Optimization. Homework 2

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1. Show that $x - \sin x = o(x^2)$, as $x \to 0$

2. Suppose that $f(\boldsymbol{x}) = o(g(\boldsymbol{x}))$. Show that $f(\boldsymbol{x}) = O(g(\boldsymbol{x}))$. Tip: Show that for any given $\epsilon > 0$, there exists $\delta > 0$ such that if $0 < \|x\| < \delta$, then $|f(\boldsymbol{x})| < \epsilon |g(\boldsymbol{x})|$.

3. Show that if functions $f : \mathbb{R}^n \to \mathbb{R}$ and $g : \mathbb{R}^n \to \mathbb{R}$ satisfy $f(\boldsymbol{x}) = -g(\boldsymbol{x}) + o(g(\boldsymbol{x}))$ and $g(\boldsymbol{x}) > 0$ for all $\boldsymbol{x} \neq \mathbf{0}$, then for all $\boldsymbol{x} \neq \mathbf{0}$ sufficiently small, we have $f(\boldsymbol{x}) < 0$.

4. Compute the stationary points of $f(x, y) = \frac{3x^4 - 4x^3 - 12x^2 + 18}{12(1+4y^2)}$ and determine their corresponding type (ie: minimum, maximum or saddle point)

5. Show that the function $f(\boldsymbol{x}) = 8x_1 + 12x_2 + x_1^2 - 2x_2^2$ has only one stationary point, and that it is neither a maximum or minimum, but a saddle point. Plot the contour lines of f.

6. Compute the gradient $\nabla f(\boldsymbol{x})$ and Hessian $\nabla^2 f(\boldsymbol{x})$ of the Rosenbrock function

$$f(\boldsymbol{x}) = \sum_{i=1}^{N-1} [100(x_{i+1} - x_i^2)^2 + (1 - x_i)^2]$$

   where $\boldsymbol{x} = [x_1, \ldots, x_N]^T \in \mathbb{R}^N$

   If $n = 2$ show that $\boldsymbol{x}^* = [1, 1]^T$ is the only local minimizer of this function, and that the Hessian matrix at that point is positive definite. Plot the contour lines of f.

7. Show, without using the optimality conditions, that $f(\boldsymbol{x}) > f(\boldsymbol{x}^*)$ for all $\boldsymbol{x} \neq \boldsymbol{x}^*$ if

$$f(\boldsymbol{x}) = \frac{1}{2}\boldsymbol{x}^T \mathbf{Q} \boldsymbol{x} - \boldsymbol{b}^T \boldsymbol{x}$$

   $\mathbf{Q} = \mathbf{Q}^T \succ 0$ and $\mathbf{Q}\boldsymbol{x}^* = \boldsymbol{b}$.

8. Let $\mathbf{A} \in \mathbb{R}^{n \times n}$ be a real symmetric matrix. Show that

$$\max\{\boldsymbol{x}^T \mathbf{A} \boldsymbol{x} : \|\boldsymbol{x}\| = 1\} = \lambda_{\max}(\mathbf{A})$$