Tensor energía momento

Teorema de Noether: cada simetría de transformación que deja a la densidad lagrangiana invariante, corresponde a una ley de conservación y entonces, a una constante de movimiento

Invariancia traslacional

$$x'_\mu = x_\mu + \varepsilon_\mu$$ ⟵ desplazamiento infinitesimal

ε_μ = infinitesimal y constante

$$\delta x_\mu = x'_\mu - x_\mu = \varepsilon_\mu \qquad (1)$$

La variación correspondiente de la densidad lagrangiana

$$\delta \mathcal{L} = \delta \mathcal{L}(\psi_\sigma(x), \partial\psi_\sigma/\partial x_\mu)$$

$$\delta \mathcal{L} = \frac{\partial \mathcal{L}}{\partial x_\mu} \underline{\delta x_\mu} = \frac{\partial \mathcal{L}}{\partial x_\mu} \cdot \epsilon_\mu$$

Si $\mathcal{L}$ es invariante traslacional, entonces $\mathcal{L}$ no depende explícitamente de las coordenadas

Entonces $\delta \mathcal{L}$

$$\delta \mathcal{L} = \sum_\sigma \left[\frac{\partial \mathcal{L}}{\partial \Psi_\sigma} \underline{\delta \Psi_\sigma} + \left[\frac{\partial \mathcal{L}}{\partial(\partial \Psi_\sigma / \partial x_\mu)} \right] \delta\left(\frac{\partial \Psi_\sigma}{\partial x_\mu}\right) \right]$$

¿Qué es $\delta \Psi_\sigma$?

$$\delta \Psi_\sigma = \delta \Psi_\sigma(x) = \frac{\partial \Psi_\sigma}{\partial x_\nu} \delta x_\nu$$ usando ec. 1

$$\boxed{\delta \Psi_\sigma = \frac{\partial \Psi_\sigma}{\partial x_\nu} \cdot \epsilon_\nu} \quad (2)$$

Usaremos las ecuaciones de E.L.

$$\text{(3)}\quad \frac{\partial \mathcal{L}}{\partial \psi_\sigma} - \frac{\partial}{\partial x^\mu}\left[\frac{\partial \mathcal{L}}{\partial(\partial \psi_\sigma/\partial x^\mu)}\right] = 0$$

Entonces (2) y (3)

$$\delta \mathcal{L} = \sum_\sigma \left[\left(\frac{\partial \mathcal{L}}{\partial \psi_\sigma}\right)\delta \psi_\sigma + \frac{\partial \mathcal{L}}{\partial(\partial \psi_\sigma/\partial x_\mu)}\,\delta\left(\frac{\partial \psi_\sigma}{\partial x_\mu}\right)\right]$$

(4)

(1)

$$\text{(4)}\quad \delta\left[\frac{\partial \psi_\sigma}{\partial x_\mu}\right] = \frac{\partial}{\partial x_\nu}\left(\frac{\partial \psi_\sigma}{\partial x_\mu}\right)\delta x_\nu = \underline{\frac{\partial}{\partial x_\nu}\left[\frac{\partial \psi_\sigma}{\partial x_\mu}\right]\epsilon_\nu}$$

$$\delta \mathcal{L} = \epsilon_\mu \frac{\partial \mathcal{L}}{\partial x_\mu} = \sum_\sigma \left[\frac{\partial}{\partial x^\mu}\left(\frac{\partial \mathcal{L}}{\partial(\partial \psi_\sigma/\partial x^\mu)}\right)\right\} \epsilon_\nu \frac{\partial \psi}{\partial x_\nu}$$

$$\left. + \frac{\partial \mathcal{L}}{\partial(\partial \psi_\sigma/\partial x_\mu)}\,\frac{\partial}{\partial x_\nu}\left(\frac{\partial \psi_\sigma}{\partial x_\mu}\right)\epsilon_\nu\right]$$

$$= \sum \left[\left\{ \frac{\partial}{\partial x^\mu} \frac{\partial \mathcal{L}}{\partial(\partial \psi_\sigma/\partial x^\mu)} \right\} \epsilon_\nu \frac{\partial \psi_\sigma}{\partial x_\nu} \right.$$

$$\left. + \frac{\partial \mathcal{L}}{\partial(\partial x_\sigma/\partial x^\mu)} \cdot \frac{\partial}{\partial x_\mu} \left(\frac{\partial \psi_\sigma}{\partial x_\nu} \right) \epsilon_\nu \right]$$

$$= \sum \frac{\partial}{\partial x^\mu} \left[\frac{\partial \mathcal{L}}{\partial(\partial \psi_\sigma/\partial x^\mu)} \epsilon_\nu \frac{\partial \psi_\sigma}{\partial x_\nu} \right]$$

$$\delta \mathcal{L} = \epsilon_\mu \frac{\partial \mathcal{L}}{\partial x_\mu} = \sum_\sigma \frac{\partial}{\partial x^\mu} \left[\frac{\partial \mathcal{L}}{\partial(\partial \psi_\sigma/\partial x^\mu)} \cdot \epsilon_\nu \frac{\partial \psi_\sigma}{\partial x_\nu} \right]$$

Si $\mathcal{L}$ es invariante ante ec. 1

$$0 = \sum \frac{\partial}{\partial x^\mu} \left[\frac{\partial \mathcal{L}}{\partial(\partial \psi_\sigma/\partial x^\mu)} \epsilon_\nu \frac{\partial \psi_\sigma}{\partial x_\nu} \right] - \frac{\partial \mathcal{L}}{\partial x_\mu} \epsilon_\mu$$

$$0 = \frac{\partial}{\partial x_\mu}\left[\sum_\sigma \frac{\partial \mathcal{L}}{\partial(\partial\psi_\sigma/\partial x_\mu)} \frac{\partial \psi_\sigma}{\partial x_\nu}\epsilon_\nu\right]$$

$$- \frac{\partial \mathcal{L}}{\partial x_\mu} g_{\mu\nu}\epsilon^\nu$$

$$0 = \frac{\partial}{\partial x_\mu}\left[\sum \frac{\partial \mathcal{L}}{\partial(\partial\psi_\sigma/\partial x_\mu)} \frac{\partial \psi_\sigma}{\partial x^\nu}\epsilon^\nu - \mathcal{L} g_{\mu\nu}\epsilon^\nu\right]$$

$$0 = \frac{\partial}{\partial x_\mu}\underbrace{\left[\sum \frac{\partial \mathcal{L}}{\partial(\partial\psi_\sigma/\partial x_\mu)} \frac{\partial \psi_\sigma}{\partial x^\nu} - \mathcal{L} g_{\mu\nu}\right]}_{}\epsilon^\nu$$

Como se satisface la ecuación para cualquier ϵ^ν

$$\Rightarrow \boxed{\frac{\partial}{\partial x_\mu} T_{\mu\nu} = 0}$$

$T_{\mu\nu}$ = Tensor energía momento

$\partial^{\mu} T_{\mu\nu} = 0$ para cada valor de ν

$\partial^{\mu} T_{\mu 0} = 0 \quad \partial^{\mu} T_{\mu 1} = 0 \quad \partial^{\mu} T_{\mu 2} = 0$

$\partial^{\mu} T_{\mu 3} = 0$

$$T_{\mu\nu} = -g_{\mu\nu} \mathcal{L} + \sum_{\sigma} \frac{\partial \mathcal{L}}{\partial(\partial \psi_\sigma / \partial x_\mu)} \frac{\partial \psi_\sigma}{\partial x^\nu}$$

Analizaremos T_{00}

Se define la densidad de momento conjugado a ψ_σ por

$$\Pi_\sigma = \frac{\partial \mathcal{L}}{\partial \dot{\psi}_\sigma}$$

$$\boxed{\dot{\psi}_\sigma = \frac{\partial \psi_\sigma}{\partial (ct)}}$$

Se define $\mathcal{H}$, la densidad Hamiltoniana

$\mathcal{H} = \mathcal{H}(\Pi_\sigma, \psi_\sigma)$ tal que

$$H = \int d^3x \, \mathcal{H}(\Pi_\sigma, \psi_\sigma)$$

Entonces $\mathcal{H} = \sum_\sigma \Pi_\sigma \dot{\psi}_\sigma - \mathcal{L}$

Definimos

$$P_\nu = \int d^3x \, T_{0\nu}$$

$$= \int d^3x \left[-g_{0\nu} \mathcal{L} + \sum_\sigma \frac{\partial \mathcal{L}}{\partial(\partial \psi_\sigma / \partial x_0)} \frac{\partial \psi_\sigma}{\partial x^\nu} \right]$$

$$= \int d^3x \left[-g_{0\nu} \mathcal{L} + \sum_\sigma \frac{\partial \mathcal{L}}{\partial \dot{\psi}_\sigma} \frac{\partial \psi_\sigma}{\partial x^\nu} \right]$$

$$\boxed{P_\nu = \int dx^3 \left[-g_{0\nu}\mathcal{L} + \sum \Pi_\sigma \frac{\partial \psi_\sigma}{\partial x^\nu} \right]}$$

Pero $\frac{\partial}{\partial x_\mu} T_{\mu\nu} = 0$

$$\partial^\mu T_{\mu\nu} = 0 = \partial^0 T_{0\nu} - \partial^i T_{i\nu} = 0$$

$$\int \partial^0 T_{0\nu}\, dx^3 = \int_V \partial^i T_{i\nu}\, dx^3$$

$$\partial^0 \int T_{0\nu}\, dx^3 = \int_V \partial^i T_{i\nu}\, dx^3$$

$$= \int_\sigma \vec{T}_\nu \cdot d\vec{S} = 0$$

↳ localización de los campos

$$\frac{\partial}{\partial (ct)} \int T_{0\nu}\, dx^3 = 0 = \frac{1}{c}\frac{\partial}{\partial t} P_\nu = 0$$

$P_\nu =$ constante.

$$T_{00} = \mathcal{H} \quad \checkmark$$

$$\underset{\uparrow}{P^0} = \int d\vec{x}^3\, T_{00} = \int d\vec{x}^3\, \mathcal{H} = H$$

$$T_{00} = \left[-g_{00}\mathcal{L} + \sum \Pi_\sigma \frac{\partial \varphi_\sigma}{\partial x^0} \right]$$

$$= -L + \sum \Pi_\sigma \dot{\varphi}_\sigma = \mathcal{H}$$

Se identifica a P^ν con el 4-vector momento

$$\frac{\partial P^i}{\partial t} = 0$$

Regresamos al lagrangiano de K.G.

$$T_\mu{}^\nu = \sum_\sigma \frac{\partial \varphi_\sigma}{\partial x^\mu} \frac{\partial \mathcal{L}}{\partial(\partial \varphi_\sigma / \partial x^\nu)} - \mathcal{L}\, g_\mu{}^\nu$$

$$T_\mu{}^\nu = \frac{\partial \psi}{\partial x^\mu}\frac{\partial \mathcal{L}}{\partial(\partial\psi/\partial x^\nu)} + \frac{\partial \psi^*}{\partial x^\mu}\frac{\partial \mathcal{L}}{\partial(\partial\psi^*/\partial x^\nu)}$$

$$- \mathcal{L}\, g_\mu{}^\nu$$

$$T_\mu{}^\nu = \frac{\hbar^2}{2m_0}\left[g^{\sigma\nu}\frac{\partial\psi^*}{\partial x^\sigma}\frac{\partial\psi}{\partial x^\mu} + g^{\sigma\nu}\frac{\partial\psi}{\partial x^\sigma}\frac{\partial\psi^*}{\partial x^\mu}\right.$$

Problema Propuesto

$$\left. - \left(g^{\sigma\rho}\frac{\partial\psi^*}{\partial x^\sigma}\frac{\partial\psi}{\partial x^\rho} - \frac{m_0^2}{\hbar} c^2 \psi^*\psi\right) g_\mu{}^\nu \right]$$

Luego

$$\mathcal{H} = T_0{}^0 = \frac{\hbar^2}{2m_0}\left[g^{\sigma 0}\frac{\partial\psi^*}{\partial x^\sigma}\frac{\partial\psi}{\partial x^0} + g^{\sigma 0}\frac{\partial\psi}{\partial x^\sigma}\frac{\partial\psi^*}{\partial x^0}\right.$$

$$\left. - \left[g^{\sigma\rho}\frac{\partial\psi^*}{\partial x^\sigma}\frac{\partial\psi}{\partial x^\rho} - \frac{m_0^2}{\hbar} c^2 \psi^*\psi\right] g_0{}^0 \right]$$

Mostrar que

$$T_0^{\,0} = \frac{\hbar^2}{2m_0}\left[\frac{1}{c^2}\frac{\partial \psi^*}{\partial t}\frac{\partial \psi}{\partial t} + (\nabla\psi^*)\cdot(\nabla\psi)\right.$$

$$\left. + \frac{m_0^2c^2}{\hbar^2}\psi^*\psi\right]$$

Las soluciones $\psi_n(\pm) = \sqrt{\dfrac{m_0c^2}{L^3 E_{p_n}}}\, \exp\left[\dfrac{i}{\hbar}\left(\vec{p}_n\cdot\vec{x} \mp E_{p_n} t\right)\right]$

$$\boxed{H = \int_{L^3} T_0^{\,0}(n,+)\, dx^3}$$

esto es, tomar $\psi_n^*(+)$, $\psi_n^*(+)$

$H(+) = \underline{E_{p_n}}$ Mostrar [propuesto]

$$H = \int T^0_0(n,-)\,dx^3$$

$$\Psi_n(-) = \sqrt{\frac{m_0 c^2}{L^3 E_{p_n}}}\,\exp\left[\frac{i}{\hbar}\left(\vec{p_n}\cdot\vec{x} + E_p \cdot t\right)\right]$$

$$H_n(-) = E_{pn} \quad \checkmark$$