Kinematic of Vehicles

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Outline

A Car's Model

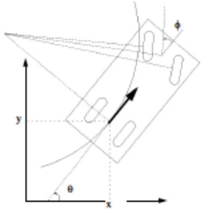


Fig. 1. The car-like model

Figure: A car-like robot

$$
\begin{aligned}
\dot{x} &= v \cos\theta \cos\phi \\
\dot{y} &= v \sin\theta \cos\phi \qquad (1)\\
\dot{\theta} &= v \sin\phi
\end{aligned}
$$

$$
\begin{aligned}
u_1 &= v \cos\phi \\
u_2 &= v \sin\phi
\end{aligned} \qquad (2)
$$

$$|u_1| \leq 1;\ |u_2| \leq 1;\ \phi < \frac{\pi}{4} \qquad (3)$$

A Car's Model

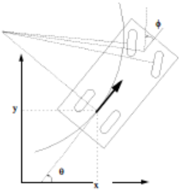


Fig. 1. The car-like model

Figure: A car-like robot

$$\dot{\mathbf{X}} = f(\mathbf{X}, \mathbf{U}) \tag{5}$$

$$\begin{pmatrix} \dot{x} \\ \dot{y} \\ \dot{\theta} \end{pmatrix} = \begin{pmatrix} \cos\theta \\ \sin\theta \\ 0 \end{pmatrix} u_1 + \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} u_2 \tag{6}$$

$$\begin{pmatrix} \dot{x} \\ \dot{y} \\ \dot{\theta} \end{pmatrix} = \begin{pmatrix} \cos\theta & 0 \\ \sin\theta & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} \tag{7}$$

A Second Car's Model

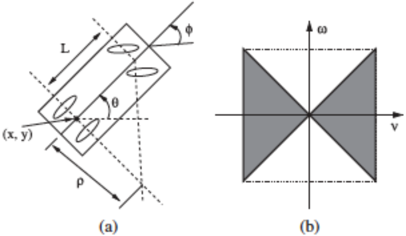
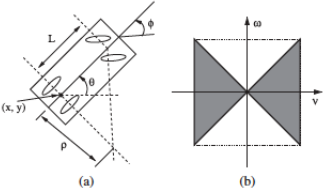


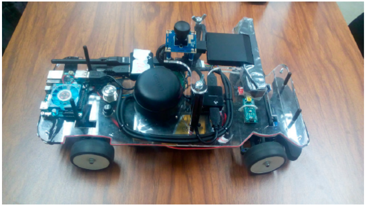
Figure: A second model.

$$
\begin{aligned}
\tan\phi &= \frac{L}{\rho} \\
\frac{dp}{dt} &= \rho\frac{d\theta}{dt} \\
\frac{d\theta}{dt} &= \frac{\tan\phi}{L}\frac{dp}{dt} \\
\dot{\theta} &= v\frac{\tan\phi}{L}
\end{aligned} \tag{8}
$$

A Second Car's Model



(a) Parameters and control space



(b) A car-like robot

Figure: A car-like robot

$$\begin{aligned} \dot{x} &= v \cos\theta \cos\phi \\ \dot{y} &= v \sin\theta \cos\phi \\ \dot{\theta} &= v \frac{\tan\phi}{L} \end{aligned} \tag{9}$$

$$u_1 = v \cos\phi; \;\; u_2 = v \frac{\tan\phi}{L} \tag{10}$$

Differential Equation Modeling the Motion

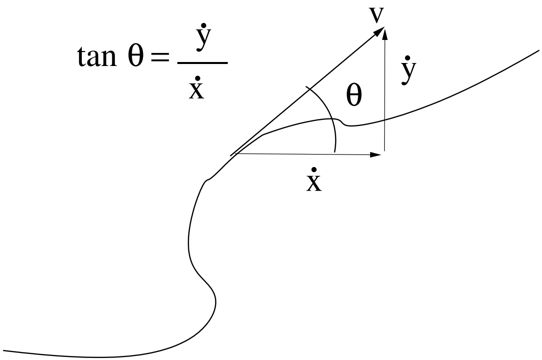
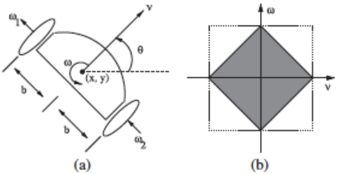


Figure: A car-like robot

$$
\begin{aligned}
\tan\theta &= \frac{\dot{y}}{\dot{x}} \\
\sin\theta\dot{x} - \cos\theta\dot{y} &= 0
\end{aligned}
\tag{11}
$$

Differential Drive Robot



(a) Differential Drive Robot

(b) Differential Drive Robot

Figure: A DDR

$$\begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = \begin{pmatrix} v \\ \omega \end{pmatrix} = \begin{pmatrix} \frac{\omega_l + \omega_r}{2} \\ \frac{\omega_l - \omega_r}{2b} \end{pmatrix}. \tag{12}$$

$$\begin{pmatrix} \dot{x} \\ \dot{y} \\ \dot{\theta} \end{pmatrix} = \begin{pmatrix} \cos\theta & 0 \\ \sin\theta & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} \tag{13}$$

$$|\omega^{max}| \leq \frac{1}{b}(V^{max} - |v|). \tag{14}$$

DDR: Second Order Dynamics

$$\dot{\mathbf{X}} = f(\mathbf{X}, \mathbf{U}) \tag{15}$$

$$\mathbf{X} = (x, y, \theta, v, \omega) \tag{16}$$

$$\begin{pmatrix} \dot{x} \\ \dot{y} \\ \dot{\theta} \\ \dot{w}_r \\ \dot{w}_l \end{pmatrix} = \begin{pmatrix} \frac{w_r + w_l}{2} \cos\theta \\ \frac{w_r + w_l}{2} \sin\theta \\ \frac{w_r - w_l}{2b} \\ 0 \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 0 \end{pmatrix} a_r + \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} a_l \tag{17}$$

Wheels Velocities w_r and w_l and Linear and Angular Velocities v and ω

$$\begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2b} & -\frac{1}{2b} \end{pmatrix} \begin{pmatrix} w_r \\ w_l \end{pmatrix} \tag{18}$$

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \tag{19}$$

$$A^{-1} = \frac{1}{ad - bc} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix} \tag{20}$$

$$A^{-1} = -\frac{1}{2b} \begin{pmatrix} -\frac{1}{2b} & -\frac{1}{2} \\ -\frac{1}{2b} & \frac{1}{2} \end{pmatrix} \tag{21}$$

$$\begin{pmatrix} w_r \\ w_l \end{pmatrix} = \frac{1}{2b} \begin{pmatrix} \frac{1}{2b} & \frac{1}{2} \\ \frac{1}{2b} & -\frac{1}{2} \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} \tag{22}$$

Tools for Kinematic Planning with RRT

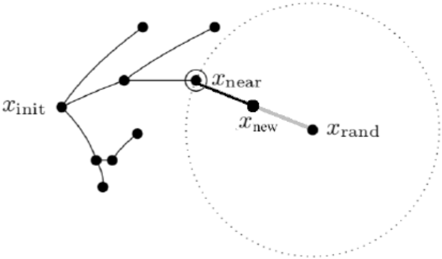


Figure: RRT

- How to generate the trajectory joining x_{near} and x_{new}?
- A Two-Point Boundary Value Problem (BVP)

Tools for Kinematic Planning with RRT

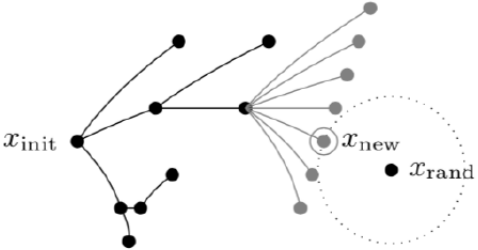


Figure: Generating Controls

- Answer: Simulate the controls
- Two options
  1. Choose u_{new} such that x_{new} is the closest to x_{rand}
  2. Sampling u_{new} from a discret o continuous domain.

Tools for Kinematic Planning with RRT

- $U = \{w_r, w_l\}$, where $w_i \in \{-1, 0, 1\}$
- Δt fixed
- (x_0, y_0, θ_0) initial conditions
- Use a numerical method to integrate the state transition equation $f(\mathbf{X}, \mathbf{U})$

$$
\begin{aligned}
x_{k+1} &= x_k + \frac{w_r + w_l}{2} \cos(\theta_k)\Delta t \\
y_{k+1} &= y_k + \frac{w_r + w_l}{2} \sin(\theta_k)\Delta t \\
\theta_{k+1} &= \theta_k + \frac{w_r - w_l}{2b}\Delta t
\end{aligned} \tag{23}
$$

Metric to measure distances between state in $\mathbb{R}^2 \times S^1$.

$$
d(X, X') = \sqrt{(x - x')^2 + (y - y')^2 + \alpha^2} \tag{24}
$$

$$
\alpha = \min\{|\theta - \theta'|, 2\pi - |\theta - \theta'|\} \tag{25}
$$

Tools for Kinematic Planning with RRT

Algorithm 1: BuildRRT$(x_{\text{init}}, \mathcal{X}_{\text{goal}})$

$V \leftarrow \{x_{\text{init}}\}$;
$E \leftarrow \emptyset$;
while $V \cap \mathcal{X}_{\text{goal}} = \emptyset$ **do**
  $x_{\text{rand}} \leftarrow \text{SampleState}()$;
  $x_{\text{near}} \leftarrow \text{NearestNeighbor}(V, x_{\text{rand}})$;
  $(x_{\text{new}}, u_{\text{new}}, \Delta t) \leftarrow \text{NewState}(x_{\text{near}}, x_{\text{rand}})$;
  if $\text{CollisionFree}(x_{near}, x_{new}, u_{new}, \Delta t)$ **then**
    $V \leftarrow V \cup \{x_{\text{new}}\}$;
    $E \leftarrow E \cup \{(x_{\text{near}}, x_{\text{new}}, u_{\text{new}}, \Delta t)\}$;
return (V, E);

Algorithm 2: NewState$(x_{\text{near}}, x_{\text{rand}})$
(using fixed time step and best-input extension)

$\boldsymbol{u}_{\text{new}} \leftarrow \arg\min_{\boldsymbol{u} \in \mathcal{U}} \{\rho(\text{Simulate}(\boldsymbol{x}_{\text{near}}, \boldsymbol{u}, \Delta t), \boldsymbol{x}_{\text{rand}})\}$;
$\boldsymbol{x}_{\text{new}} \leftarrow \text{Simulate}(\boldsymbol{x}_{\text{near}}, \boldsymbol{u}_{\text{new}}, \Delta t)$;
return $(x_{\text{new}}, u_{\text{new}}, \Delta t)$;

Figure: Algorithms

S. M. LaValle *Planning Algorithms*, Chapter 13, Cambridge University Press, 2006.

T. Kunz and M. Stilman, Kinodynamic RRTs with fixed time step and best-input extension are not probabilistically complete, In Algorithmic foundations of robotics, pp. 233-244, Springer, 2015.