Interacción de una partícula con spin 0 con un campo electromagnético

$\boxed{A^\mu} = \{A_0, \vec{A}\}$ $\qquad$ $A_\mu = g_{\mu\nu}A^\nu = \{A_0, -\vec{A}\}$

Mecánica cuántica no relativista, el acoplamiento mínimo del campo electromagnético

$$E \to i\hbar\frac{\partial}{\partial t} - eA_0 \qquad \vec{P} \to -i\hbar\vec{\nabla} - \frac{e}{c}\vec{A}$$

$$p^\mu \to p^\mu - \frac{e}{c}A^\mu \qquad \boxed{P_\mu \to P_\mu - \frac{e}{c}A_\mu}$$

De la misma forma, se usa el acoplamiento mínimo, en la ecuación de K.G.

$$\left[\Box + \frac{m_0^2c^2}{\hbar^2} \right] \psi \longleftrightarrow p^\mu p_\mu \psi = m_0^2 c^2 \psi$$

$$\left(p^\mu - \frac{e}{c} A^\mu\right)\left(p_\mu - \frac{e}{c} A_\mu\right) \psi = m_0^2 c^2 \psi$$

$$\left(i\hbar\partial^\mu - \frac{e}{c} A^\mu\right)\left(i\hbar\partial_\mu - \frac{e}{c} A_\mu\right) \psi = m_0^2 c^2 \psi$$

$$\boxed{g^{\mu\nu}\left(i\hbar\partial_\nu - \frac{e}{c} A_\nu\right)\left(i\hbar\partial_\mu - \frac{e}{c} A_\mu\right) \psi = m_0^2 c^2 \psi}$$

$$g^{00}\left(i\hbar\partial_0 - \frac{e}{c} A_0\right)\left(i\hbar\partial_0 - \frac{e}{c} A_0\right) \psi$$

$$+ \left[g^{ii}\right]\left(i\hbar\partial_i + \frac{e}{c} A_i\right)\left(i\hbar\partial_i + \frac{e}{c} A_i\right) \psi = m_0^2 c^2 \psi$$

$$\left(i\hbar\partial_0 - \frac{e}{c} A_0\right)\left(i\hbar\partial_0 - \frac{e}{c} A_0\right) \psi$$

$$- \sum_{j=1}^{3}\left(i\hbar\partial_j + \frac{e}{c} A_j\right)\left(i\hbar\partial_j + \frac{e}{c} A_j\right) \psi = m_0^2 c^2 \psi$$

además $\partial_0 = \frac{\partial}{\partial x_0} = \frac{\partial}{\partial (ct)} = \frac{1}{c}\frac{\partial}{\partial t}$

$$\frac{1}{c^2}\left[i\hbar\frac{\partial}{\partial t} - eA_0\right]^2\psi = \left[\underbrace{\sum_{j=1}^{3}\left(i\hbar\partial_j + \frac{e}{c}A_j\right)^2}_{\mathrm{I}} + m_0^2c^2\right]\psi$$

Escribimos el término I

$$\sum_j = \left(i\hbar\frac{\partial}{\partial x^1} + \frac{e}{c}A_1\right)^2 + \left(i\hbar\frac{\partial}{\partial x^2} + \frac{e}{c}A_2\right)^2$$

$$+ \left(i\hbar\frac{\partial}{\partial x^3} + \frac{e}{c}A_3\right)^2$$

$$= \left(i\hbar\frac{\partial}{\partial x_1}\right)\left(i\hbar\frac{\partial}{\partial x^1}\right) + i\hbar\frac{\partial}{\partial x^1}\left[\frac{e}{c}A_1\right] + \frac{e}{c}A_1\, i\hbar\frac{\partial}{\partial x^1}$$

$$+ \left(\frac{e}{c}\right)A_1\left(\frac{e}{c}\right)A_1 + \dots$$

$$(i\hbar)^2\frac{\partial^2}{\partial^2 x^2} + i\hbar\left(\frac{e}{c}\right)\frac{\partial}{\partial x^2}A_2 + i\hbar\frac{e}{c}A_2\frac{\partial}{\partial x^2}$$

$$+ \frac{e^2}{c^2}A_2^2 + \dots$$

$$\sum_{\nu=1}^{3}[\quad]\Psi = (i\hbar)^2\left[\frac{\partial^2}{\partial^2 x_1}\Psi + \frac{\partial^2}{\partial^2 x_2}\Psi + \frac{\partial^2}{\partial^2 x_3}\Psi\right] \checkmark$$

$$+ \frac{e^2}{c^2}\left[A_1^2 + A_2^2 + A_3^2\right]\Psi + \cdots$$

$$= (i\hbar)^2\,\vec{\nabla}\cdot\vec{\nabla}\,\Psi + \frac{e^2}{c^2}[\vec{A}\cdot\vec{A}]\,\Psi$$

$$+ i\hbar\left(\frac{e}{c}\right)\left[\frac{\partial}{\partial x^1}(A_1\Psi) + \frac{\partial}{\partial x^2}(A_2\Psi)\right.$$

$$\left. + \frac{\partial}{\partial x_3}(A_3\Psi)\right]$$

$$+ i\hbar\left(\frac{e}{c}\right)\left[A_1\frac{\partial}{\partial x_1}\Psi + A_2\frac{\partial\Psi}{\partial x_2} + A_3\frac{\partial}{\partial x_3}\Psi\right]$$

$$= (i\hbar)^2\,\vec{\nabla}\cdot\vec{\nabla}\,\Psi + \frac{e^2}{c^2}\vec{A}\cdot\vec{A}\,\Psi$$

$$+ i\hbar\left(\frac{e}{c}\right)\left[\vec{\nabla}\cdot(\vec{A}\,\Psi)\right] + i\hbar\left(\frac{e}{c}\right)(\vec{A}\cdot\vec{\nabla})\,\Psi$$

$$= \left(i\hbar\vec{\nabla} + \frac{e}{c}\vec{A}\right)^2 \psi$$

Entonces la ec. de K.G con electromagnetismo

$$\frac{1}{c^2}\left[i\hbar\frac{\partial}{\partial t} - eA_0\right]^2 \psi = \left[\left(i\hbar\vec{\nabla} + \frac{e}{c}\vec{A}\right)^2 + m_0^2c^2\right]\psi$$

Densidad de corriente

$$\psi^*\left[g^{\mu\nu}\left[i\hbar\frac{\partial}{\partial x^\nu} - \frac{e}{c}A_\nu\right]\left[i\hbar\frac{\partial}{\partial x^\mu} - \frac{e}{c}A_\mu\right]\psi\right]$$

$$= \underline{\psi^* m_0^2 c^2 \psi} \quad (1)$$

Ec. de K.G., conjugada, multiplicando a la izquierda ψ

$$\psi\left[g^{\mu\nu}\left[-i\hbar\frac{\partial}{\partial x^\nu} - \frac{e}{c}A_\nu\right]\left[-i\hbar\frac{\partial}{\partial x^\mu} - \frac{e}{c}A_\mu\right]\psi^*\right]$$

$$= \underline{\psi\, m_0^2 c^2 \psi^*} \quad (2)$$

restamos 1 - 2

$$0 = g^{\mu\nu}\left[\psi^*\left(i\hbar\partial_\nu - \frac{e}{c}A_\nu\right)\left[i\hbar\partial_\mu - \frac{e}{c}A_\mu\right]\psi\right.$$

$$\left. - \psi\left[-i\hbar\partial_\nu - \frac{e}{c}A_\nu\right]\left[i\hbar\partial_\mu - \frac{e}{c}A_\mu\right]\psi^*\right.$$

$$0 = g^{\mu\nu}\left[(i\hbar)^2\psi^*\left(\partial_\nu + \frac{ie}{\hbar c}A_\nu\right)\left(\partial_\mu + \frac{ie}{\hbar c}A_\mu\right)\psi\right.$$

$$\left. - \underline{(i\hbar)^2}\,\psi\,\underline{\left[-\partial_\nu + \frac{ie}{\hbar c}A_\nu\right]\left[-\partial_\mu + \frac{ie}{\hbar c}A_\mu\right]\psi^*}\right]$$

$$0 = g^{\mu\nu}\left[-\hbar^2\psi^*\left[\partial_\nu\partial_\mu\psi + \frac{ie}{\hbar c}\partial_\nu(A_\mu\psi)\right.\right.$$

$$\left. + \frac{ie}{\hbar c}A_\nu\partial_\mu\psi + \left(\frac{ie}{\hbar c}\right)^2\cancel{A_\nu A_\mu\psi}\right]$$

$$+ \hbar^2\psi\left[\partial_\nu\partial_\mu\psi^* - \frac{ie}{\hbar c}\partial_\nu(A_\mu\psi^*)\right.$$

$$\left.\left. - \frac{ie}{\hbar c}A_\nu\partial_\mu\psi^* + \left(\frac{ie}{\hbar c}\right)^2\cancel{A_\nu A_\mu\psi^*}\right]\right]$$

$$0 = g^{\mu\nu}\Big[\psi\,\partial_\mu\partial_\nu\psi^* - \frac{ie}{\hbar c}\psi\,\partial_\nu(A_\mu\psi^*)$$

$$- \psi A_\nu\,\partial_\mu\psi^*\frac{ie}{\hbar c}$$

$$- \psi^*\partial_\mu\partial_\nu\psi - \frac{ie}{\hbar c}\psi^*\partial_\nu(A_\mu\psi) - \frac{ie}{\hbar c}\psi^* A_\nu\partial_\mu\psi\Big]$$

$$0 = g^{\mu\nu}\Big[\partial_\mu[\psi\,\partial_\nu\psi^*] - (\partial_\nu\psi\,\partial_\mu\psi^*)$$

$$- \partial_\mu[\psi^*\partial_\nu\psi] + (\partial_\mu\psi^*\,\partial_\nu\psi)\Big]$$

$$- g^{\mu\nu}\frac{ie}{\hbar c}\Big[\psi A_\nu\partial_\mu\psi^* + \psi^*\partial_\nu(A_\mu\psi)\Big]$$

$$- g^{\mu\nu}\frac{ie}{\hbar c}\Big[\psi\,\partial_\nu(A_\mu\psi^*) + \psi^* A_\nu(\partial_\mu\psi)\Big]$$

$$0 = g^{\mu\nu} \partial_\mu \left[\psi \partial_\nu \psi^* - \psi^* \partial_\nu \psi \right]$$

$$- \frac{ie}{\hbar c} \left[g^{\mu\nu} \psi A_\nu \partial_\mu \psi^* + g^{\nu\mu} \psi^* \partial_\mu (A_\nu \psi) \right]$$

$$- \frac{ie}{\hbar c} \left[g^{\mu\nu} \psi \partial_\nu (A_\mu \psi^*) + g^{\nu\mu} \psi^* A_\mu (\partial_\nu \psi) \right]$$

$$0 = g^{\mu\nu} \partial_\mu \left[\psi \partial_\nu \psi^* - \psi^* \partial_\nu \psi \right]$$

$$- \frac{ie}{\hbar c} \left[g^{\mu'\nu'} \partial_{\mu'} \left[\psi A_{\nu'} \psi^* \right] \right]$$

$$- \frac{ie}{\hbar c} \left[g^{\mu'\nu'} \partial_{\nu'} \left[\psi A_{\mu'} \psi^* \right] \right]$$

$$\mu \leftrightarrow \nu$$

$$0 = g^{\mu\nu} \Big[\partial_\mu (\psi \partial_\nu \psi^* - \psi^* \partial_\nu \psi] - 2 \partial_\mu \left[\frac{ie}{\hbar c} \psi \partial_\nu \psi^* \right] \Big]$$

$$0 = g^{\mu\nu}\partial_\mu\left[\frac{i\hbar e}{2m_0}\{\psi^*\partial_\nu\psi - \psi\partial_\nu\psi^*\} - \frac{e^2}{m_0 c}A_\nu\psi\psi^*\right]$$

So define $j'_\nu = \frac{i\hbar e}{2m_0}\{\psi^*\partial_\nu\psi - \psi\partial_\nu\psi^*\}$

$$- \frac{e^2}{m_0 c^2}A_\nu\psi\psi^*$$

$$0 = g^{\mu\nu}\partial_\mu j'_\nu = 0$$

$$\boxed{\partial^\nu j'_\nu = 0}$$

$$j'^\nu = \{c\rho', -\vec{j}'\}$$

$\nu = 0 \quad j'_0 = c\rho' \qquad \nu = 1,2,3 \rightarrow \vec{j}'$

$$c\rho' = \frac{i\hbar e}{2m_0}\left[\Psi^*\frac{\partial}{\partial x^0}\Psi - \Psi\frac{\partial}{\partial x^0}\Psi^*\right] - \frac{e^2}{m_0 c}A_0\Psi\Psi^*$$

$$\rho' = \frac{i\hbar e}{2m_0 c^2}\left[\Psi^*\frac{\partial}{\partial t}\Psi - \Psi\frac{\partial}{\partial t}\Psi^*\right]$$

$$-\frac{e^2}{m_0 c^2}A_0\Psi\Psi^*$$

Mostrar (propuesto)

$$\vec{J}' = -\frac{i\hbar e}{2m_0}\left[\Psi^*\nabla\Psi - \Psi\nabla\Psi^*\right]$$

$$-\frac{e^2}{m_0 c}\vec{A}\,\Psi\Psi^*$$