$$\cdot\, u = \frac{\sqrt{u'^2 + v^2 + 2u'v\cos\theta' - \left[\frac{u'v\,\mathrm{Sen}\,\theta'}{c}\right]^2}}{\left[1 + \frac{u'v}{c^2}\cos\theta'\right]}$$

Mostrar $\gamma_u = \gamma_v \gamma_{u'}\left[1 + \frac{\vec{v}\cdot\vec{u}'}{c^2}\right]$

$$u\left[1 + \frac{u'v}{c^2}\cos\theta'\right] = \sqrt{[\quad]}$$

$$[\quad] = u'^2 + 2u'v\cos\theta' - \frac{u'^2v^2}{c^2}\left[1 - \cos^2\theta'\right] + v^2$$

$$= u'^2\left[1 - \frac{v^2}{c^2}\right] + v^2 + 2u'v\cos\theta' + \frac{u'^2v^2}{c^2}\cos^2\theta' + c^2 - c^2$$

$$= u'^2\left[1 - \frac{v^2}{c^2}\right] + (v^2 - c^2) + \underbrace{c^2 + 2u'v\cos\theta' + \frac{u'^2v^2\cos^2\theta'}{c^2}}$$

$$= u'^2\left[1 - \frac{v^2}{c^2}\right] - c^2\left[1 - \frac{v^2}{c^2}\right] + \left[c + \frac{u'v\cos\theta}{c}\right]^2$$

$$= (u'^2 - c^2)\left[1 - \frac{v^2}{c^2}\right] + \left[c + \frac{u'v\cos\theta}{c}\right]^2$$

$$= -c^2\left[1 - \frac{u'^2}{c^2}\right]\left[1 - \frac{v^2}{c^2}\right] + c^2\left[1 + \frac{u'v\cos\theta}{c^2}\right]^2$$

$$= c^2\left[\left(1 + \frac{u'v\cos\theta}{c^2}\right)^2 - \frac{1}{\gamma_{u'}^2}\cdot\frac{1}{\gamma_v^2}\right] \qquad \gamma = \frac{1}{\left[1 - \frac{u^2}{c^2}\right]^{1/2}}$$

Regresamos a $u\left[1 + \frac{u'v\cos\theta'}{c^2}\right] = \sqrt{c^2\left[\left(1 + \frac{u'v\cos\theta'}{c^2}\right)^2 - \frac{1}{\gamma_{u'}^2}\cdot\frac{1}{\gamma_v^2}\right]}$

$$\frac{u^2}{c^2}\left[1 + \frac{u'v\cos\theta'}{c^2}\right]^2 = \left[1 + \frac{u'v\cos\theta'}{c^2}\right]^2 - \frac{1}{\gamma_{u'}^2}\cdot\frac{1}{\gamma_v^2}$$

$$\left[\frac{u^2}{c^2} - 1\right]\left[1 + \frac{u'v\cos\theta'}{c^2}\right]^2 = -\frac{1}{\gamma_{u'}^2\gamma_v^2}$$

$$\left[1 + \frac{u'v\cos\theta'}{c^2}\right]^2 = \frac{\gamma_u^2}{\gamma_{u'}^2\gamma_v^2}$$

$$\left\{\gamma_u = \gamma_v\gamma_{u'}\left[1 + \frac{u'v\cos\theta'}{c^2}\right]\right\} \cdot c$$

$$c\gamma_u = \gamma_v[\gamma_{u'}c]\left[1 + \frac{vu'\cos\theta'}{c^2}\right]$$

$$c\gamma_u = \gamma_v\left[(\gamma_{u'}c) + \frac{v}{c}(\gamma_{u'}u')\cos\theta'\right] \ldots \text{①}$$

Ademas

$$U_{\parallel} = \frac{u_{\parallel}' + v}{1 + \frac{\vec{v}\cdot\vec{u}'}{c^2}} \qquad u_{\perp} = \frac{u_{\perp}'}{\gamma_v\left[1 + \frac{\vec{v}\cdot\vec{u}'}{c^2}\right]}$$

usando $\gamma_u = \gamma_v\gamma_{u'}\left[1 + \frac{\vec{v}\cdot\vec{u}'}{c^2}\right]$

$$\gamma_u u_{\parallel} = \gamma_v\left[\gamma_{u'}u_{\parallel}' + v\gamma_{u'}\right]$$

$$\boxed{\gamma_u u_{\parallel} = \gamma_v\left[\gamma_{u'}u_{\parallel}' + \frac{v}{c}[c\gamma_{u'}]\right]} \ldots \text{②}$$

$$\boxed{\gamma_u u_{\perp} = \gamma_{u'}u_{\perp}'} \ldots \text{③}$$

Las transformaciones inversas

$$\begin{cases} x_0 = \gamma\left[x_0' + \vec{\beta}\cdot\vec{x}'\right] & \longleftrightarrow \quad \gamma_u c = \gamma_v\left[\gamma_{u'}c + \vec{\beta}\cdot(\gamma_{u'}u)\right] \\ x_{\parallel} = \gamma\left[x_{\parallel}' + \beta x_0'\right] & \longleftrightarrow \quad \gamma_u u_{\parallel} = \gamma_v\left[\gamma_{u'}u_{\parallel}' + \beta(c\gamma_{u'})\right] \\ x_{\perp} = x_{\perp}' & \longleftrightarrow \quad \gamma_u u_{\perp} = \gamma_{u'}u_{\perp}' \end{cases}$$

Entonces $(\gamma_u c, \gamma_u\vec{u})$ transforman de la misma forma que $(x_0, \vec{x}) \rightarrow$ ✓

La velocidad ordinaria $\frac{d\vec{x}}{dt} = \vec{v}$ $\quad \vec{v}' = \frac{d\vec{x}'}{dt}$ $\quad dt = dt'$

Construimos lo siguiente

$$U_0 = \frac{dx_0}{d\tau} = \frac{dx_0}{dt}\left[\frac{dt}{d\tau}\right] \qquad \text{pero } d\tau = \frac{dt}{\gamma}$$

$$\boxed{U_0 = \frac{d}{dt}[ct]\,\gamma_u = c\gamma_u}$$

$$\boxed{\vec{U} = \frac{d\vec{x}}{d\tau} = \frac{d\vec{x}}{dt}\frac{dt}{d\tau} = \gamma_u\vec{u}}$$

Generalización del momento lineal y la energía relativista

Velocidades pequeñas $\vec{p} = m\vec{u}$ $\quad E_c: E = \frac{1}{2}mu^2 + E(0)$ Ⓘ

Postulamos, caso relativista

$$\vec{p} = \mu(u)\vec{u} \qquad E = \xi(u) \quad \text{Ⓘ Ⓘ}$$

Observando Ⓘ Ⓘ y Ⓘ

$$\mu(0) = m \qquad \frac{\partial}{\partial u^2}\xi(u)\bigg|_{u=0} = \frac{\partial\xi(0)}{\partial u^2} = \frac{\partial E}{\partial u^2} = \frac{m}{2}$$

Asumiremos que $\mu(u)$, $\xi(u)$ son funciones monótonas en su argumento

Consideremos una colisión elástica [conservación de momento / conservación de E_k] vista en 2 sistemas inerciales K, K'.

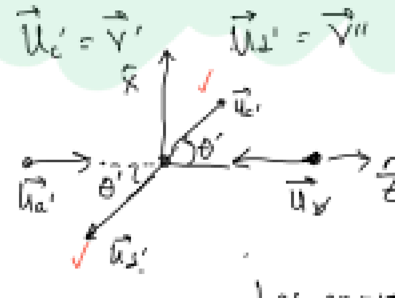
El sistema K' es el centro de masa, y la colisión es de 2 partículas con misma masa [idénticas]

a tiene una velocidad $\vec{u}_{a'} = \vec{v}$

b tiene una velocidad $\vec{u}_{b'} = -\vec{v}$

Colisionan y se dispersan con velocidades finales

$\vec{u}_{c'} = \vec{v}'$ $\quad \vec{u}_{d'} = \vec{v}''$



En K'

$\vec{p}_{a'} + \vec{p}_{b'} = \vec{p}_{c'} + \vec{p}_{d'}$ $\quad$ i)

$E_{a'} + E_{b'} = E_{c'} + E_{d'}$ $\quad$ ii)

Las expresiones relativistas

$$\mu(|\vec{u}_{a'}| = v)\vec{v} - \mu(v)\vec{v} = \mu(v')\vec{v}' + \mu(v'')\vec{v}'' \quad \text{iii)}$$

$$\xi(v) + \xi(v) = \xi(v') + \xi(v'') \quad \text{iv)}$$

Como las partículas son idénticas, $\xi(v') = \xi(v'')$

$v' = v''$ ✓

$\Rightarrow$ iv) $\xi(v) + \xi(v) = \xi(v') + \xi(v')$

$\Rightarrow \boxed{v = v' = v''}$

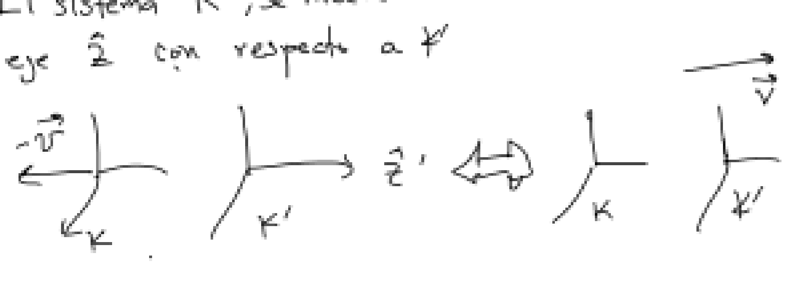
Regresamos a iii)

$$\mu(v)\vec{v} - \mu(v)\vec{v} = \mu(v)\vec{v}' + \mu(v)\vec{v}''$$

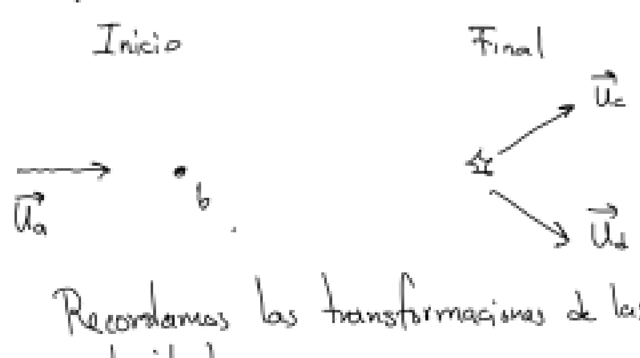
$$\Rightarrow 0 = \vec{v}' + \vec{v}'' \rightarrow \boxed{\vec{v}' = -\vec{v}''}$$

Recordamos $\vec{u}_{a'} = \vec{v}$ $\quad \vec{u}_{b'} = -\vec{v}$

El sistema K, se mueve a velocidad $-\vec{v}$ en eje $\hat{z}$ con respecto a K'



Lo que observa el sistema K



Recordamos las transformaciones de las velocidades

$$u_{\parallel} = \frac{u_{\parallel}' + V_{REL}}{1 + \frac{\vec{V}_r\cdot\vec{u}'}{c^2}} \qquad u_{\perp} = \frac{u_{\perp}'}{\gamma_v\left[1 + \frac{\vec{V}_r\cdot\vec{u}'}{c^2}\right]}$$

K $\quad K'$

V_{REL} = velocidad con la cuál K' se mueve con respecto a K

$V_{REL} = v$

$u_{\perp}' = 0$ al inicio

INICIO en K

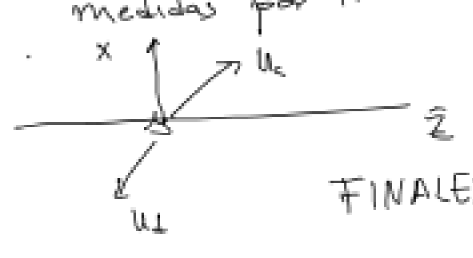
$$u_{\parallel}^{(b)} = \frac{u_{\parallel}'^{(b)} + V_{rel}}{1 + \frac{\vec{v}\cdot\vec{u}'}{c^2}} \qquad \text{pero } u_{\parallel}'^{(b)} = -v \quad V_{rel} = +v$$

$$\boxed{u_{\parallel}^{(b)} = 0}$$

$$u_{\parallel}^{(a)} = \frac{u_{\parallel}'^{(a)} + V_{REL}}{1 + \frac{V_{REL}u'^{(a)}}{c^2}} = \frac{v + v}{1 + \frac{v^2}{c^2}} = \frac{2v}{1 + \frac{v^2}{c^2}} \qquad \beta = \frac{v}{c}$$

$$\boxed{u_{\parallel}^{(a)} = \frac{2c\beta}{1 + \beta^2}}$$

Después de la colisión, las velocidades medidas por K



Componentes x, z

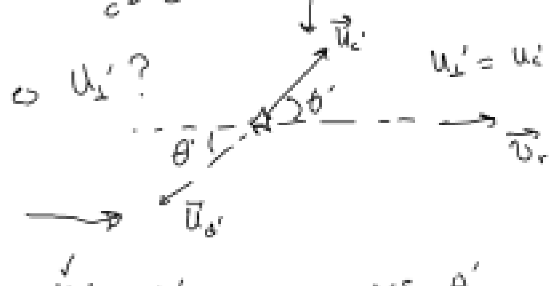
FINALES

$$(u_c)_x = u_{c\perp} \qquad (u_c)_{\parallel} = (u_c)_z$$

$$(u_c)_x = \frac{u_{\perp}'}{\gamma_v\left[1 + \frac{\vec{v}_r\cdot\vec{u}'}{c^2}\right]} \qquad (u_c)_z = \frac{u_{\parallel}' + v_r}{1 + \frac{\vec{v}_r\cdot\vec{u}'}{c^2}}$$

¿Que es $u_{\perp}'$?

$u_{\perp}' = u_c'\,\mathrm{Sen}\,\theta'$



$$(u_c)_x = \frac{u_c'\,\mathrm{Sen}\,\theta'}{\gamma_v\left[1 + v\frac{u_c'}{c^2}\cos\theta'\right]} = \frac{v\,\mathrm{Sen}\,\theta'}{\gamma_v\left[1 + \frac{v^2}{c^2}\cos\theta'\right]}$$

$$\boxed{(u_c)_x = \frac{c\beta\,\mathrm{Sen}\,\theta'}{\gamma_v\left[1 + \beta^2\cos\theta'\right]}}$$

$$(u_c)_z = \frac{u_{cz}' + V_{rel}}{1 + \frac{\vec{V}_{rel}\cdot\vec{u}_c'}{c^2}} = \frac{v\cos\theta' + v}{1 + \frac{v^2}{c^2}\cos\theta'}$$

$$\boxed{(u_c)_z = \frac{c\beta(1 + \cos\theta')}{1 + \beta^2\cos\theta'}}$$

Ejercicio propuesto: Mostrar, de manera similar

$$\boxed{\begin{aligned} (u_d)_x &= (u_d)_{\perp} = \frac{-c\beta\,\mathrm{Sen}\,\theta'}{\gamma(1 - \beta^2\cos\theta')} \\ (u_d)_z &= (u_d)_{\parallel} = \frac{c\beta(1 - \cos\theta')}{1 - \beta^2\cos\theta'} \end{aligned}}$$

$$\rightarrow \boxed{\mu(u) = ?} \; ✓ \rightarrow \vec{p} = \frac{m\vec{u}}{\sqrt{1 - u^2/c^2}}$$

$$\mu(u) = \frac{m}{\sqrt{1 - u^2/c^2}}$$