Densidad lagrangiana de K.G y la invariancia de Gauge.

$$\mathcal{L} = \frac{\hbar^2}{2m_0} \partial_\mu \phi (\partial^\mu \phi)^* - \frac{m_0 c^2}{2} \phi \phi^*$$

Sea la transformación $\phi' = e^{i\alpha} \phi$ (1)

con $\underline{\alpha}$ = constante.

Veamos la invariancia de la densidad lagrangiana

$$\mathcal{L}' = \frac{\hbar^2}{2m_0} \partial_\mu \phi' (\partial^\mu \phi')^* - \frac{m_0 c^2}{2} \phi' \phi'^*$$

usando (1) $\rightarrow$ $\partial_\mu \phi' = \partial_\mu [e^{i\alpha} \phi]$

$$= e^{i\alpha} \partial_\mu \phi$$

$$\mathcal{L}' = \frac{\hbar^2}{2m_0} \partial_\mu (e^{i\alpha}\phi) [\partial^\mu (e^{i\alpha}\phi)]^*$$

$$- \frac{m_0 c^2}{2} [e^{i\alpha}\phi][e^{i\alpha}\phi]^*$$

$$= \frac{\hbar^2}{2m_0} \underline{e^{i\alpha} e^{-i\alpha}} \partial_\mu \phi (\partial^\mu \phi)^*$$

$$- \frac{m_0 c^2}{2} \underline{e^{i\alpha} e^{-i\alpha}} \phi \phi^*$$

$$= \frac{\hbar^2}{2m_0} \partial_\mu \phi (\partial^\mu \phi)^* - \frac{m_0 c^2}{2} \phi \phi^* = \mathcal{L}$$

$$\mathcal{L}' = \mathcal{L}$$

$\mathcal{L}$ es invariante ante la transformación global (1).

Sea ahora $\alpha = \alpha(x_\mu)$

$$\phi' = e^{\frac{ie}{\hbar c}\alpha(x)}\phi \quad (2)$$

¿Será $\mathcal{L}$ invariante ante las transformación local dada en ec. (2)?

el término

$$-\frac{m_0 c^2}{2}\phi'\phi'^* = -\frac{m_0 c^2}{2}\left(e^{\frac{ie}{\hbar c}\alpha(x)}\phi\right) \times \left[e^{-\frac{ie}{\hbar c}\alpha(x)}\phi\right]^*$$

$$= -\frac{m_0 c^2}{2} e^{ie\alpha/\hbar c} e^{\frac{-ie\alpha}{\hbar c}}\phi\phi^*$$

$$= -\frac{m_0 c^2}{2}\phi\phi^*$$

Observemos

$$\partial_\mu \phi' = \partial_\mu \left[e^{\frac{ie}{\hbar c}\alpha(x)} \phi \right]$$

$$= \left[\frac{ie}{\hbar c} \partial_\mu \alpha(x) \right] e^{\frac{ie}{\hbar c}\alpha(x)} \phi$$

$$+ e^{\frac{ie}{\hbar c}\alpha(x)} \partial_\mu \phi$$

$$\partial_\mu \phi' = e^{\frac{ie}{\hbar c}\alpha(x)} \left[\frac{ie}{\hbar c} \partial_\mu \alpha(x) \phi + \partial_\mu \phi \right]$$

$$\left[\partial^\mu \phi' \right]^* = e^{-\frac{ie}{\hbar c}\alpha(x)} \left[-\frac{ie}{\hbar c} \partial^\mu \alpha \, \phi^* + \partial_\mu \phi^* \right]$$

$$\mathcal{L}' = \frac{\hbar^2}{2m_0} \left\{ e^{\frac{ie}{\hbar c}\alpha(x)} \left[\frac{ie}{\hbar c} \partial_\mu \alpha(x) \phi + \partial_\mu \phi \right] \right\}$$

$$\times \left\{ e^{-\frac{ie}{\hbar c}\alpha(x)} \left[-\frac{ie}{\hbar c} \partial^\mu \alpha \, \phi^* + \partial^\mu \phi \right] \right\}$$

$$- \frac{m_0 c^2}{2} \phi \phi^*$$

$$\mathcal{L}' = \frac{\hbar^2}{2m_0}\left[\frac{ie}{\hbar c}\partial_\mu \alpha\, \phi + \partial_\mu \phi\right]\left[\frac{-ie}{\hbar c}\partial^\mu \alpha\, \phi^* + \partial^\mu \phi^*\right] - \frac{m_0}{2} c\, \phi\phi^*$$

$$\neq \mathcal{L}$$

Supongamos que $\boxed{\partial_\mu \phi \to \left(\partial_\mu + \frac{ie}{\hbar c} A_\mu'\right)\phi}$

$$\boxed{\partial^\mu \phi^* = \left(\partial^\mu \phi^* - \frac{ie}{\hbar c} A_\mu' \phi^*\right)}$$

$$\mathcal{L}' = \frac{\hbar^2}{2m_0}\left[\frac{ie}{\hbar c}(\partial_\mu \alpha)\phi + \partial_\mu \phi + \frac{ie}{\hbar c} A_\mu' \phi\right] \times$$

$$\left[-\frac{ie}{\hbar c}(\partial^\mu \alpha)\phi^* + \partial^\mu \phi^* - \frac{ie}{\hbar c} A^{\mu\prime} \phi^*\right]$$

$+ \ldots$

$$\mathcal{L}' = \frac{\hbar^2}{2m} \left[\partial_\mu \phi + \frac{ie}{\hbar c} \left[A_\mu' + \partial_\mu \alpha(x) \right] \phi \right] \times$$

$$\times \left[\partial^\mu \phi^* - \frac{ie}{\hbar c} \left[A^{\mu\prime} + \partial^\mu \alpha(x) \right] \phi^* \right]$$

$$+ \ldots$$

Sea la transformación

$$\boxed{A_\mu = A_\mu' + \partial_\mu \alpha(x)} \quad (3)$$

$$\mathcal{L}' = \frac{\hbar^2}{2m} \left[\partial_\mu \phi + \frac{ie}{\hbar c} A_\mu \phi \right] \left[\partial^\mu \phi^* - \frac{ie}{\hbar c} A^\mu \phi^* \right]$$

$$+ \ldots$$

Si consideramos una transformación local

$$\phi' = e^{\frac{ie}{\hbar c}\alpha(x)}\phi \qquad (2)$$

entonces es necesario introducir el campo

A_μ tal que (3) $A_\mu \to A_\mu' + \partial_\mu \alpha(x)$

y la derivada covariante

$$\partial^\mu \phi \longrightarrow \partial^\mu \phi + \frac{ie}{\hbar c} A^\mu \phi$$

$$\mathcal{L} = \frac{\hbar^2}{2m_0}\left[\partial_\mu \phi + \frac{ie}{\hbar c} A_\mu \phi\right]\left[\partial^\mu \phi^* - \frac{ie}{\hbar c} A^\mu \phi^*\right]$$

$$- \frac{m_0 c}{2}\phi\phi^*$$

es invariante ante ec. (2), con ec (3)

Es necesario introducir un término cinético para A_μ

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$$

$-\frac{1}{4} F_{\mu\nu} F^{\mu\nu}$ ← invariante de Gauge

$$\boxed{A_\mu = A_\mu' + \partial_\mu \alpha}$$

$$-\frac{1}{4} F_{\mu\nu}' F'^{\mu\nu} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu}$$

$\mathcal{L}(\phi, \phi^*, A_\mu)$ es invariante ante transformaciones locales

¿Y el término de masa para A_μ?

$\propto m_\gamma^2 A_\mu A^\mu$ no es invariante de Gauge

$$m_\gamma^2 A'_\mu A'^\mu \neq m_\gamma^2 A_\mu A^\mu$$

Límite no relativista de la ecuación de K.G con electromagnetismo

Supongamos que la solución se puede separar

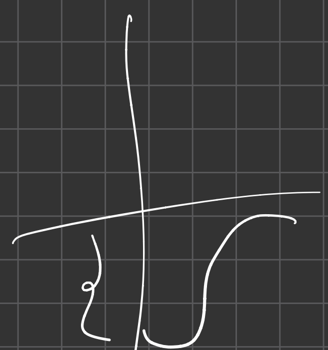
$$\Psi(x,t) = \underline{\varphi(x,t)}\, e^{\{-\frac{i}{\hbar} m_0 c^2 t\}}$$

La energía no relativista es pequeña

$$\left| i\hbar \frac{\partial}{\partial t} \varphi(x,t) \right| \ll m_0 c^2 |\varphi|$$

También se asumirá

$$|e A_0 \varphi| \ll m_0 c^2 |\varphi|$$



Analizamos la parte temporal de la ec. de K.G

$$\left(i\hbar\partial_\nu - \frac{e}{c}A_\nu\right) = \underbrace{i\hbar\partial_0 - \frac{e}{c}A_0}_{} + \ldots$$

$$\underline{i\hbar}\ \underline{\partial_0}\psi - \frac{e}{c}A_0\psi =$$

$$= i\hbar\frac{1}{c}\frac{\partial}{\partial t}\psi - \frac{e}{c}A_0\psi$$

$$= \frac{1}{c}\left[i\hbar\frac{\partial}{\partial t} - eA_0\right]\left[\underbrace{\varphi\, e^{-i\frac{m_0c^2}{\hbar}t}}_{\psi}\right]$$

$$= \frac{1}{c}\left[i\hbar\frac{\partial}{\partial t}\varphi + i\hbar\left[-i\frac{m_0c^2}{\hbar}\right]\varphi - eA_0\varphi\right]\times$$

$$\times\ e^{-i\frac{m_0c^2}{\hbar}t}$$

$[\ \iota$

Pero en la ec. de K.G.

$$\boxed{\begin{aligned}&\left[i\hbar\frac{\partial}{\partial t}-eA_0\right]^2\psi = \quad \text{Propuesto, mostrar}\\ &\cong\Big[-ie\hbar\frac{\partial}{\partial t}A_0\varphi+2i\hbar m_0c^2\frac{\partial}{\partial t}\varphi\\ &\quad -2eA_0m_0c^2\varphi+m_0^2c^4\varphi\Big]e^{-\frac{im_0c^2t}{\hbar}}\end{aligned}}$$

se han omitido $\left|i\hbar A_0\frac{\partial}{\partial t}\varphi\right| \ll A_0m_0c^2|\varphi|$

$|A_0\varphi_e| \ll m_0c^2|\varphi|$

La ec. de K.G.

$$\frac{1}{c^2}\left[i\hbar\frac{\partial}{\partial t}-eA_0\right]^2\psi=\left[\left(i\hbar\vec{\nabla}+\frac{e}{c}\vec{A}\right)^2+m_0^2c^2\right]\psi$$

Expandiendo la ecuación anterior

$$e^{-\frac{i m_0 c^2 t}{\hbar}}\left[\cancel{m_0 c^2} - 2 m_0 e A_0 + \boxed{2 m_0 i \hbar \frac{\partial}{\partial t}} - \frac{ie}{c^2}\frac{\partial}{\partial t}A_0\right]\varphi =$$

$$= e^{-\frac{i m_0 c^2 t}{\hbar}}\left[\left(i\hbar\vec{\nabla}+\frac{e}{c}\vec{A}\right)^2 + \cancel{m_0^2 c^2}\right]\varphi$$

$$\boxed{i\hbar\frac{\partial}{\partial t}\varphi = \left[\frac{1}{2m_0}\left(i\hbar\vec{\nabla}+\frac{e}{c}\vec{A}\right)^2 + \frac{i\hbar e}{2m_0 c^2}\frac{\partial}{\partial t}A_0 + eA_0\right]\varphi}$$

Propuesto: mostrar que al desarrollar $\left(i\hbar\vec{\nabla}+\frac{e}{c}\vec{A}\right)^2\varphi$, la ecuación anterior se escribe como sigue:

$$i\hbar \frac{\partial \varphi}{\partial t} = \Big[\frac{\vec{p}^{\,2}}{2m} - \frac{e}{m_0 c} \vec{A}\cdot\vec{p} + eA_0$$

$$+ \frac{i\hbar e}{2m_0} [\vec{\nabla}\cdot\vec{A}] + \frac{i\hbar e}{2m_0 c^2} \frac{\partial A_0}{\partial t} \Big] \varphi$$

$$\vec{p} = -i\hbar \vec{\nabla}$$

En la norma de Lorentz

$$\boxed{\frac{1}{c}\frac{\partial}{\partial t} A_0 + \nabla\cdot\vec{A} = 0}$$

$$\boxed{i\hbar \frac{\partial \varphi}{\partial t} = \Big[\frac{\vec{p}^{\,2}}{2m_0} - \frac{e}{m_0 c} \vec{A}\cdot\vec{p} + eA_0 \Big] \varphi}$$

$$\vec{A} = 0$$

$$A_0 \propto \frac{1}{r}$$