La ecuación de Klein-Gordon

La ec. de Schrödinger

$$i\hbar \frac{\partial}{\partial t}\psi = \left[-\frac{\hbar^2}{2m}\nabla^2 + V(x)\right]\psi$$

Mecánica Clásica

$$\checkmark\ E = \frac{p^2}{2m} + V$$

$$\boxed{\hat{E} = i\hbar\frac{\partial}{\partial t}}\ \checkmark \qquad \boxed{\hat{p} = -i\hbar\nabla}\ \checkmark$$

$$\hat{E}\psi = \left[\frac{\hat{p}^2}{2m} + \hat{V}\right]\psi$$

$$i\hbar\frac{\partial}{\partial t}\psi = \left[\frac{(-i\hbar)^2\nabla^2}{2m} + V\right]\psi$$

Ahora, para un sistema relativista

$$P_\mu P^\mu = \frac{E^2}{c^2} - \vec{p}\cdot\vec{p} = m_o c^2$$

Se remplaza $p^\mu = i\hbar \frac{\partial}{\partial x_\mu}$

$$p^\mu = i\hbar\left[\partial_0, -\vec{\nabla}\right] \checkmark$$

$$\hat{P}_\mu \hat{p}^\mu \psi = [m_o c^2]\psi$$

$$\left[i\hbar[\partial_0, -\vec{\nabla}]\right] i\hbar[\partial_0, \vec{\nabla}]\psi = m_o c^2 \psi$$

$$[-\hbar^2 \partial_0 \partial^0 + \hbar^2 \nabla^2]\psi = m_o c^2 \psi$$

$$[\partial_0 \partial^0 - \nabla^2]\psi = -\frac{m_o c^2}{\hbar^2}\psi$$

$$\Box = \partial_0 \partial^0 - \nabla^2 = \frac{1}{c^2}\frac{\partial^2}{\partial t^2} - \nabla^2$$

$$\hat{\Box}\Psi + \frac{m_0 c^2}{\hbar^2}\bar{\Psi} = 0$$

$$\checkmark \quad \boxed{\hat{P}_\mu \hat{p}^\mu \psi = m_0 c^2 \psi} \quad (1)$$

Ec. de Klein-Gordon

Ψ = función escalar de Lorentz.

$$\left[\frac{1}{c^2}\frac{\partial^2}{\partial t^2} - \nabla^2\right]\psi = -\frac{m_0 c^2}{\hbar^2}\psi$$

Se propone solución

$$\Psi_s = \exp\left[-\frac{i}{\hbar} P_\mu x^\mu\right]$$

$$= \exp\left[-\frac{i}{\hbar}\left[P_0 x^0 - \vec{P}\cdot\vec{X}\right]\right]$$

$$\Psi_s = \exp\left[\frac{i}{\hbar}(\vec{p}\cdot\vec{x} - Et)\right] \quad (2)$$

Usamos ec. (2) en (1)

$$P_\mu P^\mu \psi = \left(i\hbar \frac{\partial}{\partial x^\mu}\right)\left(i\hbar\frac{\partial}{\partial x_\mu}\right)\psi_s$$

$$= i\hbar \frac{\partial}{\partial x^\mu}\left[i\hbar \frac{\partial}{\partial x_\mu} \exp\left[-\frac{i}{\hbar} P_\alpha x^\alpha\right]\right]$$

$$= i\hbar \frac{\partial}{\partial x^\mu}\left[i\hbar \exp\left[-\frac{i}{\hbar} P_\alpha x^\alpha\right] \cdot \frac{\partial}{\partial x_\mu}\left[-\frac{i}{\hbar} P_\alpha x^\alpha\right]\right]$$

$$= i\hbar \frac{\partial}{\partial x^\mu}\left[i\hbar \exp\left[-\frac{i}{\hbar} P_\alpha x^\alpha\right] \left(-\frac{i}{\hbar}\right) \frac{\partial}{\partial x_\mu}\left[P^\alpha x_\alpha\right]\right]$$

$$= i\hbar \frac{\partial}{\partial x^\mu}\left[i\hbar \exp\left[-\frac{i}{\hbar} P_\alpha x^\alpha\right] \left(-\frac{i}{\hbar}\right) P^\alpha \left(\frac{\partial x_\alpha}{\partial x_\mu}\right)\right]$$

$$= i\hbar \frac{\partial}{\partial x^\mu}\left[i\hbar \exp\left[-\frac{i}{\hbar} P_\alpha x^\alpha\right] \left(-\frac{i}{\hbar}\right) P^\alpha \delta_\alpha^{\ \mu}\right]$$

$$= i\hbar \frac{\partial}{\partial x^\mu}\left[i\hbar \exp\left[-\frac{i}{\hbar} P_\alpha x^\alpha\right] \left(-\frac{i}{\hbar}\right) P^\mu\right]$$

$$\Rightarrow \hat{P}_\mu \hat{P}^\mu \Psi_s = P_\mu P^\mu \Psi_s \overset{\text{derecho}}{=\!=} m_o c^2 \Psi_s$$

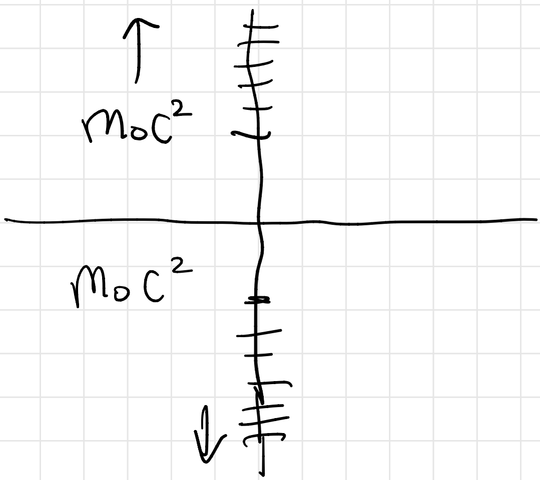
Entonces $P_\mu P^\mu = m_o c^2$

$$\boxed{\frac{E^2}{c^2} - \vec{p}\cdot\vec{p} = m_o^2 c^2}$$

$$E = \pm\sqrt{m_o^2 c^4 + \vec{p}^{\,2} c^2} = \pm c\sqrt{m_o^2 c^2 + p^2}$$

$$E_+ = c\sqrt{m_o^2 c^2 + \vec{p}\cdot\vec{p}}$$

$$E_- = -c\sqrt{m_o^2 c^2 + \vec{p}\cdot\vec{p}}$$



Partimos de

$$[P_\mu P^\mu - m_0^2 c^2]\psi = 0 \quad (3) \qquad P_\mu^* = ?$$

$$[P_\mu P^\mu - m_0^2 c^2]\psi^* = 0 \quad (4)$$

$$\psi^* (3) \Rightarrow \psi^* [P_\mu P^\mu - m_0 c^2]\psi = 0 \quad (5)$$

$$\psi (4) \Rightarrow \psi [P_\mu P^\mu - m_0 c^2]\psi^* = 0 \quad (6)$$

(5) - (6) y usando $p^\mu = i\hbar[\partial_0, -\vec{\nabla}] = i\hbar\nabla^\mu$

$$P_\mu = i\hbar(\partial_0, \vec{\nabla}] = i\hbar\nabla_\mu$$

$$(5) = \psi^*[i\hbar\vec{\nabla}_\mu \, i\hbar\nabla^\mu - m_0 c^2]\psi =$$

$$= -\psi^*[\hbar^2 \nabla_\mu \nabla^\mu + m_0^2 c^2]\psi = 0$$

$$(6) \quad 0 = \psi[-\hbar^2 \nabla_\mu \nabla^\mu - m_0^2 c^2]\psi^*$$

$$⑤ - ⑥ = -\psi^*[\hbar^2 \nabla_\mu \nabla^\mu + m_0^2 c^2]\psi$$

$$+ \psi[\hbar^2 \nabla_\mu \nabla^\mu + m_0^2 c^2]\psi^* = 0$$

$$\underline{-\psi^* \nabla_\mu \nabla^\mu \psi \hbar^2 + \psi \hbar^2 \nabla_\mu \nabla^\mu \psi^*}$$

$$+ \cancel{\underline{m_0^2 c^2 \psi\psi^*}} - \cancel{\underline{m_0^2 c^2 \psi^*\psi}} = 0$$

$$\psi\psi^* = |\psi|^2 \qquad \psi^*\psi = |\psi|^2$$

$$\underline{-\psi^* \nabla_\mu \nabla^\mu \psi} + \psi \nabla_\mu \nabla^\mu \psi^* = 0$$

$$= \underline{\nabla_\mu[(\psi^*) \nabla^\mu \psi]} + \cancel{(\nabla_\mu \psi^*)(\nabla^\mu \psi)}$$

$$+ \nabla_\mu[\psi \nabla^\mu \psi^*] - \cancel{(\nabla_\mu \psi)(\nabla^\mu \psi^*)} = 0$$

$$\nabla_\mu [\underbrace{\psi \nabla^\mu \psi^* - \psi^* \nabla^\mu \psi}] = 0$$

$$\nabla_\mu J^\mu = 0$$

Se define $J_\mu = \left(\frac{i\hbar}{2m_0}\right)\left[\psi^* \nabla_\mu \psi - \psi \nabla_\mu \psi^*\right]$

Se justifica, para que J_0 tenga las dimensiones de una densidad de probabilidad.

$$\nabla_\mu J^\mu = 0$$

$$\frac{1}{c}\frac{\partial}{\partial t}\left[\frac{i\hbar}{2m_0}\left[\psi^* \frac{1}{c}\frac{\partial}{\partial t}\psi - \psi \frac{1}{c}\frac{\partial}{\partial t}\psi^*\right]\right]$$

$$- \nabla \cdot \left[\frac{i\hbar}{2m_0}\left(\psi^* \nabla \psi - \psi (\nabla \psi^*)\right)\right] = 0$$

Recordemos la ec. de continuidad

$$\boxed{\frac{\partial}{\partial t}\rho + \nabla \cdot \vec{J} = 0}$$

En mecánica cuántica, no relativista,

$$J = \frac{i\hbar}{2m}\left[\psi^* \nabla \psi - \psi \nabla \psi^*\right]$$

$$\rho = \psi\psi^* = |\psi|^2 > 0 \quad ✓$$

Entonces, es natural identificar

$$\rho_r = \frac{i\hbar}{2m_0c^2}\left[\underbrace{\psi^* \frac{\partial}{\partial t}\psi - \psi \frac{\partial}{\partial t}\psi^*}\right] = \text{densidad de probabilidad.}$$

$\therefore$ Límite no relativista de la ec. de K.G.

$$\Psi = \exp\left[\frac{i}{\hbar}\left[\vec{p}\cdot\vec{x} - Et\right]\right]$$

Asumimos que $\boxed{\Psi = \varphi(\vec{r}, t)\exp\left[\frac{-i}{\hbar} m_0c^2 t\right]}$

En el límite no relativista, la diferencia entre la energía total y su masa en reposo es pequeña

$$E' = E - m_0 c^2 = T \ll m_0 c^2$$

$$\Rightarrow \left\| \left[i\hbar \frac{\partial}{\partial t} \varphi(\vec{r}, t) \right] \right\| = E' \varphi \ll (m_0 c^2) \checkmark$$

Aplicando $\frac{\partial}{\partial t} \Psi = \frac{\partial}{\partial t} \left[\varphi(\vec{r}, t) \exp\left[-\frac{i}{\hbar} m_0 c^2 t \right] \right]$

$$= \left[\frac{\partial}{\partial t} \varphi(\vec{r}, t) \exp\left[-\frac{i}{\hbar} m_0 c^2 t \right] + \varphi(\vec{r}, t) \frac{\partial}{\partial t} \exp[\ ] \right]$$

$$= \left[\frac{\partial}{\partial t} \varphi - \frac{i m_0 c^2}{\hbar} \varphi \right] \exp\left[-\frac{i}{\hbar} m_0 c^2 t \right]$$

$$= -\frac{i m_0 c^2}{\hbar} \left[\varphi + \frac{i\hbar}{m_0 c^2} \frac{\partial}{\partial t} \varphi \right] \exp\left[-\frac{i}{\hbar} m_0 c^2 t \right]$$

$\rightarrow 0$

$$\frac{\partial}{\partial t}\Psi \cong -i\frac{m_o c^2}{\hbar}\varphi(\vec{r},t)\exp\left[-\frac{i m_o c^2}{\hbar}t\right]$$

Mostrar que

$$\frac{\partial^2}{\partial t^2}\Psi = -\left[i\frac{2m_o c^2}{\hbar}\frac{\partial}{\partial t}\varphi + \frac{m_o^2 c^4}{\hbar^2}\varphi\right]\times$$

$$\times\exp\left[-\frac{i}{\hbar}m_o c^2 t\right]$$

Usando este resultado en ec. K.6

$$\boxed{i\hbar\frac{\partial}{\partial t}\varphi(\vec{r},t) = -\frac{\hbar^2}{2m_o}\left[\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}+\frac{\partial^2}{\partial z^2}\right]\varphi(\vec{r},t)}$$

Ec. de Schrödinger para $\underline{\varphi(\vec{r},t)}$

Partículas con spin cero.

¿Cómo interpretamos ahora ρ y $\vec{J}$?

$$\rho = \frac{i\hbar}{2m_0c^2}\left[\psi^*\frac{\partial}{\partial t}\psi - \psi\frac{\partial}{\partial t}\psi^*\right]$$

$$\vec{J} = \frac{-i\hbar}{2m_0}\left[\psi^*\nabla\psi - \psi\nabla\psi^*\right]$$

Multiplicamos por $|e| = e$

$$J'_\mu = \frac{ie\hbar}{2m_0}\left[\psi^*\nabla_\mu\psi - \psi\nabla_\mu\psi^*\right]$$

$$= [c\rho', -\vec{j}']$$

$$\Rightarrow \rho' = \frac{i\hbar e}{2m_0c^2}\left[\psi^*\frac{\partial}{\partial t}\psi - \psi\frac{\partial}{\partial t}\psi^*\right] \rightarrow \text{densidad de carga}$$

$$\vec{J}' = \frac{ie\hbar}{2m_0}\left[\psi^*\nabla\psi - \psi\nabla\psi^*\right] \rightarrow \text{densidad de corriente}$$

La ec. de K.G para partícula libre

Propuesta $\Psi_s = A \exp\left[\frac{i}{\hbar}(\vec{p}\cdot\vec{x} - E_p t)\right]$

$$E_p = \pm c\sqrt{p^2 + m^2c^2}$$

$$\Psi_+ = A_+ \exp\left[\frac{i}{\hbar}(\vec{p}\cdot\vec{x} - |E_p| t)\right] \checkmark$$

$$\Psi_- = A_- \exp\left[\frac{i}{\hbar}(\vec{p}\cdot\vec{x} + |E_p| t)\right]$$

Usando Ψ_+, Ψ_-, podemos mostrar

$$\left.\begin{aligned} \rho'_+ &= \frac{+e\,|E_p|}{m_0c^2}\Psi_+^*\Psi_+ \\ \rho'_- &= \frac{-e\,|E_p|}{m_0c^2}\Psi_-^*\Psi_- \end{aligned}\right\} \text{Problema propuesto}$$

Esta nueva interpretación sugiere

$\Psi_+ \rightarrow$ partículas de masa m_0
y carga $+e$ y energía E_p

$\Psi_- \rightarrow$ partículas de masa m_0
y carga $-e$ y energía E_p

La solución general a Ec. K.G, es una combinación de Ψ_+, Ψ_-