UNIVERSIDAD AUTÓNOMA DE NUEVO LEÓN
FACULTAD DE CIENCIAS FÍSICO MATEMÁTICAS

Mécaninca Cuántica Relativista
Ejercicios propuestos
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Ejercicio 1

Mostrar que:

$$U_\perp = \frac{u_i}{\gamma_v \left[1 + \frac{v \cdot v}{c^2}\right]}$$

De las transformaciones:

$$\begin{aligned} r_\parallel &= \gamma_v [{r_\parallel}' + vt'] \\ r_\perp &= {r_\perp}' \\ t &= \gamma_v \left[t' + \frac{v \cdot v}{c^2}\right] \end{aligned}$$

tomando los diferenciales:

$$\begin{aligned} dr_\parallel &= \gamma_v \left[dr'_\parallel + vdt\right] \\ dr_\perp &= dr'_perp \\ dt &= \gamma_v \left[dt + \frac{vdr}{c^2}\right] \end{aligned}$$

entonces:

$$\begin{aligned} \frac{dr_\perp}{dt} &= \frac{dr'_\perp}{\gamma_v dt' \left[1 + \frac{v}{c^2}\frac{dr'}{dt'}\right]} \\ u_\perp &= \frac{dr'_\perp}{\gamma_v dt' \left[1 + \frac{v}{c^2}\frac{dr'}{dt'}\right]} \\ &= \frac{u'_\perp}{\gamma_v \left[1 + \frac{v \cdot u'}{c^2}\right]} \end{aligned}$$

por lo tanto:

$$u_\perp = \frac{u'_\perp}{\gamma_v \left[1 + \frac{v \cdot u'}{c^2}\right]} \tag{1}$$

Ejercicio 2

Mostrar que

$$\begin{aligned} (U_d)_x &= \frac{-c\beta \sin(\theta')}{\gamma_v (1 - \beta^2 \cos(\theta'))} \\ (U_d)_z &= \frac{c\beta(1 - \cos(\theta'))}{1 - \beta^2 \cos(\theta')} \end{aligned}$$

Se sabe que por convención:

$$\begin{aligned} (U_d)_\perp = (U_d)_x &= \frac{(U'_d)_\perp}{\gamma_v \left(1 + \frac{v \cdot u'}{c^2}\right)} \\ (U_d)_\parallel = (U_c)_z &= \frac{(U'_d)_\parallel + v}{1 + \frac{v \cdot u'_d}{c^2}} \end{aligned}$$

pero, del diagrama

$$(U'_d)_\perp = U'_d \sin(\theta')$$
$$(U'_d)_\parallel = U'_d \cos(\theta')$$

por lo tanto:

$$(U_d)_x = \frac{U'_d \sin(\theta')}{\gamma_v \left(1 + \frac{|v||u_d|\cos(\theta)}{c^2}\right)}$$
$$(U_d)_z = \frac{U'_d \cos(\theta'_d) + v}{\gamma_v \left(1 + \frac{|v||u_d|\cos(\theta)}{c^2}\right)}$$

pero $U'_d = -v$

$$\begin{aligned}(U_d)_x &= \frac{-v \sin(\theta')}{\gamma_v \left(1 - \frac{v^2}{c^2}\cos(\theta')\right)} \\ &= \frac{-c\beta \sin(\theta')}{\gamma_v(1 - \beta^2 \cos(\theta'))} \\ (U_d)_z &= \frac{c\beta(1 - \cos(\theta'))}{1 - \beta^2 \cos(\theta')}\end{aligned}$$

Ejercicio 3

Mostrar que

$$u_c^2 = u_a^2 - \frac{\eta}{\gamma_a} \tag{2}$$

$$\begin{aligned}(U_c)_x &= \frac{c\beta \sin(\theta')}{\gamma_v \left[1 + \beta^2 \cos(\theta')\right]} \\ &\approx \frac{c\beta\theta}{\gamma_v \left[1 + \beta^2 \left(1 - \frac{\theta^2}{2}\right)\right]} \\ (U_c)_z &\approx \frac{c\beta(1 + \left(1 - \frac{\theta^2}{2}\right))}{1 + \beta^2 \left(1 - \frac{\theta^2}{2}\right)}\end{aligned}$$

realizando el calculo para ángulos pequeños, tomando en cuenta que $\cos(\theta) = 1-\theta^2/2$ y $\sin(\theta) = \theta$

$$
\begin{aligned}
(U_c)_z^2 &= \frac{c^2\beta^2\left(4 - 2\theta^2 + \frac{\theta^4}{4}\right)}{(1+\beta^2)\left(1 - \frac{\beta^2\theta^2}{2(1+\beta^2)}\right)^2} \\
&= \frac{c^2\beta^2(4 - 2\theta^2)}{(1+\beta^2)^2\left(1 - \frac{\beta^2\theta^2}{2(1+\beta^2)}\right)^2} \\
&\approx \frac{c^2\beta^2(4-2\theta^2)}{(1+\beta^2)^2}\left(1 + \frac{\beta^2}{1+\beta^2}\theta^2\right) \\
&\approx \frac{4c^2\beta^2}{(1+\beta^2)^2} - \frac{2c^2\beta^2\theta^2}{(1+\beta^2)^2} + \frac{4c^2\beta^4\theta^2}{(1+\beta^2)^3} \\
&\approx u_a^2 - \frac{2c^2\beta^2\theta^2}{(1+\beta^2)^2} + \frac{4c^2\beta^4\theta^2}{(1+\beta^2)^3}
\end{aligned}
$$

$$
\begin{aligned}
(U_c)_x^2 &= \frac{c^2\beta^2\theta^2}{\gamma_v^2(1+\beta^2)^2\left(1 - \frac{\beta^2\theta^2}{2(1+\beta^2)}\right)^2} \\
&\approx \frac{c^2\beta^2\theta^2}{\gamma_v^2(1+\beta^2)}\left(1 + \frac{\beta}{1+\beta^2}\theta^2\right) \\
&\approx \frac{c^2\beta^2\theta^2}{\gamma^2(1+\beta^2)^2} + \frac{c^2\beta^4\theta^4}{\gamma^2(1+\beta^2)^3} \\
&\approx \frac{c^2\beta^2\theta^2}{\gamma^2(1+\beta^2)^2}
\end{aligned}
$$

se tiene que:

$$u_a = \frac{2\beta c}{1+\beta^2} \qquad\qquad \gamma_a = \frac{1+\beta^2}{1-\beta^2}$$

por lo tanto:

$$
\begin{aligned}
u_c^2 &= (u_c)_x^2 + (u_c)_z^2 \\
&= u_a^2 - \frac{2c^2\beta^2\theta^2}{(1+\beta^2)^2} + \frac{4c^2\beta^4\theta^2}{(1+\beta^2)^3} + \frac{c^2\beta^2\theta^2}{\gamma^2(1+\beta^2)^2} \\
&= u_a^2 + \frac{c^2\beta^2\theta^2}{(1+\beta^2)^2}\left(1 - \beta^2 - 2 + \frac{4\beta^2}{1+\beta^2}\right) \\
&= u_a^2 + \frac{c^2\beta^2\theta^2}{(1+\beta^2)^2}\left(\frac{1 - 2\beta^2 - \beta^4}{1+\beta^2}\right) \\
&= u_a^2 - \frac{c^2\beta^2\theta^2}{(1+\beta^2)^2}\left(\frac{(1-\beta^2)^2}{1+\beta^2}\right) \\
&= u_a^2 - \frac{c^2\beta^2\theta^2}{1-\beta^2}\left(\frac{(1-\beta^2)^3}{(1+\beta^2)^3}\right) \\
&= u_a^2 - \eta\frac{1}{\gamma_a^3}
\end{aligned}
$$

Ejercicio 4

Muestre que:

$$\partial_\alpha A^\alpha = \partial^\alpha A_\alpha$$

Se tiene que:

$$x_\alpha = g_{\alpha\beta} x^\beta \qquad\qquad x^\alpha = g^{\alpha\beta} x_\beta$$

por lo tanto:

$$g^{\alpha\beta}\partial_\alpha = \partial^\alpha$$
$$g_{\alpha\beta}\partial^\alpha = \partial_\alpha$$

calculando $\partial^\alpha A_\alpha$

$$\begin{aligned}\partial^\alpha A_\alpha &= \left(\frac{\partial A_0}{\partial x_0}\right) - \left(\frac{\partial A_1}{\partial x_1}\right) - \left(\frac{\partial A_2}{\partial x_2}\right) - \left(\frac{\partial A_3}{\partial x_3}\right) \\ &= \frac{\partial A_0}{\partial x_0} - \nabla A\end{aligned}$$

por lo que se encuentra que:

$$\begin{aligned}A_0 &= A^0 \\ A_1 &= -A^1 \\ A_2 &= -A^2 \\ A_3 &= -A^3\end{aligned}$$

$$\begin{aligned}\partial^\alpha A_\alpha &= (g^{\alpha\beta}\partial_\beta)(g_{\alpha\gamma}A^\gamma) \\ \delta^\beta_\gamma &= \partial_\beta A^\gamma\end{aligned}$$

Ejercicio 5

Por verificar que:

$$\partial^\alpha = \left(\frac{\partial}{\partial x_0}, -\nabla\right)$$

Sea A^α un tensor covariante, entonces:

$$\begin{aligned}\partial^\alpha A_\alpha &= \left(\frac{\partial A_0}{\partial x_0}\right) - \left(\frac{\partial A_1}{\partial x_1}\right) - \left(\frac{\partial A_2}{\partial x_2}\right) - \left(\frac{\partial A_3}{\partial x_3}\right) \\ &= \left(\frac{\partial}{\partial x_0}, -\frac{\partial}{\partial x_1}, -\frac{\partial}{\partial x_2}, -\frac{\partial}{\partial x_3}\right) \cdot (A_0, A_1, A_2, A_3) \\ &= \left(\frac{\partial}{\partial x_0}, -\nabla\right) \cdot A_\alpha\end{aligned}$$

por lo tanto:

$$\partial^\alpha = \left(\frac{\partial}{\partial x_0}, -\nabla\right)$$

Ejercicio 6

Probar que las matrices S_1^2, S_2^2, S_3^2 son diagonales con -1 y que las matrices K_1^2, K_2^2, K_3^2 son diagonales con 1: Se tiene la matriz S_1 igual a:

$$S_1 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

entonces, calculando S_1^2, se tiene que:

$$\begin{aligned} S_1^2 &= \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{pmatrix}^2 \\ &= \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{pmatrix} \\ &= \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \end{aligned}$$

Se tiene la matriz S_2 igual a:

$$S_2 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

entonces, calculando S_2^2, se tiene que:

$$\begin{aligned} S_2^2 &= \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix}^2 \\ &= \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix} \\ &= \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \end{aligned}$$

Se tiene la matriz S_3 igual a:

$$S_3 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & -0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

entonces, calculando S_3^2, se tiene que:

$$\begin{aligned} S_3^2 &= \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}^2 \\ &= \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \\ &= \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \end{aligned}$$

por lo tanto las matrices S_μ^2 son diagonales con -1 Se tiene la matriz K_1 igual a:

$$S_1 = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

entonces, calculando K_1^2, se tiene que:

$$\begin{aligned} K_1^2 &= \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}^2 \\ &= \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \\ &= \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \end{aligned}$$

Se tiene la matriz K_2 igual a:

$$K_2 = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

entonces, calculando k_2^2, se tiene que:

$$
\begin{aligned}
K_2^2 &= \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}^2 \\
&= \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \\
&= \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}
\end{aligned}
$$

Se tiene la matriz K_3 igual a:

$$
K_3 = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}
$$

entonces, calculando K_3^2, se tiene que:

$$
\begin{aligned}
K_3^2 &= \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}^2 \\
&= \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix} \\
&= \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}
\end{aligned}
$$

por lo tanto las matrices K_μ^2 son diagonales con 1

Ejercicio 7

Mostrar que $F_{\alpha\gamma} = g_{\alpha\gamma} F^{\gamma\delta} g_{\delta\beta}$
Se sabe que:

$$
F^{\gamma\delta} = \begin{pmatrix} 0 & -Ex & -Ey & -Ez \\ E_x & 0 & -B_z & B_y \\ E_y & B_z & 0 & -B_x \\ E_z & -B_y & B_x & 0 \end{pmatrix} \qquad g_{\alpha\gamma} = g_{\delta\beta} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}
$$

realizando la multiplicacion $F^{\gamma\delta}g_{\delta\beta}$

$$F^{\gamma\delta}g_{\delta\beta} = \begin{pmatrix} 0 & -Ex & -Ey & -Ez \\ E_x & 0 & -B_z & B_y \\ E_y & B_z & 0 & -B_x \\ E_z & -B_y & B_x & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$$
$$= \begin{pmatrix} 0 & Ex & Ey & Ez \\ E_x & 0 & B_z & -B_y \\ E_y & -B_z & 0 & B_x \\ E_z & B_y & -B_x & 0 \end{pmatrix}$$

por lo tanto:

$$F^{\gamma}_{\beta} = \begin{pmatrix} 0 & Ex & Ey & Ez \\ E_x & 0 & B_z & -B_y \\ E_y & -B_z & 0 & B_x \\ E_z & B_y & -B_x & 0 \end{pmatrix}$$

realizando la multiplicacion $g_{\alpha\gamma}F^{\gamma}_{\beta}$ se obtiene que:

$$g_{\alpha\gamma}F^{\gamma}_{\beta} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \begin{pmatrix} 0 & Ex & Ey & Ez \\ E_x & 0 & B_z & -B_y \\ E_y & -B_z & 0 & B_x \\ E_z & B_y & -B_x & 0 \end{pmatrix}$$
$$= \begin{pmatrix} 0 & Ex & Ey & Ez \\ -E_x & 0 & -B_z & B_y \\ -E_y & B_z & 0 & -B_x \\ -E_z & -B_y & B_x & 0 \end{pmatrix}$$

por lo tanto:

$$F_{\alpha\beta} = \begin{pmatrix} 0 & Ex & Ey & Ez \\ -E_x & 0 & -B_z & B_y \\ -E_y & B_z & 0 & -B_x \\ -E_z & -B_y & B_x & 0 \end{pmatrix}$$

Ejercicio 8

Compruebe la forma de L, que cumple $L^T g = -gL$ donde L tiene diagonal de ceros y g es la representación matricial de $g_{\mu\nu\rho}$ Se tiene que:

$$g = diag(1, -1, -1, -1)$$

y que

$$g^T = g = g^{-1}$$

por lo tanto:

$$\begin{aligned} c^T &= (gL)^T \\ &= L^T g \\ &= -gL \\ &= -c \end{aligned}$$

por lo tanto:

$$c_{ij} = -c_{ji}$$

si $i = j$, entonces $c_{ii} = 0$, por lo tanto:

$$gL = C = \begin{pmatrix} 0 & C_{12} & C_{13} & C_{14} \\ -C_{12} & 0 & C_{23} & C_{24} \\ -C_{13} & -C_{23} & 0 & C_{34} \\ -C_{14} & -C_{24} & -C_{34} & 0 \end{pmatrix}$$

realizando la operación $gc = ggL$, se tiene que:

$$\begin{aligned} gc &= g(gL) \\ &= (gg)L \\ &= L \end{aligned}$$

por lo tanto $gc = L$

Ejercicio 9

Formule la matriz de rotación respecto a $\hat{z}$ por medio de la transformación de Lorentz partiendo de la invarianza de S^2 se obtiene la forma (dependiente de 6 parámetros) de L, tal que $A = e^L$ es la transformación de Lorentz.
Se tiene la base, S_μ, K_μ, donde $L = -\vec{\omega} \cdot \vec{s} - \vec{\xi} \cdot \vec{k}$, para rotar con respecto $\hat{z}$, se tiene que cumplir que: $\vec{\xi} = 0, \vec{\omega} = \omega_z \hat{z}$, entonces:

$$L = -\vec{\omega} \cdot \vec{s} = -\omega_z s_3$$

por lo tanto:

$$\begin{aligned} A &= e^L \\ &= e^{-\omega s_3} \\ &= \sum_{i=0}^{\infty} \frac{(-1)^i (\omega s_3)^i}{i!} \end{aligned}$$

donde se cumple que:

$$\begin{aligned} s_3^3 &= -s_3 \\ s_3^4 &= -s_3^2 \\ s_3^5 &= s_3 \end{aligned}$$

entonces:

$$\begin{aligned} A &= 1 - \omega s_3 + \frac{\omega^2}{2!} s_3^2 + \frac{\omega^3}{3!} s_3 - \frac{\omega^4}{4!} s_3^2 \\ &= (1 + s_3^2) - s_3^2 \left[1 - \frac{\omega^2}{2!} + \frac{\omega^4}{4!} - \dots \right] - s_3 \left[\omega - \frac{\omega^3}{3!} + \frac{\omega^5}{5!} - \dots \right] \\ &= \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} - \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} cos(\omega) - \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} sen(\omega) \end{aligned}$$

entonces:

$$A = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & cos(\omega) & sen(\omega) & 0 \\ 0 & -sen(\omega) & cos(\omega) & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

Ejercicio 10

Determina la energía threshold para las siguientes reacciones, asumiendo que el proton blanco está en reposo. Consulta en la página de Partice Data Group las masas de las partııculas.

- $p + p \to p + p + \pi^0$
- $p + p \to p + p + \pi^+ + \pi^-$
- $\pi^- + p \to p + \bar{p} + n$
- $\pi^- + p \to K^0 + \sum^0$

Ejercicio 11

Una partícula A en reposo, decae en 2 partículas B y C ($A \to B + C$). Mostrar que la energía de la partícula que emergió es

$$E_B = \frac{m_A^2 + m_B^2 - m_C^2}{2m_A} c^2$$

De la conservación de la energía se tiene que:

$$m_a = m_b\gamma_b + m_c\gamma_c$$

y de la conservación del momento lineal:

$$0 = p_b + p_c$$

por lo tanto $p_b = -p_c$, entonces:

$$\begin{aligned} 0 =& p_b + p_c \\ 0 =& m_b\gamma_b v_b - m_c\gamma_c v_c \\ 0 =& m_b\sqrt{\gamma_b^2 - 1} - m_c\sqrt{\gamma_c^2 - 1} \\ m_b^2\gamma_b^2 - m_c^2\gamma_c^2 =& m_b^2 - m_c^2 \end{aligned}$$

nombando a $\chi_b \equiv m_b\gamma_b$ y a $\chi_c \equiv m_c\gamma_c$, entonces se tiene el siguiente sistema de ecuaciones

$$\chi_b^2 - \chi_c^2 = m_b^2 - m_c^2 \qquad \chi_b + \chi_c = m_a$$

resolviendo ese sistema se obtine que χ_b es igual a

$$\chi_b = \frac{m_a^2 + m_b^2 - m_c^2}{2m_a}$$

pero $E_b = c^2\chi_b$, por lo tanto:

$$E_b = \frac{m_a^2 + m_b^2 - m_c^2}{2m_a}c^2$$

Ejercicio 12

En una dispersión de 2 cuerpos $A + B \to C + D$, es conveniente introducir las variables de Mandelstam

$$\begin{aligned} s &= (p_A + p_B)^2/c^2 \\ t &= (p_A - p_C)^2/c^2 \\ u &= (p_A - p_D)^2/c^2 \end{aligned}$$

1. Mostrar que $s + t + u = m_A^2 + m_B^2 + m_c^2 + m_D^2$
   Realizando la suma de $s + t + u$, se tiene que:

$$\begin{aligned} s + t + u &= \frac{(p_A + p_B)^2 + (p_A - p_C)^2 + (p_A - p_D)^2}{c^2} \\ &= \frac{p_A^2 + 2p_Ap_B + p_B^2 + p_A^2 - 2p_Ap_C + p_C^2 + p_A^2 - 2p_Ap_D + p_D^2}{c^2} \\ &= \frac{3p_A^2 + 2p_A(p_B - p_C - p_D) + p_B^2 + p_C^2 + p_D^2}{c^2} \\ &= \frac{3p_A^2 - 2p_A^2 + p_B^2 + p_C^2 + p_D^2}{c^2} \\ &= \frac{p_A^2 + p_B^2 + p_C^2 + p_D^2}{c^2} \\ &= m_A^2 + m_B^2 + m_C^2 + m_D^2 \end{aligned}$$

2. Mostrar que la energía de centro de masa de A es $E_A^{CM} = (s + m_A^2 - m_B^2)c^2/2\sqrt{s}$

3. Mostrar que la energía de A en el sistema de laboratorio (B en reposo) es $E_A^{LAB} = (s - m_A^2 - m_B^2)c^2/2m_B$

Ejercicio 13

Mostrar que:

$$\begin{aligned} \rho'_+ &= \frac{e|E_p|}{m_0c^2}\psi_+^*\psi_+ \\ \rho'_- &= \frac{e|E_p|}{m_0c^2}\psi_-^*\psi_- \end{aligned}$$

considerando que:

$$\psi_+ = A_+ exp\left[\frac{i}{\hbar}\left(\vec{p}\cdot\vec{x} - |E_p t|\right)\right]$$
$$\psi_- = A_- exp\left[\frac{i}{\hbar}\left(\vec{p}\cdot\vec{x} + |E_p t|\right)\right]$$

Sea

$$\rho' = \frac{i\hbar e}{2m_0c^2}\left[\psi^*\frac{\partial}{\partial t}\psi - \psi\frac{\partial}{\partial t}\psi^*\right]$$

Usando a ψ_+ se calcularan las derivadas partiales

$$\frac{\partial}{\partial t}\psi_+ = -\frac{|E_p|i}{\hbar}\psi_+ \qquad \frac{\partial}{\partial t}\psi_+^* = \frac{|E_p|i}{\hbar}\psi_+^*$$
$$\psi_+^*\frac{\partial}{\partial t}\psi_+ = -\frac{|E_p|i}{\hbar}\psi_+\psi_+^* \qquad \psi_+\frac{\partial}{\partial t}\psi_+^* = \frac{|E_p|i}{\hbar}\psi_+^*\psi_+$$

entonces:

$$\psi^*\frac{\partial}{\partial t}\psi - \psi\frac{\partial}{\partial t}\psi^* = -\frac{2|E_p|i}{\hbar}\psi_+^*\psi_+$$
$$\frac{i\hbar e}{2m_0c^2}\left[\psi^*\frac{\partial}{\partial t}\psi - \psi\frac{\partial}{\partial t}\psi^*\right] = \frac{e|E_p|}{m_0c^2}\psi_+^*\psi_+$$

por lo tanto:

$$\rho'_+ = \frac{e|E_p|}{m_0c^2}\psi_+^*\psi_+$$

Usando a ψ_- se calcularan las derivadas partiales

$$\frac{\partial}{\partial t}\psi_- = \frac{|E_p|i}{\hbar}\psi_- \qquad \frac{\partial}{\partial t}\psi_-^* = -\frac{|E_p|i}{\hbar}\psi_-^*$$
$$\psi_-^*\frac{\partial}{\partial t}\psi_- = \frac{|E_p|i}{\hbar}\psi_-\psi_-^* \qquad \psi_-\frac{\partial}{\partial t}\psi_-^* = -\frac{|E_p|i}{\hbar}\psi_-^*\psi_-$$

entonces:

$$\psi^*\frac{\partial}{\partial t}\psi - \psi\frac{\partial}{\partial t}\psi^* = \frac{2|E_p|i}{\hbar}\psi_-^*\psi_-$$
$$\frac{i\hbar e}{2m_0c^2}\left[\psi^*\frac{\partial}{\partial t}\psi - \psi\frac{\partial}{\partial t}\psi^*\right] = -\frac{e|E_p|}{m_0c^2}\psi_-^*\psi_-$$

por lo tanto:

$$\rho'_- = -\frac{e|E_p|}{m_0c^2}\psi_-^*\psi_-$$

Ejercicio 14

Usar la ecuación de Euler-Lagrande para ψ^* y obtener la ecuación de Klein Gordon para ψ.
Sea

$$\frac{\mathcal{L}}{\partial \psi_0} - \frac{\partial}{\partial x_\beta}\left[\frac{\partial \mathcal{L}}{\partial\left(\frac{\partial \psi_\sigma}{\partial x_\mu}\right)}\right] = 0$$

y la densidad lagrangiana

$$\mathcal{L}\left(\psi, \psi^*, \frac{\partial \psi}{\partial x^\beta}, \frac{\partial \psi^*}{\partial x^\beta}\right) = \frac{\hbar^2}{2m_0}\left[g^{\beta\nu}\frac{\partial \psi^*}{\partial x^\mu}\frac{\partial \psi}{\partial x^\nu} - \frac{m_0^2 c^2}{\hbar^2}\psi^*\psi\right]$$

Para el campo $\psi_\sigma = \psi^*$, calculando $\frac{\mathcal{L}}{\partial \psi^*}$

$$\frac{\partial \mathcal{L}}{\partial \psi^*} = \frac{\hbar^2}{2m_0}\left[-\frac{m_0^2 c^2}{\hbar}\psi\right]$$

calculando $\frac{\partial \mathcal{L}}{\partial\left(\frac{\partial \psi^*}{\partial x_\mu}\right)}$

$$\begin{aligned}
\frac{\partial \mathcal{L}}{\partial\left(\frac{\partial \psi^*}{\partial x_\mu}\right)} &= \frac{\hbar^2}{2m_0}\frac{\partial}{\partial\left(\frac{\partial \psi^*}{\partial x_\mu}\right)}\left(g^{\mu\nu}\frac{\partial \psi}{\partial x^\nu}\frac{\partial \psi^*}{\partial x^\mu}\right)\\
&= \frac{\hbar^2}{2m_0}\left(g^{\mu\nu}\frac{\partial \psi}{\partial x^\nu}\delta^\mu_\beta\right)\\
&= \frac{\hbar^2}{2m_0}\left(g_{\mu\nu}\frac{\partial \psi}{\partial x_\nu}\delta^\mu_\beta\right)\\
&= \frac{\hbar^2}{2m_0}\left(g_{\beta\nu}\frac{\partial \psi}{\partial x_\nu}\right)
\end{aligned}$$

calculando $\frac{\partial}{\partial x_\beta}\left(\frac{\partial \mathcal{L}}{\partial\left(\frac{\partial \psi^*}{\partial x_\mu}\right)}\right)$:

$$\begin{aligned}
\frac{\partial}{\partial x_\beta}\left(\frac{\partial \mathcal{L}}{\partial\left(\frac{\partial \psi^*}{\partial x_\mu}\right)}\right) &= \frac{\partial}{\partial x_\beta}\left(\frac{\hbar^2}{2m_0}\left(g_{\beta\nu}\frac{\partial \psi}{\partial x_\nu}\right)\right)\\
&= \frac{\hbar^2}{2m_0}\left(\partial^\beta\left(g_{\beta\nu}\frac{\partial \psi}{\partial x_\nu}\right)\right)\\
&= \frac{\hbar^2}{2m_0}\partial^\beta\left(g_{\beta\nu}\partial^\mu\psi\right)\\
&= \frac{\hbar^2}{2m_0}\partial_\mu\partial^\mu\psi
\end{aligned}$$

por lo tanto:

$$\left[\frac{m_0^2 c^2}{\hbar^2} + \partial_\mu\partial^\mu\right]\psi = 0$$

Ejercicio 15

Mostrar que el tensor de energía momento para la densidad lagrangiana es

$$T_\mu^\nu = \frac{\hbar^2}{2m_0}\left[g^{\sigma\nu}\frac{\partial\psi^*}{\partial x^\sigma}\frac{\partial\psi}{\partial x^\mu} + g^{\sigma\nu}\frac{\partial\psi}{\partial x^\sigma}\frac{\partial\psi^*}{\partial x^\mu} - \left(g^{\sigma\rho}\frac{\partial\psi^*}{\partial x^\sigma}\frac{\partial\psi}{\partial x^\rho} - \frac{m_0^2c^2}{\hbar}\psi^*\psi\right)g_\mu^\nu\right]$$

Sea el ternsor energía momento definido por:

$$T_\mu^\nu = \sum_\sigma \frac{\partial\psi_\sigma}{\partial x^\mu}\frac{\partial\mathcal{L}}{\partial\left[\partial\psi_\sigma/\partial x^\nu\right]} - \mathcal{L}g_\mu^\nu$$

Utilizando la siguiente densidad lagrangiana

$$\mathcal{L}\left(\psi,\psi^*,\frac{\partial\psi}{\partial x^\beta},\frac{\partial\psi^*}{\partial x^\alpha}\right) = \frac{\hbar^2}{2m_0}\left[g^{\beta\nu}\frac{\partial\psi^*}{\partial x^\alpha}\frac{\partial\psi}{\partial x^\beta} - \frac{m_0^2c^2}{\hbar^2}\psi^*\psi\right]$$

Calculando $\partial\mathcal{L}/\partial(\partial\psi/\partial x^\mu)$

$$\begin{aligned}\frac{\partial\mathcal{L}}{\partial(\partial\psi/\partial x^\mu)} &= \frac{\hbar^2}{2m_0}\left(g^{\sigma\beta}\frac{\partial\psi^*}{\partial x^\sigma}\delta_\nu^\beta\right)\\ &= \frac{\hbar^2}{2m_0}\left(g^{\sigma\nu}\frac{\partial\psi^*}{\partial x^\sigma}\right)\end{aligned}$$

Calculando $\partial\mathcal{L}/\partial(\partial\psi^*/\partial x^\mu)$

$$\begin{aligned}\frac{\partial\mathcal{L}}{\partial(\partial\psi^*/\partial x^\mu)} &= \frac{\hbar^2}{2m_0}\left(g^{\sigma\beta}\frac{\partial\psi}{\partial x^\sigma}\delta_\nu^\beta\right)\\ &= \frac{\hbar^2}{2m_0}\left(g^{\sigma\nu}\frac{\partial\psi}{\partial x^\sigma}\right)\end{aligned}$$

$$T_\mu^\nu = \frac{\hbar^2}{2m_0}\left[g^{\sigma\nu}\frac{\partial\psi^*}{\partial x^\sigma}\frac{\partial\psi}{\partial x^\mu} + g^{\sigma\nu}\frac{\partial\psi}{\partial x^\sigma}\frac{\partial\psi^*}{\partial x^\mu} - \left(g^{\sigma\rho}\frac{\partial\psi^*}{\partial x^\sigma}\frac{\partial\psi}{\partial x^\rho} - \frac{m_0^2c^2}{\hbar}\psi^*\psi\right)g_\mu^\nu\right]$$

Ejercicio 16

Mostrar que:

$$T_0^0 = \frac{\hbar^2}{2m_0}\left[\frac{1}{c^2}\frac{\partial\psi^*}{\partial t}\frac{\partial\psi}{\partial t} + (\nabla\psi^*)\cdot(\nabla\psi) + \frac{m_0^2c^2}{\hbar^2}\psi^*\psi\right]$$

Sea el tensor energía momento:

$$T_\mu^\nu = \frac{\hbar^2}{2m_0}\left[g^{\sigma\nu}\frac{\partial\psi^*}{\partial x^\sigma}\frac{\partial\psi}{\partial x^\mu} + g^{\sigma\nu}\frac{\partial\psi}{\partial x^\sigma}\frac{\partial\psi^*}{\partial x^\mu} - \left(g^{\sigma\rho}\frac{\partial\psi^*}{\partial x^\sigma}\frac{\partial\psi}{\partial x^\rho} - \frac{m_0^2c^2}{\hbar}\psi^*\psi\right)g_\mu^\nu\right]$$

Tomando el caso de $\mu = 0, \nu = 0$ y el tensor metrico $g_{\mu\nu}$ tal que

$$g^{\mu\nu} = \begin{bmatrix} \frac{1}{c^2} & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix}$$

entonces:

$$\begin{aligned} T_0^0 &= \frac{\hbar^2}{2m_0}\left[\frac{1}{c^2}\frac{\partial\psi^*}{\partial t}\frac{\partial\psi}{\partial t} + \frac{1}{c^2}\frac{\partial\psi^*}{\partial t}\frac{\partial\psi}{\partial t} - \frac{1}{c^2}\frac{\partial\psi^*}{\partial t}\frac{\partial\psi}{\partial t} + \frac{\partial\psi^*}{\partial x^1}\frac{\partial\psi}{\partial x^1} + \frac{\partial\psi^*}{\partial x^2}\frac{\partial\psi}{\partial x^2} + \frac{\partial\psi^*}{\partial x^3}\frac{\partial\psi}{\partial x^3} + \frac{m_0^2c^2}{\hbar^2}\psi^*\psi\right] \\ &= \frac{\hbar^2}{2m_0}\left[\frac{1}{c^2}\frac{\partial\psi^*}{\partial t}\frac{\partial\psi}{\partial t}, \left(\frac{\partial\psi^*}{\partial x^1}, \frac{\partial\psi^*}{\partial x^2}, \frac{\partial\psi^*}{\partial x^3}\right)\cdot\left(\frac{\partial\psi}{\partial x^1}, \frac{\partial\psi}{\partial x^2}, \frac{\partial\psi}{\partial x^3}\right), \frac{m_0^2c^2}{\hbar^2}\psi^*\psi\right] \\ &= \frac{\hbar^2}{2m_0}\left[\frac{1}{c^2}\frac{\partial\psi^*}{\partial t}\frac{\partial\psi}{\partial t} + (\nabla\psi^*)\cdot(\nabla\psi) + \frac{m_0^2c^2}{\hbar^2}\psi^*\psi\right] \end{aligned}$$

por lo tanto:

$$T_0^0 = \frac{\hbar^2}{2m_0}\left[\frac{1}{c^2}\frac{\partial\psi^*}{\partial t}\frac{\partial\psi}{\partial t} + (\nabla\psi^*)\cdot(\nabla\psi) + \frac{m_0^2c^2}{\hbar^2}\psi^*\psi\right]$$

Ejercicio 17

Mostras que

- $H(+) = E_{pn}$
- $H(-) = E_{pn}$

Se tiene que:

$$\psi_n(\pm) = \sqrt{\frac{m_0c^2}{L^3E_{\rho m}}}exp\left[\frac{i}{\hbar}\left(\vec{p}\cdot\vec{x} \mp E_{\rho n}t\right)\right]$$

y la operación H es definida como:

$$H = \int_{L^B} T_0^0(n, \pm)dx^3$$

donde

$$T_0^0 = \frac{\hbar^2}{2m_0}\left[\frac{1}{c^2}\frac{\partial\psi^*}{\partial t}\frac{\partial\psi}{\partial t} + (\nabla\psi^*)\cdot(\nabla\psi) + \frac{m_0^2c^2}{\hbar^2}\psi^*\psi\right]$$

Para $H(+)$, calculando $\frac{\partial\psi^*}{\partial t}$ y $\frac{\partial\psi}{\partial t}$:

$$\frac{\partial\psi^*}{\partial t} = \frac{i}{\hbar}E_{\rho n}\psi^* \qquad \frac{\partial\psi}{\partial t} = \frac{-i}{\hbar}E_{\rho n}\psi$$

$$\begin{aligned} \frac{1}{c^2}\frac{\partial\psi^*}{\partial t}\frac{\partial\psi}{\partial t} &= \frac{E_{\rho n}^2}{\hbar^2c^2}\psi^*\psi \\ &= \frac{E_{\rho n}m_0}{\hbar^2L^3} \end{aligned}$$

Calculando $\nabla\psi^*$ y $\nabla\psi$

$$\nabla\psi^* = \frac{-i\vec{P}}{\hbar}\psi^* \qquad \nabla\psi = \frac{i\vec{P}}{\hbar}\psi$$

$$\begin{aligned}(\nabla\psi^*)\cdot(\nabla\psi) &= \frac{\vec{p}\cdot\vec{p}}{\hbar^2}\psi^*\psi \\ &= \frac{P^2}{\hbar^2}\left(\frac{m_0c^2}{L^3E_{\rho n}}\right) \\ &= \frac{1}{c^2\hbar^2}\left(\frac{m_0c^2}{L^3E_{\rho n}}\right)(E_{\rho n}^2 - m_0^2c^4) \\ &= \left(\frac{m_0}{L^3E_{\rho n}\hbar^2}\right)(E_{\rho n}^2 - m^2c^4)\end{aligned}$$

por lo tanto

$$\begin{aligned}T_0^0 &= \frac{\hbar^2}{2m_0}\left[\frac{1}{c^2}\frac{\partial\psi^*}{\partial t}\frac{\partial\psi}{\partial t} + (\nabla\psi^*)\cdot(\nabla\psi) + \frac{m_0^2c^2}{\hbar^2}\psi^*\psi\right] \\ &= \frac{\hbar^2}{2m_0}\left[\frac{E_{\rho n}m_0}{\hbar^2L^3} + \left(\frac{m_0}{L^3E_{\rho n}\hbar^2}\right)(E_{\rho n}^2 - m_0^2c^4) + \left(\frac{m_0}{L^3E_{\rho n}\hbar^2}\right)(m_0^2c^4)\right] \\ &= \frac{E_{\rho n}}{L^3}\end{aligned}$$

entonces

$$\begin{aligned}\int\limits_{L^B} T_0^0(n,+)dx^3 &= \int\limits_{L^B}\frac{E_{\rho n}}{L^3}dV \\ &= \frac{E_{\rho n}}{L^3}\int\limits_{L^B} dV \\ &= E_{\rho n}\end{aligned}$$

por lo tanto

$$H(+) = E_{\rho n}$$

Para $H(-)$, calculando $\frac{\partial\psi^*}{\partial t}$ y $\frac{\partial\psi}{\partial t}$:

$$\frac{\partial\psi^*}{\partial t} = \frac{-i}{\hbar}E_{\rho n}\psi^* \qquad \frac{\partial\psi}{\partial t} = \frac{i}{\hbar}E_{\rho n}\psi$$

$$\begin{aligned}\frac{1}{c^2}\frac{\partial\psi^*}{\partial t}\frac{\partial\psi}{\partial t} &= \frac{E_{\rho n}^2}{\hbar^2c^2}\psi^*\psi \\ &= \frac{E_{\rho n}m_0}{\hbar^2L^3}\end{aligned}$$

Calculando $\nabla\psi^*$ y $\nabla\psi$

$$\nabla\psi^* = \frac{-i\vec{P}}{\hbar}\psi^* \qquad \nabla\psi = \frac{i\vec{P}}{\hbar}\psi$$

$$\begin{aligned}
(\nabla\psi^*)\cdot(\nabla\psi) &= \frac{\vec{p}\cdot\vec{p}}{\hbar^2}\psi^*\psi \\
&= \frac{P^2}{\hbar^2}\left(\frac{m_0c^2}{L^3E_{\rho n}}\right) \\
&= \frac{1}{c^2\hbar^2}\left(\frac{m_0c^2}{L^3E_{\rho n}}\right)(E_{\rho n}^2 - m_0^2c^4) \\
&= \left(\frac{m_0}{L^3E_{\rho n}\hbar^2}\right)(E_{\rho n}^2 - m^2c^4)
\end{aligned}$$

por lo tanto

$$\begin{aligned}
T_0^0 &= \frac{\hbar^2}{2m_0}\left[\frac{1}{c^2}\frac{\partial\psi^*}{\partial t}\frac{\partial\psi}{\partial t} + (\nabla\psi^*)\cdot(\nabla\psi) + \frac{m_0^2c^2}{\hbar^2}\psi^*\psi\right] \\
&= \frac{\hbar^2}{2m_0}\left[\frac{E_{\rho n}m_0}{\hbar^2L^3} + \left(\frac{m_0}{L^3E_{\rho n}\hbar^2}\right)(E_{\rho n}^2 - m_0^2c^4) + \left(\frac{m_0}{L^3E_{\rho n}\hbar^2}\right)(m_0^2c^4)\right] \\
&= \frac{E_{\rho n}}{L^3}
\end{aligned}$$

entonces

$$\begin{aligned}
\int\limits_{L^B} T_0^0(n,-)dx^3 &= \int\limits_{L^B}\frac{E_{\rho n}}{L^3}dV \\
&= \frac{E_{\rho n}}{L^3}\int\limits_{L^B} dV \\
&= E_{\rho n}
\end{aligned}$$

por lo tanto

$$H(-) = E_{\rho n}$$

Ejercicio 18

Obtener la constante de la función de onda para $E = -E_p$.
Se tiene que:

$$\psi^{(-)}(\rho) = A_{(-)}\begin{pmatrix}\varphi_0^{(-)}\\ \chi_0^{(-)}\end{pmatrix} exp\left[i\left(\vec{p}\cdot\vec{x} + Et\right)\right] \equiv \begin{bmatrix}\varphi^{(-)}(\rho)\\ \chi^{(-)}(\rho)\end{bmatrix}$$

con

$$\begin{bmatrix}\rho_0^{(-)}\\ \chi_0^{(-)}\end{bmatrix} = \begin{bmatrix}m_0c^2 - E_\rho\\ m_0c^2 + E_\rho\end{bmatrix}$$

como se sabe que esta función se encuentra normalizada, entonces se tiene que cumplir que:

$$\int \psi^{(-)*}\hat{\tau}_3\psi^{(-)}dx^3 = -1$$

realizando la integral se tiene que:

$$\begin{aligned}
\int \psi^{(-)*}\hat{\tau}_3\psi^{(-)}dx^3 &= \int (\varphi\varphi^* - \chi\chi^*)\, dV \\
&= \int \left(A_{(-)}\varphi_0 e^{i\xi}A^*_{(-)}\varphi_0^* e^{-i\xi} - A_{(-)}\chi_0 e^{i\xi}A^*_{(-)}\chi_0^* e^{-i\xi}\right) dV \\
&= \int \left(|A_{(-)}|^2\varphi_0\varphi_0^* - |A_{(-)}|^2\xi_0\xi_0^*\right) dV \\
&= |A_{(-)}|^2 \int (\varphi_0\varphi_0^* - \xi_0\xi_0^*) \\
&= |A_{(-)}|^2 \int \left((m_0c^2 - E_\rho)^2 - (m_0c^2 + E_\rho)^2\right) dV \\
&= |A_{(-)}|^2(-4m_0c^2E_\rho) \int dV \\
&= |A_{(-)}|^2(-4m_0c^2E_\rho)L^3
\end{aligned}$$

entonces:

$$|A_{(-)}|^2(-4m_0c^2E_\rho)L^3 = -1$$

por lo tanto

$$A_{(-)} = \frac{1}{\sqrt{4m_0c^2}}\frac{1}{\sqrt{L^3E_\rho}}$$

Ejercicio 19

Mostrar que:

$$\vec{J'} = -\frac{i\hbar e}{2m_0}\left[\psi^*\nabla\psi - \psi\nabla\psi^*\right] - \frac{e^2}{m_0c}\vec{A}\psi\psi^*$$

Ejercicio 20

Sea

$$\mathcal{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} \qquad F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu \tag{3}$$

- Determinar las ecuaciones que satisface el campo A_ν
  Se tiene que la ecuación de Euler-Lagrande para campos es la siguiente:

$$\partial_\nu\left(\frac{\partial\mathcal{L}}{\partial(\partial_\nu A_\mu)}\right) = \frac{\partial\mathcal{L}}{\partial A_\mu} \tag{4}$$

Como la ecuación 3 no depende del campo A_μ, entonces:

$$\frac{\partial \mathcal{L}}{\partial A_\mu} = 0$$

por lo tanto, la ecuación 4 se escribe de la siguiente manera:

$$\partial_\nu \left(\frac{\partial \mathcal{L}}{\partial(\partial_\nu A_\mu)} \right) = 0 \tag{5}$$

Calculando $\frac{\partial \mathcal{L}}{\partial(\partial_\nu A_\mu)}$ se tiene que :

$$\begin{aligned}\frac{\partial \mathcal{L}}{\partial(\partial_\nu A_\mu)} &= \frac{\partial}{\partial(\partial_\nu A_\mu)} \left(\frac{1}{4} F_{\mu\nu} F^{\mu\nu} \right) \\ &= \frac{\partial}{\partial(\partial_\nu A_\mu)} ((\partial_\mu A_\nu - \partial_\nu A_\mu) F^{\mu\nu}) \\ &= F^{\mu\nu}\end{aligned}$$

entonces

$$\partial_\nu \left(\frac{\partial \mathcal{L}}{\partial(\partial_\nu A_\mu)} \right) = \partial_\nu F^{\mu\nu}$$

por lo tanto el campo A_ν debe cumplir la siguiente ecuación:

$$\partial_\nu F^{\mu\nu} = 0$$

- Determinar el tensor $T^\mu_\nu T^{\mu\nu}$
  Se tiene que

$$T^\mu{}_\nu = \frac{\partial \mathcal{L}}{\partial(\partial_\mu A_\nu)} - g^\mu_\nu \mathcal{L} \qquad T^{\mu\nu} = \frac{\partial \mathcal{L}}{\partial(\partial_\mu A_\nu)} - g^{\mu\nu} \mathcal{L}$$

  Calculando $T^\mu{}_\nu(T^{\mu\nu})$

$$\begin{aligned}T^\mu{}_\nu(T^{\mu\nu}) = &\left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu A_\nu)} - g^\mu_\nu \mathcal{L} \right) \left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu A_\nu)} - g^{\mu\nu} \right) \\ =&(F^\mu{}_\nu \partial_\nu A_\mu)(F^{\mu\nu} \partial_\nu A_\mu) - (F^\mu{}_\nu \partial_\nu A_\mu)(g^{\mu\nu} \mathcal{L}) \\ &- (g^\mu{}_\nu \mathcal{L})(F^{\mu\nu} \partial_\nu A_\mu) + (g^\mu{}_\nu \mathcal{L})(g^{\mu\nu} \mathcal{L}) \\ =&(F^\mu{}_\nu \partial_\nu A_\mu)(F^{\mu\nu} \partial_\nu A_\mu)\end{aligned}$$

  por lo tanto

$$T^\mu{}_\nu(T^{\mu\nu}) = (F^\mu{}_\nu \partial_\nu A_\mu)(F^{\mu\nu} \partial_\nu A_\mu)$$

Ejercicio 21

- Usando la ecuación de Euler-Lagrange para campos, determinar que ψ satisface

$$\left(p^\mu - \frac{e}{c} A^\mu \right) \left(p_\mu - \frac{e}{c} A_\mu \right) \psi = m_0^2 c^2 \psi$$

- Mostrar que la ecuación para A_μ

$$\partial^\mu F_{\mu\nu} = J_\nu = \frac{ie\hbar}{2m_0} \begin{bmatrix} \psi^* \left[\partial_\nu + \frac{ie}{\hbar c} A_\nu \right] \psi \\ -\psi \left[\partial_\nu - \frac{ie}{\hbar c} A_\nu \right] \psi^* \end{bmatrix}$$

Ejercicio 22

Mostrar que:

$$\mathcal{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu}$$

es invariante ante la transformacion

$$A'_{\mu} = A_{\mu} + \partial_{\mu}\xi(x)$$

Se tiene que:

$$F'_{\mu\nu} = \partial_{\mu}A'_{\nu} - \partial_{\nu}A'_{\mu}$$

entonces:

$$\begin{aligned}
F'_{\mu\nu} &= \partial_{\mu}A'_{\nu} - \partial_{\nu}A'_{\mu} \\
&= \partial_{\mu}\left[A_{\nu} + \partial_{\nu}\xi(x)\right] - \partial_{\nu}\left[A_{\mu} + \partial_{\mu}\xi(x)\right] \\
&= \partial_{\mu}A_{\nu} + \partial_{\mu}\partial_{\nu}\xi(x) - \partial_{\nu}A_{\mu} - \partial_{\nu}\partial_{\mu}\xi(x) \\
&= \partial_{\mu}A_{\nu} + \partial_{\mu}\partial_{\nu}\xi(x) - \partial_{\nu}A_{\mu} - \partial_{\mu}\partial_{\nu}\xi(x) \\
&= \partial_{\mu}A_{\nu} - \partial_{\nu}A_{\mu} \\
&= F_{\mu\nu}
\end{aligned}$$

$$\begin{aligned}
F'^{\mu\nu} &= \partial^{\mu}A'^{\nu} - \partial^{\nu}A'^{\mu} \\
&= \partial^{\mu}\left[A^{\nu} + \partial^{\nu}\xi(x)\right] - \partial^{\nu}\left[A^{\mu} + \partial^{\mu}\xi(x)\right] \\
&= \partial^{\mu}A^{\nu} + \partial^{\mu}\partial^{\nu}\xi(x) - \partial^{\nu}A^{\mu} - \partial^{\nu}\partial^{\mu}\xi(x) \\
&= \partial^{\mu}A^{\nu} + \partial^{\mu}\partial^{\nu}\xi(x) - \partial^{\nu}A^{\mu} - \partial^{\mu}\partial^{\nu}\xi(x) \\
&= \partial^{\mu}A^{\nu} - \partial^{\nu}A^{\mu} \\
&= F^{\mu\nu}
\end{aligned}$$

por lo tanto:

$$\begin{gathered}
-\frac{1}{4}F'_{\mu\nu}F'^{\mu\nu} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} \\
\mathcal{L}' = \mathcal{L}
\end{gathered}$$

por lo tanto $\mathcal{L}$ es invariante ante la transformacion.

Ejercicio 23

Mostrar que

$$\left[i\hbar\frac{\partial}{\partial t} - eA_0\right]^2\psi \approx \left[-ie\hbar\frac{\partial}{\partial t}A_0\varphi + 2i\hbar m_0c^2\frac{\partial}{\partial t}\varphi - 2eA_0m_0c^2\varphi + m_0^2c^4\varphi\right]e^{\frac{-im_0c^2t}{\hbar}}$$

se han omitido

$$(i\hbar A_0\frac{\partial}{\partial t}\varphi) << A_0m_0c^2|\varphi| \qquad |A_0\varphi_e| << m_0c^2|\varphi|$$

tomando en cuenta que:

$$\psi = \varphi e^{\frac{-i}{\hbar} m_0 c^2 t}$$

Calculando $\left[i\hbar\frac{\partial}{\partial t} - eA_0\right]^2$

$$\left[i\hbar\frac{\partial}{\partial t} - eA_0\right]^2 \psi = \left[-\hbar^2\frac{\partial^2}{\partial t^2} - 2i\hbar e A_0 \frac{\partial}{\partial t} + e^2 A_0\right]\psi$$

calculando $\frac{\partial \psi}{\partial t}$

$$\frac{\partial \psi}{\partial t} = \left[\frac{-i}{\hbar} m_0 c^2 \varphi + \frac{\partial \varphi}{\partial t}\right] e^{\frac{-i}{\hbar} m_0 c^2 t}$$

calculando $\frac{\partial^2 \psi}{\partial t^2}$

$$\frac{\partial^2 \psi}{\partial t^2} = \left[\frac{\partial^2 \varphi}{\partial t^2} - \frac{2i}{\hbar} m_0 c^2 \frac{\partial \varphi}{\partial t} - \frac{1}{\hbar^2} m_0^2 c^4 \varphi\right] e^{\frac{-i}{\hbar} m_0 c^2 t}$$

por lo tanto:

$$\begin{aligned}
\left[i\hbar\frac{\partial}{\partial t} - eA_0\right]^2 \psi &= \left[-\hbar^2\frac{\partial^2}{\partial t^2} - 2i\hbar e A_0 \frac{\partial}{\partial t} + e^2 A_0\right]\psi \\
&= \left[-\hbar^2\frac{\partial^2 \varphi}{\partial t^2} + 2i\hbar m_0 c^2 \frac{\partial \varphi}{\partial t} + m_0^2 c^4 \varphi - 2m_0 c^2 e A_0 \varphi - 2i\hbar e A_0 \frac{\partial \varphi}{\partial t}\right] e^{\frac{-i}{\hbar} m_0 c^2 t} \\
&\approx \left[2i\hbar m_0 c^2 \frac{\partial \varphi}{\partial t} + m_0^2 c^4 \varphi - 2m_0 c^2 e A_0 \varphi - 2i\hbar e A_0 \frac{\partial \varphi}{\partial t}\right] e^{\frac{-i}{\hbar} m_0 c^2 t}
\end{aligned}$$

Ejercicio 24

Mostrar que al desarrollar

$$\left[i\hbar\vec{\nabla} + \frac{e}{c}\vec{A}\right]^2 \varphi$$

de la ecuación:

$$i\hbar\frac{\partial}{\partial t}\varphi = \left[\frac{1}{2m_0}\left[i\hbar\vec{\nabla} + \frac{e}{c}\vec{A}\right]^2 + \frac{i\hbar e}{2m_0 c^2}\frac{\partial}{\partial t}A_0 + eA_0\right]\varphi$$

se escribe como:

$$i\hbar\frac{\partial}{\partial t}\varphi = \left[\frac{\vec{P}^2}{2m} - \frac{e}{m_0 c}\vec{A}\cdot\vec{P} + eA_0 + \frac{i\hbar e}{2m_0}\left[\nabla\cdot\vec{A}\right] + \frac{i\hbar e}{2m_0 c^2}\frac{\partial}{\partial t}A_0\right]\varphi$$

Desarrollando el termino $\left[i\hbar\vec{\nabla} + \frac{e}{c}\vec{A}\right]^2 \varphi$, se tiene que:

$$\begin{aligned}
\frac{1}{2m_0}\left[i\hbar\vec{\nabla} + \frac{e}{c}\vec{A}\right]^2 \varphi =& \frac{1}{2m_0}\left[-\hbar^2\vec{\nabla}^2 + \frac{i\hbar e}{c}\vec{\nabla}\cdot\vec{A} + \frac{ei\hbar}{c}\vec{A}\cdot\vec{\nabla} + \frac{e^2}{c^2}(\vec{A}\cdot\vec{A})\right] \\
=& \frac{\vec{P}}{2m_0} + \frac{i\hbar e}{2m_0 c}\vec{\nabla}\cdot\vec{A} + \frac{e^2}{2m_0 c^2}\vec{A}\cdot\vec{A} + \frac{ei\hbar}{2m_0 c}\vec{A}\cdot\vec{\nabla} \\
=& \frac{\vec{P}}{2m_0} + \frac{i\hbar e}{2m_0 c}\vec{\nabla}\cdot\vec{A} + \frac{e^2}{2m_0 c^2}\vec{A}\cdot\vec{A} - \frac{e}{2m_0 c}\vec{A}\cdot\vec{P}
\end{aligned}$$

entonces:

$$\begin{aligned}
\frac{1}{2m_0}\left[i\hbar\vec{\nabla}+\frac{e}{c}\vec{A}\right]^2+\frac{i\hbar e}{2m_0c^2}\frac{\partial}{\partial t}A_0+eA_0 =&\frac{\vec{P}}{2m_0}+\frac{i\hbar e}{2m_0c}\vec{\nabla}\cdot\vec{A}+\frac{e^2}{2m_0c^2}\vec{A}\cdot\vec{A}-\frac{e}{2m_0c}\vec{A}\cdot\vec{P}+\frac{i\hbar e}{2m_0c^2}\frac{\partial}{\partial t}A_0\\
=&\frac{\vec{P}}{2m_0}-\frac{e}{2m_0c}\vec{A}\cdot\vec{P}+\frac{i\hbar e}{2m_0c}\left(\frac{\partial}{\partial t}A_0+\vec{\nabla}\cdot\vec{A}\right)+eA_0\\
=&\frac{\vec{P}}{2m_0}-\frac{e}{2m_0c}\vec{A}\cdot\vec{P}+eA_0
\end{aligned}$$

por lo tanto:

$$i\hbar\frac{\partial}{\partial t}\varphi=\left[\frac{\vec{P}}{2m_0}-\frac{e}{2m_0c}\vec{A}\cdot\vec{P}+eA_0\right]\varphi$$

Ejercicio 25

Mostrar que se cumple:

$$\alpha_i\alpha_j+\alpha_j\alpha_i=2\delta_{ij}\mathbb{I}$$

Calculando $\alpha_i\alpha_j$

$$\alpha_i\alpha_j=\begin{bmatrix}0 & \sigma_i\\ \sigma_i & 0\end{bmatrix}\begin{bmatrix}0 & \sigma_j\\ \sigma_j & 0\end{bmatrix}=\begin{bmatrix}\sigma_i\sigma_j & 0\\ 0 & \sigma_i\sigma_j\end{bmatrix}$$

Calculando $\alpha_j\alpha_i$

$$\alpha_j\alpha_i=\begin{bmatrix}0 & \sigma_j\\ \sigma_j & 0\end{bmatrix}\begin{bmatrix}0 & \sigma_i\\ \sigma_i & 0\end{bmatrix}=\begin{bmatrix}\sigma_j\sigma_i & 0\\ 0 & \sigma_j\sigma_i\end{bmatrix}$$

entonces

$$\alpha_i\alpha_j+\alpha_j\alpha_i=\begin{bmatrix}\sigma_i\sigma_j+\sigma_j\sigma_i & 0\\ 0 & \sigma_i\sigma_j+\sigma_j\sigma_i\end{bmatrix}$$

donde

$$\sigma_1=\begin{bmatrix}0 & 1\\ 1 & 0\end{bmatrix}\qquad\sigma_2=\begin{bmatrix}0 & -i\\ i & 0\end{bmatrix}\qquad\sigma_3=\begin{bmatrix}1 & 0\\ 0 & -1\end{bmatrix}$$

Calculando todas las permutaciones se tiene que:

$$\sigma_1\sigma_2 = \begin{bmatrix}0 & 1\\ 1 & 0\end{bmatrix}\begin{bmatrix}0 & -i\\ i & 0\end{bmatrix} = \begin{bmatrix}i & 0\\ 0 & -i\end{bmatrix}$$
$$\sigma_2\sigma_1 = \begin{bmatrix}0 & -i\\ i & 0\end{bmatrix}\begin{bmatrix}0 & 1\\ 1 & 0\end{bmatrix} = \begin{bmatrix}-i & 0\\ 0 & i\end{bmatrix}$$
$$\sigma_1\sigma_3 = \begin{bmatrix}0 & 1\\ 1 & 0\end{bmatrix}\begin{bmatrix}1 & 0\\ 0 & -1\end{bmatrix} = \begin{bmatrix}0 & -1\\ 1 & 0\end{bmatrix}$$
$$\sigma_3\sigma_1 = \begin{bmatrix}1 & 0\\ 0 & -1\end{bmatrix}\begin{bmatrix}0 & 1\\ 1 & 0\end{bmatrix} = \begin{bmatrix}0 & 1\\ -1 & 0\end{bmatrix}$$
$$\sigma_2\sigma_3 = \begin{bmatrix}0 & -i\\ i & 0\end{bmatrix}\begin{bmatrix}1 & 0\\ 0 & -1\end{bmatrix} = \begin{bmatrix}0 & i\\ i & 0\end{bmatrix}$$
$$\sigma_3\sigma_2 = \begin{bmatrix}1 & 0\\ 0 & -1\end{bmatrix}\begin{bmatrix}0 & -i\\ i & 0\end{bmatrix} = \begin{bmatrix}0 & -i\\ -i & 0\end{bmatrix}$$
$$\sigma_1\sigma_1 = \begin{bmatrix}0 & 1\\ 1 & 0\end{bmatrix}\begin{bmatrix}0 & 1\\ 1 & 0\end{bmatrix} = \begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}$$
$$\sigma_2\sigma_2 = \begin{bmatrix}0 & -i\\ i & 0\end{bmatrix}\begin{bmatrix}0 & -i\\ i & 0\end{bmatrix} = \begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}$$
$$\sigma_3\sigma_3 = \begin{bmatrix}1 & 0\\ 0 & -1\end{bmatrix}\begin{bmatrix}1 & 0\\ 0 & -1\end{bmatrix} = \begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}$$

con lo cual se puede observar que :

$$\{\sigma_i, \sigma_j\} = 2\delta_{ij}\mathbb{I}$$

para $i \neq j$, y

$$\sigma_i\sigma_j = \mathbb{I}$$

para $i = j$, por lo tanto:

$$\alpha_i\alpha_j + \alpha_j\alpha_i = \begin{bmatrix}\sigma_i\sigma_j + \sigma_j\sigma_i & 0\\ 0 & \sigma_i\sigma_j + \sigma_j\sigma_i\end{bmatrix} = \begin{bmatrix}2\delta_{ij}\mathbb{I} & 0\\ 0 & 2\delta_{ij}\mathbb{I}\end{bmatrix} = 2\delta_{ij}\mathbb{I}$$

Ejercicio 26

Determinar los eigenvalores de σ_i
Para σ_1, se tiene que:

$$\sigma_1 = \begin{bmatrix}0 & 1\\ 1 & 0\end{bmatrix}$$

entonces,

$$|\sigma_1 - \lambda\mathbb{I}| = \begin{vmatrix}-\lambda & 1\\ 1 & -\lambda\end{vmatrix} = \lambda^2 - 1 = \begin{cases}\lambda_1 = 1\\ \lambda_2 = -1\end{cases}$$

Para σ_2, se tiene que:

$$\sigma_2 = \begin{bmatrix}0 & -i\\ i & 0\end{bmatrix}$$

entonces,

$$|\sigma_2 - \lambda\mathbb{I}| = \begin{vmatrix} -\lambda & i \\ -i & -\lambda \end{vmatrix} = \lambda^2 - 1 = \begin{cases} \lambda_1 = 1 \\ \lambda_2 = -1 \end{cases}$$

Para σ_3, se tiene que:

$$\sigma_3 = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

entonces,

$$|\sigma_3 - \lambda\mathbb{I}| = \begin{vmatrix} 1-\lambda & 0 \\ 0 & -1-\lambda \end{vmatrix} = \lambda^2 - 1 = \begin{cases} \lambda_1 = 1 \\ \lambda_2 = -1 \end{cases}$$

Ejercicio 27

A partir de la ecuación

$$i\hbar\frac{\partial\psi}{\partial t} = \left[\frac{\hbar c}{i}\sum \alpha_i \frac{\partial}{\partial x_i} + \beta m_0 c^2\right]\psi$$

mostrar que

$$\frac{\partial\rho}{\partial t} + \sum \frac{\partial}{\partial x_i}\left[c\psi^*\alpha_i\psi\right] = 0$$

Multiplicando a la ecuación de Schrödinger que propuso Dirac por ψ^* por la izquierda, se tiene que:

$$i\hbar\psi^*\frac{\partial\psi}{\partial t} = \psi^*\left[\frac{\hbar c}{i}\sum \alpha_i \frac{\partial}{\partial x_i} + \beta m_0 c^2\right]\psi$$

Multiplicando ψ por la izquierda a la ecuación de Schrödinger para ψ^*, se tiene que:

$$i\hbar\psi\frac{\partial\psi^*}{\partial t} = \psi\left[\frac{\hbar c}{i}\sum \alpha_i \frac{\partial}{\partial x_i} + \beta m_0 c^2\right]\psi^*$$

restando estas dos ecuaciones se tiene que:

$$i\hbar\left[\psi^*\frac{\partial\psi}{\partial t} - \psi\frac{\partial\psi^*}{\partial t}\right] = \frac{\hbar c}{i}\left[\sum \alpha_i\psi^*\frac{\partial\psi}{\partial x_i} - \alpha_i\psi\frac{\partial\psi^*}{\partial x_i}\right] + \beta m_0 c^2\left[\psi^*\psi - \psi\psi^*\right]$$

de la ecuación de Schrödinger se tiene que:

$$\psi\frac{\partial\psi^*}{\partial t} = \frac{\psi}{\hbar i}\left[\frac{\hbar c}{i}\sum \alpha_i \frac{\partial}{\partial x_i} + \beta m_0 c^2\right]\psi^*$$

y tomando en cuenta que:

$$\frac{\partial\rho}{\partial t} = \psi\frac{\partial\psi^*}{\partial t} + \psi^*\frac{\partial\psi}{\partial t}$$

entonces:

$$i\hbar\left[\frac{\partial\rho}{\partial t} - 2\psi\frac{\partial\psi^*}{\partial t}\right] = \frac{\hbar c}{i}\left[\sum \alpha_i\psi^*\frac{\partial\psi}{\partial x_i} - \alpha_i\psi\frac{\partial\psi^*}{\partial x_i}\right] + \beta m_0 c^2\left[\psi^*\psi - \psi\psi^*\right]$$

$$i\hbar\frac{\partial\rho}{\partial t} = \frac{\hbar c}{i}\left[\sum \alpha_i\psi^*\frac{\partial\psi}{\partial x_i} - \alpha_i\psi\frac{\partial\psi^*}{\partial x_i}\right] + \beta m_0 c^2\left[\psi^*\psi - \psi\psi^*\right] + 2i\hbar\psi\frac{\partial\psi^*}{\partial t}$$

$$
\begin{aligned}
i\hbar\frac{\partial\rho}{\partial t} &= \frac{\hbar c}{i}\left[\sum\psi^*\alpha_i\frac{\partial\psi}{\partial x_i}+\psi\alpha_i\frac{\partial\psi^*}{\partial x_i}\right] \\
\frac{\partial\rho}{\partial t} &= -\sum c\alpha_i\left(\psi^*\alpha_i\frac{\partial\psi}{\partial x_i}+\psi\alpha_i\frac{\partial\psi^*}{\partial x_i}\right) \\
\frac{\partial\rho}{\partial t} &= -\sum\frac{\partial c\psi^*\alpha_i\psi}{\partial x_i}
\end{aligned}
$$

por lo tanto

$$
\frac{\partial\rho}{\partial t}+\sum\frac{\partial}{\partial x_i}\left[c\psi^*\alpha_i\psi\right]=0
$$

Ejercicio 28

Mostrar la identidad

$$
\left(\vec{\sigma}\cdot\vec{A}\right)\left(\vec{\sigma}\cdot\vec{B}\right)=\vec{A}\cdot\vec{B}\mathbb{I}+i\vec{\sigma}\cdot\left(\vec{A}\times\vec{B}\right)
$$

tomando en cuenta que

$$
\{\sigma_i\sigma_j\}=2\delta_{ij}\mathbb{I}\qquad[\sigma_i,\sigma_j]=2i\epsilon_{ijk}\sigma_k
$$

calculando $\{\sigma_i\sigma_j\}+[\sigma_i,\sigma_j]$, se tiene que:

$$
\begin{aligned}
[\sigma_i,\sigma_j] &= 2\delta_{ij}\mathbb{I}+2i\epsilon_{ijk}\sigma_k \\
2\sigma_i\sigma_j &= 2\delta_{ij}\mathbb{I}+2i\epsilon_{ijk}\sigma_k \\
\sigma_i\sigma_j &= \delta_{ij}\mathbb{I}+i\epsilon_{ijk}\sigma_k
\end{aligned}
$$

multiplicando por a_ib_j

$$
\begin{aligned}
\sigma_i\sigma_ja_ib_j &= \left(\delta_{ij}\mathbb{I}+i\epsilon_{ijk}\sigma_k\right)a_ib_j \\
\sigma_ia_i\sigma_jb_j &= \delta_{ij}\mathbb{I}a_ib_j+i\epsilon_{ijk}\sigma_ka_ib_j \\
\sigma_ia_i\sigma_jb_j &= \delta_{ij}a_ib_j\mathbb{I}+i\epsilon_{ijk}a_ib_j\sigma_k
\end{aligned}
$$

tomando en cuenta que

$$
\begin{aligned}
\sigma_ia_i &= \vec{\sigma}\cdot\vec{A} \\
\sigma_jB_j &= \vec{\sigma}\cdot\vec{B} \\
\delta_{ij}a_ib_j &= \vec{A}\cdot\vec{B} \\
\epsilon_{ijk}a_ib_j &= \vec{A}\times\vec{B} \\
\epsilon_{ijk}a_ib_j\sigma_k &= \vec{A}\times\vec{B}\cdot\vec{\sigma}
\end{aligned}
$$

por lo tanto:

$$
\left(\vec{\sigma}\cdot\vec{A}\right)\left(\vec{\sigma}\cdot\vec{B}\right)=\vec{A}\cdot\vec{B}\mathbb{I}+i\vec{\sigma}\cdot\left(\vec{A}\times\vec{B}\right)
$$

Ejercicio 29

Mostrar que

$$\left[\hat{\Sigma}\cdot\hat{P},\hat{\vec{H}}\right]\psi_{p,\lambda}=0$$

Reescribiendo se tiene que:

$$\begin{aligned}\left[\hat{\Sigma}\cdot\hat{P},\hat{\vec{H}}\right]&=\left[\delta_{ij}\Sigma_iP_j,c\delta_{ij}\alpha_iP_j+m_0c^2\beta\right]\\&=c\left[\delta_{ij}\Sigma_iP_j,\delta_{ij}\alpha_iP_j\right]+m_0c^2\left[\delta_{ij}\Sigma_iP_j,\beta\right]\end{aligned}$$

calculando $[\delta_{ij}\Sigma_iP_j,\beta]$:

$$\begin{aligned}\left[\delta_{ij}\Sigma_iP_j,\beta\right]&=\delta_{ij}\Sigma_iP_j\beta-\beta\delta_{ij}\Sigma_iP_j\\&=\left(\Sigma_i\beta-\beta\Sigma_i\right)P_i\end{aligned}$$

calculando $\Sigma_i\beta$, se tiene que:

$$\Sigma_i\beta=\begin{bmatrix}\sigma_i&0\\0&\sigma_i\end{bmatrix}\begin{bmatrix}\mathbb{I}&0\\0&-\mathbb{I}\end{bmatrix}=\begin{bmatrix}\sigma_i&0\\0&-\sigma_i\end{bmatrix}$$

calculando $\beta\Sigma_i$, se tiene que:

$$\beta\Sigma_i=\begin{bmatrix}\mathbb{I}&0\\0&-\mathbb{I}\end{bmatrix}\begin{bmatrix}\sigma_i&0\\0&\sigma_i\end{bmatrix}=\begin{bmatrix}\sigma_i&0\\0&-\sigma_i\end{bmatrix}$$

por lo tanto:

$$\left[\delta_{ij}\Sigma_iP_j,\beta\right]=0$$

calculando $[\delta_{ij}\Sigma_iP_j,\delta_{ij}\alpha_iP_j]$

$$\begin{aligned}\left[\delta_{ij}\Sigma_iP_j,\delta_{ij}\alpha_iP_j\right]&=\Sigma_iP_i\alpha_iP_i-\alpha_iP_i\Sigma_iP_i\\&=\left(\Sigma_i\alpha_i-\alpha_i\Sigma_i\right)P_i^2\end{aligned}$$

calculando $\Sigma_i\alpha_i$:

$$\Sigma_i\alpha_i=\begin{bmatrix}\sigma_i&0\\0&\sigma_i\end{bmatrix}\begin{bmatrix}0&\sigma_i\\\sigma_i&0\end{bmatrix}=\begin{bmatrix}0&\mathbb{I}\\\mathbb{I}&0\end{bmatrix}$$

calculando $\alpha_i\Sigma_i$:

$$\alpha_i\Sigma_i=\begin{bmatrix}0&\sigma_i\\\sigma_i&0\end{bmatrix}\begin{bmatrix}\sigma_i&0\\0&\sigma_i\end{bmatrix}=\begin{bmatrix}0&\mathbb{I}\\\mathbb{I}&0\end{bmatrix}$$

entonces:

$$\left[\delta_{ij}\Sigma_iP_j,\delta_{ij}\alpha_iP_j\right]=0$$

por lo tanto:

$$\left[\hat{\Sigma}\cdot\hat{P},\hat{\vec{H}}\right]\psi_{p,\lambda}=0$$

Ejericio 30

Mostrar que

$$T^i{}_j = -\psi^* \alpha^i \hat{P}_j c\psi$$

Se tiene que el lagrangiano es:

$$\mathcal{L} = \psi^* \left[i\hbar c\gamma^\mu \partial_\mu + m_0 c^2 \right] \psi$$

y el tensor energía momento es

$$T^\mu{}_\nu = \frac{\partial \mathcal{L}}{\partial(\partial_\mu \psi)} \partial_\nu \psi + \frac{\partial \mathcal{L}}{\partial(\partial_\mu \psi^*)} \partial_\nu \psi^* - \delta^\mu_\nu \mathcal{L}$$

calculando $\frac{\partial \mathcal{L}}{\partial(\partial_\mu \psi)} \partial_\nu \psi$:

$$\frac{\partial \mathcal{L}}{\partial(\partial_\mu \psi)} \partial_\nu \psi = \psi^* c i\hbar \gamma^\mu \psi$$

como la lagrangiana no depende de $\partial_\mu \psi^*$, entonces

$$\frac{\partial \mathcal{L}}{\partial(\partial_\mu \psi^*)} \partial_\nu \psi^* = 0$$

por lo tanto, el tensor energía momento es

$$T^\mu{}_\nu = \psi^* c i\hbar \gamma^\mu \delta_\nu \psi - \delta^\mu_\nu \left[\psi^* \left[i\hbar c\gamma^\mu \partial_\mu + m_0 c^2 \right] \psi \right]$$

Tomando el caso de $\mu = i, \nu = j$, se tiene que:

$$T^i{}_j = \psi^* c i\hbar \alpha^i \partial_j \psi$$

Como el operador del momento lineal es $\hat{P}_j = -i\hbar\partial_j$, entonces

$$\begin{aligned} T^i{}_j &= -\psi^* c\alpha^i (-i\hbar\partial_j)\psi \\ &= \psi^* c\alpha^i \hat{P}_j \psi \end{aligned}$$

por lo tanto:

$$T^i{}_j = -\psi^* \alpha^i \hat{P}_j c\psi$$

Ejercicio 31

Comprobar que

$$\sigma_{\alpha\beta} = \frac{i}{2} \left[\gamma_\alpha, \gamma_\beta \right]$$

satisface

$$\left[\sigma_{\alpha\beta}, \gamma^\nu \right] = -2i \left[\delta^\nu_\alpha \gamma_\beta - \delta^\nu_\beta \gamma_\alpha \right]$$

se tiene que:

$$\begin{aligned}
[\sigma_{\alpha\beta},\gamma^{\nu}] =& \sigma_{\alpha\beta}\gamma^{\nu}-\gamma^{\nu}\sigma_{\alpha\beta}\\
=& \frac{i}{2}\left[\gamma_{\alpha}\gamma_{\beta}-\gamma_{\beta}\gamma_{\alpha},\gamma^{\nu}\right]\\
=& \frac{i}{2}\left(\left(\gamma_{\alpha}\gamma_{\beta}-\gamma_{\beta}\gamma_{\alpha}\right)\gamma^{\nu}-\gamma^{\nu}\left(\gamma_{\alpha}\gamma_{\beta}-\gamma_{\beta}\gamma_{\alpha}\right)\right)\\
=& \frac{i}{2}\left(\gamma_{\alpha}\gamma_{\beta}\gamma^{\nu}-\gamma_{\beta}\gamma_{\alpha}\gamma^{\nu}-\gamma^{\nu}\gamma_{\alpha}\gamma_{\beta}+\gamma^{\nu}\gamma_{\beta}\gamma_{\alpha}\right)
\end{aligned}$$

se tiene que

$$\gamma^{\nu}\gamma_{\alpha}\gamma_{\beta}=-4\delta^{\nu}_{\alpha}\gamma_{\beta}$$

y que

$$\begin{aligned}
\gamma_{\alpha}\gamma_{\beta}\gamma^{\nu} =& \gamma_{\alpha}g_{\theta\beta}\gamma^{\theta}g^{\theta\alpha}\gamma_{\theta}\\
=& \gamma_{\alpha}g_{\theta\beta}g^{\theta\alpha}\gamma^{\theta}g_{\theta}\\
=& \gamma_{\alpha}\delta^{\alpha}_{\beta}\gamma^{\theta}g_{\theta}\\
=& 4\gamma_{\alpha}\delta^{\alpha}_{\beta}
\end{aligned}$$

entonces

$$\begin{aligned}
\gamma_{\alpha}\gamma_{\beta}\gamma^{\nu}-\gamma_{\beta}\gamma_{\alpha}\gamma^{\nu} =& 4\gamma_{\alpha}\delta^{\alpha}_{\beta}-4\gamma_{\beta}\delta^{\alpha}_{\beta}\\
=& 4\gamma_{\alpha}-4\gamma_{\alpha}\\
=& 0
\end{aligned}$$

y tambien

$$-\gamma^{\nu}\gamma_{\alpha}\gamma_{\beta}+\gamma^{\nu}\gamma_{\beta}\gamma_{\alpha} = 4\delta^{\nu}_{\beta}\gamma_{\alpha}-4\delta^{\nu}_{\alpha}$$

por lo tanto

$$\begin{aligned}
[\sigma_{\alpha\beta},\gamma^{\nu}] =& \frac{i}{2}\left(\gamma_{\alpha}\gamma_{\beta}\gamma^{\nu}-\gamma_{\beta}\gamma_{\alpha}\gamma^{\nu}-\gamma^{\nu}\gamma_{\alpha}\gamma_{\beta}+\gamma^{\nu}\gamma_{\beta}\gamma_{\alpha}\right)\\
=& \frac{i}{2}\left(4\delta^{\nu}_{\beta}\gamma_{\alpha}-4\delta^{\nu}_{\alpha}\right)\\
=& -2i\left(\delta^{\nu}_{\alpha}\gamma_{\beta}-\delta^{\nu}_{\beta}\gamma_{\alpha}\right)
\end{aligned}$$

Ejercicio 32

Mostrar que

$$\omega_{0}{}^{1}=-\omega^{0}{}_{1}$$

Obteniendo el termino $\omega_{0}{}^{1}$

$$\begin{aligned}
\omega_{0}{}^{1} =& g_{0\beta}\omega^{\beta 1}\\
=& g_{00}\omega^{01}+g_{01}\omega^{11}+g_{02}\omega^{21}+g_{03}\omega^{31}\\
=& g_{00}\omega^{01}\\
=& \omega^{01}
\end{aligned}$$

obtiendo el termino ${\omega^0}_1$

$$\begin{aligned}{\omega^0}_1 =& g_{1\beta}\omega^{0\beta}\\ =& g_{10}\omega^{00} + g_{11}\omega^{01} + g_{21}\omega^{02} + g_{31}\omega^{03}\\ =& g_{11}\omega^{01}\\ =& -\omega^{01}\end{aligned}$$

por lo tanto

$${\omega_0}^1 = -{\omega^0}_1$$

Ejercicio 33

Mostrar que las matrices $\hat{\Gamma}^m$ son linealmente independientes

Ejercicio 34

Mostrar que

$$\gamma^\mu\gamma^5 + \gamma^5\gamma^\mu = 0$$

Ejercicio 35

Mostrar para 2, 3, 4, 5

$$\bar{\psi}'\gamma^5\gamma^\nu\psi' = -a^\nu_\mu\bar{\psi}\gamma^5\gamma^0\psi$$