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UNIVERSIDAD AUTÓNOMA DE NUEVO LEÓN

**FCFM**

FACULTAD DE CIENCIAS FÍSICO MATEMÁTICAS



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**Mécaninca Cuántica Relativista**  
**Problemas propuestos**  
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1. Mostrar que:

$$U_{\perp} = \frac{u_i}{\gamma_v \left[ 1 + \frac{v \cdot v}{c^2} \right]}$$

De las transformaciones:

$$\begin{aligned} r_{\parallel} &= \gamma_v [r'_{\parallel} + vt'] \\ r_{\perp} &= r'_{\perp} \\ t &= \gamma_v \left[ t' + \frac{v \cdot v}{c^2} \right] \end{aligned}$$

tomando los diferenciales:

$$\begin{aligned} dr_{\parallel} &= \gamma_v [dr'_{\parallel} + v dt] \\ dr_{\perp} &= dr'_{\perp} \\ dt &= \gamma_v \left[ dt + \frac{v dr}{c^2} \right] \end{aligned}$$

entonces:

$$\begin{aligned} \frac{dr_{\perp}}{dt} &= \frac{dr'_{\perp}}{\gamma_v dt' \left[ 1 + \frac{v}{c^2} \frac{dr'}{dt'} \right]} \\ u_{\perp} &= \frac{dr'_{\perp}}{\gamma_v dt' \left[ 1 + \frac{v}{c^2} \frac{dr'}{dt'} \right]} \\ &= \frac{u'_{\perp}}{\gamma_v \left[ 1 + \frac{v \cdot u'}{c^2} \right]} \end{aligned}$$

por lo tanto:

$$u_{\perp} = \frac{u'_{\perp}}{\gamma_v \left[ 1 + \frac{v \cdot u'}{c^2} \right]} \quad (1)$$

2. Mostrar que

$$\begin{aligned} (U_d)_x &= \frac{-c\beta \sin(\theta')}{\gamma_v(1 - \beta^2 \cos(\theta'))} \\ (U_d)_z &= \frac{c\beta(1 - \cos(\theta'))}{1 - \beta^2 \cos(\theta')} \end{aligned}$$

Se sabe que por convención:

$$\begin{aligned} (U_d)_{\perp} &= (U_d)_x = \frac{(U'_d)_{\perp}}{\gamma_v \left( 1 + \frac{v \cdot u'}{c^2} \right)} \\ (U_d)_{\parallel} &= (U_c)_z = \frac{(U'_d)_{\parallel} + v}{1 + \frac{v \cdot u'_d}{c^2}} \end{aligned}$$

pero, del diagrama

$$\begin{aligned} (U'_d)_{\perp} &= U'_d \sin(\theta') \\ (U'_d)_{\parallel} &= U'_d \cos(\theta') \end{aligned}$$

por lo tanto:

$$(U_d)_x = \frac{U'_d \sin(\theta')}{\gamma_v \left(1 + \frac{|v||u_d| \cos(\theta)}{c^2}\right)}$$

$$(U_d)_z = \frac{U'_d \cos(\theta') + v}{\gamma_v \left(1 + \frac{|v||u_d| \cos(\theta)}{c^2}\right)}$$

pero  $U'_d = -v$

$$(U_d)_x = \frac{-v \sin(\theta')}{\gamma_v \left(1 - \frac{v^2}{c^2} \cos(\theta')\right)}$$

$$= \frac{-c\beta \sin(\theta')}{\gamma_v (1 - \beta^2 \cos(\theta'))}$$

$$(U_d)_z = \frac{c\beta (1 - \cos(\theta'))}{1 - \beta^2 \cos(\theta')}$$

3. Mostrar que

$$u_c^2 = u_a^2 - \frac{\eta}{\gamma_a} \quad (2)$$

$$(U_c)_x = \frac{c\beta \sin(\theta')}{\gamma_v [1 + \beta^2 \cos(\theta')]}$$

$$\approx \frac{c\beta \theta}{\gamma_v [1 + \beta^2 (1 - \frac{\theta^2}{2})]}$$

$$(U_c)_z \approx \frac{c\beta (1 + (1 - \frac{\theta^2}{2}))}{1 + \beta^2 (1 - \frac{\theta^2}{2})}$$

realizando el calculo para ángulos pequeños, tomando en cuenta que  $\cos(\theta) = 1 - \theta^2/2$  y  $\sin(\theta) = \theta$

$$(U_c)_z^2 = \frac{c^2 \beta^2 \left(4 - 2\theta^2 + \frac{\theta^4}{4}\right)}{(1 + \beta^2) \left(1 - \frac{\beta^2 \theta^2}{2(1 + \beta^2)}\right)^2}$$

$$= \frac{c^2 \beta^2 (4 - 2\theta^2)}{(1 + \beta^2)^2 \left(1 - \frac{\beta^2 \theta^2}{2(1 + \beta^2)}\right)^2}$$

$$\approx \frac{c^2 \beta^2 (4 - 2\theta^2)}{(1 + \beta^2)^2} \left(1 + \frac{\beta^2}{1 + \beta^2} \theta^2\right)$$

$$\approx \frac{4c^2 \beta^2}{(1 + \beta^2)^2} - \frac{2c^2 \beta^2 \theta^2}{(1 + \beta^2)^2} + \frac{4c^2 \beta^4 \theta^2}{(1 + \beta^2)^3}$$

$$\approx u_a^2 - \frac{2c^2 \beta^2 \theta^2}{(1 + \beta^2)^2} + \frac{4c^2 \beta^4 \theta^2}{(1 + \beta^2)^3}$$

$$\begin{aligned}
(U_c)_x^2 &= \frac{c^2 \beta^2 \theta^2}{\gamma_v^2 (1 + \beta^2)^2 \left(1 - \frac{\beta^2 \theta^2}{2(1 + \beta^2)}\right)^2} \\
&\approx \frac{c^2 \beta^2 \theta^2}{\gamma_v^2 (1 + \beta^2)} \left(1 + \frac{\beta}{1 + \beta^2} \theta^2\right) \\
&\approx \frac{c^2 \beta^2 \theta^2}{\gamma^2 (1 + \beta^2)^2} + \frac{c^2 \beta^4 \theta^4}{\gamma^2 (1 + \beta^2)^3} \\
&\approx \frac{c^2 \beta^2 \theta^2}{\gamma^2 (1 + \beta^2)^2}
\end{aligned}$$

se tiene que:

$$u_a = \frac{2\beta c}{1 + \beta^2} \qquad \gamma_a = \frac{1 + \beta^2}{1 - \beta^2}$$

por lo tanto:

$$\begin{aligned}
u_c^2 &= (u_c)_x^2 + (u_c)_z^2 \\
&= u_a^2 - \frac{2c^2 \beta^2 \theta^2}{(1 + \beta^2)^2} + \frac{4c^2 \beta^4 \theta^2}{(1 + \beta^2)^3} + \frac{c^2 \beta^2 \theta^2}{\gamma^2 (1 + \beta^2)^2} \\
&= u_a^2 + \frac{c^2 \beta^2 \theta^2}{(1 + \beta^2)^2} \left(1 - \beta^2 - 2 + \frac{4\beta^2}{1 + \beta^2}\right) \\
&= u_a^2 + \frac{c^2 \beta^2 \theta^2}{(1 + \beta^2)^2} \left(\frac{1 - 2\beta^2 - \beta^4}{1 + \beta^2}\right) \\
&= u_a^2 - \frac{c^2 \beta^2 \theta^2}{(1 + \beta^2)^2} \left(\frac{(1 - \beta^2)^2}{1 + \beta^2}\right) \\
&= u_a^2 - \frac{c^2 \beta^2 \theta^2}{1 - \beta^2} \left(\frac{(1 - \beta^2)^3}{(1 + \beta^2)^3}\right) \\
&= u_a^2 - \eta \frac{1}{\gamma_a^3}
\end{aligned}$$

4. Muestre que:

$$\partial_\alpha A^\alpha = \partial^\alpha A_\alpha$$

Se tiene que:

$$x_\alpha = g_{\alpha\beta} x^\beta \qquad x^\alpha = g^{\alpha\beta} x_\beta$$

por lo tanto:

$$\begin{aligned}
g^{\alpha\beta} \partial_\alpha &= \partial^\alpha \\
g_{\alpha\beta} \partial^\alpha &= \partial_\alpha
\end{aligned}$$

calculando  $\partial^\alpha A_\alpha$

$$\begin{aligned}
\partial^\alpha A_\alpha &= \left(\frac{\partial A_0}{\partial x_0}\right) - \left(\frac{\partial A_1}{\partial x_1}\right) - \left(\frac{\partial A_2}{\partial x_2}\right) - \left(\frac{\partial A_3}{\partial x_3}\right) \\
&= \frac{\partial A_0}{\partial x_0} - \nabla A
\end{aligned}$$

por lo que se encuentra que:

$$\begin{aligned}A_0 &= A^0 \\A_1 &= -A^1 \\A_2 &= -A^2 \\A_3 &= -A^3\end{aligned}$$

$$\begin{aligned}\partial^\alpha A_\alpha &= (g^{\alpha\beta} \partial_\beta)(g_{\alpha\gamma} A^\gamma) \\ \delta_\gamma^\beta &= \partial_\beta A^\gamma\end{aligned}$$

5. Por verificar que:

$$\partial^\alpha = \left( \frac{\partial}{\partial x_0}, -\nabla \right)$$

Sea  $A^\alpha$  un tensor covariante, entonces:

$$\begin{aligned}\partial^\alpha A_\alpha &= \left( \frac{\partial A_0}{\partial x_0} \right) - \left( \frac{\partial A_1}{\partial x_1} \right) - \left( \frac{\partial A_2}{\partial x_2} \right) - \left( \frac{\partial A_3}{\partial x_3} \right) \\ &= \left( \frac{\partial}{\partial x_0}, -\frac{\partial}{\partial x_1}, -\frac{\partial}{\partial x_2}, -\frac{\partial}{\partial x_3} \right) \cdot (A_0, A_1, A_2, A_3) \\ &= \left( \frac{\partial}{\partial x_0}, -\nabla \right) \cdot A_\alpha\end{aligned}$$

por lo tanto:

$$\partial^\alpha = \left( \frac{\partial}{\partial x_0}, -\nabla \right)$$

6. Probar que las matrices  $S_1^2, S_2^2, S_3^2$  son diagonales con -1 y que las matrices  $K_1^2, K_2^2, K_3^2$  son diagonales con 1: Se tiene la matriz  $S_1$  igual a:

$$S_1 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

entonces, calculando  $S_1^2$ , se tiene que:

$$\begin{aligned}S_1^2 &= \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{pmatrix}^2 \\ &= \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{pmatrix} \\ &= \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}\end{aligned}$$

Se tiene la matriz  $S_2$  igual a:

$$S_2 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

entonces, calculando  $S_2^2$ , se tiene que:

$$\begin{aligned} S_2^2 &= \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix}^2 \\ &= \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix} \\ &= \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \end{aligned}$$

Se tiene la matriz  $S_3$  igual a:

$$S_3 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & -0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

entonces, calculando  $S_3^2$ , se tiene que:

$$\begin{aligned} S_3^2 &= \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}^2 \\ &= \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \\ &= \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \end{aligned}$$

por lo tanto las matrices  $S_\mu^2$  son diagonales con -1 Se tiene la matriz  $K_1$  igual a:

$$S_1 = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

entonces, calculando  $K_1^2$ , se tiene que:

$$\begin{aligned}
 K_1^2 &= \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}^2 \\
 &= \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \\
 &= \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}
 \end{aligned}$$

Se tiene la matriz  $K_2$  igual a:

$$K_2 = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

entonces, calculando  $K_2^2$ , se tiene que:

$$\begin{aligned}
 K_2^2 &= \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}^2 \\
 &= \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \\
 &= \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}
 \end{aligned}$$

Se tiene la matriz  $K_3$  igual a:

$$K_3 = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}$$

entonces, calculando  $K_3^2$ , se tiene que:

$$\begin{aligned}
 K_3^2 &= \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}^2 \\
 &= \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix} \\
 &= \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}
 \end{aligned}$$

por lo tanto las matrices  $K_\mu^2$  son diagonales con 1

7. Compruebe la forma de L, que cumple  $L^T g = -gL$  donde L tiene diagonal de ceros y g es la representación matricial de  $g_{\mu\nu\rho}$
8. Mostrar que  $F_{\alpha\gamma} = g_{\alpha\gamma} F^{\gamma\delta} g_{\delta\beta}$   
Se sabe que:

$$F^{\gamma\delta} = \begin{pmatrix} 0 & -Ex & -Ey & -Ez \\ Ex & 0 & -Bz & By \\ Ey & Bz & 0 & -Bx \\ Ez & -By & Bx & 0 \end{pmatrix} \quad g_{\alpha\gamma} = g_{\delta\beta} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$$

realizando la multiplicacion  $F^{\gamma\delta} g_{\delta\beta}$

$$\begin{aligned}
 F^{\gamma\delta} g_{\delta\beta} &= \begin{pmatrix} 0 & -Ex & -Ey & -Ez \\ Ex & 0 & -Bz & By \\ Ey & Bz & 0 & -Bx \\ Ez & -By & Bx & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \\
 &= \begin{pmatrix} 0 & Ex & Ey & Ez \\ Ex & 0 & Bz & -By \\ Ey & -Bz & 0 & Bx \\ Ez & By & -Bx & 0 \end{pmatrix}
 \end{aligned}$$

por lo tanto:

$$F_\beta^\gamma = \begin{pmatrix} 0 & Ex & Ey & Ez \\ Ex & 0 & Bz & -By \\ Ey & -Bz & 0 & Bx \\ Ez & By & -Bx & 0 \end{pmatrix}$$



realizando la multiplicación  $g_{\alpha\gamma}F_{\beta}^{\gamma}$  se obtiene que:

$$\begin{aligned} g_{\alpha\gamma}F_{\beta}^{\gamma} &= \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \begin{pmatrix} 0 & Ex & Ey & Ez \\ Ex & 0 & B_z & -B_y \\ Ey & -B_z & 0 & B_x \\ Ez & B_y & -B_x & 0 \end{pmatrix} \\ &= \begin{pmatrix} 0 & Ex & Ey & Ez \\ -Ex & 0 & -B_z & B_y \\ -Ey & B_z & 0 & -B_x \\ -Ez & -B_y & B_x & 0 \end{pmatrix} \end{aligned}$$

por lo tanto:

$$F_{\alpha\beta} = \begin{pmatrix} 0 & Ex & Ey & Ez \\ -Ex & 0 & -B_z & B_y \\ -Ey & B_z & 0 & -B_x \\ -Ez & -B_y & B_x & 0 \end{pmatrix}$$

9. Formule la matriz de rotación respecto a  $\hat{z}$  por medio de la transformación de Lorentz partiendo de la invarianza de  $S^2$  se obtiene la forma (dependiente de 6 parámetros) de L, tal que  $A = e^L$  es la transformación de Lorentz.