HOMEWORK 2. NUMERICAL METHODS

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(1) Find the fourth Taylor polynomial $P_4(x)$ for the function $f(x) = xe^{x^2}$ about $x_0 = 0$.
   a: Find an upper bound for $|f(x) - P_4(x)|$, for $0 \leq x \leq 0.4$, ie find an upper bound of $|R_4(x)|$ for $0 \leq x \leq 0.4$
   b: Approximate $\int_0^{0.4} f(x)dx$ using $\int_0^{0.4} P_4(x)dx$

(2) Implement a function to compute

$$f(x) \;=\; \frac{1}{\sqrt{x^2+1}-x}.$$

When evaluating the previous function we can lose accuracy, transform the right hand side to avoid error (or improve the accuracy). Implement the transformed expression and compare the results with the original function. Note: use values of x greater than 10000.

(3) Implement a function to compute the exponential function by using the Taylor/Maclaurin series

$$e^x \;=\; 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + ...$$

Since we cannot add infinite terms, we can approximate this expansion by

$$e^x \;\approx\; \sum_{k=0}^{n} \frac{x^k}{k!}$$

(4) Riemann sums can be used to estimate the area under the curve $y = f(x)$ in the interval $[a, b]$. Left- and right-endpoint approximations, with subintervals of the same width, are special kinds of Riemann sums, ie,

Left-Endpoint Approximation:

$$\begin{aligned} L_n &= f(x_0)\Delta x + f(x_1)\Delta x + \cdots + f(x_{n-1})\Delta x \\ &= \sum_{i=1}^{n} f(x_{i-1})\Delta x \end{aligned}$$

Right-Endpoint Approximation:

$$\begin{aligned} R_n &= f(x_1)\Delta x + f(x_2)\Delta x + \cdots + f(x_n)\Delta x \\ &= \sum_{i=1}^{n} f(x_i)\Delta x \end{aligned}$$

where $\Delta x = \frac{b-a}{n}$ and $x_i = a + i\Delta x$, $i = 0, 1, \cdots, n$.

Implement functions to compute L_n and R_n. Using the function $f(x) = \sin x$ over the interval $[0, \frac{\pi}{2}]$, compute L_n and R_n for $n = 10$. Compare the previous results with

$$\int_0^{\frac{\pi}{2}} f(x)dx$$