Ecuación de Dirac

El factor exponencial $e^{-\frac{i}{\hbar}\epsilon t}$

$\epsilon = \pm E_p$

La densidad lagrangiana para Dirac

$$\boxed{\mathcal{L} = \overline{\Psi}\left[\underline{c i \hbar \gamma^{\mu}\partial_{\mu}} - m_0 c^2\right]\Psi}$$

Se ha definido el espinor adjunto a Ψ

$$\overline{\Psi} := \Psi^{\dagger}\gamma^{0}$$

donde $\gamma^{0} = \hat{\beta}$ $\quad \gamma^{i} = \hat{\beta}\hat{\alpha}^{i} =$

$$\gamma^{\mu} = (\gamma^{0}, \beta\alpha^{i})$$

$$\mathcal{L} = \bar{\Psi} c i\hbar \underline{\gamma^0 \partial_0} \psi + c\bar{\Psi} i\hbar \gamma^i \partial_i \psi - \bar{\Psi} m_0 c^2 \psi$$

$$\gamma^0 = \hat{\beta} \qquad \underline{\hat{\beta}^2 = \mathbb{1}}$$

$$\mathcal{L} = c i\hbar\, \psi^\dagger \underline{\gamma^0 \gamma^0}\, \underline{\partial_0 \psi} + i\hbar c\, \psi^\dagger \underline{\gamma^0}\, \underline{\gamma^i} \partial_i \psi - \psi^\dagger \gamma^0 m c^2 \psi$$

$$= \cancel{c} i\hbar\, \psi^\dagger \mathbb{1} \frac{\partial}{\partial(\cancel{c}t)} \psi + i\hbar c\, \psi^\dagger \underbrace{\hat{\beta}\beta}_{1} \underline{\alpha^i \partial_i} \psi - \psi^\dagger m_0 c^2 \psi$$

$$\mathcal{L} = c i\hbar\, \psi^\dagger \frac{\partial}{\partial t} \psi + \underline{i\hbar} c\, \psi^\dagger \vec{\alpha} \cdot \vec{\nabla} \psi - \psi^\dagger m_0 c^2 \psi$$

$$\mathcal{L} = \psi^{+}\left[i\hbar \frac{\partial}{\partial t}\psi - c\vec{\alpha}\cdot\hat{\vec{p}}\,\psi - m_0 c^2 \gamma_0 \right]\psi$$

$$\mathcal{L} = \psi^{+}\left[i\hbar\, \partial_t - c\vec{\alpha}\cdot\vec{p} - m_0 c^2 \gamma_0 \right]\psi$$

Obtenemos la ecuación de movimiento

$$\frac{\partial \mathcal{L}}{\partial \bar{\psi}} - \partial_\mu \left[\cancelto{0}{\frac{\partial \mathcal{L}}{\partial(\partial_\mu \bar{\psi})}} \right] = 0$$

$$\frac{\partial}{\partial \bar{\psi}}\left[\underbrace{\bar{\psi}\left[c i\hbar \gamma^\mu \partial_\mu - m_0 c^2 \right]\psi} \right] = 0$$

$$\left[c i\hbar \gamma^\mu \partial_\mu - m_0 c^2 \right]\psi = 0$$

Ecuación de Dirac

Expandimos la ec. de Dirac

$$\left[ci\hbar\, \underline{\gamma^0} \partial_0 + ci\hbar\, \gamma^i \partial_i - m_0 c^2\right]\psi = 0$$

multiplicando a la izq. por $\hat{\beta}$

$$\left[\hat{\beta}\, \underline{ci\hbar\, \hat{\beta}}\, \partial_0 + ci\hbar\, \underline{\hat{\beta}\,\hat{\beta}}\, \underline{\alpha^i \partial_i} - m_0 c^2 \hat{\beta}\right]\psi = 0$$

$$\left[\mathbb{1}\, i\hbar \cancel{c} \frac{\partial}{\partial(\cancel{c}t)} + \underline{ci\hbar\, \mathbb{1}}\, \vec{\alpha}\cdot\vec{\nabla} - m_0 c^2 \hat{\beta}\right]\psi = 0$$

$$i\hbar \frac{\partial}{\partial t}\psi - c\,\vec{\alpha}\cdot\hat{p}\,\psi - m_0 c^2 \hat{\beta}\,\psi = 0$$

$$i\hbar \frac{\partial}{\partial t}\psi = \left[c\,\vec{\alpha}\cdot\hat{p} + \hat{\beta} m_0 c^2\right]\psi$$

¿Y las ec. Euler-Lagrange respecto al espinor ψ?

Ahora $\dfrac{\partial \mathcal{L}}{\partial \psi} - \partial_\mu \left[\dfrac{\partial \mathcal{L}}{\partial(\partial_\mu \psi)} \right] = 0$

$$\mathcal{L} = \bar{\psi}\left[c i \hbar \gamma^\mu \partial_\mu - m_0 c^2 \right] \psi$$

$$\frac{\partial \mathcal{L}}{\partial \psi} = \frac{\partial}{\partial \psi}\left[-\bar{\psi} m_0 c^2 \psi \right] = -m_0 c^2 \bar{\psi}$$

$$\frac{\partial \mathcal{L}}{\partial(\partial_\mu \psi)} = \frac{\partial}{\partial(\partial_\mu \psi)}\left[\bar{\psi} i \hbar c \gamma^\alpha \partial_\alpha \psi \right]$$

$$= \bar{\psi} i \hbar c \gamma^\alpha \frac{\partial(\partial_\alpha \psi)}{\partial(\partial_\mu \psi)}$$

diferente de cero $\alpha = \mu$

$$\frac{\partial \mathcal{L}}{\partial(\partial_\mu \psi)} = \bar{\psi} i \hbar c \gamma^\alpha \delta_\alpha{}^\mu = \bar{\psi} i \hbar c \gamma^\mu$$

El término $\underline{\partial_\mu \left[\frac{\partial \mathcal{L}}{\partial(\partial_\mu \psi)} \right]} =$

$$= \partial_\mu \left[\bar{\Psi} c i \hbar \gamma^\mu \right] = i\hbar c \bar{\Psi} \overleftarrow{\partial_\mu} \gamma^\mu$$

Entonces

$$\underbrace{-\bar{\Psi} m_0 c^2 - i\hbar c \; \bar{\Psi} \overleftarrow{\partial_\mu} \gamma^\mu = 0}$$

$$\boxed{\bar{\Psi} \left[i\hbar c \gamma^\mu \overleftarrow{\partial_\mu} + m_0 c^2 \right] = 0}$$

Calculamos el tensor energía momento

$$T^\mu_{\ \nu} = \underline{\frac{\partial \mathcal{L}}{\partial(\partial_\mu \psi)}} \partial_\nu \psi + \underline{\frac{\partial \mathcal{L}}{\partial(\partial_\mu \bar{\Psi})}} \partial_\nu \bar{\Psi} - \delta^\mu_\nu \mathcal{L}$$

$$\frac{\partial \mathcal{L}}{\partial(\partial_\mu \psi)} = \frac{\partial}{\partial(\partial_\mu \psi)} \left[\underline{\bar{\psi}\, i\hbar c\, \gamma^\alpha} \partial_\alpha \psi - \underline{\bar{\psi} m_0 c^2 \psi} \right]$$

$$= \bar{\psi}\, c i\hbar\, \gamma^\mu$$

$$\frac{\partial \mathcal{L}}{\partial(\partial_\mu \bar{\psi})} = 0$$

Entonces

$$T^\mu{}_\nu = \bar{\psi}\, c i\hbar\, \gamma^\mu \partial_\nu \psi$$

$$- \underline{\delta^\mu_\nu}\, \bar{\psi}\, i\hbar\, \gamma^\alpha \underline{\partial_\alpha} \psi + \delta^\mu_\nu\, m_0 c^2 \bar{\psi} \psi$$

$$T^0{}_0 = \bar{\psi}\, i\hbar c\, \gamma^0 \partial_0 \psi - \underline{\delta^0_0} \bar{\psi}\, \underline{\gamma^\alpha \partial_\alpha} \psi\, i\hbar c$$

$$+ \underline{\delta^0_0}\, m_0 c^2 \bar{\psi} \psi$$

$$= \underline{\cancel{\bar{\psi}\, i\hbar c\, \gamma^0 \partial_0 \psi}} - \underline{\cancel{\bar{\psi} \gamma^0 \partial_0 \psi}}\, i\hbar c$$

$$- \bar{\psi} \gamma^i \partial_i \psi\, i\hbar c + m_0 c^2 \bar{\psi} \psi$$

$$T^0_0 = -\psi^\dagger i\hbar c\, \underline{\gamma^0 \gamma^i}\, \partial_i \psi + m_0 c^2 \bar{\psi} \psi$$

pero $\gamma^0 = \hat{\beta}$ $\quad \gamma^i = \hat{\beta}\hat{\alpha}^i$

$$T^0_0 = -\underline{\psi^\dagger i\hbar c\, \alpha^i \partial_i \psi} + m_0 c^2 \bar{\psi} \psi$$

$$= -c\, i\hbar\, \psi^\dagger \vec{\alpha} \cdot \vec{\nabla} \psi + m_0 c^2 \bar{\psi} \psi$$

$$= c\, \psi^\dagger \vec{\alpha} \cdot \hat{\vec{p}}\, \psi + m_0 c^2 \psi^\dagger \beta \psi$$

$$\boxed{T^0_0 = \psi^\dagger \left[\vec{\alpha} \cdot \hat{\vec{p}}\, c + m_0 c^2 \beta \right] \psi}$$

$$\underline{\left[\vec{\alpha} \cdot \vec{p}\, c + m_0 c^2 \hat{\beta} \right]} = \hat{H}_f$$

$$i\hbar \frac{\partial}{\partial t} \psi = [\, \hat{H}_f \,] \psi$$

$$T^0_0 = \psi^\dagger \hat{H}_F \psi$$

Integrando $\int T^0_0 \, dx^3 = \int \Psi^\dagger \hat{H}_f \Psi \, dx^3$

$= \langle \Psi | \hat{H}_f | \Psi \rangle$ = valor de expectación del eigen valor de $\hat{H}_f$, en el estado Ψ

Por analogía

$$\left.\begin{aligned} T^0_i &= \bar{\Psi} \, i\hbar c \gamma^0 \partial_i \Psi \\ &= \Psi^\dagger (\hat{P})_i \, c \, \Psi \end{aligned}\right\} \text{Mostrar}$$

Luego $\frac{1}{c} \int T^0_i \, dx^3 = \langle \Psi | (\hat{P})_i | \Psi \rangle$

= Valor de expectación del operador de momento lineal [componente iesima] en el estado Ψ

Mostrar que

$$T^i_j = -\psi^\dagger \alpha^i \hat{p}_j c \psi$$

Evaluemos $E = \langle\psi | \hat{H}_f | \psi\rangle = \int T^0_0 \, d x^3$

$$T^0_0 = \psi^\dagger (\vec{\alpha}\cdot\hat{\vec{p}}c + \beta m_0 c^2)\psi$$

Usamos

$$\psi_{p,\lambda,1/2} = N \begin{bmatrix} \begin{pmatrix}1\\0\end{pmatrix} \\ \dfrac{c\sigma_z p}{m_0 c^2 + \lambda E_p}\begin{pmatrix}1\\0\end{pmatrix} \end{bmatrix} \times$$

$$\times \exp\left[\frac{i}{\hbar}(pz - \lambda E_p t)\right]$$

En el espinor:

$$\cdot\begin{bmatrix} \begin{pmatrix}1\\0\end{pmatrix} \\ \dfrac{c\sigma_z P}{m_0c^2+\lambda E_p}\begin{pmatrix}1\\0\end{pmatrix}\end{bmatrix} = \begin{bmatrix}1\\0\\ \dfrac{cP}{m_0c^2+\lambda E_p}\\0\end{bmatrix} \checkmark$$

$$\sigma_z\begin{bmatrix}0\\1\end{bmatrix} = \begin{bmatrix}1 & 0\\0 & -1\end{bmatrix}\begin{bmatrix}1\\0\end{bmatrix} = \begin{bmatrix}1\\0\end{bmatrix}$$

$$E = \int N'\left[1, 0, \frac{cP}{m_0c^2+\lambda E_p}, 0\right]\left[\underline{\vec{\alpha}\cdot\vec{P}c + \hat{\beta}m_0c^2}\right]\cdot N' \times$$

$$\times\begin{bmatrix}1\\0\\ \dfrac{Pc}{m_0c^2+\lambda E_p}\\0\end{bmatrix}\cdot dx^3$$

$$\vec{P} = (0, 0, P) \Rightarrow \vec{\alpha}\cdot\vec{P} = \underline{\alpha_z P_z}$$

$$E = N^2 V \left[1, 0, \frac{pc}{m_0c^2+\lambda E_p}, 0\right] \times \left[pc \begin{bmatrix} 0 & \sigma_z \\ \sigma_z & 0 \end{bmatrix} + m_0 c^2 \begin{bmatrix} 1\!\!1 & 0 \\ 0 & -1\!\!1 \end{bmatrix} \right] \begin{bmatrix} 1 \\ 0 \\ \frac{pc}{m_0c^2+\lambda E_p} \\ 0 \end{bmatrix}$$

$$= N^2 V [\ldots\ldots] \times \left[pc \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{bmatrix} + m_0c^2 \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix} \right]$$

$$\times \begin{bmatrix} 1 \\ 0 \\ pc/m_0c^2+\lambda E_p \\ 0 \end{bmatrix}$$

$$= N^2 V \left[1, 0, \frac{pc}{m_0c^2+\lambda E_p}, 0\right] \left[pc \begin{bmatrix} \frac{pc}{m_0c^2+\lambda E_p} \\ 0 \\ 1 \\ 0 \end{bmatrix} + m_0 c^2 \begin{bmatrix} 1 \\ 0 \\ -pc/m_0c^2+\lambda E_p \\ 0 \end{bmatrix} \right]$$

$$E = N^2 V \left[\frac{2p^2c^2}{m_0c^2+\lambda E_p} + m_0c^2 - \frac{[m_0c^2p^2c^2]}{(m_0c^2+\lambda E_p)^2} \right]$$

$\lambda = \pm 1 \qquad \lambda^2 = 1$

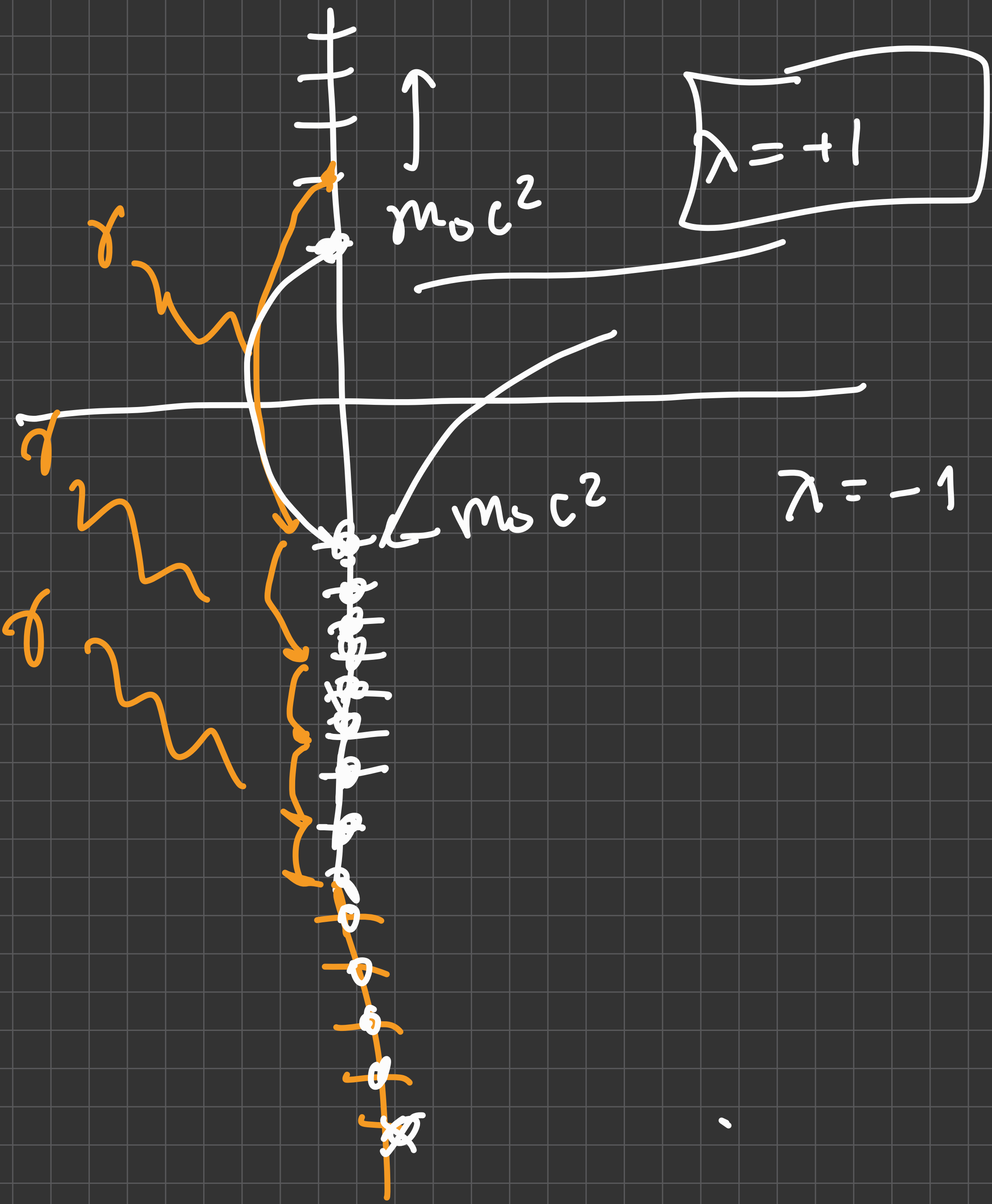
$$E = \underline{N}^2 V \left[m_0c^2 + \lambda E_p \right]^{-1} 2\lambda^2 E_p^2$$

El factor de normalización de la solución a la ec. de Dirac.

$$N = \sqrt{\frac{m_0c^2+\lambda E_p}{2\lambda E_p}} \cdot \frac{1}{\sqrt{V}}$$

$$\boxed{E = \lambda E_p = \pm E_p}$$

$$E_p = \sqrt{p^2c^2 + m_0^2c^4}$$



$m_0 c^2$
$\lambda = +1$
γ
$m_0 c^2$
$\lambda = -1$
γ
γ