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Heat equation and its comparative solutions

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Abstract

In this paper we will conduct an analytic comparative study between the powerful Adomian method and the traditional separation of variables method. This is achieved by handling homogeneous and non-homogeneous boundary value problem for one-dimensional heat equation. The study shows the reliability and efficiency of Adomian method. Adomian method provides the solution in a rapidly convergent series through evaluating elegantly computable components.

Keywords: Adomian decomposition method; Separation of variables method; Heat equations; Partial differential equations; Homogeneous equations; Non-homogeneous equations

1. Introduction

The Heat equation is a partial differential equation that describes the variation of temperature in a given region over a period of time. Traditionally, the heat equations are often solved by classic methods such as Separation of variables and Fourier series methods. However, these methods suffer from tedious work and the use of transformation formulae for non-homogeneous cases. Adomian decomposition method [1,2] is an efficient method that provides a practical alternative to handle differential equations, ordinary or partial, and homogeneous or inhomogeneous. The method gives the solution in a fast convergent series. It was formally shown by many researchers that if an exact solution exists for the problem, then the obtained solution converges very rapidly to that solution. The convergence concept of the decomposition series was thoroughly investigated by many researchers to confirm the rapid convergence of the resulting series. Cherruautt examined the convergence of Adomian method in [3], Abbaoui, et al. [4] formally proved the convergence of Adomian method when applied to differential equations in general.

Adomian decomposition method was first developed by Adomian [1,2] and used by many such as in [3–8] and the references therein. It is a non-numerical method for solving linear and nonlinear differential equations, both ordinary, and partial. As stated before, it often yields convergent series solution by using few iterations only. The advantage of this method is that, it solves the problem directly without the need for linearization, perturbation, or any other transformation, and also, reduces the massive computation works required by most other methods [3–8].

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It is the aim of this work to emphasize the fact that Adomian method gives a reasonable improvements over the existing traditional techniques. To support this comparative study, we will consider the heat equation,

$$u_t = ku_{xx}, \tag{1}$$

where k is the diffusion factor, and the initial and boundary conditions are usually prescribed. Of course, the study can be extended to other physical models for making further progress.

2. Separation of variables method

In this section we will apply the separation of variables method to solve both the homogeneous, and non-homogeneous initial boundary value problem (IBVP) of heat flow equations. The approach will be presented in a simplified form.

2.1. Homogeneous heat equation

We will consider first the heat equation of the form,

$$u_t = u_{xx}, \quad 0 < x < \pi,\ t > 0, \tag{2}$$

subject to the following boundary conditions,

$$u(0, t) = u(\pi, t) = 0, \quad t > 0, \tag{3}$$

and the initial condition,

$$u(x, 0) = x^2. \tag{4}$$

Physically, problem (2)–(4) represent a model for the heat flow in an insulated wire whose ends are kept at 0 °C, and the initial temperature distribution given as x^2. The separation of variables method admits the use of

$$u(x, t) = f(t)g(x), \tag{5}$$

that will carry Eq. (2) into,

$$f'(t)g(x) = g''(x)f(t), \tag{6}$$

or equivalently

$$\frac{f'(t)}{f(t)} = \frac{g''(x)}{g(x)}. \tag{7}$$

This means that each side of (7) equals a constant, therefore we set

$$\frac{f'(t)}{f(t)} = -\lambda^2, \tag{8}$$

and,

$$\frac{g''(x)}{g(x)} = -\lambda^2. \tag{9}$$

Each equation can be solved to get

$$f(t) = c\mathrm{e}^{-\lambda^2 t}, \tag{10}$$

and,

$$g(x) = c_1 \sin(\lambda x) + c_2 \cos(\lambda x). \tag{11}$$

Then substituting both $f(t)$, and $g(x)$ into Eq. (5) gives

$$u(x,t) = e^{-\lambda^2 t}[A\sin(\lambda x) + B\cos(\lambda x)], \tag{12}$$

where A, B are constants. The boundary conditions can be used to determine the constants A and B to find

$$u_n(x,t) = B_n \sin(nx) e^{-n^2 t}, \quad n = 1, 2, \ldots, t > 0. \tag{13}$$

Using the superposition principle, we can write the formal solution as,

$$u(x,t) = \sum_{n=1}^{\infty} u_n(x,t) = \sum_{n=1}^{\infty} B_n e^{-n^2 t} \sin(nx), \quad n = 1, 2, \ldots, t > 0, \tag{14}$$

where, B_n is an arbitrary constant to be determined using the initial conditions,

$$u(x,0) = \sum_{n=1}^{\infty} B_n \sin(nx) = x^2, \quad 0 < x < \pi. \tag{15}$$

We have thus reduced the problem (2)–(4) of heat flow in a thin wire to the problem of determining an expression for $f(x)$ of the form,

$$x^2 = \sum_{n=1}^{\infty} B_n \sin(nx), \tag{16}$$

such an expression is called a Fourier Sine Series, where x^2 represents the given initial conditions, using this condition we could determine the value of the arbitrary constant B_n as follows,

$$B_n = \frac{2}{\pi}\int_0^{\pi} x^2 \sin(nx) dx, \quad 0 < x < \pi. \tag{17}$$

To calculate the coefficients, we use integration by parts twice, which leads to,

$$B_n = \frac{2\pi}{n}(-1)^{n+1} + \frac{4}{\pi n^3}[(-1)^n - 1]. \tag{18}$$

This in turn gives,

$$u(x,t) = \sum_{n=1}^{\infty} e^{-n^2 t} \left\{ \frac{2\pi}{n}(-1)^{n+1} + \frac{4}{\pi n^3}[(-1)^n - 1] \right\} \sin(nx). \tag{19}$$

2.2. *Non-homogeneous heat equation*

Consider the heat equation of the form,

$$u_t = u_{xx} + g(x), \quad 0 < x < \pi, t > 0, \tag{20}$$

and for the given non-homogeneous boundary conditions,

$$u(0,t) = A, \qquad u(\pi,t) = B, \quad t > 0, \tag{21}$$

and the following initial conditions,

$$u(x,0) = x^2, \quad 0 < x < \pi. \tag{22}$$

Where, $g(x)$ is the heat source independent on time. The solution to the heat flow problem (20) with non-homogeneous boundary conditions will consist of a steady-state solution $v(x)$ that satisfies the non-homogeneous boundary conditions Plus a transient solution $w(x,t)$ so that

$$u(x,t) = v(x) + w(x,t), \tag{23}$$

where, $w(x,t)$ and its partial derivatives tend to zero as $t \to \infty$. The function $w(x,t)$ will then satisfy the homogeneous boundary conditions. Substituting this assumption into (21) yields,

$$\begin{aligned}
&u_t = w_t = v''(x) + w_{xx} + g(x), \quad 0 < x < \pi,\ t > 0,\\
&v(0) + w(0,t) = A, \qquad v(\pi) + w(\pi,t) = B, \quad t > 0, \\
&v(x) + w(x,0) = f(x) = x^2, \quad 0 < x < \pi.
\end{aligned} \tag{24}$$

If we allow $t \to \infty$, assuming $w(x,t)$ is a transient solution, we obtain the steady-state boundary value problem.

$$\begin{aligned}
&v''(x) = -g(x), \quad 0 < x < \pi,\\
&v(0) = A, \qquad v(\pi) = B,
\end{aligned} \tag{25}$$

the solution to this boundary value problem can be obtained by two integration by parts using the boundary coefficients to determine the constants of integration. The solution $v(x)$ is given by the formula,

$$v(x) = \left[B - A + \int_0^\pi \left(\int_0^z g(s)\mathrm{d}s\right)\mathrm{d}z\right]\frac{x}{\pi} + A - \int_0^x \left(\int_0^z g(x)\mathrm{d}s\right)\mathrm{d}z. \tag{26}$$

With this choice for $v(x)$, the initial boundary value problem (21) reduces to the following initial boundary value problem for $w(x,t)$;

$$\begin{aligned}
&w_t = w_{xx}, \quad 0 < x < \pi,\ t > 0,\\
&w(0,t) = w(\pi,t) = 0, \quad t > 0,\\
&w(x,0) = x^2 - v(x), \quad 0 < x < \pi.
\end{aligned} \tag{27}$$

Recall, that a formal solution to the initial and boundary value problem is given by,

$$w(x,t) = \sum_{n=1}^{\infty} c_n \mathrm{e}^{-n^2 t} \sin(nx), \tag{28}$$

where the $c_n's$ are constants determined from the Fourier Sine series expansion of $x^2 - v(x)$;

$$x^2 - v(x) = \sum_{n=1}^{\infty} c_n \sin(nx). \tag{29}$$

Thus the formal solution to (20) is given by

$$u(x,t) = v(x) + w(x,t), \tag{30}$$

where, both $v(x)$ and $w(x,t)$ are given above.

3. Adomian decomposition method

It remain now to employ Adomian method to the homogeneous and non-homogeneous heat equation. This will enable us to present a comparative study between the two proposed schemes. In this section we will demonstrate the main algorithm of Adomian Decomposition Method on solving both homogeneous, and non-homogeneous heat equation problem.

3.1. Homogeneous IBVP of heat equation

The mechanism of the method begins by rewriting Eq. (2) in an operator form as follows,

$$L_t u(x,t) = L_{xx} u(x,t), \tag{31}$$

where, L_t, and L_{xx} are operators defined as, $L_t = \frac{\partial}{\partial t}$, and $L_{xx} = \frac{\partial^2}{\partial x^2}$, assuming that both L_t, and L_{xx} are invertible, therefore their inverse operators are,

$$L_t^{-1}(.) = \int_0^t (.)\mathrm{d}t, \tag{32}$$

and,

$$L_{xx}^{-1}(.) = \int_0^x \int_0^x (.)\mathrm{d}x\mathrm{d}x. \tag{33}$$

Applying the inverse on (32) to both sides of Eq. (31) we get,

$$u(x,t) - u(x,0) = L_t^{-1}(L_{xx}u(x,t)). \tag{34}$$

Then with the initial conditions given in (4) this will be written as,

$$u(x,t) = x^2 + L_t^{-1}(L_{xx}u(x,t)), \tag{35}$$

substituting the decomposition series,

$$u(x,t) = \sum_{n=0}^{\infty} u_n(x,t), \tag{36}$$

on both sides of (35) we will get,

$$\sum_{n=0}^{\infty} u_n(x,t) = x^2 + L_t^{-1}\left(L_{xx}\left(\sum_{n=0}^{\infty} u_n(x,t)\right)\right). \tag{37}$$

The decomposition method suggests that the zeroth component $u_0(x,t)$ is identified by the terms arising from the initial, or boundary conditions and from the source terms. The remaining components of $u(x,t)$ are determined in a recursive manner as shown,

$$\begin{aligned} &u_0(x,t) = x^2 \\ &u_{k+1}(x,t) = L_t^{-1}(L_{xx}(u_k(x,t))), \quad k \geq 0. \end{aligned} \tag{38}$$

This gives

$$\begin{aligned} &u_0(x,t) = x^2 \\ &u_1(x,t) = L_t^{-1}(L_{xx}(u_0)) = 2t \\ &u_2(x,t) = 0, \\ &u_3(x,t) = 0, \\ &\vdots \end{aligned} \tag{39}$$

Then this gives the solution

$$u(x,t) = x^2 + 2t, \tag{40}$$

obtained by adding the components u_0 and u_1.

3.2. Non-homogeneous IBVP of heat equation

In this section we will demonstrate how Adomian decomposition method can be easily applied to non-homogeneous initial boundary value problem also.

We begin by rewriting Eq. (20) in operator form,

$$L_t u(x,t) = L_{xx} u(x,t) + g(x). \tag{41}$$

Applying the inverse operator (32) to both sides of Eq. (41), and using the initial values (22) yields,

$$u(x,t) = tg(x) + x^2 + L_t^{-1}(L_{xx} u(x,t)). \tag{42}$$

Writing the solution in a sum of coefficients form as defined by (48) gives,

$$u_n(x,t) = tg(x) + x^2 + L_t^{-1}(L_{xx} u(x,t)). \tag{43}$$

This will lead to the following terms,

$$u_0(x,t) = tg(x) + x^2. \tag{44}$$

$$u_1(x,t) = L_t^{-1}(L_{xx}(u_0(x,t))) = \frac{t^2}{2} g''(x) + 2t, \tag{45}$$

where,

$$g''(x) = \frac{\partial^2}{\partial x^2} g(x), \tag{46}$$

and,

$$u_2(x,t) = L_t^{-1}(L_{xx}(u_1(x,t))) = \frac{t^3}{6} g^{iv}(x). \tag{47}$$

Then in series form the final solution will take this form,

$$u(x,t) = u_0 + u_1 + u_2 + \cdots. \tag{48}$$

3.2.1. Example-1

Consider the heat equation of the form,

$$u_t = u_{xx}, \quad 0 < x < \pi,\ t > 0, \tag{49}$$

subject to the following non-homogeneous boundary conditions,

$$u(0,t) = 4 - e^{-t}, \qquad u(\pi,t) = 4 - e^{-t}, \tag{50}$$

and given initial conditions,

$$u(x,0) = 4 - \cos x. \tag{51}$$

Applying the decomposition method, we will rewrite the last equation in the operator form as follows,

$$L_t u(x,t) = L_{xx} u(x,t), \tag{52}$$

where,

$$L_t = \frac{\partial}{\partial t}, \qquad L_{xx} = \frac{\partial^2}{\partial x^2}. \tag{53}$$

Assuming that the operators are invertible, we could apply L_t^{-1} to both sides of Eq. (44),

$$L_t^{-1} L_t u(x,t) = L_t^{-1} L_{xx} u(x,t), \tag{54}$$

where, $L_t^{-1} = \int_0^t (.) \mathrm{d}t$. Using the initial condition given in the problem, we can write the solution as,

$$u(x,t) = 4 - \cos x + L_t^{-1}(L_{xx} u(x,t)). \tag{55}$$

Substituting the series,

$$u(x,t)=\sum_{n=0}^{\infty}u_n(x,t) \tag{56}$$

into (46) yields

$$u(x,t)=\sum_{n=0}^{\infty}u_n(x,t)=4-\cos x+L_t^{-1}\left(L_{xx}\left(\sum_{n=0}^{\infty}u_n(x,t)\right)\right). \tag{57}$$

Identifying the zeroth component as the initial condition, and using the recursive algorithm, see [5], we obtain the following,

$$\begin{aligned}
u_0 &= 4-\cos x\\
u_1 &= -\cos x\frac{t}{1!}\\
u_2 &= \cos x\frac{t^2}{2!}\\
u_3 &= -\cos x\frac{t^3}{3!}\\
u_4 &= \cos x\frac{t^4}{4!}.
\end{aligned}$$

Consequently, the solution $u(x,t)$ in a series form is given by,

$$\begin{aligned}
u(x,t) &= u_0+u_1+u_2+u_3+u_4+\cdots\\
&= 4-\cos x\left\{1-t+\frac{t^2}{2!}-\frac{t^3}{3!}+\frac{t^4}{4!}+\cdots\right\},
\end{aligned}$$

and in closed form by,

$$u(x,t)=4-\mathrm{e}^{-t}\cos x, \tag{58}$$

where the terms in between the braces are the Taylor expansion for e^{-t}.

3.2.2. Example-2

Consider the following non-homogeneous initial and boundary value problem of heat equation,

$$u_t(x,t)=u_{xx}(x,t)+u(x,t),\quad 0<x<\pi,\ t>0, \tag{59}$$

with the following boundary conditions,

$$u(0,t)=0,\qquad u(\pi,t)=\mathrm{e}^{-t}\sinh(\pi),\quad t>0, \tag{60}$$

and the initial conditions,

$$u(x,0)=\sinh(x). \tag{61}$$

Proceeding as before, Eq. (59) becomes,

$$L_tu(x,t)=L_{xx}u(x,t)+u(x,t),\quad 0<x<\pi,\ t>0, \tag{62}$$

where, $L_t=\frac{\partial}{\partial t}$, and $L_{xx}=\frac{\partial^2}{\partial x^2}$.

Then applying the inverse operator L_t^{-1} to both sides of Eq. (62), yields,

$$L_t^{-1}L_tu(x,t)=L_t^{-1}\left(L_{xx}u(x,t)+u(x,t)\right),\quad \text{where } L_t^{-1}=\int_0^t(.)\mathrm{d}t, \tag{63}$$

applying the initial conditions, the solution can be written as,

$$u(x,t) = \sinh x + L_t^{-1}\left(L_{xx}u(x,t) + u(x,t)\right) \tag{64}$$

using the series form,

$$u(x,t) = \sum_{n=0}^{\infty} u_n(x,t). \tag{65}$$

We obtain

$$\sum_{n=0}^{\infty} u_n(x,t) = \sinh x + L_t^{-1}\left(L_{xx}\left(\sum_{n=0}^{\infty} u_n(x,t)\right) + \sum_{n=0}^{\infty} u_n(x,t)\right). \tag{66}$$

Identifying the first component to be the initial condition, and using the recursive algorithm, we can write the solution as,

$$\begin{aligned}
u_0 &= \sinh x,\\
u_1 &= L_t^{-1}\left(L_{xx}u_0 + u_0\right) = 2t \sinh xt,\\
u_2 &= L_t^{-1}\left(L_{xx}u_1 + u_1\right) = 2t^2 \sinh x,\\
u_3 &= L_t^{-1}\left(L_{xx}u_2 + u_2\right) = \frac{4}{3}t^3 \sinh x,\\
u_4 &= L_t^{-1}\left(L_{xx}u_4 + u_3\right) = \frac{2}{3}t^4 \sinh x,\\
&\ldots.
\end{aligned}$$

Substituting the above components, and using the Taylor expansion for e^{2t}, we find that in a closed form the solution can be written as,

$$u(x,t) = e^{2t} \sinh x. \tag{67}$$

4. Discussion

The main goal of this work is to carry out a comparative study between the traditional separation of variables method and the Adomian method. The work shows that Adomian has significant advantages over the existing techniques. The method does not require restrictive assumption or transformation formulae. The method provides fast convergent series that gives exact solutions. The goal of this work has been achieved.

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