La transformación del espinor

① $\hat{S}_R(\omega_{ij}) = \exp\left[-\frac{i}{4}\hat{\sigma}_{ij}\,\omega^{ij}\right]$

$$i, j = 1, 2, 3$$

ya que deseamos

$$[\not{p}' - m_0 c]\,\psi'(x') = 0$$

$$[\not{p} - m_0 c]\,\psi(x) = 0$$

$$x'^{\nu} = a^{\nu}{}_{\mu}\,x^{\mu}$$

① es unitaria

$$\hat{S}_R^\dagger(\omega_{ij}) = \left[\exp\left[-\frac{i}{4}\hat{\sigma}_{ij}\,\omega^{ij}\right]\right]^\dagger$$

$$= \exp\left[\frac{+i}{4}\hat{\sigma}_{ij}^\dagger\,\omega^{ij}\right]$$

$$= \exp\left[\frac{i}{4}\hat{\sigma}_{ij}\;\omega^{ij}\right]$$

$$\hat{S}_R^\dagger\;\hat{S}_R = \exp\left[\frac{i}{4}\hat{\sigma}_{ij}\,\omega^{ij}\right]\exp\left[-\frac{i}{4}\sigma_{ij}\,\omega^{ij}\right]$$

$$= \exp\left[\frac{i}{4}\hat{\sigma}_{ij}\,\omega^{ij} - \frac{i}{4}\sigma_{ij}\,\omega^{ij}\right]$$

$$= \mathbb{1}$$

$$\boxed{\hat{S}_R^\dagger = \hat{S}_R^{-1}}$$

Por otro lado, para transformaciones que involucran boost, pasar de un sistema O a O' ($\vec{v}$ respecto a O)

$$\Psi'(x') = \exp\left[-\frac{i}{2}\omega\hat{\sigma}_{01}\right]\Psi(x)$$

pero de la definición de $\hat{\sigma}_{\mu\nu}$

$$\hat{\sigma}_{01} = i\hat{\alpha}_1$$

$$\Psi'(x') = \exp\left[-\frac{i}{2}\omega(+i\alpha_1)\right]\Psi(x)$$

$$= \underbrace{\exp\left[\frac{1}{2}\omega\hat{\alpha}_1\right]}\Psi(x)$$

$$\hat{S}_L = \exp\left[\frac{1}{2}\omega\hat{\alpha}_1\right]$$

$$\hat{S}^{L\dagger} = \exp\left[\frac{1}{2}\omega\hat{\alpha}_1^{\dagger}\right]$$

$$= \exp\left[\frac{1}{2}\omega\alpha_1\right] = \hat{S}^L \neq \hat{S}_L^{-1}$$

Necesitamos determinar la transformación inversa

Si $\boxed{\hat{S} = \exp\left[-\frac{i}{4}\hat{\sigma}_{\mu\nu}(I_n)^{\mu\nu}\right]}$

entonces la inversa

$$\boxed{\hat{S}^{-1} = \gamma^0 \hat{S}^\dagger \gamma^0}$$

i) Rotaciones : $\hat{I}^{0\mu}, \hat{I}^{0\nu} = 0$

$$\hat{S} = \exp\left[-\frac{i}{4}\omega^{ij}\sigma_{ij}\right] \quad \underline{i,j = 1,2,3}$$

$$\hat{S}^\dagger = \exp\left[\frac{i}{4}\omega^{ij}\hat{\sigma}_{ij}\right]$$

Recordamos $[\hat{\sigma}_{\alpha\beta}, \gamma^\nu] = -2i[g^\nu_\alpha \gamma_\beta - g^\nu_\beta \gamma_\alpha]$

$\alpha, \beta = 0,1,2,3.$

$$[\hat{\sigma}_{ij}, \gamma^0] = -2i[\underline{g^0_{\ i}}\gamma_j - g^0_{\ \underline{j}}\gamma_i]$$

$$= -2i[0 - 0]$$

$$\underline{[\hat{\sigma}_{ij}, \gamma^0] = 0}$$

Entonces $\quad S^{+}\mathbb{1} = S^{+}\gamma^0\gamma^0$

$$S^{+}\mathbb{1} = \gamma^0 S^{+}\gamma^0$$

$$\boxed{S^{+} = \hat{S}^{-1} = \gamma^0 S^{+}\gamma^0}$$

ii) Supongamos un boost que coincide con el eje $\hat{x}$

$$\hat{S} = \exp\left[-\frac{i}{2}\omega\hat{\sigma}_{01}\right]$$

$$S^{\dagger} = \exp\left[+ \frac{i}{2} \, \omega \, \sigma_{01}^{\dagger} \right]$$

pero $\sigma_{01} = \frac{i}{2} \{ \gamma_0 \gamma_1 - \gamma_1 \gamma_0 \}$

$$\sigma_{01}^{\dagger} = -\frac{i}{2} \left[(\gamma_0 \gamma_1)^{\dagger} - (\gamma_1 \gamma_0)^{\dagger} \right]$$

$$= -\frac{i}{2} \left[\gamma_1^{\dagger} \gamma_0^{\dagger} - \gamma_0^{\dagger} \gamma_1^{\dagger} \right]$$

pero $\gamma^{i\dagger} = -\gamma^{i} \qquad \gamma^{0\dagger} = \gamma^{0}$

$$\sigma_{01}^{\dagger} = -\frac{i}{2} \left[-\gamma_1 \gamma_0 + \gamma_0 \gamma_1 \right]$$

$$= -\sigma_{01}$$

Construimos $\gamma_0 \hat{\sigma}_{01}$

$$\gamma_0 \hat{\sigma}_{01} = \frac{i}{2} \gamma_0 [\gamma_0 \gamma_1 - \gamma_1 \gamma_0]$$

$$= \frac{i}{2} [\underline{\gamma_0 \gamma_0 \gamma_1} - \gamma_0 \gamma_1 \gamma_0]$$

pero $\gamma^\mu \gamma^\nu + \gamma^\nu \gamma^\mu = 2 g^{\mu\nu}$

$\mu = 0, \quad \nu = 1$

$$\gamma^0 \gamma^1 + \gamma^1 \gamma^0 = 2 \cancel{g^{01}}^{\,0} = 0$$

$$\gamma^0 \gamma^1 = - \gamma^1 \gamma^0$$

$$\gamma_0 \hat{\sigma}_{01} = \frac{i}{2} [\gamma_1 \gamma_0 \underline{\gamma_0} - \gamma_0 \gamma_1 \underline{\gamma_0}]$$

$$= \frac{i}{2} [\gamma_1 \gamma_0 - \gamma_0 \gamma_1] \gamma_0$$

$$= \hat{\sigma}_{10} \gamma_0 = - \hat{\sigma}_{01} \gamma_0$$

Luego

$$\boxed{\gamma_0 \hat{S}_{\frac{1}{2}}^{+} \gamma_0} = \gamma_0 \left[\sum_{n=0}^{\infty} \frac{\left(-\frac{i}{2} \omega \sigma_{01}\right)^n}{n!} \right] \gamma_0$$

$$= \sum_{n=0}^{\infty} \gamma_0 \frac{\left(-\frac{i}{2} \omega \sigma_{01}\right)^n}{n!} \Big] \gamma_0$$

$$= \ldots + \frac{\gamma_0}{n!} \left(-\frac{i}{2} \omega \sigma_{01}\right) \left(-\frac{i}{2} \omega \sigma_0\right) \ldots \gamma_0$$

$$= \ldots + \frac{1}{n!} \left(\frac{i}{2} \omega \sigma_{01}\right) \gamma_0 \left(-\frac{i}{2} \omega \sigma_0\right) \ldots \gamma_0$$

$$= \ldots + \frac{1}{n!} \left(\frac{i}{2} \omega \sigma_{01}\right) \ldots \left(\frac{i}{2} \omega \sigma_{01}\right) \gamma_0^2 + \ldots$$

$$= \sum_{n=0}^{\infty} \frac{1}{n!} \left(\frac{i}{2} \omega \sigma_{01}\right)^n = e^{\left\{\frac{i}{2} \omega \sigma_{01}\right\}}$$

$$= \hat{S}^{-1}$$

Entonces, la transformación inversa $\hat{S}$, es

$$\hat{S}^{-1} = \gamma_0 \hat{S}^{+} S_0$$

Ahora probaremos la covarianza de

$$\boxed{\frac{\partial}{\partial t}\rho + \nabla \vec{J} = 0}$$

con $\rho = \underline{\psi^{+}\psi}$ $\quad \vec{j} = c\psi^{+}\hat{\alpha}\psi$

$$j^{\mu} = (j^{0}, \vec{j}) = (\underline{c\psi^{+}\psi}, \underline{c\psi^{+}\hat{\alpha}\psi})$$

$$= \underline{c\psi^{+}\gamma^{0}\gamma^{\mu}\psi}$$

si $\mu = 0$ $\quad \underline{j^{0}} = c\psi^{+}\gamma^{0}\gamma^{0}\psi = c\psi^{+}\psi$

Veamos la transformación ante Lorentz

$$j^{\mu\prime}(x') = c\,\underline{\psi'^{\dagger}(x')}\,\gamma^0\gamma^\mu\,\underline{\psi'(x')}$$

$$\psi'(x') = \hat{S}\,\psi(x)$$

$$\psi'^{\dagger}(x) = [\hat{S}\,\psi(x)]^{\dagger} = \psi^{\dagger}(x)\,\hat{S}^{\dagger}$$

$$j^{\mu\prime}(x') = c\,\psi^{\dagger}(x)\,\hat{S}^{\dagger}\,\gamma^0\gamma^\mu\,\hat{S}\,\psi(x)$$

pero $\hat{S}^{-1} = \gamma_0\,\hat{S}^{\dagger}\,\gamma_0$

$$j^{\mu\prime}(x') = c\,\psi^{\dagger}\gamma^0\,\underbrace{\gamma^0\,\hat{S}^{\dagger}\,\gamma^0}\,\gamma^\mu\,\hat{S}\,\psi$$

$$= c\,\psi^{\dagger}\gamma^0\,\hat{S}^{-1}\,\gamma^\mu\,S\,\psi$$

Ademas $\hat{S}^{-1} \gamma^{\mu} S = a^{\mu}{}_{\nu} \gamma^{\nu}$

$$j^{\mu'}(x') = c \Psi^{\dagger} \gamma_0 \, \underline{a^{\mu}{}_{\nu}} \, \gamma^{\nu} \Psi(x)$$

$$j^{\mu'}(x') = a^{\mu}{}_{\nu} \left[c \, \Psi^{\dagger}(x) \gamma_0 \gamma^{\nu} \Psi(x) \right]$$

$$\boxed{j^{\mu'}(x') = a^{\mu}{}_{\nu} \, j^{\mu}(x)}$$

j^{μ} es un 4-vector

Entonces la ec. de continuidad es

$$\frac{\partial j^{\mu}}{\partial x^{\mu}} = 0 = \partial_{\mu} j^{\mu} = 0$$

Se define el espinor adjunto

$$\overline{\Psi} \equiv \psi^{\dagger} \gamma^{0} \leftarrow$$

¿Cómo transforma el espinor adjunto?

$$\overline{\Psi}'(x') = \psi'^{\dagger}(x')\, \gamma^{0\prime}$$

$$= [\underline{\Psi'(x')}]^{\dagger} \gamma^{0}$$

$$= [\hat{S}\, \Psi(x)]^{\dagger} \gamma^{0}$$

$$= \psi^{\dagger} \hat{S}^{\dagger} \gamma^{0}$$

$$\mathbb{1} = \gamma^{0} \gamma^{0}$$

$$= \underbrace{\psi^{\dagger} \gamma^{0}}\; \underbrace{\gamma^{0} \hat{S}^{\dagger} \gamma^{0}}$$

$$\boxed{\overline{\Psi}'(x') = \overline{\Psi}(x)\, \hat{S}^{-1}}$$

Luego

$$\overline{\Psi}(x)\,\Psi(x) \to \text{visto en un sistema } O'$$

$$\overline{\Psi'}(x')\,\Psi'(x') = \overline{\Psi}(x)\,\hat{S}^{-1}\hat{S}\,\Psi(x)$$

$$= \overline{\Psi}(x)\,\Psi(x)$$

escalar de Lorentz

Covariantes bilineales de espinores de Dirac.

Existen 16 matrices 4x4 linealmente independientes.

i) $\hat{\Gamma}^S = \mathbb{1}$

$S =$ escalar

1 sola matriz

ii) $\hat{\Gamma}^V_\mu = \gamma_\mu$ $\mu = 0,1,2,3$

$V =$ vector

4 matrices

iii) $\Gamma^T_{\mu\nu} = \sigma_{\mu\nu} = -\sigma_{\nu\mu}$

$T =$ tensor

6 matrices

iv) $\hat{\Gamma}^P = i\gamma^0\gamma^1\gamma^2\gamma^3 = \gamma^5$

1 pseudo vector

v) $\hat{\Gamma}_\mu{}^A = \gamma_5\gamma_\mu$ $\mu = 0,1,2,3$

Axial

4

Propiedades

1.- Para cada $\hat{\Gamma}^n$ $\quad n = S, V, T, P, A$

$$[\hat{\Gamma}^n]^2 = \pm \mathbb{1}$$

2.- Para cada $\hat{\Gamma}^n$, excepto $\underline{\hat{\Gamma}^S}$

existe una correspondiente $\hat{\Gamma}^m$

tal que

$$\boxed{\hat{\Gamma}^n \hat{\Gamma}^m = -\hat{\Gamma}^m \hat{\Gamma}^n}$$

multiplicamos a la derecha por $\hat{\Gamma}^m$

$$\hat{\Gamma}^n(\hat{\Gamma}^m \hat{\Gamma}^m) = -\hat{\Gamma}^m \hat{\Gamma}^n \hat{\Gamma}^m$$

$$\pm \hat{\Gamma}^n \mathbb{1} = \pm \hat{\Gamma}^n = -\hat{\Gamma}^m \hat{\Gamma}^n \hat{\Gamma}^m$$

Calculamos la traza de la ecuación anterior

$$\mathrm{Tr}[\pm\hat{\Gamma}^n] = \pm\mathrm{Tr}[\Gamma^n] = -\mathrm{Tr}[\hat{\Gamma}^m\hat{\Gamma}^n\hat{\Gamma}^m]$$

$$\boxed{\begin{aligned}\pm\mathrm{Tr}[\hat{\Gamma}^n] &= -\mathrm{Tr}[\Gamma^n\Gamma^m\Gamma^m]\\ &= -\mathrm{Tr}[\Gamma^n(\Gamma^m)^2]\end{aligned}}$$

pero $\pm\hat{\Gamma}^n = -\hat{\Gamma}^m\hat{\Gamma}^n\hat{\Gamma}^m$

$$= -(\Gamma^m)(-\Gamma^m\Gamma^n)$$

$$= +(\Gamma^m)^2\Gamma^n$$

$$\pm\mathrm{Tr}[\hat{\Gamma}^n] = \underline{+\mathrm{Tr}[(\Gamma^m)^2\Gamma^n]} =$$

$$\underline{= -\mathrm{Tr}[\Gamma^n(\Gamma^m)^2]}$$

$\mathrm{Tr}[\hat{\Gamma}^n] = 0$

3.- Para $\hat{\Gamma}^a , \hat{\Gamma}^b \quad a \neq b$

existe $\hat{\Gamma}^m \neq \hat{\Gamma}^s \leftarrow$

$$\hat{\Gamma}^a \hat{\Gamma}^b = f^n_{ab} \hat{\Gamma}^n$$

donde $f^n_{ab} \in \mathbb{C}$

$\boxed{\text{Ejercicio Propuesto: Mostrar que las matrices } \hat{\Gamma}^m \text{ son L. I.}}$

Las expresiones bilineales

$\overline{\Psi}\,\hat{\Gamma}\,\Psi \rightarrow$ forma bilineal

Se cumple $\gamma^\mu\gamma^5 + \gamma^5\gamma^\mu = 0$. I

Ejercicio propuesto: Mostrar

Recordamos $\hat{S}(a) = \exp\left[-\frac{i}{4}\hat{\sigma}_{\mu\nu}\omega^{\mu\nu}\right]$

si se satisface I, entonces

se cumple $[\gamma_5, \sigma_{\mu\nu}] = 0$ ✓

$\gamma_5\sigma_{\mu\nu} - \sigma_{\mu\nu}\gamma_5 = 0$

Entonces $[\hat{S}(a), \gamma_5]_- = 0$

1.- $\bar{\Psi}'(x)\,\Psi'(x') = \bar{\Psi}(x)\,\Psi(x)$

$\bar{\Psi}'(x)\,\mathbb{1}\,\Psi'(x') = \bar{\Psi}'\,\hat{\Gamma}^{s}\,\Psi(s)$ ✓ ✓

Escalar de Lorentz

2.- $\bar{\Psi}'(x')\,\gamma_5\,\Psi'(x')$ es un pseudo escalar,

$\bar{\Psi}'(x')\,\gamma^5\,\Psi'(x') = -\bar{\Psi}(x)\gamma^5\Psi$

3.- $\bar{\Psi}'(x')\,\gamma^\nu\,\Psi'(x') = a^\nu{}_\mu\,\bar{\Psi}\gamma^\mu\Psi$

?

4.- $\bar{\Psi}'(x')\gamma^5\gamma^\nu\,\Psi'(x')$ = pseudo vector

$= -\,a^\nu{}_\mu\,\bar{\Psi}\gamma^5\gamma^\mu\Psi$

$\boxed{\text{Propuestos: Mostrar 2, 3, 4, 5.}}$

5. $\bar{\Psi}'(x')\sigma^{\mu\nu}\psi'(x') =$

$$= a^{\mu}{}_{\alpha}\, a^{\nu}{}_{\beta}\, \bar{\Psi}\sigma^{\alpha\beta}\psi$$

Tensor de rango 2

$$\bar{A}^{\mu\nu} = a^{\mu}{}_{\beta}\, a^{\nu}{}_{\alpha}\, A^{\beta\alpha}$$

$A_{\mu\nu} A^{\mu\nu}$ = invariante de Lorentz.

$$\left(\bar{\Psi}(x)\sigma^{\mu\nu}\psi\right)\left(\bar{\Psi}\sigma_{\mu\nu}\Psi\right)$$