La fuerza de Lorentz.

S.I $\quad \vec{F} = q[\vec{E} + \vec{v}\times\vec{B})$

S. Gaussiano $\quad \frac{1}{4\pi\epsilon_0} = 1$

$q \quad \frac{\mu_0}{4\pi} = \frac{1}{c^2}$

$$\vec{F} = q\left[\vec{E} + \frac{\vec{v}}{c}\times\vec{B}\right] \cdots (1)$$

$$\vec{F} = \frac{d\vec{P}}{dt} = q\left[\vec{E} + \frac{\vec{v}}{c}\times\vec{B}\right]$$

$$\frac{d\vec{P}}{dt} = \frac{d\vec{P}}{d\tau}\left[\frac{d\tau}{dt}\right] = \frac{d\vec{P}}{d\tau}\frac{1}{\gamma} = q\left[\vec{E} + \frac{\vec{v}}{c}\times\vec{B}\right]$$

$$\frac{d\vec{P}}{d\tau} = q\left[\gamma\vec{E} + \frac{\gamma\vec{v}}{c}\times\vec{B}\right]$$

$$= \frac{q}{c}\left[(\gamma c)\vec{E} + (\gamma\vec{v})\times\vec{B}\right]$$

$$\boxed{\frac{d\vec{P}}{d\tau} = \frac{q}{c}\left[U_0\vec{E} + \vec{U}\times\vec{B}\right] \cdots (2)}$$

En S.I.

$$\frac{d\vec{P}}{d\tau} = \frac{q}{c}\left[U_0\vec{E} + c\vec{U}\times\vec{B}\right]$$

La energía de una carga q en presencia de campos electromagnéticos

$$\boxed{\frac{d\mathcal{E}}{dt} = q\vec{v}\cdot\vec{E}}$$

Recordamos $P^\alpha = (P_0, \vec{P}) = (mU_0, m\vec{U})$

$P_0 = \mathcal{E}/c$

$$\left[\frac{dP_0}{dt}\right] = \frac{dP_0}{d\tau}\cdot\left(\frac{d\tau}{dt}\right) = \frac{dP_0}{d\tau}\cdot\gamma = \frac{d}{dt}\left[\frac{\mathcal{E}}{c}\right] = \frac{1}{c}\frac{d\mathcal{E}}{dt}$$

$$\frac{dP_0}{d\tau} = \frac{\gamma}{c}q\vec{v}\cdot\vec{E} = \frac{q}{c}[\underbrace{\gamma\vec{v}}_{\vec{U}}]\cdot\vec{E}$$

$$\frac{dP_0}{d\tau} = \frac{q}{c}\vec{U}\cdot\vec{E} \cdots (3)$$

$$\frac{d\vec{P}}{d\tau} = \frac{q}{c}\left[U_0\vec{E} + \vec{U}\times\vec{B}\right] \quad (2)$$

$$\frac{dP_0}{d\tau} = \frac{q}{c}\vec{U}\cdot\vec{E} \quad (3)$$

q = carga electrica = constante
= escalar de Lorentz.

Las ecuaciones del electromagnetismo

La ec. de continuidad

$$\boxed{\frac{\partial\rho}{\partial t} + \nabla\cdot\vec{J} = 0} \qquad \partial^\alpha A_\alpha = \frac{\partial A^0}{\partial x^0} + \nabla\cdot\vec{A}$$

Definiendo $\rho, \vec{J} \rightarrow (c\rho, \vec{J}) = J^\alpha$

$$\boxed{\partial^\alpha J_\alpha = \partial_\alpha J^\alpha = 0}$$

$$\partial'_\alpha J^{\alpha'} = \partial_\alpha J^\alpha = 0$$

Los potenciales : escalar $\varphi \rightarrow \vec{E} = -\nabla\varphi$
Vectorial $\vec{A} \rightarrow \vec{B} = \nabla\times\vec{A}$

$$\frac{1}{c^2}\frac{\partial^2}{\partial t^2}\vec{A} - \nabla^2\vec{A} = \frac{4\pi}{c}\vec{J} \quad ✓$$

$$\frac{1}{c^2}\frac{\partial^2}{\partial t^2}\varphi - \nabla^2\varphi = 4\pi\rho. \quad ✓$$

Condición de Lorentz $\boxed{\frac{1}{c}\frac{\partial}{\partial t}\varphi + \nabla\cdot\vec{A} = 0}$

El operador $\Box = \partial_\alpha\partial^\alpha = \frac{\partial^2}{\partial x^{0\,2}} - \vec{\nabla}^2$

Si se propone $A^\alpha = (\varphi, \vec{A})$

$$\boxed{\Box A^\alpha = \frac{4\pi}{c}J^\alpha}$$

La condición de Lorentz $\boxed{\partial_\alpha A^\alpha = 0}$ ✓

Los campos $\vec{E}, \vec{B}$

$$\vec{E} = -\frac{1}{c}\frac{\partial\vec{A}}{\partial t} - \nabla\varphi \qquad \vec{B} = \nabla\times\vec{A}$$

Componente x

$$E_x = -\frac{1}{c}\frac{\partial A_x}{\partial t} - \frac{\partial\varphi}{\partial x} = -(\partial^0 A^1 - \partial^1 A^0)$$

$$B_x = \frac{\partial}{\partial y}A_z - \frac{\partial}{\partial z}A_y = -(\partial^2 A^3 - \partial^3 A^2)$$

Con $\partial^\alpha = \left(\frac{\partial}{\partial x_0}, -\vec{\nabla}\right)$

Las seis componentes son elementos

$$\boxed{F^{\alpha\beta} = \partial^\alpha A^\beta - \partial^\beta A^\alpha} \rightarrow \text{Contravariante}$$

$$F^{\alpha\beta} = \begin{pmatrix} 0 & -E_x & -E_y & -E_z \\ E_x & 0 & -B_z & B_y \\ E_y & B_z & 0 & -B_x \\ E_z & -B_y & B_x & 0 \end{pmatrix}$$

$$F_{\alpha\beta} = g_{\alpha\gamma}F^{\gamma\delta}g_{\delta\beta} = \begin{pmatrix} 0 & E_x & E_y & E_z \\ -E_x & 0 & -B_z & B_y \\ -E_y & B_z & 0 & -B_x \\ -E_z & -B_y & B_x & 0 \end{pmatrix}$$

↳ Propuesto: Demostrarlo

Se define

$$\epsilon^{\alpha\beta\gamma\delta} = \begin{cases} 1 & \text{si } \alpha=0,\ \beta=1,\ \gamma=2,\ \delta=3 \text{ y cualquier permutación par} \\ & \epsilon^{0123} = 1 \\ & \epsilon^{1032} = 1 \\ -1 & \text{permutaciones impares} \\ & \epsilon^{1023} = -1 \\ & \epsilon^{1230} = -1 \\ 0 & \text{si algun índice es igual a otro} \\ & \alpha=\beta \quad \alpha=\beta=\gamma \\ & \alpha=\gamma \end{cases}$$

$$\boxed{\epsilon_{\alpha\beta\gamma\delta} = -\epsilon^{\alpha\beta\gamma\delta}} \rightarrow \text{Propuesto}$$

Se define el tensor dual

$$\tilde{\mathcal{F}}^{\alpha\beta} = \frac{1}{2}\epsilon^{\alpha\beta\gamma\delta}F_{\gamma\delta} = \begin{pmatrix} 0 & -B_x & -B_y & -B_z \\ B_x & 0 & E_z & -E_y \\ B_y & -E_z & 0 & E_x \\ B_z & E_y & -E_x & 0 \end{pmatrix}$$

Las ecuaciones de Maxwell

$$\nabla\cdot\vec{E} = 4\pi\rho \quad ✓ \qquad \nabla\times\vec{B} - \frac{1}{c}\frac{\partial\vec{E}}{\partial t} = \frac{4\pi}{c}\vec{J} \quad ✓$$

$$\boxed{\partial_\alpha F^{\alpha\beta} = \frac{4\pi}{c}J^\beta} \quad ✓$$

$$\nabla\cdot\vec{B} = 0 \qquad \nabla\times\vec{E} + \frac{1}{c}\frac{\partial\vec{B}}{\partial t} = 0$$

$$\boxed{\partial_\alpha\tilde{\mathcal{F}}^{\alpha\beta} = 0} \quad ✓$$

Otra forma de escribir las ec. homogéneas

$$\boxed{\partial_\alpha F_{\beta\gamma} + \partial_\beta F_{\gamma\alpha} + \partial_\gamma F_{\alpha\beta} = 0}$$

Regresando a la fuerza de Lorentz

$$\boxed{\frac{dp^\alpha}{d\tau} = m\frac{d}{d\tau}U^\alpha = \frac{q}{c}F^{\alpha\beta}U_\beta}$$

$$F_{\alpha\beta}F^{\alpha\beta} = F'_{\alpha\beta}F'^{\alpha\beta}$$

$$\boxed{\mathcal{L} = a\,F_{\alpha\beta}F^{\alpha\beta}}$$