UNIVERSIDAD AUTÓNOMA DE NUEVO LEÓN
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Relatividad General
Símbolos de Christoffel

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Calcule los símbolos de Christoffel para las coordenadas esfericas (r, θ, ϕ), partiendo del elemento diferencial de desplazamiento:

$$dS^2 = dr^2 + r^2 d\theta^2 + r^2 \sin^2(\theta) d\phi^2$$

A partir del elemento diferencial de desplazamiento se tiene que el tensor metrico es:

$$g_{ij} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & r^2 & 0 \\ 0 & 0 & r^2 \sin^2(\theta) \end{pmatrix}, \qquad g^{ij} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{r^2} & 0 \\ 0 & 0 & \frac{1}{r^2 \sin^2(\theta)} \end{pmatrix}$$

Se tiene que los símbolos de Christoffel estan definidos por:

$$\Gamma^i_{kl} = \frac{1}{2} g^{im} \left(\frac{\partial g_{mk}}{\partial x^l} + \frac{\partial g_{ml}}{\partial x^k} - \frac{\partial g_{kl}}{\partial x^m} \right)$$

como el tensor metrico g es diferente de cero en su diagonal, entonces, los símbolos de Christoffel puede ser distinto de cero para $i = m$, y para $i \neq m$ iguales a cero, calculando para $i = 1$, se tiene que:

$$\Gamma^i_{kl} = \frac{1}{2} g^{ii} \left(\frac{\partial g_{ik}}{\partial x^l} + \frac{\partial g_{il}}{\partial x^k} - \frac{\partial g_{kl}}{\partial x^i} \right)$$

calculando sus permutaciones, se tiene que:

$$\begin{aligned} \Gamma^1_{11} &= \frac{1}{2} \left(\frac{\partial 1}{\partial r} + \frac{\partial 1}{\partial r} - \frac{\partial 1}{\partial r} \right) \\ &= 0 \end{aligned} \qquad \begin{aligned} \Gamma^1_{21} &= \frac{1}{2} \left(\frac{\partial 0}{\partial r} + \frac{\partial 1}{\partial \theta} - \frac{\partial 1}{\partial r} \right) \\ &= 0 \end{aligned} \qquad \begin{aligned} \Gamma^1_{31} &= \frac{1}{2} \left(\frac{\partial 0}{\partial r} + \frac{\partial 1}{\partial \phi} - \frac{\partial 1}{\partial r} \right) \\ &= 0 \end{aligned}$$

$$\begin{aligned} \Gamma^1_{12} &= \frac{1}{2} \left(\frac{\partial 1}{\partial \theta} + \frac{\partial 0}{\partial r} - \frac{\partial 0}{\partial r} \right) \\ &= 0 \end{aligned} \qquad \begin{aligned} \Gamma^1_{22} &= \frac{1}{2} \left(\frac{\partial 0}{\partial \theta} + \frac{\partial 0}{\partial \theta} - \frac{\partial r^2}{\partial r} \right) \\ &= -r \end{aligned} \qquad \begin{aligned} \Gamma^1_{32} &= \frac{1}{2} \left(\frac{\partial 0}{\partial \theta} + \frac{\partial 0}{\partial \phi} - \frac{\partial 0}{\partial r} \right) \\ &= 0 \end{aligned}$$

$$\begin{aligned} \Gamma^1_{13} &= \frac{1}{2} \left(\frac{\partial 1}{\partial \phi} + \frac{\partial 0}{\partial r} - \frac{\partial 0}{\partial r} \right) \\ &= 0 \end{aligned} \qquad \begin{aligned} \Gamma^1_{23} &= \frac{1}{2} \left(\frac{\partial 0}{\partial \phi} + \frac{\partial 0}{\partial \theta} - \frac{\partial 0}{\partial r} \right) \\ &= 0 \end{aligned} \qquad \begin{aligned} \Gamma^1_{33} &= \frac{1}{2} \left(\frac{\partial 0}{\partial \phi} + \frac{\partial 0}{\partial \phi} - \frac{\partial r^2 \sin^2(\theta)}{\partial r} \right) \\ &= -2r \sin^2(\theta) \end{aligned}$$

calculando para $i = 2$, se tiene que:

$$\begin{aligned} \Gamma^2_{11} &= \frac{1}{2r^2} \left(\frac{\partial 0}{\partial r} + \frac{\partial 0}{\partial r} - \frac{\partial 1}{\partial \theta} \right) \\ &= 0 \end{aligned} \qquad \begin{aligned} \Gamma^2_{21} &= \frac{1}{2r^2} \left(\frac{\partial r^2}{\partial r} + \frac{\partial 0}{\partial \theta} - \frac{\partial 1}{\partial \theta} \right) \\ &= \frac{1}{r} \end{aligned} \qquad \begin{aligned} \Gamma^2_{31} &= \frac{1}{2r^2} \left(\frac{\partial 0}{\partial r} + \frac{\partial 0}{\partial \phi} - \frac{\partial 1}{\partial \theta} \right) \\ &= 0 \end{aligned}$$

$$\begin{aligned} \Gamma^2_{12} &= \frac{1}{2r^2} \left(\frac{\partial 0}{\partial \theta} + \frac{\partial r^2}{\partial r} - \frac{\partial 0}{\partial \theta} \right) \\ &= \frac{1}{r} \end{aligned} \qquad \begin{aligned} \Gamma^2_{22} &= \frac{1}{2r^2} \left(\frac{\partial r^2}{\partial \theta} + \frac{\partial r^2}{\partial \theta} - \frac{\partial r^2}{\partial \theta} \right) \\ &= 0 \end{aligned} \qquad \begin{aligned} \Gamma^2_{32} &= \frac{1}{2r^2} \left(\frac{\partial 0}{\partial \theta} + \frac{\partial r^2}{\partial \phi} - \frac{\partial 0}{\partial \theta} \right) \\ &= 0 \end{aligned}$$

$$\begin{aligned} \Gamma^2_{13} &= \frac{1}{2r^2} \left(\frac{\partial 0}{\partial \phi} + \frac{\partial 0}{\partial r} - \frac{\partial 0}{\partial \theta} \right) \\ &= 0 \end{aligned} \qquad \begin{aligned} \Gamma^2_{23} &= \frac{1}{2r^2} \left(\frac{\partial r^2}{\partial \phi} + \frac{\partial 0}{\partial \theta} - \frac{\partial 0}{\partial \theta} \right) \\ &= 0 \end{aligned} \qquad \begin{aligned} \Gamma^2_{33} &= \frac{1}{2r^2} \left(\frac{\partial 0}{\partial \phi} + \frac{\partial 0}{\partial \phi} - \frac{\partial r^2 \sin^2(\theta)}{\partial \theta} \right) \\ &= -2 \sin(\theta) \cos(\theta) \end{aligned}$$

calculando para $i = 3$, se tiene que:

$$\begin{aligned}
\Gamma^3_{11} &= \frac{1}{2r^2\sin^2(\theta)}\left(\frac{\partial 0}{\partial r} + \frac{\partial 0}{\partial r} - \frac{\partial 1}{\partial \phi}\right) \\
&= 0 \\
\Gamma^3_{12} &= \frac{1}{2r^2\sin^2(\theta)}\left(\frac{\partial 0}{\partial \theta} + \frac{\partial 0}{\partial r} - \frac{\partial 0}{\partial \phi}\right) \\
&= 0 \\
\Gamma^3_{13} &= \frac{1}{2r^2\sin^2(\theta)}\left(\frac{\partial 0}{\partial \phi} + \frac{\partial r^2\sin^2(\theta)}{\partial r} - \frac{\partial 0}{\partial \phi}\right) \\
&= \frac{1}{r}
\end{aligned}$$

$$\begin{aligned}
\Gamma^3_{21} &= \frac{1}{2r^2\sin^2(\theta)}\left(\frac{\partial 0}{\partial r} + \frac{\partial 0}{\partial \theta} - \frac{\partial 1}{\partial \phi}\right) \\
&= 0 \\
\Gamma^3_{22} &= \frac{1}{2r^2\sin^2(\theta)}\left(\frac{\partial 0}{\partial \theta} + \frac{\partial 0}{\partial \theta} - \frac{\partial r^2}{\partial \phi}\right) \\
&= 0 \\
\Gamma^3_{23} &= \frac{1}{2r^2\sin^2(\theta)}\left(\frac{\partial 0}{\partial \phi} + \frac{\partial r^2\sin^2(\theta)}{\partial \theta} - \frac{\partial 0}{\partial \phi}\right) \\
&= \cot(\theta)
\end{aligned}$$

$$\begin{aligned}
\Gamma^3_{31} &= \frac{1}{2r^2\sin^2(\theta)}\left(\frac{\partial r^2\sin^2(\theta)}{\partial r} + \frac{\partial 0}{\partial \phi} - \frac{\partial 1}{\partial \phi}\right) \\
&= \frac{1}{r} \\
\Gamma^3_{32} &= \frac{1}{2r^2\sin^2(\theta)}\left(\frac{\partial r^2\sin^2(\theta)}{\partial \theta} + \frac{\partial 0}{\partial \phi} - \frac{\partial 0}{\partial \phi}\right) \\
&= \cot(\theta) \\
\Gamma^3_{33} &= \frac{1}{2r^2\sin^2(\theta)}\left(\frac{\partial r^2\sin^2(\theta)}{\partial \phi} + \frac{\partial r^2\sin^2(\theta)}{\partial \phi} - \frac{\partial r^2\sin^2(\theta)}{\partial \phi}\right) \\
&= 0
\end{aligned}$$

dando asi como resultado que los símbolos de Christoffel diferentes a cero son:

$$\begin{aligned}
\Gamma^1_{22} &= -r \\
\Gamma^1_{33} &= -2r\sin^2(\theta)
\end{aligned}
\qquad
\begin{aligned}
\Gamma^2_{12} &= \frac{1}{r} \\
\Gamma^2_{32} &= \frac{1}{r} \\
\Gamma^2_{33} &= -2\sin(\theta)\cos(\theta)
\end{aligned}
\qquad
\begin{aligned}
\Gamma^3_{13} &= \frac{1}{r} \\
\Gamma^3_{31} &= \frac{1}{r} \\
\Gamma^3_{23} &= \cot(\theta) \\
\Gamma^3_{32} &= \cot(\theta)
\end{aligned}$$