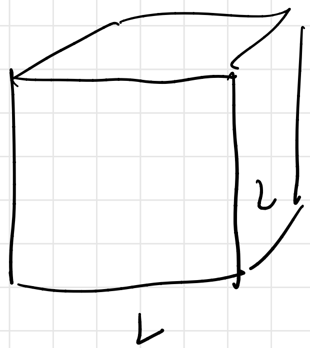
Normalización

Confinamos las soluciones en una caja cúbica



Se demandan condiciones periódicas

$\psi_\pm$ se anulan $x=0$ $x=L$

$y=0$ $y=L$

$z=0$ $z=L$

Entonces $\psi_{m\pm} = A_m^\pm \exp\left[\frac{i}{\hbar}\left(\vec{P}_m\cdot\vec{x} \mp E_{P_m} t\right)\right]$

donde $\underline{\underline{\vec{P}_m}} = \frac{2\pi}{L}\vec{m}$ $\vec{m}=\{m_x, m_y, m_z\}$

$m_x, m_y, m_z \in \mathbb{N}$

y $E_{P_m} = c\sqrt{P_m^2 + m^2c^2} = E_m$

Para determinar los coeficientes, se demanda

$$\int_{L^3} \rho'(\pm)\, dx^3 = \pm e = \pm \frac{e\, E_{p_m}}{m_0 c^2} \frac{|A_m(\pm)|^2}{} L^3$$

$$\Rightarrow A_m(\pm) = \sqrt{\frac{m_0 c^2}{L^3 E_{p_m}}}$$

si se asume que las amplitudes sean reales

Así entonces

$$\boxed{\Psi_m(\pm) = \left[\frac{m_0 c^2}{L^3 E_{p_m}}\right]^{1/2} \exp\left[\frac{i}{\hbar}\left(\vec{p}_m \cdot \vec{x} \mp E_{p_m} t\right)\right]}$$

Ψ_m^+, Ψ_m^- se distinguen por $\mp E_{p_m} t$

La solución general

$$\checkmark \quad \Psi_{(+)} = \sum a_m \Psi_{m(+)} = \sum a_m \sqrt{\frac{m_0 c^2}{L^3 E_{pm}}} \exp\left[\frac{i}{\hbar}\left[\vec{p}_m \cdot \vec{x} - E_{pm} t\right]\right]$$

$$\checkmark \quad \Psi_{(-)} = \sum b_m \Psi_{m(-)} = \sum b_m \sqrt{\frac{m_0 c^2}{L^3 E_{pm}}} \exp\left[\frac{i}{\hbar}\left[\vec{p}_m \cdot \vec{x} + E_{pm} \cdot t\right]\right]$$

¿Y partículas sin spin y sin carga?

La construcción del campo $\Psi_{(0)}$ debe cumplir que

$$\int \rho_{(0)} \, dx^3 = 0 \quad \checkmark$$

En otras palabras $\Psi_m(0)^* = \Psi_m(0)$ ✓

$$\Psi_m(0) = \frac{1}{\sqrt{2}}\left[\Psi_+^m(p_m) + \Psi_-^m(-p_m)\right]$$

$$= \left[\frac{m_0 c^2}{2L^3 E_{pm}}\right] 2 \cos\left[\frac{\vec{p}_m \cdot \vec{x} - E_{pm} t}{\hbar}\right]$$

En el caso no relativista, partículas sin spin se propagan con un momento $\vec{P}$

En el caso relativista, partículas sin spin, pero existen 3 soluciones que corresponden a partículas con carga [+, -, 0], por cada momento $\vec{p}$

El campo complejo de Klein-Gordon

$$j^{\mu} = \frac{ie\hbar}{2m_0}\left[\varphi^{*} \nabla^{\mu}\varphi - \varphi \nabla^{\mu}\phi^{*}\right]$$

$$\partial_{\mu} J^{\mu} = 0 \qquad \dot{\psi} = \frac{\partial \psi}{\partial (ct)}$$

La carga $Q = \frac{ie\hbar}{2m_0c^2}\int d^3x \left[\psi^{*}\dot{\psi} - \varphi\dot{\varphi}^{*}\right]$

Asumiremos que es posible realizar la descomposición

$$\varphi = \frac{1}{\sqrt{2}}\left[\underline{\varphi_1(x) + i\,\varphi_2(x)}\right]$$

donde $\varphi_1(x), \varphi_2(x) \in \mathbb{R}$

Si φ satisface la ec. de K.G., entonces φ_1, φ_2 satisfacen la ec. de K.G.

$$\left[\Box + \frac{m_0^2 c^2}{\hbar^2}\right]\varphi = 0 \rightarrow \begin{array}{l} \left[\Box + \frac{m_0 c^2}{\hbar^2}\right]\underline{\varphi_1(x) = 0} \\ \left[\Box + \frac{m_0 c^2}{\hbar^2}\right]\underline{\varphi_2(x) = 0} \end{array}$$

De manera inversa, si $\varphi_1(x)$ y $\varphi_2(x)$ satisfacen la ec. de K.G, con la misma masa, entonces las 2 ecuaciones pueden ser remplazadas, si definimos

$$\varphi = \frac{1}{\sqrt{2}}\left[\varphi_1 + i\varphi_2\right] \qquad \varphi^* = \frac{1}{\sqrt{2}}\left[\varphi_1 - i\varphi_2\right]$$

$$\hookrightarrow \left[\Box + \frac{m_0^2c^2}{\hbar^2}\right]\varphi = 0 \quad y \quad \left[\Box + \frac{m_0^2c^2}{\hbar^2}\right]\varphi^* = 0$$

Para $\varphi \longrightarrow Q_1 = \frac{i\hbar e}{2m_0c^2}\int dx^3\left[\varphi^*\dot{\varphi} - \varphi\dot{\varphi}^*\right]$

Para $\varphi^* \longrightarrow Q_2 =$ se obtiene cambiando $\varphi \leftrightarrow \varphi^*$

$$Q_2 = \frac{ie\hbar}{2m_0c^2}\int dx^3\left[\varphi\dot{\varphi}^* - \varphi^*\dot{\varphi}\right]$$

$$= -\frac{ie\hbar}{2m_0c^2}\int dx^3\left[\varphi^*\dot{\varphi} - \varphi\dot{\varphi}^*\right]$$

$$Q_2 = -Q_1$$

Así entonces φ y φ^* caracterizan partículas con carga opuesta

Los piones π^+, π^-, π^0

Se puede definir $\varphi_{\pi^+} = \varphi^* = \frac{1}{\sqrt{2}}[\varphi_1 - i\varphi_2]$

$$\varphi_{\pi^-} = \varphi = \frac{1}{\sqrt{2}}[\varphi_1 + i\varphi_2]$$

$$\varphi_{\pi^0} = \varphi_0^* = \varphi_0$$

La teoría de campo está basado en un concepto similar al clásico.

$$S = \int \mathcal{L}\, d^4x \qquad \int \mathcal{L}\, dx^3 = L$$

$$\boxed{\mathcal{L} = \mathcal{L}\left(\Psi_\sigma, \frac{\partial \Psi_\sigma}{\partial x_\mu}\right)}$$

$$\delta S = 0 \longrightarrow \boxed{\frac{\partial}{\partial x^\mu}\left[\frac{\partial \mathcal{L}}{\partial(\partial \Psi_\sigma / \partial x^\mu)}\right] - \frac{\partial \mathcal{L}}{\partial \Psi_\sigma} = 0}$$

$$\delta \int_V \mathcal{L}\left(\Psi_\sigma, \frac{\partial \Psi_\sigma}{\partial x_\mu}\right) d^4x = 0$$

$$= \int \left[\frac{\partial \mathcal{L}}{\partial \Psi_\sigma} \delta \Psi_\sigma + \frac{\partial \mathcal{L}}{\partial\left(\frac{\partial \Psi_\sigma}{\partial x_\mu}\right)} \delta \left[\frac{\partial \Psi_\sigma}{\partial x_\mu}\right]\right] d^4x$$

Pero

$$\delta\left[\frac{\partial \Psi_\sigma}{\partial x_\mu}\right] = \frac{\partial}{\partial x_\mu}\left[\Psi_\sigma + \delta \Psi_\sigma\right] - \frac{\partial \Psi_\sigma}{\partial x_\mu}$$

$$= \cancel{\frac{\partial \Psi_\sigma}{\partial x_\mu}} + \frac{\partial}{\partial x_\mu}\left[\delta \Psi_\sigma\right] - \cancel{\frac{\partial \Psi_\sigma}{\partial x_\mu}}$$

$$\boxed{\delta\left[\frac{\partial \Psi_\sigma}{\partial x_\mu}\right] = \frac{\partial}{\partial x_\mu}\left[\delta \Psi_\sigma\right]}$$

$$= \int \left[\frac{\partial \mathcal{L}}{\partial \psi_\sigma} \delta \psi_\sigma + \left(\frac{\partial \mathcal{L}}{\partial \left(\frac{\partial \psi_\sigma}{\partial x_\mu} \right)} \cdot \frac{\partial}{\partial x_\mu} (\delta \psi_\sigma) \right) \right] dx^4$$

el segundo término

$$\int d(fg) = fg|$$

frontera

$$\int \left(\frac{\partial \mathcal{L}}{\partial \left(\frac{\partial \psi_\sigma}{\partial x_\mu} \right)} \right) \left(\frac{\partial}{\partial x_\mu} \right) [\delta \psi_\sigma] \, dx^4 =$$

$$= \int \left(\frac{\partial}{\partial x_\mu} \left[\frac{\partial \mathcal{L}}{\partial \left(\frac{\partial \psi_\sigma}{\partial x_\mu} \right)} \cdot \delta \psi_\sigma \right] - \delta \psi_\sigma \frac{\partial}{\partial x_\mu} \left[\frac{\partial \mathcal{L}}{\partial \left(\frac{\partial \psi_\sigma}{\partial x_\mu} \right)} \right] \right) \times dx^4$$

$$= \int \frac{\partial}{\partial x_\mu} \left[\frac{\partial \mathcal{L}}{\partial \left(\frac{\partial \psi_\sigma}{\partial x_\mu} \right)} \cdot \delta \psi_\sigma \right] dx^4 - \int \left(\frac{\partial}{\partial x_\mu} \left[\frac{\partial \mathcal{L}}{\partial \left(\frac{\partial \psi_\sigma}{\partial x_\mu} \right)} \right] \right) \cdot \delta \psi_\sigma \, dx^4$$

√

$$= \underbrace{\frac{\partial \mathcal{L}}{\partial(\partial \psi_\sigma/\partial x_\mu)}\, \delta \psi_\sigma \Big|}_{\text{Frontera}} - \int \frac{\partial}{\partial x_\mu}\left[\frac{\partial \mathcal{L}}{\partial\left(\frac{\partial \psi_\sigma}{\partial x_\mu}\right)}\right] \delta \psi_\sigma \, d x^4$$

$$= -\int \frac{\partial}{\partial x_\mu}\left[\frac{\partial \mathcal{L}}{\partial\left(\frac{\partial \psi_\sigma}{\partial x_\mu}\right)}\right] \delta \psi_\sigma \, d x^4$$

Entonces

$$\delta \int \mathcal{L}\, d x^4 = 0 = \int \delta \psi_\sigma \left[\frac{\partial \mathcal{L}}{\partial \psi_\sigma} - \frac{\partial}{\partial x_\mu}\left(\frac{\partial \mathcal{L}}{\partial\left(\frac{\partial \psi_\sigma}{\partial x_\mu}\right)}\right)\right] d x^4$$

$$\Rightarrow \boxed{\frac{\partial \mathcal{L}}{\partial \psi_\sigma} - \frac{\partial}{\partial x_\mu}\left[\frac{\partial \mathcal{L}}{\partial(\partial \psi_\sigma/\partial x_\mu)}\right] = 0}$$

La densidad lagrangiana para el campo de Klein-Gordon

$$\mathcal{L}\left(\psi, \psi^*, \frac{\partial \psi}{\partial x^\mu}, \frac{\partial \psi^*}{\partial x^\nu}\right) =$$

$$= \left(\frac{\hbar^2}{2m_0}\right)\left[g^{\mu\nu}\frac{\partial \psi^*}{\partial x^\mu}\frac{\partial \psi}{\partial x^\nu} - \frac{m_0^2 c^2}{\hbar^2}\psi^*\psi\right]$$

$$g^{\mu\nu}\frac{\partial \psi^*}{\partial x^\mu}\frac{\partial \psi}{\partial x^\nu} = g^{\mu\nu}\partial_\mu \psi^* \partial_\nu \psi$$

$$= g_{\mu\nu}\partial^\mu \psi^* \partial^\nu \psi$$

$$= \partial_\nu \psi^* \partial^\nu \psi$$

Para campo ψ ~~ψ^*~~

$$\frac{\partial \mathcal{L}}{\partial \psi} - \frac{\partial}{\partial x_\mu}\left[\frac{\partial \mathcal{L}}{\partial(\partial\psi/\partial x_\mu)}\right] =$$

$$= \frac{\hbar^2}{2m_0}\left[(-1)\frac{m_0^2c^2}{\hbar^2}\psi^*\right]$$

?

$$-\frac{\partial}{\partial x_\beta}\frac{\hbar^2}{2m_0}\left[\frac{\partial}{\partial(\partial\psi/\partial x_\beta)}\left[g_{\mu\nu}\left(\frac{\partial\psi^*}{\partial x_\mu}\right)\left(\frac{\partial\psi}{\partial x_\nu}\right)\right]\right]$$

$$= \frac{-c^2}{2m_0}\psi^* - \frac{\hbar^2}{2m_0}\frac{\partial}{\partial x_\beta}\left[g_{\mu\nu}\frac{\partial\psi^*}{\partial x_\mu}\delta^\nu_\beta\right] = 0$$

$$= -\frac{c^2}{2m_0}\psi^* - \frac{\hbar^2}{2m_0}\frac{\partial}{\partial x_\beta}\left[g_{\mu\beta}\frac{\partial\psi^*}{\partial x_\mu}\right] = 0$$

$$-\frac{c^2}{2m_0}\psi^* - \frac{\hbar^2}{2m_0}\underbrace{\frac{\partial}{\partial x_\beta}}\left[g_{\mu\beta}\frac{\partial\psi^*}{\partial x_\mu}\right]=0$$

$$-\frac{c^2}{2m_0} - \frac{\hbar^2}{2m_0}\underbrace{\partial^\beta\left[g_{\mu\beta}\partial^\mu\right.}\left.\psi^*\right]=0$$

$$\frac{\hbar^2}{2m_0}\underbrace{\left[\frac{c^2 m_0^2}{\hbar^2}\psi^* + \partial_\mu\partial^\mu\psi^*\right]}=0$$

Ejercicio propuesto. Usar la ec. de E.L.

para ψ^* y obtener la ec. de K.G

para ψ.

$$\psi = \psi(x_\mu) \leftarrow$$ Escalar de Lorentz

$$\psi'(x')=\psi(x)$$