Transformación del espinor

$$\hat{S}(\hat{a}) = \hat{S}(\Delta\omega_{\omega}^{\ \nu}) = \hat{S}(\Delta\omega^{\mu\nu})$$

$\Delta\omega^{\mu\nu}$ elementos infinitesimales

$$\hat{S}(\Delta\omega^{\mu\nu}) = \mathbb{1} - \frac{i}{4}\underline{\sigma_{\mu\nu}}\,\underline{\Delta\omega^{\mu\nu}} \quad (1)$$

parámetro infinitesimal

$$\hat{S}^{-1}(\Delta\omega^{\mu\nu}) = \mathbb{1} + \frac{i}{4}\tilde{\sigma}_{\mu\nu}\Delta\omega^{\mu\nu} \quad (2)$$

$\sigma_{\mu\nu} = -\sigma_{\nu\mu} \leftarrow$ determinar

Condición para ec. de Dirac sea covariante

$$\hat{S}(a)\,\gamma^{\nu}\,\hat{S}^{-1}(a) = a_{\mu}^{\ \nu}\gamma^{\mu} \quad (3)$$

Usando 1, 2 y

$$a_\mu{}^\nu = \delta_\mu^\nu + \Delta\omega_\mu{}^\nu$$

en la ec. 3

$$a_\mu{}^\nu \gamma^\mu = \hat{S}(a)\gamma^\nu \hat{S}^{-1}(a)$$

$$(\delta_\mu^\nu + \Delta\omega_\mu{}^\nu)\gamma^\mu = \left(\mathbb{1} - \frac{i}{4}\hat{\sigma}_{\alpha\beta}\Delta\omega^{\alpha\beta}\right)\gamma^\nu \left(\mathbb{1} + \frac{i}{4}\hat{\sigma}_{\alpha\beta}\Delta\omega^{\alpha\beta}\right)$$

$$\delta_\mu^\nu\gamma^\mu + \Delta\omega_\mu{}^\nu\gamma^\mu \approx \mathbb{1}\gamma^\nu + \frac{i}{4}\gamma^\nu\sigma_{\alpha\beta}\Delta\omega^{\alpha\beta} - \frac{i}{4}\hat{\sigma}_{\alpha\beta}\Delta\omega^{\alpha\beta}\gamma^\nu$$

i

$$\boxed{\Delta\omega_\mu{}^\nu\gamma^\mu} = \frac{i}{4}\left[\gamma^\nu\sigma_{\alpha\beta}\Delta\omega^{\alpha\beta} - \sigma_{\alpha\beta}\Delta\omega^{\alpha\beta}\gamma^\nu\right]$$

$$\Delta\omega_{\mu}{}^{\nu}\gamma^{\mu} = \underline{\Delta\omega^{\mu\nu}}\gamma_{\mu}$$

$$= -\Delta\omega^{\nu\mu}\gamma_{\mu}$$

$$= -\underline{\Delta\omega^{\nu\beta}\gamma_{\beta}}$$

$$= -\Delta\omega^{\alpha\beta}\gamma_{\beta}\delta_{\alpha}{}^{\nu}$$

$$= -\Delta\omega^{\alpha\beta}\left[\underline{\delta_{\alpha}{}^{\nu}\gamma_{\beta}}\right]$$

$$= -\frac{1}{2}\Delta\omega^{\alpha\beta}\left[\delta_{\alpha}{}^{\nu}\gamma_{\beta}+\delta_{\alpha}{}^{\nu}\gamma_{\beta}\right]$$

$$= -\frac{1}{2}\Delta\omega^{\alpha\beta}\delta_{\alpha}{}^{\nu}\gamma_{\beta} - \frac{1}{2}\Delta\omega^{\alpha\beta}\delta_{\alpha}{}^{\nu}\gamma_{\beta}$$

$$= -\frac{1}{2}\Delta\omega^{\alpha\beta}\delta_{\alpha}{}^{\nu}\gamma_{\beta} - \underline{\Delta\omega^{\beta\alpha}}\delta_{\beta}{}^{\nu}\gamma_{\alpha}$$

$$= -\frac{1}{2}\Delta\omega^{\alpha\beta}\delta_{\alpha}{}^{\nu}\gamma_{\beta} + \Delta\omega^{\alpha\beta}\delta_{\beta}{}^{\nu}\gamma_{\alpha}$$

$$= -\frac{1}{2}\Delta\omega^{\alpha\beta}\left[\delta_{\alpha}{}^{\nu}\gamma_{\beta} - \delta_{\beta}{}^{\nu}\gamma_{\alpha}\right] \quad \text{iii}$$

Uniendo (i) y con (ii)

$$-\frac{1}{2}\Delta\omega^{\alpha\beta}\left[\delta_\alpha^{\ \nu}\gamma_\beta - \delta_\beta^{\ \nu}\gamma_\alpha\right] =$$

$$= -\frac{i}{4}\Delta\omega^{\alpha\beta}\left[\sigma_{\alpha\beta}\gamma^\nu - \gamma^\nu\sigma_{\alpha\beta}\right]$$

$$\underline{\sigma_{\alpha\beta}\gamma^\nu - \gamma^\nu\sigma_{\alpha\beta}} = -2i\left[\delta_\alpha^{\ \nu}\gamma_\beta - \delta_\beta^{\ \nu}\gamma_\alpha\right]$$

$$\boxed{\left[\sigma_{\alpha\beta}, \gamma^\nu\right] = -2i\left[\delta_\alpha^{\ \nu}\gamma_\beta - \delta_\beta^{\ \nu}\gamma_\alpha\right]}$$

(A)

Como $\Delta\omega^{\mu\nu}$ es antísimétrico,

$\sigma_{\alpha\beta}$ es antisimétrico = 6 parámetros

Se propone $\boxed{\sigma_{\alpha\beta} = \frac{i}{2}\left[\gamma_\alpha, \gamma_\beta\right]}$

y satisface la ec. (A) Ej. propuesto.

El operador S

$$\hat{S}(\Delta\omega^{\mu\nu}) = \mathbb{1} - \frac{i}{4}\sigma_{\mu\nu}\,\underline{\Delta\omega^{\mu\nu}}$$

$$= \mathbb{1} + \frac{1}{8}[\gamma_\mu,\gamma_\nu]\,\Delta\omega^{\mu\nu}$$

$\Delta\omega^{\nu}{}_{\mu}$ contiene la información de las transformaciones de Lorentz

$$\Delta\omega^{\nu}{}_{\mu} = \underline{\underline{\Delta\omega}}\,(\hat{I}_n)^{\nu}{}_{\mu}$$

$\Delta\omega$ = parámetro infinitesimal

$(\hat{I}_n)^{\nu}{}_{\mu}$ = representación matricial 4x4 para una rotación de Lorentz unitaria alrededor de $\underline{\hat{n}}$

Analizamos las transformaciones para los coordenadas

$$x^{\nu'} = a^{\nu}{}_{\mu} x^{\mu}$$

$$a^{\nu}{}_{\mu} = \delta^{\nu}{}_{\mu} + \underline{\Delta \omega^{\nu}{}_{\mu}} \quad \checkmark$$

Debido a que $\underline{dx^{\mu} dx_{\mu}}$ = invariante de Lorentz

$$a^{\mu}{}_{\nu}\, a_{\mu}{}^{\sigma} = \underline{\underline{\delta^{\sigma}{}_{\nu}}} = (\delta^{\mu}{}_{\nu} + \Delta \omega^{\mu}{}_{\nu})(\delta_{\mu}{}^{\sigma} + \Delta \omega_{\mu}{}^{\sigma})$$

$$\approx \underline{\underline{\delta^{\sigma}{}_{\nu} + \Delta \omega_{\nu}{}^{\sigma} + \Delta \omega^{\sigma}{}_{\nu}}}$$

$$\underline{\Delta \omega_{\nu}{}^{\sigma}} + \underline{\Delta \omega^{\sigma}{}_{\nu}} = \underline{g_{\nu\beta} \Delta \omega^{\beta\sigma}}$$

$$+ g_{\nu\beta} \Delta \omega^{\sigma\beta} = 0$$

$$= g_{\nu\beta} [\Delta \omega^{\beta\sigma} + \Delta \omega^{\sigma\beta}] = 0$$

$$\Delta \omega^{\beta\sigma} = -\Delta \omega^{\sigma\beta}$$

Construimos una transformación infinitesimal

Sea $\Delta \omega^{10} = -\Delta \omega^{01} = -\Delta\beta \neq 0$

$$\boxed{\Delta \omega^{01} = \Delta \beta \qquad \Delta \omega^{10} = -\Delta\beta}$$

$$\Delta \omega_1{}^0 = g_{1\beta}\, \Delta \omega^{\beta 0} \qquad \beta = 0,1,2,3$$

$$= g_{11}\, \Delta \omega^{10} + \cancel{g_{12}}\, \cancel{\Delta \omega^{20}} + \cancel{g_{13}}\, \cancel{\Delta \omega^{30}} + \cancel{g_{10}}\, \cancel{\Delta \omega^{00}}$$

$$\Delta \omega_1{}^0 = g_{11}\, \Delta \omega^{10} = (-1)\, \Delta \omega^{10}$$

$$\Delta \omega_1{}^0 = -\Delta \omega^{10} = -[-\Delta\beta] = \Delta\beta$$

$$\Delta \omega_1{}^0 = \Delta\beta \qquad \Delta \omega^{01} = \Delta\beta$$

Ej. propuesto: Mostrar que $\omega_0{}^1 = -\omega^0{}_1$ y

$$\Delta\omega_0{}^1 = -\Delta\omega^0{}_1 = \Delta\beta$$

Por lo tanto $\boxed{\Delta\omega_i{}' = g_{i\sigma}\,\Delta\omega^{\sigma\prime} = 0}$

$i = 1, 2, 3$

Entonces

$$x'^{\nu} = a^{\nu}{}_{\mu}\, x^{\mu} = [\delta^{\nu}{}_{\mu} + \Delta\omega^{\nu}{}_{\mu}]\, x^{\mu}$$

$$x'^{\nu} = [\delta^{\nu}{}_{\mu} + \underset{\substack{\nu=1\\ \mu=0}}{\underline{\qquad}} + \underset{\substack{\nu=0\\ \mu=1}}{\underline{\qquad}}]\, x^{\mu}$$

$$= [\delta^{\nu}{}_{\mu} + \Delta\omega^1{}_0\, \delta^{\nu}{}_1\, \delta^0{}_{\mu} + \Delta\omega^0{}_1\, \delta^{\nu}{}_0\, \delta^1{}_{\mu}]$$

$$\times\, x^{\mu}$$

Ya que $\Delta\omega^{10} = -\Delta\beta = \underline{g^{0\beta}\Delta\omega^{1}{}_{\beta}}$

$$= g^{00}\Delta\omega^{1}{}_{0} = \Delta\omega^{1}{}_{0}$$

$$\underline{\Delta\omega^{1}}{}_{0} = \underline{-\Delta\beta}$$

Así entonces

$$\boxed{x'^{\nu} = [\delta^{\nu}_{\mu} - \Delta\beta\,\delta^{\nu}_{1}\delta^{0}_{\mu} - \Delta\beta\,\delta^{\nu}_{0}\delta^{1}_{\mu}]x^{\mu}}$$

$\nu = 0,1,2,3 \qquad \mu = 0,1,2,3$

$\nu = 0$ — $\qquad \nu = 1, \quad \nu = 2, \quad \nu = 3$

$$x^{0\prime} = [\underline{\delta^{0}_{\mu}} - \Delta\beta\,\delta^{0}_{1}\delta^{0}_{\mu} - \Delta\beta\,\delta^{0}_{0}\delta^{1}_{\mu}]x^{\mu}$$

$$x^{0\prime} = [\delta^{0}_{\underline{\mu}}x^{\mu} - \cancel{\Delta\beta\,\delta^{0}_{\underline{1}}\delta^{0}_{\mu}x^{\mu}}^{\,0}$$
$$- \Delta\beta\,\delta^{0}_{0}\underline{\delta^{1}_{\mu}x^{\mu}}]$$

$$\boxed{x^{0\prime} = x^{0} - \Delta\beta\,x^{1}}$$

Para construir la transformacion finita.

$$\Delta \omega^{\nu}{}_{\mu} = \Delta \omega \left(\hat{I}_n\right)^{\nu}{}_{\mu}$$

$\left(\hat{I}_n\right)^{\nu}{}_{\mu}$ = matriz 4x4, para una transformacion der rotación unitaria alrededor del eje $\hat{n}$.

En el caso de la transformación previamente analizada

$$x'^{\nu} = \underbrace{\left[\delta^{\nu}_{\mu} - \Delta\beta\, \delta_1{}^{\nu} \delta^{0}_{\mu} - \Delta\beta\, \delta_0{}^{\nu} \delta^{1}_{\mu}\right]}_{a_{\mu}{}^{\nu}} x^{\mu}$$

corresponde a una transformación de Lorentz a lo largo del eje x, con velocidad

$\Delta v = c\, \Delta\beta$

Problema Propuesto

Verificar $x^{1'}, x^{2'}, x^{3'}$

$$a^{\nu}{}_{\mu} = \delta^{\nu}{}_{\mu} + \Delta\omega\, (\hat{I}_x)^{\nu}{}_{\mu}$$

Con $\underline{(\hat{I}_x)^{\nu}{}_{\mu} = -\left(\delta_1^{\nu}\,\delta^{0}_{\mu} + \delta_0^{\nu}\,\delta^{1}_{\mu}\right)}$

$$\underline{(\hat{I}_x)^{\nu}{}_{\mu}} = \begin{array}{cccc|c} 0 & 1 & 2 & & \\ \hline \begin{bmatrix} 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} & & & & \begin{matrix} 0 \\ 1 \\ 2 \\ 3 \end{matrix} \end{array}$$

$$(\hat{I}_x)^{\nu=1}{}_{\underline{\mu=0}} = -\left[\delta_1^{1}\,\delta^{0}_{0} + \cancel{\delta_0^{1}\,\delta^{0}_{\mu}}\right]$$

$$= -1$$

$$(I_x)^{0}{}_{1} = -1 = (I_x)^{01}\, g_{11} = I_x^{01}(-1)$$

Se cumple $(I_x)^2 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$

$$(I_x)^3 = \begin{pmatrix} 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} = I_x$$

Se define $\Delta w = \frac{w}{N}$

una sucesión de transformaciones infinitesimales,

$$(X')^{\nu} = \lim_{N \to \infty} \overbrace{\underline{\left(\mathbb{1} + \frac{\omega}{N} I_x\right)^{\nu}_{\nu_1}}}\left(I + \frac{\omega}{N} I_x\right)^{\nu_1}_{\nu_2}$$

$$\ldots\ldots X^{\nu_N}$$

$$= \lim_{N \to \infty} \left[\left(\mathbb{1} + \frac{\omega}{N} I_x\right)^{N}\right]^{\nu}_{\mu} X^{\mu}$$

$$= \left[e^{\omega I_x}\right]^{\nu}_{\mu} X^{\mu}$$

$$(X^{\nu})^{1} = \left[\mathbb{1} - (I_x)^2 + \operatorname{Cosh} \omega (I_x)^2 + \operatorname{Senh} \omega\, \hat{I}_x\right]^{\nu}_{\mu} X^{\mu}$$

En el caso del espinor

$$\Psi'(x') = \hat{S}(\hat{a})\,\Psi(x)$$

$$= \lim_{N\to\infty}\left[\mathbb{1} - \frac{i}{4}\frac{\omega}{N}\hat{\sigma}_{\mu\nu}(I_n)^{\mu\nu}\right]^N \Psi$$

$$= e^{-\frac{i}{4}\omega \frac{\tilde{\sigma}_{\mu\nu}}{4}(\hat{I}_n)^{\mu\nu}}\,\Psi(x)$$

En particular si $\boxed{(\hat{I}_x)^{\nu}_{\mu} = -(\delta_1^{\nu}\delta_{\mu}^{0} + \delta_0^{\nu}\delta_{\mu}^{1})}$

$$\Psi'(x') = \exp\left[-\frac{i}{4}\omega\,\sigma_{\mu\nu}\underline{(I_x)^{\mu\nu}}\right]\Psi(x)$$

$$\underline{(I_x)^{\nu}_{\mu}\, g^{\mu\beta}} = \underline{(I_x)^{\nu\beta}}$$

$$= -[\delta_1^{\nu}\delta_{\mu}^{0}g^{\mu\beta} + \delta_0^{\nu}\delta_{\mu}^{1}g^{\mu\beta}]$$

$$= -[\delta_1^{\nu}\, g^{0\beta} + \delta_0^{\nu}\, g^{1\beta}]$$

Por otro lado

$$\sigma_{\nu\beta}\,(I_x)^{\nu\beta} = -\sigma_{\nu\beta}\left[\delta^{\nu}_{1}\, g^{0\beta} + \delta^{\nu}_{0}\, g^{1\beta}\right]$$

$$= -\sigma_{1}{}^{0} - \sigma_{0}{}^{1}$$

$$= -\sigma_{10}\,\underline{g^{00}} - \sigma_{01}\,\underline{g^{11}}$$

$$= \underline{-\sigma_{10}} \quad + \sigma_{01}$$

$$= \sigma_{01} \quad - \sigma_{10}$$

$$= \sigma_{01} + \sigma_{01} = 2\sigma_{01}$$

$$\psi'(x') = \exp\left[\frac{-i\omega}{2}\sigma_{01}\right]\psi(x)$$

De manera similar, una rotación alrededor del eje z, con angulo φ

I_1, I_2, I_3

$$\Delta\omega_{\mu}^{\ \nu} = \Delta\varphi (\hat{I}_3)_{\mu}^{\ \nu}$$

$$(I_3)_{\mu}^{\ \nu} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

Unicos elementos diferentes de cero,

$$\boxed{(I_3)^{2}{}_{1} = (I_3)^{21} g_{11} = -(I_3)^{21}}$$

Transformación finita

$$\Psi'(x') = \exp\left[-\frac{i}{4}\varphi\, \hat{\sigma}_{\mu\nu}\, (I_3)^{\mu\nu}\right]\Psi(x)$$

Problema propuesto: obtener que

$$\Psi'(x') = \exp\left[\frac{i}{2}\varphi\, \sigma_{12}\right]\Psi(x)$$

Se identifica quien es σ_{12}

$$\underline{\sigma}_{12} = \frac{i}{2}\left[\gamma^1, \gamma^2\right] = \begin{bmatrix} \sigma_p^3 & 0 \\ 0 & \sigma_p^3 \end{bmatrix} = \hat{\Sigma}_3$$

$$\sigma_p^3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

$$\boxed{\Psi'(x') = \exp\left[i(\varphi/2)\hat{\Sigma}_3 \right]\Psi(x)}$$

S_1

En una rotación dada por la ec. anterior obtendremos el mismo espinor, siempre que $\varphi = 4\pi$

$$\exp\left[i\left[(4\pi)/2\right]\hat{\Sigma}_3\right]$$

$$= \exp\left[i(2\pi)\hat{\Sigma}_3\right]$$

Prob. propuesto: Tomar $\varphi = 2\pi, 4\pi$ y obtener $\Psi'(x')$ a partir de S_1

Las observables físicas deben consistir de formas bilineales, es decir, un número par de espinores

Para rotaciones de espinores

$$\hat{S}_R(\omega_{ij}) = \exp\left[-\frac{i}{4}\hat{\sigma}_{ij}\,\omega^{ij}\right]$$

$$i,j = 1,2,3$$

$$S_R^{\dagger}(\omega_{ij}) = \exp\left[\frac{i}{4}\sigma_{ij}^{\dagger}\,\omega^{ij}\right]$$

$$= \exp\left[\frac{i}{4}\sigma_{ij}\,\omega_{ij}\right] = S_R^{-1}$$