Ecuación de Dirac

$$i\hbar \underline{\gamma^\mu \partial_\mu} \Psi - mc\Psi = 0$$

$$i\hbar\partial_\mu = P_\mu \quad , \quad \gamma^\mu P_\mu = \not{P}$$

$$[\not{P} - mc]\Psi = 0$$

$$[i\hbar \gamma^\mu \partial_\mu \Psi - m_0 c \Psi]^\dagger = 0$$

$$-i\hbar\, \partial_\mu \psi^\dagger \gamma^{\mu\dagger} - m_0 c \psi^\dagger = 0$$

$$\gamma^{\mu\dagger} = \gamma^0 \gamma^\mu \gamma^0$$

$$-i\hbar\, \partial_\mu \underline{\psi^\dagger \gamma^0} \gamma^\mu \gamma^0 - m_0 c \psi^\dagger = 0$$

$$[-i\hbar\, \partial_\mu \overline{\Psi}\, \gamma^\mu \gamma^0 - m_0 c \psi^\dagger = 0]\gamma^0$$

$$i\hbar\, \partial_\mu \overline{\Psi}\, \gamma^\mu \underline{\gamma^0 \gamma^0} - m_0 c \psi^\dagger \gamma^0 = 0$$

$$i\hbar\,\partial_\mu \bar{\Psi}\gamma^\mu + mc\bar{\Psi} = 0 \quad \checkmark$$

$$\bar{\Psi}\, i\hbar \overleftarrow{\partial_\mu} \gamma^\mu + mc\bar{\Psi} = 0$$

$$\bar{\Psi}\left[\cancel{p} + mc\right] = 0$$

La densidad lagrangiana de Dirac

$$① \quad \mathcal{L} = c\bar{\Psi}\left[i\hbar\partial_\mu\gamma^\mu - mc\right]\Psi$$

$\mathcal{L}$ es invariante de Lorentz

$$② \quad \mathcal{L}' = c\bar{\Psi}'\left[i\hbar\gamma^{\mu\prime}\partial_\mu' - mc\right]\Psi'$$

$$\mathcal{L} = \mathcal{L}'$$

el segundo término, por mostrar

$$\underline{mc^2}\ \underline{\overline{\Psi}'\psi'} = mc^2\overline{\Psi}\Psi$$

Usando las transformaciones

$$\psi' = \hat{S}\Psi \quad (3)$$

$$\psi'^{\dagger} = (\hat{S}\Psi)^{\dagger} = \Psi^{\dagger}\hat{S}^{\dagger} \quad (4)$$

Ademas $\overline{\Psi} = \psi^{\dagger}\gamma^0$ (5) ✓

$$\overline{\Psi}'\psi' = \psi'^{\dagger}\gamma^0\psi'$$

$$= (S\Psi)^{\dagger}\gamma^0(S\Psi)$$

$$= \psi^{\dagger}S^{\dagger}\gamma^0 S\Psi$$

$$= \psi^{\dagger}(\gamma^0\gamma^0)S^{\dagger}\gamma^0 S\Psi$$

$$\overline{\Psi}'\psi' = (\psi^{+}\gamma^{0})\left[\gamma^{0}S^{+}S^{0}\right] S\Psi$$

$$\text{pero} \quad \gamma^{0}S^{+}\gamma^{0} = \hat{S}^{-1}$$

$$\overline{\Psi}'\psi' = \overline{\Psi}\,\hat{S}^{-1}S\Psi = \overline{\Psi}\Psi$$

El primer término

$$i\hbar\overline{\Psi}'\gamma^{\mu}\partial_{\mu}'\Psi' = \psi'^{+}\gamma^{0}i\hbar\gamma^{\mu}a_{\mu}^{\beta}\partial_{\beta}$$

$$\cdot(S\Psi)$$

$$= i\hbar\left[S\Psi\right]^{+}\gamma^{0}\gamma^{\mu}a_{\mu}^{\beta}\partial_{\beta}S\Psi$$

$$= i\hbar\,\psi^{+}S^{+}\gamma^{0}\gamma^{\mu}a_{\mu}^{\beta}\partial_{\beta}S\Psi$$

$$= i\hbar\,\psi^{+}\gamma^{0}\gamma^{0}\hat{S}^{+}\gamma^{0}\gamma^{\mu}a_{\mu}^{\beta}\partial_{\beta}S\Psi$$

$$= i\hbar\,\overline{\Psi}\,\underbrace{\gamma^{0}S^{+}\gamma^{0}}\,\underbrace{\overset{\times}{a_{\mu}^{\beta}}\overset{\times}{\partial_{\beta}}}\,S\Psi$$

$$= i\hbar \bar{\Psi} \underbrace{\hat{S}^{-1} \gamma^\mu S}\, a_\mu{}^\beta \partial_\beta \Psi$$

$$\hat{S}^{-1} \gamma^\mu S = a^\mu{}_\alpha \gamma^\alpha$$

$$= i\hbar \bar{\Psi}\, \underline{a}^\mu{}_\alpha \gamma^\alpha\, \underline{a_\mu}{}^\beta \partial_\beta \Psi$$

$$= i\hbar \underbrace{a^\mu{}_\alpha a_\mu{}^\beta}_{\delta_\alpha{}^\beta} \bar{\Psi} \gamma^\alpha \partial_\beta \Psi$$

$$= i\hbar \bar{\Psi} \gamma^\beta \partial_\beta \Psi$$

$$i\hbar \bar{\Psi}' \gamma^\beta \partial_\beta' \Psi' = i\hbar \bar{\Psi} \gamma^\beta \partial_\beta \Psi$$

La densidad lagrangiana es invariante de Lorentz.

La lagrangiana

$$\mathcal{L} = c\bar{\Psi}[i\hbar\gamma^{\mu}\partial_{\mu} - mc]\Psi$$

Sea la tranformación

(6) $$\Psi' = e^{ig\alpha/\hbar c}\Psi$$

α = escalar de Lorentz.

g = escalar de Lorentz.

¿Es $\mathcal{L}$ invariante ante la transformación de la ec. (6)?

$$\mathcal{L}' = c\bar{\Psi}'[i\hbar\gamma^{\mu}\partial_{\mu} - mc]\Psi'$$

$$\bar{\Psi}' = e^{-ig\alpha/\hbar c}\bar{\Psi} \quad (7)$$

$$\mathcal{L}' = c\,[e^{-ig\cancel{\alpha}/\hbar c}]^* \psi^{\dagger} \gamma^0 [i\hbar\gamma^\mu \partial_\mu - mc] \times e^{ig\cancel{\alpha}/\hbar c}\,\psi$$

$$\underbrace{\phantom{e^{ig\alpha}}}_{1}$$

$$\mathcal{L}' = c\,\overline{\Psi}\,[i\hbar\gamma^\mu\partial_\mu - mc]\,\Psi = \mathcal{L}$$

Se le conoce como transformación de norma, Gauge, global [ec. 6].

¿Y si $\alpha = \alpha(x)$?

Revisemos si $\mathcal{L}$ es invariante ante

(8) $$\Psi' = e^{ig\alpha(x)/\hbar c}\,\Psi$$

Transformación de norma local

$$\mathcal{L}' = c\, e^{-ig\alpha(x)/\hbar c}\, \bar{\Psi}\left[i\hbar\gamma^\mu \partial_\mu - mc\right] \times$$

$$\times \left[e^{ig\alpha(x)/\hbar c}\, \Psi(x)\right]$$

$$= c\, e^{-ig\alpha(x)/\hbar c}\, \bar{\Psi} \Big[i\hbar\gamma^\mu \left[\partial_\mu \Psi(x)\right] e^{ig\alpha(x)/\hbar c}$$

$$+ i\hbar\gamma^\mu \left(\partial_\mu e^{ig\alpha(x)/\hbar c}\right)\Psi$$

$$- mc\,\Psi(x) \Big]$$

$$= c\,\bar{\Psi}\left[i\hbar\gamma^\mu \partial_\mu - mc\right]\Psi$$

$$+ \cancel{c}\, e^{-ig\alpha(x)/\hbar c}\, \bar{\Psi}\left[i\cancel{\hbar}\gamma^\mu e^{ig\alpha(x)/\hbar c}\, \partial_\mu \left[\frac{ig\alpha(x)}{\cancel{\hbar c}}\right]\Psi\right]$$

$$\mathcal{L}' = \mathcal{L} + \underline{\bar{\Psi}\left[i^2 g\,\gamma^\mu \partial_\mu \alpha(x)\right]\Psi}$$

¿Cómo se obtiene una densidad lagrangiana que sea invariante ante las transformaciones de norma local?

Se introduce un objeto A_μ, tal que

$\mathcal{L} = \mathcal{L}'$ con $\psi' = e^{ig\alpha(x)/\hbar c}$

solicitando

$$\partial_\mu \longrightarrow \partial_\mu - \frac{ig}{\hbar c} A_\mu \qquad (9)$$

Además $A'_\mu = A_\mu + \partial_\mu \alpha(x)$ (10)

$$\mathcal{L} = c\bar{\psi}\left[i\hbar\gamma^\mu\left[\partial_\mu - \frac{ig}{\hbar c}A_\mu\right] - mc\right]\psi$$

$$\mathcal{L}' = c\bar{\Psi}\, e^{-ig\alpha(x)/\hbar c} \,*$$

$$\times \left[i\hbar\gamma^\mu\left(\partial_\mu - \frac{ig}{\hbar c} A'_\mu\right) - mc \right] \underline{\Psi'}$$

$\checkmark$

$$= c\bar{\Psi}\, \cancel{e^{-ig\alpha(x)/\hbar c}} \;\times$$

$$\Big[\underline{i\hbar\gamma^\mu\left(\cancel{e^{ig\alpha(x)/\hbar c}}\,\partial_\mu\Psi\right)} + i\hbar\gamma^\mu\partial_\mu\left(\frac{ig}{\hbar c}\alpha(x)\right.$$

$$\times\, \Psi\, \underline{e^{ie\alpha(x)/\hbar c}}$$

$$- i^2 e\gamma^\mu A'_\mu\, \underline{e^{i\alpha(x)g/\hbar c}}\,\Psi - \underline{mc\,\cancel{e^{ig\alpha(x)/\hbar c}}\,\Psi} \Big]$$

$$= c\bar{\Psi}\, i\hbar\gamma^\mu(\partial_\mu - mc)\Psi$$

$$+ c\bar{\Psi}\, e^{-ig\alpha(x)/\hbar c}\left[i\hbar\gamma^\mu e^{i\alpha(x)g/\hbar c}\right] \times$$

$$\left[\frac{ig}{\hbar c}\partial_\mu\alpha(x) - \frac{ie}{\hbar c}A'_\mu\right]\Psi$$

$$\mathcal{L}' = c\bar{\Psi} i\hbar\gamma^\mu (\partial_\mu - mc\Psi)$$

$$+ c\bar{\Psi} i\hbar\gamma^\mu \left(-\frac{ig}{\hbar c}\right)\left[\underline{A_\mu' - \partial_\mu \alpha(x)}\right]\Psi$$

$$\mathcal{L}' = c\acute{\bar{\Psi}} i\hbar\gamma^\mu (\partial_\mu - \underset{\uparrow}{mc})\acute{\Psi}$$

$$+ c\bar{\Psi} i\hbar\gamma^\mu \left(-\frac{ig}{\hbar c}\right) A_\mu \Psi$$

$$\mathcal{L}' = c\bar{\Psi}\left[i\hbar\gamma^\mu\left(\partial_\mu - \frac{ig}{\hbar c}A_\mu\right] - mc\right]\Psi$$

$$= \mathcal{L}$$

$g = e$

Se define $D_\mu = \partial_\mu - \frac{ie}{\hbar c} A_\mu$

$$\mathcal{L} = c\bar{\Psi}\left[i\hbar\gamma^\mu D_\mu - mc\right]\Psi$$

A_μ corresponde al 4-vector electromagnético

A_μ es un campo de norma (Gauge)

La transformación de gauge

$$\boxed{\psi' = e^{ie\alpha(x)/\hbar c}}$$

se le llama $U(1)$

$$\hat{\theta} = e^{ie\alpha(x)/\hbar c}$$

$$\theta^+ = e^{-ie\alpha(x)/\hbar c}$$

$$\theta^+ \hat{\theta} = \mathbb{1} \qquad \theta^+ = \theta^{-1}$$

¿Existen terminos adicionales a $\mathcal{L}$, derivados de la existencia de A_μ?

$m^2 A_\mu A^\mu$ = invariante de Lorentz

No es invariante ante U(1)

$$A'_\mu A'^\mu = [A_\mu + \partial_\mu \alpha][A^\mu + \partial^\mu \alpha]$$

$$\cancel{=} A_\mu A^\mu + 2m A_\mu \partial^\mu \alpha$$

$$+ \partial_\mu \alpha \, \partial^\mu \alpha$$

y el término

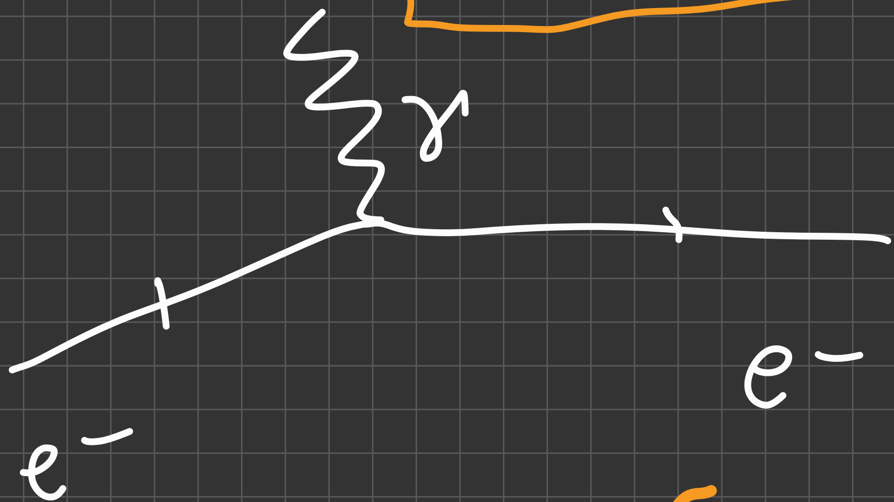
$$-\frac{1}{4} F_{\mu\nu} F^{\mu\nu}$$

Problema. Mostrar que este término es invariante ante U(1)

$$\mathcal{L} = c\bar{\Psi}\left[i\hbar\gamma^{\mu}D_{\mu} - mc\right]\psi - \frac{1}{4}F_{\mu\nu}F^{\mu\nu}$$

Interacción electromagnetica

$$\mathcal{L} = \dots + c\bar{\Psi}\, i\hbar\gamma^{\mu}\left[-\frac{ie}{\hbar c}A_{\mu}\right]\psi + \dots$$



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