [2], it's possible to start with the following statement:

> *"Supermarket lines may not be the longest, just the most loathed. Two years ago, in 20 out of 25 major U.S. cities, the average wait time at grocery stores was under five minutes."*

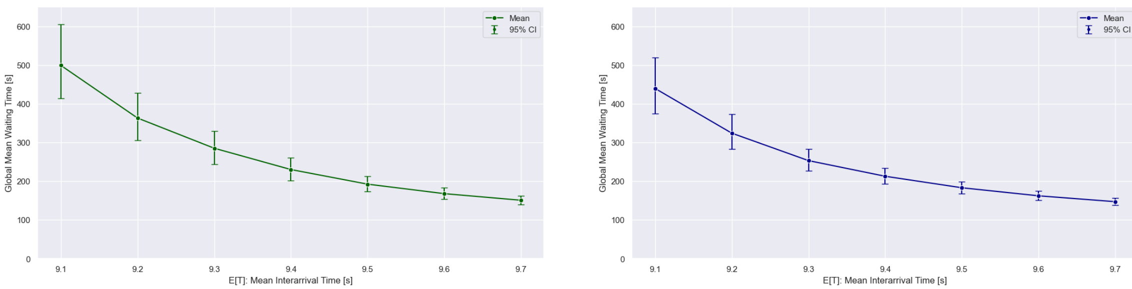
It will be considered **the worst-case scenario**, where **the Mean Waiting Time is 5 minutes**.

$$E[W] = 5 \text{ m} = 300\text{s}$$

To achieve this, $E[T] = 9.3$ is selected in the case where M is taken from an exponential distribution (Figure 6a) and $E[T] = 9.2$ for the lognormal case (Figure 6b), because those are the only values in which the confidence interval contains the value 300, obtained with $n = 30$.

9.0.1 Warmup-Period and Simulation Time

To estimate the *warm-up time*, the **Mean Service Time** (i.e. $E[t_S]$) is chosen as the performance index under observation, because considering the Mean Waiting Time, as in previous tests, resulted in **instability due to the possibility of it being zero several times**.



(a) Exponential distribution of M.

(b) Lognormal distribution of M.

Figure 6: Calibration of $E[T]$.

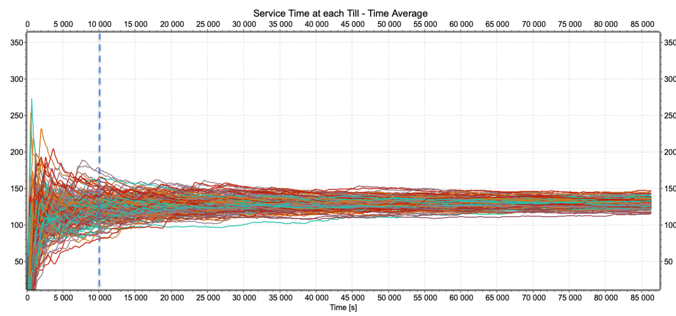


Figure 7: Estimation of the Warmup-Period.

Running with **24 hours** (simulation time) and plotting the *time average* of the Mean Service Time at each Till, the graph in Figure 7 is obtained.

After the initial 10000 seconds (i.e. approximately 2 hours and 47 minutes), **the Mean Service Time surpasses the initial transient phase**, beginning to stabilize around the value that will maintain in the steady-state.

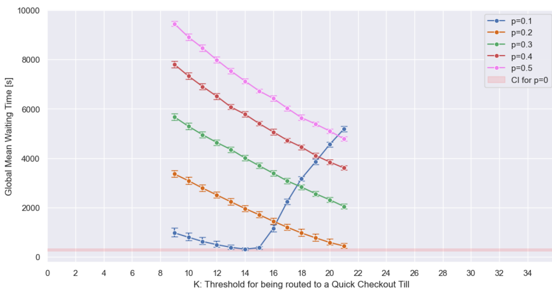
At this point, it is also clear that **a simulation duration of** 24 **hours (simulation time) is excessive**. Consequently, the total simulation time for each run was reduced to 16 **hours** (i.e. 57600 seconds, 7:00 to 23:00 in a real world scenario).

10 Experiments

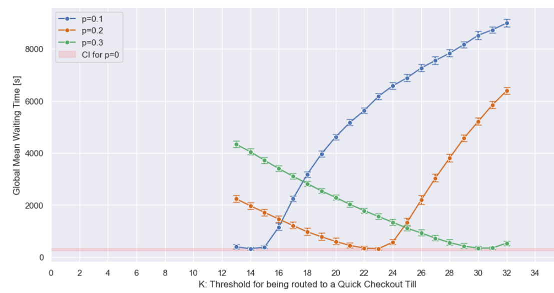
10.1 Experiment Design and Simulation Run

Since there are just two factors (i.e. p and K), **full-factorial analysis** will be exploited. All the runs were simulated with $n = 30$.

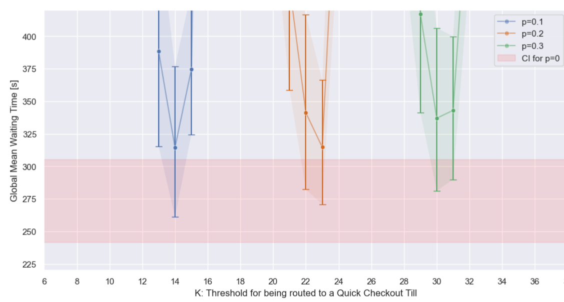
10.2 Result Analysis



(a) First Experiment.



(b) Second Experiment.



(c) Zoomed version of the Second Experiment.

Figure 8: Global Mean Waiting Time (M - Exponential).

The experiment began by examining several values of K for p ranging from 0.1 to 0.5. For subsequent analysis, only specific values of p — notably 0.1, 0.2, and 0.3 — were considered, as others did not exhibit desirable behaviors within a feasible range of K (Figure 8a).

Upon observing Figure 8b and the zoomed version in Figure 8c, attention will be directed to K values of 14 ($p = 0.1$), 23 ($p = 0.2$), and 31 ($p = 0.3$) due to their **lowest Confidence Interval lower bound** for every p configuration. Other values, despite having overlapping CIs, displayed *sub-optimal* behaviors in the single queue analysis. In the case of $K = 31$, even though the lower bound of the CI was higher than $K = 30$, the single queue analysis demonstrated superior performance with $K = 31$.

The analysis of Figure 9 highlights that Q0 (quick queue in all configurations) for $p = 0.1$ exhibits a **significantly lower Mean Waiting Time** compared to scenarios without quick tills (i.e. $p = 0$) and the other two considered configurations.

It is evident that $p = 0.1$ does not substantially worsen the Mean Waiting Time for customers with items over the threshold. The other two configurations did not exhibit sufficiently favorable behavior in quick tills to justify worse performance in normal tills.

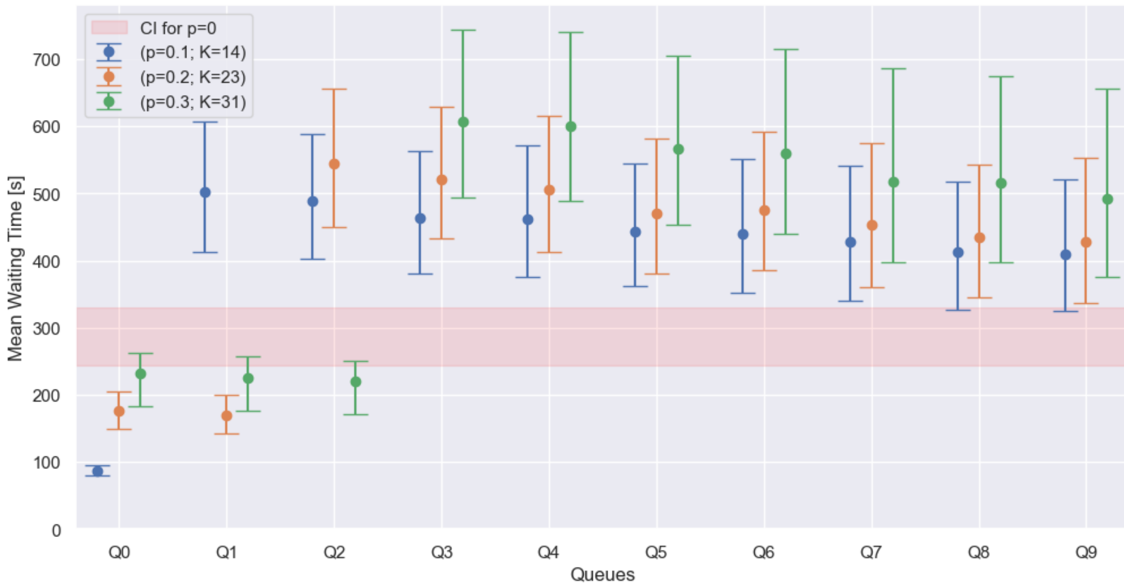


Figure 9: Mean Waiting Time of the different Queues, with exponential distribution of M.

Analyzing the lognormal experiment across various combinations of p and K, assessed in a manner consistent with the previous evaluation (Figure 10), none of these configurations demonstrated a sufficient improvement in Mean Waiting Time for Quick Tills to warrant the deterioration in Normal Tills, as Figure 11 shows.

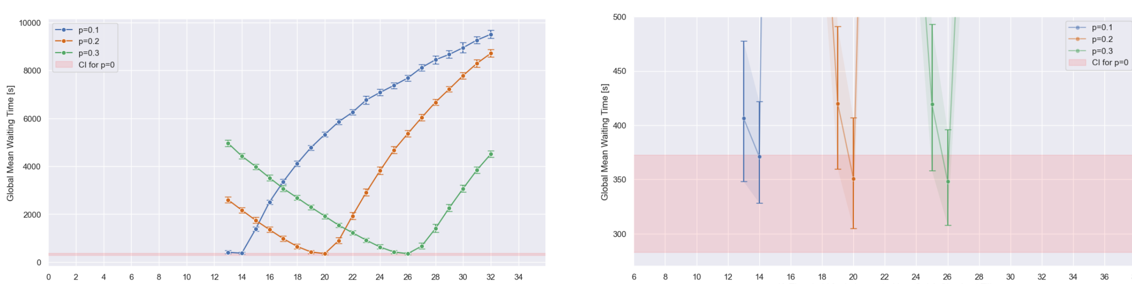


Figure 10: Global Mean Waiting Time (M - Lognormal).

Regarding **Mean Response Time**, the same results can be observed since it follows a similar trend to Mean Waiting Time, appearing as a shifted graph.

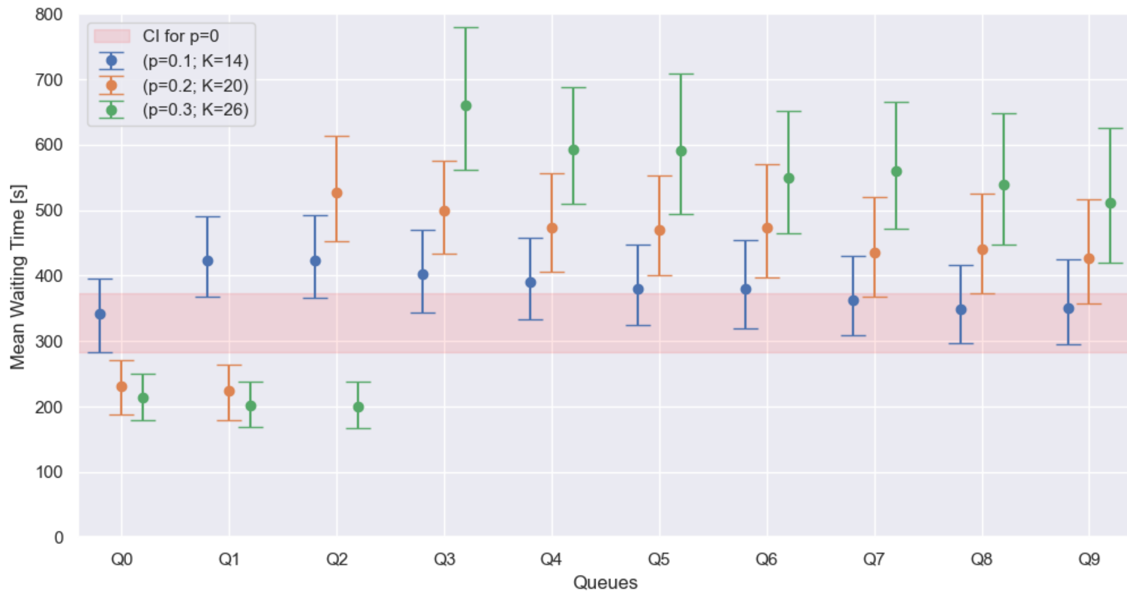


Figure 11: Mean Waiting Time of the different Queues, with lognormal distribution of M.

11 Conclusion

Thanks to the introduction of quick tills in the optimal configuration (exponential case), the Mean Waiting Time for customers with few items is reduced by **more than half** compared to the scenario without quick tills. Meanwhile, the Mean Waiting Time for customers with items exceeding the threshold increases by **less than double**. These customers, however, are more willing to wait longer in line due to having more items.

In conclusion, the behavior of quick tills is evidently influenced by the distributions of items (i.e. M). Specifically:

- In the exponential case, *one* quick till was able to reduce its Mean Waiting Time when K was equal to 14, without a significant increase in the waiting time for customers in normal tills.
- In the lognormal case, **no configuration** achieved the same result.

Therefore, in a real system, determining the **distribution of cart items** before introducing quick tills is crucial, as it has proven to be the key aspect of the analysis.

A Appendix

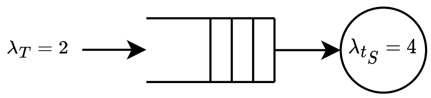


Figure 12: M/G/1 system with $\lambda_T = 2$ and $\lambda_{t_S} = 4$.

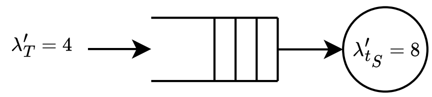


Figure 13: M/G/1 system with $\lambda'_T = 4$ and $\lambda'_{t_S} = 8$.

Let $E[T]$, $E[t_S]$ and $Var(t_S)$ be respectively the Mean inter-arrival time, the Mean service time and the Variance of the service time of an M/G/1 (Figure 12):

$$E[T] = \frac{1}{2} \qquad E[t_S] = \frac{1}{4} \qquad Var(t_S) = \frac{1}{16}$$

Taking their reciprocals, the arrival rate λ_T and the service rate λ_{t_S}:

$$\lambda_T = \frac{1}{E[T]} = 2 \qquad \lambda_{t_S} = \frac{1}{E[t_S]} = 4$$

In this case, the performance indices are the following:

- $\rho = \lambda_T \cdot E[t_S] = \frac{2}{4} = \frac{1}{2}$
- $E[N] = \rho + \frac{\rho^2 + \lambda_T^2 \cdot Var(t_S)}{2 \cdot (1-\rho)} = \frac{1}{2} + \frac{\frac{1}{4} + 4 \cdot \frac{1}{16}}{2 \cdot \left(1 - \frac{1}{2}\right)} = \frac{1}{2} + \frac{\frac{1}{4} + \frac{1}{4}}{2 \cdot \frac{1}{2}} = \frac{1}{2} + \frac{1}{2} = 1$
- $E[N_q] = E[N] - \rho = 1 - \frac{1}{2} = \frac{1}{2}$
- $E[W] = \frac{E[N_q]}{\lambda_T} = \frac{\frac{1}{2}}{2} = \frac{1}{4}$

Now, by halving the Mean inter-arrival time and the Mean service time:

$$E[T'] = E\left[\frac{T}{2}\right] = \frac{1}{4} \qquad E[t'_S] = E\left[\frac{t_S}{2}\right] = \frac{1}{8} \qquad Var(t'_S) = Var\left(\frac{t_S}{2}\right) = \frac{1}{64}$$

This translates into doubling the arrival rate and the service rate:

$$\lambda'_T = 2 \cdot \lambda_T = 4 \qquad \lambda'_{t_S} = 2 \cdot \lambda_{t_S} = 8$$

Considering this new M/G/1 system (Figure 13), the performance indices are the following:

- $\rho' = \lambda'_T \cdot E[t'_S] = \frac{4}{8} = \frac{1}{2}$
- $E[N'] = \rho' + \frac{(\rho')^2 + (\lambda'_T)^2 \cdot Var(t'_S)}{2 \cdot (1 - \rho')} = \frac{1}{2} + \frac{\frac{1}{4} + 16 \cdot \frac{1}{64}}{2 \cdot \left(1 - \frac{1}{2}\right)} = 1$
- $E[N'_q] = E[N'] - \rho' = 1 - \frac{1}{2} = \frac{1}{2}$
- $E[W'] = \frac{E[N'_q]}{\lambda'_T} = \frac{\frac{1}{2}}{4} = \frac{1}{8} = \frac{E[W]}{2}$

It's clear that **the Mean Waiting Time is halved**.

B Verification Tables

Table 6: (Exponential): Mean Number of Customers in Queue $E[N_q]$ - Confidence Interval at 95%.

Queue	**Theoretical**	**Mean**	**CI - LB**	**CI - UB**	
Queues[0]	0.003123	0.002995	0.002761	0.003230	✓
Queues[1]	0.003123	0.003402	0.003119	0.003684	✓
Queues[2]	0.334318	0.325612	0.304830	0.346395	✓
Queues[3]	0.334318	0.346722	0.322885	0.370560	✓

Table 7: (Exponential): Utilization ρ - Confidence Interval at 95%.

Queue	**Theoretical**	**Mean**	**CI - LB**	**CI - UB**	
Queues[0]	0.068452	0.068394	0.067328	0.069459	✓
Queues[1]	0.068452	0.068459	0.066834	0.070084	✓
Queues[2]	0.493988	0.492145	0.485497	0.498793	✓
Queues[3]	0.493988	0.496031	0.486992	0.505071	✓

Table 8: (Exponential): Mean Service Time $E[t_S]$ $[s]$ - Confidence Interval at 95%.

Queue	**Theoretical**	**Mean**	**CI - LB**	**CI - UB**	
Queues[0]	13.203587	13.226692	13.111825	13.341558	✓
Queues[1]	13.203587	13.106028	13.003293	13.208762	✓
Queues[2]	67.524951	67.018288	66.366837	67.669739	✓
Queues[3]	67.524951	67.810909	67.239446	68.382372	✓

Table 9: (Exponential): Mean Waiting Time $E[W]$ $[s]$ - Confidence Interval at 95%.

Queue	**Theoretical**	**Mean**	**CI - LB**	**CI - UB**	
Queues[0]	0.602429	0.593371	0.538174	0.658087	✓
Queues[1]	0.602429	0.641725	0.587174	0.705769	✓
Queues[2]	45.699115	44.582823	41.946378	47.430331	✓
Queues[3]	45.699115	47.040153	44.243622	50.086134	✓

Table 10: (Lognormal): Mean Number of Customers in Queue $E[N_q]$ - Confidence Interval at 95%.

Queue	**Theoretical**	**Mean**	**CI - LB**	**CI - UB**	
Queues[0]	0.003569	0.003838	0.003404	0.004271	✓
Queues[1]	0.003569	0.003812	0.003512	0.004112	✓
Queues[2]	0.351523	0.344048	0.311578	0.376518	✓
Queues[3]	0.351523	0.358798	0.334710	0.382886	✓

Table 11: (Lognormal): Utilization ρ - Confidence Interval at 95%.

Queue	**Theoretical**	**Mean**	**CI - LB**	**CI - UB**	
Queues[0]	0.077617	0.078654	0.077059	0.080249	✓
Queues[1]	0.077617	0.078522	0.076994	0.080050	✓
Queues[2]	0.484940	0.480816	0.473381	0.488252	✓
Queues[3]	0.484940	0.486472	0.478377	0.494567	✓

Table 12: (Lognormal): Mean Service Time $E[t_S]$ $[s]$ - Confidence Interval at 95%.

Queue	**Theoretical**	**Mean**	**CI - LB**	**CI - UB**	
Queues[0]	16.808667	16.764889	16.668415	16.861364	✓
Queues[1]	16.808667	16.833326	16.746425	16.920226	✓
Queues[2]	61.522346	61.435091	60.817201	62.052981	✓
Queues[3]	61.522346	61.817371	61.173318	62.461425	✓

Table 13: (Lognormal): Mean Waiting Time $E[W]$ $[s]$ - Confidence Interval at 95%.

Queue	**Theoretical**	**Mean**	**CI - LB**	**CI - UB**	
Queues[0]	0.772947	0.802983	0.725393	0.891695	✓
Queues[1]	0.772947	0.788791	0.718848	0.869397	✓
Queues[2]	44.596290	43.378735	39.780986	47.346844	✓
Queues[3]	44.596290	45.722288	42.696147	49.034860	✓

C Appendix

In order to convert continuous numbers drawn from M into discrete ones, the following transformation will be employed:

$$M' = round(M) = floor(M + 0.5)$$

Now, the first major change will clearly be in the probability of being routed to the quick checkout subsystem, indeed:

$$P\{M' \leq K\} = P\{floor(M + 0.5) \leq K\}$$

The question that needs to be answered is then *"what are the values of M such that M' is **less than or equal** to K?"* Figure 14 can help in answering this question. As it shows, the values that are in $[K - 0.5;\ K + 0.5)$ are mapped to K. This means that M should be **less than** $K + 0.5$ for routing a customer in the quick checkout subsystem. Remembering that M is in the continuous world, it's also possible to write the following:

$$P\{M \leq (K + 0.5)\} = F((K + 0.5); \lambda_M) = 1 - e^{-\lambda_M \cdot (K+0.5)}$$

as the probability of drawing an exact number is negligible.

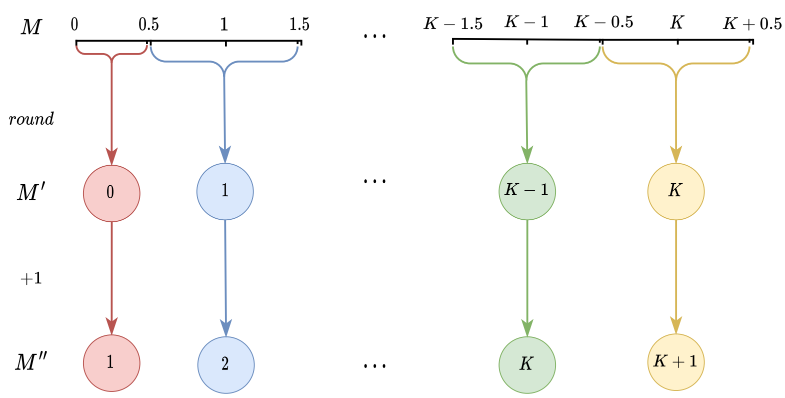


Figure 14: Converting continuous numbers drawn from M into discrete ones through the application of the *round* operation (M'). Subsequently, 1 is added to these discrete values to prevent 0 from being a possible outcome (M'').

The **mean number of items** in a quick checkout till is computed as (Appendix D):

$$E[floor(M+0.5) \mid M \leq (K+0.5)] = \frac{1}{F((K+0.5);\lambda_M)} \cdot \left[\int_0^{(K+0.5)} floor(x+0.5)\cdot \lambda_M e^{-\lambda_M \cdot x}\ dx\right] =$$

$$= E[M_i']$$

To ensure that customers have at least 1 item in their cart adding 1 to M' is sufficient. This adjustment ensures that the new discrete RV, call it M'', will have a minimum possible value of 1.

$$M'' = M' + 1 = round(M) + 1 = floor(M + 1.5)$$

Repeating the aforementioned logic, the initial adjustment is expressed as follows:

$$P\{M'' \leq K\} = P\{floor(M + 1.5) \leq K\}$$

Consulting Figure 14 once more, here M must be **less than** $K - 0.5$, since when M resides in $[K - 1.5;\ K - 0.5)$, M'' is mapped to K.

Consequently, we have:

$$P\{M \leq (K - 0.5)\} = F((K-0.5);\lambda_M) = 1 - e^{-\lambda_M \cdot (K-0.5)}$$

once again emphasizing that M remains within the continuous domain.

Considering the mean value of M as $E[M] = \dfrac{1}{\lambda_M}$, a relevant question arises: *"How does this mean value change after the specified transformations?"*

The system exhibits different behaviour since it's influenced by the distribution of M'' instead of M. By numerically computing the integral of the mean value with Python's `quad` function from `scipy.integrate`, it becomes evident that the transformation leading to M' does not alter the mean value of M, remaining approximately $\dfrac{1}{\lambda_M}$. This is not the case for M'' since 1 is added.

By applying the linearity of the expectation operator:

$$E[M''] = E[M' + 1] = E[M'] + 1 \Rightarrow E[M''] \approx \frac{1}{\lambda_M} + 1$$

This discrepancy indicates that the mean value of the item in a customer's cart, as perceived by the *overall system*, is shifted by one. To impose a specific mean value, denoted as m, on the system: $m \approx E[M] + 1 \Rightarrow E[M] \approx m - 1$. This implies providing the system with an exponential distribution whose mean value is $m - 1$.

In summary, and for completeness, the final formulas for one specific subsystem (acknowledging no significant differences in the other subsystem) are as follows: by imposing $\lambda_M = \dfrac{1}{m-1}$ and considering the earlier observation it is possible to obtain:

$$P\{M \leq (K-0.5)\} = F((K-0.5);\lambda_M) = 1 - e^{-\frac{K-0.5}{m-1}}$$

Following with **probabilities**:

$$\pi_i = \frac{P\{M \leq (K-0.5)\}}{C \cdot p}$$

and **rate**:

$$\lambda_{T_i} = \lambda_T \cdot \pi_i$$

The **mean value of** M:

$$E[floor(M+1.5) \mid M \leq (K-0.5)] = \frac{1}{F((K-0.5);\lambda_M)} \cdot \left[\int_0^{(K-0.5)} floor(x+1.5) \cdot \frac{e^{-\frac{x}{m-1}}}{m-1}\; dx\right] =$$

$$= E[M_i]$$

which leads to:

$$E[t_{S_i}] = E[M_i] \cdot 3$$

To evaluate the variance, the second moment is needed:

$$E[(floor(M+1.5))^2 \mid M \leq (K-0.5)] = \frac{1}{F((K-0.5);\lambda_M)} \cdot \left[\int_0^{(K-0.5)} (floor(x+1.5))^2 \cdot \frac{e^{-\frac{x}{m-1}}}{m-1}\; dx\right] =$$

$$= E[M_i^2]$$

The **variance of** M is:

$$Var(M_i) = E[M_i^2] - E[M_i]^2 \text{ (Appendix D).}$$

by the *non-linearity* of the variance:

$$Var(t_{S_i}) = Var(M_i) \cdot 3^2$$

Finally:

$$E[N_i] = \rho_i + \frac{\rho_i^2 + \lambda_{T_i}^2 \cdot Var(t_{S_i})}{2 \cdot (1-\rho_i)}$$

with

$$\rho_i = \lambda_{T_i} \cdot E[t_{S_i}]$$

D Appendix

Expectation of a truncated random variable [3], which is:

$$\begin{aligned} E[M|M \leq K] &= \frac{1}{F(K;\lambda_M)} \cdot \left[\int_0^K x \cdot f(x)\ dx \right] = \\ &= \frac{1}{F(K;\lambda_M)} \cdot \left[\int_0^K x \cdot \lambda_M e^{-\lambda_M \cdot x}\ dx \right] = \\ &= E[M_i] \end{aligned}$$

$$\begin{aligned} E[M|M > K] &= \frac{1}{1 - F(K;\lambda_M)} \cdot \left[\int_K^{+\infty} x \cdot f(x)\ dx \right] = \\ &= \frac{1}{1 - F(K;\lambda_M)} \cdot \left[\int_K^{+\infty} x \cdot \lambda_M e^{-\lambda_M \cdot x}\ dx \right] = \\ &= E[M_j] \end{aligned}$$

For computing $E[M^2|M \leq K]$ it is sufficient to remember that:

$$E[g(X)] = \int_{-\infty}^{+\infty} g(x) f_X(x)\ dx$$

$$E[M^2|M \leq K] = \frac{1}{F(K;\lambda_M)} \cdot \left[\int_0^K x^2 \cdot \lambda_M e^{-\lambda_M \cdot x}\ dx \right] = E[M_i^2]$$

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