UNIVERSIDAD AUTÓNOMA DE NUEVO LEÓN
FACULTAD DE CIENCIAS FÍSICO MATEMÁTICAS

Mécaninca Cuántica Relativista
Problemas propuestos

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1. Mostrar que:

$$U_\perp = \frac{u_i}{\gamma_v \left[1 + \frac{v \cdot v}{c^2}\right]}$$

De las transformaciones:

$$\begin{aligned} r_\parallel &= \gamma_v [{r_\parallel}' + vt'] \\ r_\perp &= {r_\perp}' \\ t &= \gamma_v \left[t' + \frac{v \cdot v}{c^2}\right] \end{aligned}$$

tomando los diferenciales:

$$\begin{aligned} dr_\parallel &= \gamma_v \left[dr'_\parallel + vdt\right] \\ dr_\perp &= dr'_p erp \\ dt &= \gamma_v \left[dt + \frac{vdr}{c^2}\right] \end{aligned}$$

entonces:

$$\begin{aligned} \frac{dr_\perp}{dt} &= \frac{dr'_\perp}{\gamma_v dt' \left[1 + \frac{v}{c^2}\frac{dr'}{dt'}\right]} \\ u_\perp &= \frac{dr'_\perp}{\gamma_v dt' \left[1 + \frac{v}{c^2}\frac{dr'}{dt'}\right]} \\ &= \frac{u'_\perp}{\gamma_v \left[1 + \frac{v \cdot u'}{c^2}\right]} \end{aligned}$$

por lo tanto:

$$u_\perp = \frac{u'_\perp}{\gamma_v \left[1 + \frac{v \cdot u'}{c^2}\right]} \tag{1}$$

2. Mostrar que

$$\begin{aligned} (U_d)_x &= \frac{-c\beta \sin(\theta')}{\gamma_v (1 - \beta^2 \cos(\theta'))} \\ (U_d)_z &= \frac{c\beta(1 - \cos(\theta'))}{1 - \beta^2 \cos(\theta')} \end{aligned}$$

Se sabe que por convención:

$$\begin{aligned} (U_d)_\perp = (U_d)_x &= \frac{(U'_d)_\perp}{\gamma_v \left(1 + \frac{v \cdot u'}{c^2}\right)} \\ (U_d)_\parallel = (U_c)_z &= \frac{(U'_d)_\parallel + v}{1 + \frac{v \cdot u'_d}{c^2}} \end{aligned}$$

pero, del diagrama

$$\begin{aligned} (U'_d)_\perp &= U'_d \sin(\theta') \\ (U'_d)_\parallel &= U'_d \cos(\theta') \end{aligned}$$

por lo tanto:

$$(U_d)_x = \frac{U'_d \sin(\theta')}{\gamma_v \left(1 + \frac{|v||u_d|\cos(\theta)}{c^2}\right)}$$

$$(U_d)_z = \frac{U'_d \cos(\theta'_d) + v}{\gamma_v \left(1 + \frac{|v||u_d|\cos(\theta)}{c^2}\right)}$$

pero $U'_d = -v$

$$\begin{aligned}(U_d)_x &= \frac{-v\sin(\theta')}{\gamma_v\left(1 - \frac{v^2}{c^2}\cos(\theta')\right)} \\ &= \frac{-c\beta\sin(\theta')}{\gamma_v(1-\beta^2\cos(\theta'))} \\ (U_d)_z &= \frac{c\beta(1-\cos(\theta'))}{1-\beta^2\cos(\theta')}\end{aligned}$$

3. Mostrar que

$$u_c^2 = u_a^2 - \frac{\eta}{\gamma_a} \tag{2}$$

$$\begin{aligned}(U_c)_x &= \frac{c\beta\sin(\theta')}{\gamma_v\left[1+\beta^2\cos(\theta')\right]} \\ &\approx \frac{c\beta\theta}{\gamma_v\left[1+\beta^2\left(1-\frac{\theta^2}{2}\right)\right]} \\ (U_c)_z &\approx \frac{c\beta(1+\left(1-\frac{\theta^2}{2}\right))}{1+\beta^2\left(1-\frac{\theta^2}{2}\right)}\end{aligned}$$

realizando el calculo para ángulos pequeños, tomando en cuenta que $\cos(\theta) = 1 - \theta^2/2$ y $\sin(\theta) = \theta$

$$\begin{aligned}(U_c)_z^2 &= \frac{c^2\beta^2\left(4 - 2\theta^2 + \frac{\theta^4}{4}\right)}{(1+\beta^2)\left(1 - \frac{\beta^2\theta^2}{2(1+\beta^2)}\right)^2} \\ &= \frac{c^2\beta^2(4-2\theta^2)}{(1+\beta^2)^2\left(1 - \frac{\beta^2\theta^2}{2(1+\beta^2)}\right)^2} \\ &\approx \frac{c^2\beta^2(4-2\theta^2)}{(1+\beta^2)^2}\left(1 + \frac{\beta^2}{1+\beta^2}\theta^2\right) \\ &\approx \frac{4c^2\beta^2}{(1+\beta^2)^2} - \frac{2c^2\beta^2\theta^2}{(1+\beta^2)^2} + \frac{4c^2\beta^4\theta^2}{(1+\beta^2)^3} \\ &\approx u_a^2 - \frac{2c^2\beta^2\theta^2}{(1+\beta^2)^2} + \frac{4c^2\beta^4\theta^2}{(1+\beta^2)^3}\end{aligned}$$

$$
\begin{aligned}
(U_c)_x^2 &= \frac{c^2\beta^2\theta^2}{\gamma_v^2(1+\beta^2)^2\left(1-\frac{\beta^2\theta^2}{2(1+\beta^2)}\right)^2} \\
&\approx \frac{c^2\beta^2\theta^2}{\gamma_v^2(1+\beta^2)}\left(1+\frac{\beta}{1+\beta^2}\theta^2\right) \\
&\approx \frac{c^2\beta^2\theta^2}{\gamma^2(1+\beta^2)^2}+\frac{c^2\beta^4\theta^4}{\gamma^2(1+\beta^2)^3} \\
&\approx \frac{c^2\beta^2\theta^2}{\gamma^2(1+\beta^2)^2}
\end{aligned}
$$

se tiene que:

$$
u_a = \frac{2\beta c}{1+\beta^2} \qquad\qquad \gamma_a = \frac{1+\beta^2}{1-\beta^2}
$$

por lo tanto:

$$
\begin{aligned}
u_c^2 &= (u_c)_x^2 + (u_c)_z^2 \\
&= u_a^2 - \frac{2c^2\beta^2\theta^2}{(1+\beta^2)^2} + \frac{4c^2\beta^4\theta^2}{(1+\beta^2)^3} + \frac{c^2\beta^2\theta^2}{\gamma^2(1+\beta^2)^2} \\
&= u_a^2 + \frac{c^2\beta^2\theta^2}{(1+\beta^2)^2}\left(1-\beta^2-2+\frac{4\beta^2}{1+\beta^2}\right) \\
&= u_a^2 + \frac{c^2\beta^2\theta^2}{(1+\beta^2)^2}\left(\frac{1-2\beta^2-\beta^4}{1+\beta^2}\right) \\
&= u_a^2 - \frac{c^2\beta^2\theta^2}{(1+\beta^2)^2}\left(\frac{(1-\beta^2)^2}{1+\beta^2}\right) \\
&= u_a^2 - \frac{c^2\beta^2\theta^2}{1-\beta^2}\left(\frac{(1-\beta^2)^3}{(1+\beta^2)^3}\right) \\
&= u_a^2 - \eta\frac{1}{\gamma_a^3}
\end{aligned}
$$

4. Muestre que:

$$
\partial_\alpha A^\alpha = \partial^\alpha A_\alpha
$$

Se tiene que:

$$
x_\alpha = g_{\alpha\beta}x^\beta \qquad\qquad x^\alpha = g^{\alpha\beta}x_\beta
$$

por lo tanto:

$$
\begin{aligned}
g^{\alpha\beta}\partial_\alpha &= \partial^\alpha \\
g_{\alpha\beta}\partial^\alpha &= \partial_\alpha
\end{aligned}
$$

calculando $\partial^\alpha A_\alpha$

$$
\begin{aligned}
\partial^\alpha A_\alpha &= \left(\frac{\partial A_0}{\partial x_0}\right) - \left(\frac{\partial A_1}{\partial x_1}\right) - \left(\frac{\partial A_2}{\partial x_2}\right) - \left(\frac{\partial A_3}{\partial x_3}\right) \\
&= \frac{\partial A_0}{\partial x_0} - \nabla A
\end{aligned}
$$

por lo que se encuentra que:

$$\begin{aligned} A_0 &= A^0 \\ A_1 &= -A^1 \\ A_2 &= -A^2 \\ A_3 &= -A^3 \end{aligned}$$

$$\begin{aligned} \partial^\alpha A_\alpha &= (g^{\alpha\beta}\partial_\beta)(g_{\alpha\gamma}A^\gamma) \\ \delta^\beta_\gamma &= \partial_\beta A^\gamma \end{aligned}$$

5. Por verificar que:

$$\partial^\alpha = \left(\frac{\partial}{\partial x_0}, -\nabla\right)$$

Sea A^α un tensor covariante, entonces:

$$\begin{aligned} \partial^\alpha A_\alpha &= \left(\frac{\partial A_0}{\partial x_0}\right) - \left(\frac{\partial A_1}{\partial x_1}\right) - \left(\frac{\partial A_2}{\partial x_2}\right) - \left(\frac{\partial A_3}{\partial x_3}\right) \\ &= \left(\frac{\partial}{\partial x_0}, -\frac{\partial}{\partial x_1}, -\frac{\partial}{\partial x_2}, -\frac{\partial}{\partial x_3}\right) \cdot (A_0, A_1, A_2, A_3) \\ &= \left(\frac{\partial}{\partial x_0}, -\nabla\right) \cdot A_\alpha \end{aligned}$$

por lo tanto:

$$\partial^\alpha = \left(\frac{\partial}{\partial x_0}, -\nabla\right)$$

6. Probar que las matrices S_1^2, S_2^2, S_3^2 son diagonales con -1 y que las matrices K_1^2, K_2^2, K_3^2 son diagonales con 1: Se tiene la matriz S_1 igual a:

$$S_1 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

entonces, calculando S_1^2, se tiene que:

$$\begin{aligned} S_1^2 &= \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{pmatrix}^2 \\ &= \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{pmatrix} \\ &= \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \end{aligned}$$

Se tiene la matriz S_2 igual a:

$$S_2 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

entonces, calculando S_2^2, se tiene que:

$$\begin{aligned} S_2^2 &= \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix}^2 \\ &= \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix} \\ &= \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \end{aligned}$$

Se tiene la matriz S_3 igual a:

$$S_3 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & -0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

entonces, calculando S_3^2, se tiene que:

$$\begin{aligned} S_3^2 &= \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}^2 \\ &= \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \\ &= \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \end{aligned}$$

por lo tanto las matrices S_μ^2 son diagonales con -1 Se tiene la matriz K_1 igual a:

$$S_1 = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

entonces, calculando K_1^2, se tiene que:

$$\begin{aligned}
K_1^2 &= \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}^2 \\
&= \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \\
&= \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}
\end{aligned}$$

Se tiene la matriz K_2 igual a:

$$K_2 = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

entonces, calculando k_2^2, se tiene que:

$$\begin{aligned}
K_2^2 &= \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}^2 \\
&= \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \\
&= \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}
\end{aligned}$$

Se tiene la matriz K_3 igual a:

$$K_3 = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}$$

entonces, calculando K_3^2, se tiene que:

$$\begin{aligned}
K_3^2 &= \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}^2 \\
&= \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix} \\
&= \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}
\end{aligned}$$

por lo tanto las matrices K_μ^2 son diagonales con 1

7. Compruebe la forma de L, que cumple $L^T g = -gL$ donde L tiene diagonal de ceros y g es la representación matricial de $g_{\mu\nu\rho}$ Se tiene que:

$$g = diag(1, -1, -1, -1)$$

y que

$$g^T = g = g^{-1}$$

por lo tanto:

$$\begin{aligned}
c^T &= (gL)^T \\
&= L^T g \\
&= -gL \\
&= -c
\end{aligned}$$

por lo tanto:

$$c_{ij} = -c_{ji}$$

si $i = j$, entonces $c_{ii} = 0$, por lo tanto:

$$gL = C = \begin{pmatrix} 0 & C_{12} & C_{13} & C_{14} \\ -C_{12} & 0 & C_{23} & C_{24} \\ -C_{13} & -C_{23} & 0 & C_{34} \\ -C_{14} & -C_{24} & -C_{34} & 0 \end{pmatrix}$$

realizando la operación $gc = ggL$, se tiene que:

$$\begin{aligned}
gc &= g(gL) \\
&= (gg)L \\
&= L
\end{aligned}$$

por lo tanto $gc = L$

8. Mostrar que $F_{\alpha\gamma} = g_{\alpha\gamma} F^{\gamma\delta} g_{\delta\beta}$
   Se sabe que:

$$F^{\gamma\delta} = \begin{pmatrix} 0 & -Ex & -Ey & -Ez \\ E_x & 0 & -B_z & B_y \\ E_y & B_z & 0 & -B_x \\ E_z & -B_y & B_x & 0 \end{pmatrix} \qquad g_{\alpha\gamma} = g_{\delta\beta} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$$

   realizando la multiplicacion $F^{\gamma\delta} g_{\delta\beta}$

$$\begin{aligned} F^{\gamma\delta} g_{\delta\beta} &= \begin{pmatrix} 0 & -Ex & -Ey & -Ez \\ E_x & 0 & -B_z & B_y \\ E_y & B_z & 0 & -B_x \\ E_z & -B_y & B_x & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \\ &= \begin{pmatrix} 0 & Ex & Ey & Ez \\ E_x & 0 & B_z & -B_y \\ E_y & -B_z & 0 & B_x \\ E_z & B_y & -B_x & 0 \end{pmatrix} \end{aligned}$$

   por lo tanto:

$$F^{\gamma}_{\beta} = \begin{pmatrix} 0 & Ex & Ey & Ez \\ E_x & 0 & B_z & -B_y \\ E_y & -B_z & 0 & B_x \\ E_z & B_y & -B_x & 0 \end{pmatrix}$$

   realizando la multiplicacion $g_{\alpha\gamma} F^{\gamma}_{\beta}$ se obtiene que:

$$\begin{aligned} g_{\alpha\gamma} F^{\gamma}_{\beta} &= \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \begin{pmatrix} 0 & Ex & Ey & Ez \\ E_x & 0 & B_z & -B_y \\ E_y & -B_z & 0 & B_x \\ E_z & B_y & -B_x & 0 \end{pmatrix} \\ &= \begin{pmatrix} 0 & Ex & Ey & Ez \\ -E_x & 0 & -B_z & B_y \\ -E_y & B_z & 0 & -B_x \\ -E_z & -B_y & B_x & 0 \end{pmatrix} \end{aligned}$$

   por lo tanto:

$$F_{\alpha\beta} = \begin{pmatrix} 0 & Ex & Ey & Ez \\ -E_x & 0 & -B_z & B_y \\ -E_y & B_z & 0 & -B_x \\ -E_z & -B_y & B_x & 0 \end{pmatrix}$$

9. Formule la matriz de rotación respecto a $\hat{z}$ por medio de la transformación de Lorentz partiendo de la invarianza de S^2 se obtiene la forma (dependiente de 6 parámetros) de L, tal que $A = e^L$ es la transformación de Lorentz.
   Se tiene la base, S_μ, K_μ, donde $L = -\vec{\omega} \cdot \vec{s} - \vec{\xi} \cdot \vec{k}$, para rotar con respecto $\hat{z}$, se tiene que cumplir que: $\vec{\xi} = 0, \vec{\omega} = \omega_z \hat{z}$, entonces:

$$L = -\vec{\omega} \cdot \vec{s} = -\omega_z s_3$$

por lo tanto:

$$\begin{aligned} A &= e^{L} \\ &= e^{-\omega s_3} \\ &= \sum_{i=0}^{\infty} \frac{(-1)^i(\omega s_3)^i}{i!} \end{aligned}$$

donde se cumple que:

$$\begin{aligned} s_3^3 &= -s_3 \\ s_3^4 &= -s_3^2 \\ s_3^5 &= s_3 \end{aligned}$$

entonces:

$$\begin{aligned} A &= 1 - \omega s_3 + \frac{\omega^2}{2!}s_3^2 + \frac{\omega^3}{3!}s_3 - \frac{\omega^4}{4!}s_3^2 \\ &= (1+s_3^2) - s_3^2\left[1 - \frac{\omega^2}{2!} + \frac{\omega^4}{4!} - \dots\right] - s_3\left[\omega - \frac{\omega^3}{3!} + \frac{\omega^5}{5!} - \dots\right] \\ &= \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} - \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} cos(\omega) - \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} sen(\omega) \end{aligned}$$

entonces:

$$A = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & cos(\omega) & sen(\omega) & 0 \\ 0 & -sen(\omega) & cos(\omega) & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$