Ecuación de Dirac

1928.

Dirac investiga la forma relativista de la ec. de Schrödinger

$$i\hbar \frac{\partial}{\partial t}\Psi = \hat{H}\Psi \quad (1)$$

$$(2) \quad i\hbar \frac{\partial}{\partial t}\Psi = \left\{ \frac{\hbar c}{i}\left[\alpha_1 \frac{\partial}{\partial x_1} + \alpha_2 \frac{\partial}{\partial x_2} + \alpha_3 \frac{\partial}{\partial x_3} \right] + \beta m_0 c^2 \right\} \Psi \equiv \hat{H}_f \Psi$$

Si α_i $(i=1,2,3)$ y β son escalares, la ecuación propuesta no es covariante.

Se propone es que

$$\Psi = \begin{bmatrix} \psi_1 \\ \psi_2 \\ \vdots \\ \psi_N \end{bmatrix}$$

y α_i, β son matrices

Por lo tanto $\rho = \Psi^\dagger \Psi =$

$$= [\psi_1^*, \psi_2^*, \psi_3^*, \dots \psi_N^*] \begin{bmatrix} \psi_1 \\ \psi_2 \\ \psi_3 \\ \vdots \\ \psi_N \end{bmatrix}$$

$$\rho = \sum_{i=1}^{N} \psi_i^* \psi_i > 0$$

Debe ser necesario mostrar que ρ es la componente temporal de j^μ.

Se le denomina espinor $\Psi = \begin{bmatrix} \psi_1 \\ \psi_2 \\ \psi_{\dots} \\ \psi_N \end{bmatrix}$

N = valor ?

α_i, β serán matrices $N \times N$

En términos matriciales

$$i\hbar \frac{\partial}{\partial t} \begin{bmatrix} \psi_1 \\ \vdots \\ \psi_N \end{bmatrix} = \frac{\hbar c}{i} \left[\alpha_1 \frac{\partial}{\partial x_1} + \alpha_2 \frac{\partial}{\partial x_2} + \alpha_3 \frac{\partial}{\partial x_3} \right] \begin{bmatrix} \psi_1 \\ \vdots \\ \psi_N \end{bmatrix}$$

$$+ \beta m_s c^2 \begin{bmatrix} \psi_1 \\ \vdots \\ \psi_N \end{bmatrix}$$

Tomando el elemento σ

$$i\hbar \frac{\partial}{\partial t} \Psi_\sigma = \left\{ \frac{\hbar c}{i} \sum_{\tau=1}^{N} \left(\alpha_1 \frac{\partial}{\partial x_1} + \alpha_2 \frac{\partial}{\partial x_2} + \alpha_3 \frac{\partial}{\partial x_3} \right)_{\sigma\tau} \times \Psi_\tau \right\} + m_s c^2 \sum_{\tau=1}^{N} B_{\sigma\tau} \Psi_\tau$$

Se demanda que sea posible recuperar

i) $E^2 = p^2c^2 + m^2c^4$ ✓

ii) Ecuación de continuidad $\partial_\mu J^\mu = 0$

iii) Covariancia de Lorentz.

Para satisfacer (i), cada componente debe satisfacer

$$-\hbar^2 \frac{\partial^2}{\partial t} \Psi_\sigma = \left[-\hbar^2 c^2 \vec{\nabla} + m_0^2 c^4\right] \Psi_\sigma$$

De la ec (2), iteramos

espinor

$$i\hbar \frac{\partial}{\partial t}\left[i\hbar \frac{\partial}{\partial t} \Psi \right] = -\hbar^2 \frac{\partial^2}{\partial t^2} \Psi$$

$$\left[\frac{\hbar c}{i}\left[\alpha_1 \frac{\partial}{\partial x_1} + \alpha_2 \frac{\partial}{\partial x_2} + \alpha_3 \frac{\partial}{\partial x_3}\right] + \beta m_0 c^2\right]$$

$$= \left[\frac{\hbar c}{i}\left[\alpha_1 \frac{\partial}{\partial x_1} + \alpha_2 \frac{\partial}{\partial x_2} + \alpha_3 \frac{\partial}{\partial x_3}\right] + \beta m_0 c^2\right]$$

$$\Psi$$

$$= \frac{\hbar^2 c^2}{i^2}\left[\left(\alpha_1 \frac{\partial}{\partial x_1} + \alpha_2 \frac{\partial}{\partial x_2} + \alpha_3 \frac{\partial}{\partial x_3}\right)^2\right]\Psi$$

$$+ \beta^2 m_0^2 c^4 \Psi$$

$$+ \frac{\hbar c}{i}\left[\alpha_1 \frac{\partial}{\partial x_1} + \alpha_2 \frac{\partial}{\partial x_2} + \alpha_3 \frac{\partial}{\partial x_3}\right] m_0 c^2 \beta \Psi$$

$$+ \frac{\hbar c}{i} m_0 c^2 \beta \left[\alpha_1 \frac{\partial}{\partial x_1} + \alpha_2 \frac{\partial}{\partial x_2} + \alpha_3 \frac{\partial}{\partial x_3}\right]\Psi$$

$$= -\hbar^2 c^2 \left[\left(\alpha_1 \frac{\partial}{\partial x_1}\right)\left(\alpha_1 \frac{\partial}{\partial x_1}\right)\Psi + \left(\alpha_1 \frac{\partial}{\partial x_1} \alpha_2 \frac{\partial}{\partial x_2}\right)\Psi\right.$$

$$\left. + \left(\alpha_1 \frac{\partial}{\partial x}\right)\left(\alpha_3 \frac{\partial}{\partial x_3}\right)\Psi + \cdots\right]$$

$$
\begin{aligned}
= -\hbar^2 c^2 \Big[& \alpha_1^2 \frac{\partial^2}{\partial x_1^2}\psi + \alpha_1\alpha_2 \frac{\partial^2}{\partial x_1 \partial x_2}\psi \\
& + \alpha_1\alpha_3 \frac{\partial^2}{\partial x_1 \partial x_3}\psi + \alpha_2^2 \frac{\partial^2}{\partial^2 x_2^2}\psi \\
& + \alpha_2\alpha_1 \frac{\partial^2}{\partial x_1 \partial x_2}\psi + \alpha_2\alpha_3 \frac{\partial^2}{\partial x_2 \partial x_3}\psi \\
& + \alpha_3^2 \frac{\partial^2}{\partial x_3^2}\psi + \alpha_3\alpha_1 \frac{\partial}{\partial x_3 \partial x_1}\psi \\
& + \alpha_3\alpha_2 \frac{\partial^2}{\partial x_3 \partial x_2}\psi \Big] \\
+ \frac{\hbar c}{i} m_0 c^3 \Big[& \alpha_1 \beta \frac{\partial}{\partial x_1} + \alpha_2 \beta \frac{\partial}{\partial x_2} + \alpha_3 \beta \frac{\partial}{\partial x_3} \\
& + \beta\alpha_1 \frac{\partial}{\partial x_1} + \beta\alpha_2 \frac{\partial}{\partial x_2} + \beta\alpha_3 \frac{\partial}{\partial x_3} \Big]\psi \\
+ m_0^2 c^4 \beta^2 \psi &
\end{aligned}
$$

Acomodando los términos

$$-\hbar^2 \frac{\partial^2}{\partial t^2}\Psi = -\hbar^2 c^2 \sum_{i,j=1}^{3} \frac{\alpha_i\alpha_j + \alpha_j\alpha_i}{2} \frac{\partial^2 \Psi}{\partial x_i \partial x_j}$$

$$+ \frac{\hbar m_0 c^3}{i} \sum_{i=1}^{3} (\alpha_i \beta + \beta\alpha_i) \frac{\partial \Psi}{\partial x_i}$$

$$+ \beta^2 m_0^2 c^4 \Psi$$

Pero, al comparar esta ecuación con

$$\boxed{-\hbar^2 \frac{\partial^2}{\partial t^2}\Psi_\sigma = \left[-\hbar^2 c^2 \nabla^2 + m_0^2 c^4\right] \Psi_\sigma}$$

se requiere entonces

1. $\alpha_i\alpha_j + \alpha_j\alpha_i = 2\delta_{ij} \mathbb{1}$
2. $\alpha_i\beta + \beta\alpha_i = 0$
3. $\beta^2 = \mathbb{1} = \alpha_i^2$

Por otro lado, se propuso que

$$i\hbar \frac{\partial}{\partial t}\Psi = \underbrace{\left[\frac{\hbar}{c} \sum_{j=1}^{3} \alpha_j \frac{\partial}{\partial x_j} + \beta m_0 c^2 \right]}_{} \Psi$$

$$= \hat{H}_f \Psi$$

Se requiere $\hat{H}_f^{\dagger} = \hat{H}_f$ $[E \in \mathbb{R}]$

entonces $\alpha_i^{\dagger} = \alpha_i$ $\qquad \beta^{\dagger} = \beta$

$$\alpha_i v = \lambda v \quad \text{I}$$

aplicando nuevamente α_i

$$\alpha_i(\alpha_i v) = \alpha_i(\lambda v) \qquad \lambda \text{ no es matriz}$$

$$\underline{\alpha_i^2} v = \lambda \alpha_i v$$

$$\mathbb{1} v = \lambda(\lambda v) = \lambda^2 v$$

$$\lambda^2 = 1 \qquad \boxed{\lambda = \pm 1}$$

Así entonces

$$\alpha_i = \begin{bmatrix} A_1 & 0 & 0 & 0 \\ & A_2 & & \\ & & \ddots & 0 \\ 0 & & & A_N \end{bmatrix}$$

$$\alpha_i^2 = \begin{bmatrix} A_1^2 & & 0 \\ & A_2^2 & \\ 0 & & \ddots & A_N^2 \end{bmatrix} = \mathbb{1} = \begin{bmatrix} 1 & & 0 \\ & 1 & \\ 0 & & 1 \end{bmatrix}$$

$$A_i^2 = 1 \rightarrow A_i = \pm 1$$

Veamos la traza

Condición: $\alpha_i \beta + \beta \alpha_i = 0$

$$\alpha_i \beta = -\beta \alpha_i$$

$$\alpha_i \beta \beta = -\beta \alpha_i \beta$$

$$\alpha_i \beta^2 = \boxed{\alpha_i = -\beta \alpha_i \beta}$$

$$\mathrm{Tr}\,\alpha_i = \mathrm{Tr}\,[-\beta\,\alpha_i\,\beta] = -\mathrm{Tr}\,[\beta\,\alpha_i\,\beta]$$

$$= -\mathrm{Tr}\,[\alpha_i\,\beta\cdot\beta] = -\mathrm{Tr}\,[\alpha_i\,\beta_i^{2}]$$

$$\mathrm{Tr}\,\alpha_i = -\mathrm{Tr}\,(\alpha_i)$$

$$\boxed{\mathrm{Tr}\,\alpha_i = 0}$$

$\boxed{\text{Mostrar que } \mathrm{Tr}\,\beta = 0}$

Puesto que los eigenvalores de α_i y β son ± 1, entonces las matrices deben contener un número igual de eigenvalores positivos y negativos

$$\begin{bmatrix} -1 & & & & 0 \\ & \ddots & & & \\ & & +1 & & \\ & & & +1 & \\ & & & & \ddots \\ 0 & & & & -1 \end{bmatrix}$$

<u>dimension par</u>

Para $N=2$ $\rightarrow$ solo existen 3 matrices que no tienen traza

Para $N=4$ $\rightarrow$ si existen 4 matrices sin traza [mínima representación]

$$\boxed{\alpha_i = \begin{bmatrix} 0 & \sigma_i \\ \sigma_i & 0 \end{bmatrix}} \qquad \beta = \begin{bmatrix} \mathbb{1} & 0 \\ 0 & -\mathbb{1} \end{bmatrix}$$

$$\mathbb{1} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \qquad \beta = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

σ_i = Pauli

$$\sigma_1 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \qquad \sigma_2 = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}$$

$$\sigma_3 = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

Problema propuesto:

Mostrar que se cumple $\alpha_i \alpha_j + \alpha_j \alpha_i =$
$= 2\delta_{ij} \mathbb{1}$

Determinar los eigenvalores de σ_i

Ejercicio propuesto:

Partir de ec. (2)

$$i\hbar \frac{\partial \Psi}{\partial t} = \left[\frac{\hbar c}{i} \sum \alpha_i \frac{\partial}{\partial x_i} + \beta m_0 c^2 \right] \Psi$$

Multiplicar a la izquierda por Ψ^{+}

(3) ....

Tomar + de ec. (2) y multiplicar a la derecha por Ψ

... (4) ... —

restando (3) − (2)

e identificando $\rho = \psi^{+}\psi$

mostrar que

$$\boxed{\frac{\partial \rho}{\partial t} + \sum \frac{\partial}{\partial x_i}\left[c\,\psi^{+}\alpha_i\,\psi\right] = 0}$$

Se identifica el vector

$$\vec{\alpha} = (\alpha^1, \alpha^2, \alpha^3)$$

$$= (-\alpha_1, -\alpha_2, -\alpha_3)$$

entonces $\vec{J} = c\,\psi^{+}\vec{\alpha}\,\psi$

$$\boxed{\frac{\partial \rho}{\partial t} + \nabla \cdot \vec{J} = 0}$$