Tarea 01 - Optimización
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Problema 01

Let $f_1(x_1, x_2) = x_1^2 - x_2^2$, $f_2(x_1, x_2) = 2x_1x_2$. Represent the level sets associated with $f_1(x_1, x_2) = 12$ and $f_2(x_{1,2}) = 16$ on the same figure using Python. Indicate on the figure, the points $x = [x_1, x_2]^T$ for which $f(x) = [f_1(x_1, x_2), f_2(x_1, x_2)]^T = [12, 16]^T$.

Problema 02

Consider the function $f(x) = (a^T x)(b^T x)$, where a, b and x are n-dimensional vectors. Compute the gradient $\nabla f(x)$ and the Hessian $\nabla^2 f(x)$.

Problema 03

Compute the gradient of

$$f(\theta) = \frac{1}{2}\sum_{i=1}^{n}[g(x_i) - g(Ax_i + b)]^2$$

with respect to θ, where $\theta = [a_{11}, a_{12}, a_{21}, a_{22}, b_1, b_2]^T$, $x_i \in \mathbb{R}^2$ are defined as follows:

$$A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \qquad b = [b_1, b_2]^T$$

Problema 04

Let $f(r, \theta)$ be $\mathbb{R}^2 \rightarrow \mathbb{R}$ with $r = \sqrt{x^2 + y^2}$ and $\theta = arctan(y/x)$. Compute $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$.

Se tiene que una derivada parcial puede escribir como la ecuación 1. Esto debido a la regla de la cadena.

$$\frac{\partial f}{\partial q} = \frac{\partial f}{\partial r}\frac{\partial r}{\partial q} + \frac{\partial f}{\partial \theta}\frac{\partial \theta}{\partial q} \tag{1}$$

donde q es una variable generalizada que puede ser x o y.

Calculando $\frac{\partial r}{\partial q}$ y $\frac{\partial \theta}{\partial q}$, se obtiene lo siguiente:

$$\frac{\partial r}{\partial x} = \frac{x}{\sqrt{x^2+y^2}} \qquad \frac{\partial \theta}{\partial x} = -\frac{y}{x^2+y^2}$$
$$\frac{\partial r}{\partial y} = \frac{y}{\sqrt{x^2+y^2}} \qquad \frac{\partial \theta}{\partial y} = \frac{x}{x^2+y^2}$$

Por lo tanto, $\frac{\partial f}{\partial x}$ y $\frac{\partial f}{\partial y}$ es:

$$\frac{\partial f}{\partial x} = \frac{\partial f}{\partial r}\frac{x}{\sqrt{x^2+y^2}} - \frac{\partial f}{\partial \theta}\frac{y}{x^2+y^2}$$

$$\frac{\partial f}{\partial y} = \frac{\partial f}{\partial r}\frac{y}{\sqrt{x^2+y^2}} + \frac{\partial f}{\partial \theta}\frac{x}{x^2+y^2}$$

Problema 05

The directional derivative $\frac{\partial f}{\partial v}(x_0, y_0, z_0)$ of a differentiable function f are $\frac{3}{\sqrt{2}}, \frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}$ in the directions of vectors $[0, \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}]^T$, $[\frac{1}{\sqrt{2}}, 0, \frac{1}{\sqrt{2}}]^T$ and $[\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, 0]^T$. Compute $\nabla f(x_0, y_0, z_0)$.

Se tiene el siguiente sistema:

$$[0, \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}] \begin{bmatrix} \frac{\partial f}{\partial x} \\ \frac{\partial f}{\partial y} \\ \frac{\partial f}{\partial z} \end{bmatrix} = \nabla_{v_1} f$$

$$[\frac{1}{\sqrt{2}}, 0, \frac{1}{\sqrt{2}}] \begin{bmatrix} \frac{\partial f}{\partial x} \\ \frac{\partial f}{\partial y} \\ \frac{\partial f}{\partial z} \end{bmatrix} = \nabla_{v_2} f$$

$$[\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, 0] \begin{bmatrix} \frac{\partial f}{\partial x} \\ \frac{\partial f}{\partial y} \\ \frac{\partial f}{\partial z} \end{bmatrix} = \nabla_{v_3} f$$

donde $\nabla_{v_1} f = \frac{3}{\sqrt{2}}$, $\nabla_{v_2} f = \frac{1}{\sqrt{2}}$ y $\nabla_{v_3} f = -\frac{1}{\sqrt{2}}$

Entonces, se obtiene el siguiente sistema de ecuaciones:

$$\frac{\partial f}{\partial y} + \frac{\partial f}{\partial z} = 3$$
$$\frac{\partial f}{\partial x} + \frac{\partial f}{\partial z} = 1$$
$$\frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} = -1$$

El cual puede ser llevado a ser escrito en el siguiente sistema matricial:

$$\begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix} \begin{bmatrix} \frac{\partial f}{\partial x}(x_0, y_0, z_0) \\ \frac{\partial f}{\partial y}(x_0, y_0, z_0) \\ \frac{\partial f}{\partial z}(x_0, y_0, z_0) \end{bmatrix} = \begin{bmatrix} 3 \\ 1 \\ -1 \end{bmatrix}$$

Resolviendo el sistema matricial, se obtiene que el gradiente de f evaluado en X_0 es:

$$\nabla f(x_0, y_0, z_0) = \begin{bmatrix} -\frac{3}{2} & \frac{1}{2} & \frac{5}{2} \end{bmatrix}^T$$

Problema 06

Show that the level curves of the function $f(x, y) = x^2 + y^2$ are orthogonal to the level curves of $g(x, y) = y/x$ for all (x, y).

Calculando ∇f Y ∇g, se obtiene que:

$$\nabla f = \begin{bmatrix} 2x \\ 2y \end{bmatrix} \qquad \nabla g = \begin{bmatrix} -\frac{y}{x^2} \\ \frac{1}{x} \end{bmatrix}$$

Se tiene que si el producto punto entre dos vectores es igual a 0, entonces los dos vectores son ortogonales. Entonces calculando $\nabla f^T \nabla g$, se obtiene lo siguiente:

$$\begin{aligned} \nabla f^T \nabla g &= \begin{bmatrix} 2x & 2y \end{bmatrix} \begin{bmatrix} -\frac{y}{x^2} \\ \frac{1}{x} \end{bmatrix} \\ &= -\frac{2y}{x} + \frac{2y}{x} \\ &= 0 \end{aligned}$$

Como $\nabla f^T \nabla g = 0$, entonces als curvas de nivel de f y g son ortogonales para cualquier (x, y).

Problema 07

Let f, g, h be differentiable functions, with $f : \mathbb{R}^n \to \mathbb{R}^3, g : \mathbb{R}^n \to \mathbb{R}^3$ and $h : \mathbb{R}^n \to \mathbb{R}$

$$h(x) = f(x)^T g(x)$$

show that

$$Dh(x) = f(x)^T Dg(x) + g(x)^T Df(x)$$

Se tiene que:

$$h(x) = \begin{bmatrix} f_1 & f_2 & f_3 \end{bmatrix} \begin{bmatrix} g_1 \\ g_2 \\ g_3 \end{bmatrix}$$

donde f_i, g_i son funciones que van de $\mathbb{R}^n \to \mathbb{R}$. Calculando el producto de matrices se obtiene que la función h es:

$$h(x) = f_1g_1 + f_2g_2 + f_3g_3$$

Calculando Dh se obtiene que:

$$\begin{aligned}
Dh(x) &= D(f_1g_1 + f_2g_2 + f_3g_3) \\
&= D(f_1g_1) + D(f_2g_2) + D(f_3g_3) \\
&= f_1Dg_1 + g_1Df_1 + f_2Dg_2 + g_2Df_2 + f_3Dg_3 + g_3Df_3 \\
&= g_1Df_1 + g_2Df_2 + g_3Df_3 + f_1Dg_1 + f_2Dg_2 + f_3Dg_3 \\
&= \begin{bmatrix} g_1 & g_2 & g_3 \end{bmatrix} \begin{bmatrix} Df_1 \\ Df_2 \\ Df_3 \end{bmatrix} + \begin{bmatrix} f_1 & f_2 & f_3 \end{bmatrix} \begin{bmatrix} Dg_1 \\ Dg_2 \\ Dg_3 \end{bmatrix} \\
Dh(x) &= g^T Df + f^T Dg
\end{aligned}$$

Problema 08

Consider the induced matrix norm

$$||A||_p = \max_{x \neq 0} \frac{||A_x||_p}{||x||_p}$$

where $|| \cdot ||_p$ is the l_p norm, ie

$$||x|| = (|x_1|^p + |x_2|^p + \cdots + |x_n|^p)^{\frac{1}{p}}$$

show that

$$||AB||_p \leq ||A||_p ||B||_p$$