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Comparing numerical methods for Helmholtz equation model problem

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Abstract

In this article, we implement a relatively new numerical technique, Adomian's decomposition method for solving the linear Helmholtz partial differential equations. The method in applied mathematics can be an effective procedure to obtain for the analytic and approximate solutions. A new approach to a linear or nonlinear problems is particularly valuable as a tool for Scientists and Applied Mathematicians, because it provides immediate and visible symbolic terms of analytic solution as well as its numerical approximate solution to both linear and nonlinear problems without linearization [Solving Frontier Problems of Physics: The Decomposition Method, Kluwer Academic Publishers, Boston, 1994; J. Math. Anal. Appl. 35 (1988) 501]. It does also not require discretization and consequently massive computation. In this scheme the solution is performed in the form of a convergent power series with easily computable components. This paper will present a numerical comparison with the Adomian decomposition and a conventional finite-difference method. The numerical results demonstrate that the new method is quite accurate and readily implemented.

Keywords: The Adomian decomposition method; Finite-difference method; The Helmholtz equation model problem

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1. Introduction

The Adomian's decomposition method which we present in this paper establishes symbolic and approximate solutions to (1) by using the decomposition technique [1,2]. The method is relatively new and useful for obtaining both analytic and numerical approximations of linear or nonlinear differential equations and it is also quite straightforward to write computer codes. In the literature, this method has been used to obtain approximate numerical and analytic solutions of a large class of linear or nonlinear differential equations [1–13,16–25] and references therein.

Over the past few years, several general purpose computer programs for numerically solving partial differential equations have been developed. These programs all use finite-difference methods for the spatial variables to convert the derivatives of the partial differential equation into a discretized form but in the proposed decomposition method, no discretization is necessary for obtaining the numerical results. Instead of the discretized form, the decomposition method is used to get numerical solutions in terms of the decomposition series [1,2].

Also over the past few years, many new alternatives to the use of classical finite-difference methods for the numerical solution of partial differential equations have been proposed. The basic purpose of this paper is to illustrate the advantages and simplicity of using the decomposition method over finite-difference method in terms of numerical comparisons. However, we will consider a traditional finite-difference method [14,15] and decomposition method to solve Eq. (1). In the rest of paper we introduce both the finite difference and decomposition methods. Then in Section 3 the numerical evaluations of these methods and comparisons will be given.

2. Numerical methods

Before discussing one of the finite difference and a new techniques that have been proposed for the numerical solution of the Helmholtz equation, we assume that on a given region R in the xy-plane, of the form

$$\nabla^2 u + f(x,y)u = g(x,y), \tag{1}$$

where $u(x,y)$ known on the boundary of R. Suppose that a function $u = u(x,y)$ of two variables is the solution to Eq. (1) physical problem. The boundary and initial conditions could be given by the following functions:

$$u(0,y) = \psi_1(y), \quad u_x(0,y) = \psi_2(y), \tag{2}$$

$$u(x,0) = \psi_3(x), \quad u_y(x,0) = \psi_4(x), \tag{3}$$

where $\psi_1(y)$, $\psi_2(y)$, $\psi_3(x)$, $\psi_4(x)$ are given functions.

These equations appear in such diverse phenomena as: elastic waves in solids including vibrating string, bars, membranes, sound or acoustics, electromagnetic waves, and nuclear reactors [14,15]. In order to solve this equation numerically, we use both finite difference and decomposition series methods. The comparison between two methods are also made.

In the finite-difference technique, in order to solve Eq. (1) a set of grid points is first established throughout the region occupied by the independent variables, i.e. the space of variables is discretizated. At this time, it is convenient to introduce an abbreviated notation $u_{ij} = u(x_j, y_j)$, $f_{ij} = f(x_j, y_j)$, and $g_{ij} = g(x_j, y_j)$. With it the five-point formula takes on a simple form at the gird point (x_i, y_j):

$$(\nabla^2)_{ij} \simeq \frac{1}{h^2}(u_{i+1,j} + u_{i-1,j} + u_{i,j+1} + u_{i,j-1} - 4u_{ij}). \tag{4}$$

If this approximation is made in Eq. (1), the formulae is

$$-u_{i+1,j} - u_{i-1,j} - u_{i,j+1} - u_{i,j-1} + (4 - h^2 f_{ij})u_{ij} = -h^2 g_{ij},$$

and boundary conditions (2) could be obtained u_{1j} and u_{2j}, i.e., $u_{1j} = u(0, y_j)$ and $u_{2j} = u_{1j} + h\psi_2(y_j)$. Therefore, it follows that

$$u_{i+1,j} = -h^2 g_{ij} + u_{i-1,j} + u_{i,j+1} + u_{i,j-1} + (4 - h^2 f_{ij})u_{ij}. \tag{5}$$

The coefficients of this equation can be illustrated by a n-grid points in which each point corresponds to the coefficient of u in the grid point. To be specific, we assume that the gird point has spacing h. We obtain a single linear equation of the form (first above) for each of the $n-1$ interior grid points. This $n-1$ system of equation could be solved by using a direct method such as Gaussian elimination technique or an iterative method such as Gaus–Seidel method. In this study we used to solve the system of equation through Gaus–Seidel method. A more detail of the method can be found in Burden and Faires [14] or in Gerald and Wheatley [15].

On the other hand the decomposition series method does not require this discretization and resulting massive computation. In this paper we apply the second method to obtain analytic and approximate solutions of equation given by (1) using the decomposition method. In the decomposition method, Eq. (1) is approximated by the operators in the following form:

$$L_x u = g(x, y) - f(x, y)u - L_y u, \tag{6}$$

where L_x and L_y symbolize $\partial^2/\partial x^2$ and $\partial^2/\partial y^2$, respectively. Assuming the inverse of the operator $Ł_x^{-1}$ exists, and it can conveniently be integrated with respect to x from 0 to x, i.e., $Ł_x^{-1} = \int_0^x \int_0^x (\cdot)\, dx\, dx$, then applying the inverse operator $Ł_x^{-1}$ to (6) yields

$$L_x^{-1}L_x u = L_x^{-1}(g(x,y)) - L_x^{-1}(f(x,y)u) - L_x^{-1}L_y u. \tag{7}$$

Therefore, it follows that

$$u(x,y) = u(0,y) + xu_x(0,y) + L_x^{-1}(g(x,y)) - L_x^{-1}(f(x,y)u) - L_x^{-1}L_y u. \tag{8}$$

The zeroth component is obtained, by using (2), as

$$u_0 = u(0,y) + xu_x(0,y) + L_x^{-1}(g(x,y)) \tag{9}$$

which is defined by all terms that arise from the boundary conditions. Thus, the unknown function $u(x,y)$ is computed in terms of the components defined by the decomposition series given as

$$u(x,y) = \sum_{n=0}^{\infty} u_n(x,y). \tag{10}$$

The remaining components $u_n(x,y)$, $n \geqslant 1$, can be completely determined such that each term is computed by using the previous term. Since u_0 is known,

$$\begin{aligned} u_1 &= -L_x^{-1}(f(x,y)u_0) - L_x^{-1}L_y u_0, \\ u_2 &= -L_x^{-1}(f(x,y)u_1) - L_x^{-1}L_y u_1, \\ &\vdots \\ u_n &= -L_x^{-1}(f(x,y)u_{n-1}) - L_x^{-1}L_y u_{n-1}. \end{aligned} \tag{11}$$

As a result, the series solution is given by

$$u(x,y) = u_0 - \sum_{n=1}^{\infty}\{L_x^{-1}(f(x,y)u_{n-1}) + L_x^{-1}L_y u_{n-1}\}, \tag{12}$$

where L_x^{-1} is the previously given integration operator. The solution $u(x,y)$ must satisfy the requirements imposed by the initial conditions.

The decomposition method provides a reliable technique that requires less work if compared with the traditional techniques. Moreover, the proposed method does not need discretization of the problem to obtain numerical results. We can evaluate the approximate solution ϕ_γ, by using the γ-term approximation. That is,

$$\phi_\gamma = \sum_{k=0}^{\gamma-1} u_k(x,y), \tag{13}$$

where the components are produced as

$$\begin{aligned}
\phi_1 &= u_0(x,y),\\
\phi_2 &= u_0(x,y) + u_1(x,y),\\
\phi_3 &= u_0(x,y) + u_1(x,y) + u_2(x,y),\\
&\vdots\\
\phi_\gamma &= u_0(x,y) + u_1(x,y) + u_2(x,y) + \cdots + u_{\gamma-1}(x,y).
\end{aligned}$$

To give a clear overview of the methodology, we consider several examples in the following section.

Moreover, the decomposition series solutions are generally converge very rapidly in real physical problems. The convergence of the decomposition series have investigated by several authors [16–25]. They obtained some results about the speed of convergence of this method providing us to solve linear and nonlinear functional equations. In recent work of Ngarhasta et al. [25] have proposed a new approach of convergence of the decomposition series. The authors have given a new condition for obtaining convergence of the decomposition series to the classical presentation of the ADM in [25]. Here, we will introduce a brief discussion of the convergence analysis in Hilbert space H as same manner in [25] of the ADM applied to the Helmholtz equation. Since we have

$$L_x u = g(x,y) - f(x,y)u - \frac{\partial^2 u}{\partial y^2}$$

and $\partial^2 u/\partial y^2$ is differential operators in H, then conditions (H1) and (H2) are fulfilled, see [25].

Next, we demonstrate the how approximate solutions of the Helmholtz equation model problem are close to corresponding exact solutions. The solution $u(x,y)$ must satisfy the requirements imposed by the initial conditions. The decomposition method provides a reliable technique that requires less work if compared with the traditional finite-difference techniques.

3. Experimental evaluations

In this section, we apply the above described methods (a simple finite-difference and Adomian's decomposition methods) on some examples so that the comparison are made numerically. We wish to emphasize that the purpose of the comparisons is only to give the reader insight into the relative efficiencies of the two methods and not definitive comparisons. Also we re-emphasize that the basic intends of new method is user convenience and easy as opposed to speed of computation. In order to verify numerically whether the proposed methodology lead to higher accuracy, chose two examples which were selected to show the computational accuracy.

Example 1. For comparison purposes, we consider a Helmholtz equation model problem in order to illustrate the technique discussed above. This problem is as follows:

$$\frac{\partial u^2}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}-u=0, \tag{14}$$

with the initial conditions

$$u(0,y)=y, \quad u_x(0,y)=y+\cosh(y). \tag{15}$$

In order to solve this equation by using the decomposition method, we simply taken the equation in an operator form in the same manner as given by Eq. (9) and used (8) to find the zeroth component of u_0 as

$$u_0=y+xy+x\cosh(y), \tag{16}$$

and remaining components u_1, u_2, u_3, etc. were computed by a recursive scheme either directly by hand or programmed on Matlab by using (11). Some of the symbolically computed components are as follows:

$$u_1=L_x^{-1}(u_0)-L_x^{-1}L_y u_0=\left(\frac{x^2}{2!}+\frac{x^3}{3!}\right)y, \tag{17}$$

$$u_2=L_x^{-1}(u_1)-L_x^{-1}L_y u_1=\left(\frac{x^4}{4!}+\frac{x^5}{5!}\right)y, \tag{18}$$

$$u_3=L_x^{-1}(u_2)-L_x^{-1}L_y u_2=\left(\frac{x^6}{6!}+\frac{x^7}{7!}\right)y, \tag{19}$$

$$\vdots$$

In this manner, four components of the decomposition series were obtained of which $u(x,y)$ was evaluated to have the following expansion:

$$u(x,y)=(1+x)y+x\cosh(y)+\left(\frac{x^2}{2!}+\frac{x^3}{3!}\right)y+\left(\frac{x^4}{4!}+\frac{x^5}{5!}\right)y+\left(\frac{x^6}{6!}+\frac{x^7}{7!}\right)y+\cdots \tag{20}$$

Continuing the expansion to the last term, it may be proved that the solution of the decomposition series (10) converges to the solution function of Eq. (14) as

$$u(x,y)=y\exp(x)+x\cosh(y). \tag{21}$$

Example 2. The second Helmholtz equation, we consider as follows:

$$\frac{\partial u^2}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}+5u=0, \tag{22}$$

with the initial conditions

$$u(0,y)=0, \quad u_x(0,y)=3\sinh(2y). \tag{23}$$

In the same way as Example 1, using (9) to find the zeroth component of u_0 as

$$u_0=3x\sinh(2y), \tag{24}$$

and using (11) to determine the individual terms of the decomposition series, we find

$$u_1=-5L_x^{-1}(u_0)-L_x^{-1}L_yu_0=-\frac{(3x)^3}{3!}\sinh(2y), \tag{25}$$

$$u_2=-5L_x^{-1}(u_1)-L_x^{-1}L_yu_1=\frac{(3x)^5}{5!}\sinh(2y), \tag{26}$$

$$u_3=-5L_x^{-1}(u_2)-L_x^{-1}L_yu_2=\frac{(3x)^7}{7!}\sinh(2y), \tag{27}$$

$$\vdots$$

Using (9) leads immediately to the solution of Eq. (22) given by

$$\begin{aligned} u(x,y)=3x\sinh(2y)&-\frac{(3x)^3}{3!}\sinh(2y)+\frac{(3x)^5}{5!}\sinh(2y)\\ &-\frac{(3x)^7}{7!}\sinh(2y)+\cdots \end{aligned} \tag{28}$$

and so on for other components. It can be easily observe that using (10) leads immediately to the solution of (22) given by

$$u(x,y)=\sin(3x)\sinh(2y) \tag{29}$$

which can be verified through substitution.

A comparisons of the numerical results of the absolute errors obtained by using the finite-difference and decomposition methods with those obtained by exact solution are given for the same step sizes h. Since an γ-term approximation is given by ϕ_γ as a measure of the approximation error we have taken the absolute error of the numerical solutions. It is to be noted that six terms only were used in evaluating the approximate solution by using the decomposition series solution.

We observe that the overall errors can be made smaller by adding new terms of the decomposition series in Eq. (13). Furthermore, as the decomposition method does not require discreatization of the variables, i.e. time and space, it is not effected by computation round off errors and one is not faced with necessity of large computer memory and time. The numerical results are summarized in Tables 1 and 2 (exponential and hyperbolic solution of Example 1), Tables 3 and 4 (trigonometric solution of Example 2).

Table 1
Example 1 using the finite-difference method

x_i	y_j			
	0.2	0.4	0.6	0.8
0.2	0.505149E−04	0.297427E−04	0.318288E−04	0.705710E−04
0.4	0.295043E−04	0.190734E−04	0.105261E−03	0.139474E−03
0.6	0.326037E−04	0.106096E−03	0.189660E−03	0.198602E−03
0.8	0.716447E−04	0.141024E−03	0.199794E−03	0.192880E−03

Table 2
Example 1 using the decomposition method

x_i	y_j			
	0.2	0.4	0.6	0.8
0.2	0.135437E−07	0.192644E−07	0.235942E−07	0.153364E−07
0.4	0.418424E−07	0.110191E−06	0.165229E−06	0.208902E−06
0.6	0.121109E−05	0.241663E−05	0.363755E−05	0.488794E−05
0.8	0.921738E−05	0.184878E−04	0.278620E−04	0.369526E−04

Table 3
Example 2 using the finite-difference method

x_i	y_j			
	0.2	0.4	0.6	0.8
0.2	0.866894E−02	0.162200E−01	0.204495E−01	0.173815E−01
0.4	0.142926E−01	0.266963E−01	0.336237E−01	0.285771E−01
0.6	0.146455E−01	0.273381E−01	0.344557E−01	0.293550E−01
0.8	0.933573E−02	0.174422E−01	0.220610E−01	0.189424E−01

Table 4
Example 2 using the decomposition method

x_i	y_j			
	0.2	0.4	0.6	0.8
0.2	0.143261E−08	0.259845E−07	0.148404E−07	0.447276E−08
0.4	0.347625E−07	0.418480E−07	0.878195E−07	0.389582E−07
0.6	0.193938E−07	0.163876E−08	0.606556E−07	0.320964E−07
0.8	0.141214E−06	0.319994E−06	0.499715E−06	0.825295E−06

Tables 1 and 2 give the results from the exponential and hyperbolic solution of Example 1 and illustrate the errors obtained by using the finite-difference and decomposition methods, respectively. We achieved a very good approximation with the actual solution of the equations by using six terms only.

In Table 3 we give the results from the trigonometric and hyperbolic solution of Example 2. The results show very poor accuracy in the case of Example 2 than the exponential and hyperbolic approximate solution of Example 1 by using the finite-difference method.

The numerical results illustrate very accurately by using the decomposition method even very few terms of the decomposition series in Table 4.

In general, when comparing the exact solution of Example 1 and especially Example 2 to the computed solution of the system, remember the discretization error involved in making the approximation. This error is $O(h^2)$. With h as large as $h = 1/5$, most of the errors in the computed solution are due to discretization error [14]. There is not such an error and difficulty in the case of using the decomposition method.

4. Conclusions

In this paper, we compare the above illustrated examples of the Helmholtz equation by the Adomian decomposition and finite-difference methods. For illustration purposes, chose two examples which were selected to show the computational accuracy. It may be concluded that, the Adomian methodology is very powerful and efficient in finding exact solutions for wide classes of problem. With regard to this application, the decomposition method outlined in the previous section finds quite practical analytic results with less computational work by determining the analytic and numerical solutions.

As expected the numerical solutions in the tables are clearly indicated that how the decomposition scheme obtains efficient results much closer to the actual solutions and also easier to use than the conventional finite-difference method. Numerical approximations show a high degree of accuracy and in most cases ϕ_γ, the γ-term approximation is accurate for quite low values of γ. The solution is very rapidly convergent and even in the few terms approximation is accurate by utilizing the Adomian's decomposition method [1,2]. It is also worth noting that the advantage of the decomposition methodology shows the convergence of the solutions dependent on given on insight into character and behavior of the solution.

Clearly, the series solution methodology can be applied to much more complicated nonlinear partial differential equations as well. However, if the problem becomes nonlinear, then decomposition method does not require discretization or perturbation and it does not make closure approximation, smallness assumptions or physically unrealistic white noise assumption in the nonlinear stochastic case [1,2].

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