Boundary Element Method for the Exterior Laplace Problem

In this document we cosider the solution method of the exterior Laplace problems by the boundary element method. The purpose is to solve the boundary-value problem[1] consisting of the two-dimensional Laplace equation[2]

$$\nabla^2 \varphi(\boldsymbol{p}) = 0 \qquad (\boldsymbol{p} \in D) \tag{1a}$$

in the domain D exterior to a closed boundary S with a Robin boundary condition of the form

$$\alpha(\boldsymbol{p})\boldsymbol{\varphi}(\boldsymbol{p}) + \beta(\boldsymbol{p})\frac{\partial \varphi}{\partial n_p}(\boldsymbol{p}) = f(\boldsymbol{p}) \quad (\boldsymbol{p} \in S) \tag{1b}$$

where $\boldsymbol{n}_p$ is the unit outward normal to the boundary at $\boldsymbol{p}$.

In order to apply the boundary element method the boundary is approximated by a set of n_S panels[3]

$$S \approx \tilde{S} = \sum_{j=1}^{n_S} \Delta \tilde{S}_j$$

and the boundary functions are approximated or represented by a constant value on each panel[4]. The integral equations within the boundary element method are solved by collocation[5]. By approximating the operators[6] the boundary integral equations are reduced to a linear system of equations[7]. The resulting linear system of equations is solved in order to find the solution on the boundary and this is used in turn in order to find the solution in the exterior domain.

There are a number of integral equation reformulations of the exterior Laplace problem[8]. In this document the various linear systems that arise from these formulations in this reference will be outlined. However, in practical circumstances, only one solution method is required. For simpliciity,. the same notation is used to represent the approximate or representative solutions, the different methods will generally each result in different approximate solutions.

Direct Boundary Element Method

The most straightforward direct formulation is stated in equations (3-4) in *Integral Equation Formulation of the Exterior Laplace Problem*[8]

$$\{ M\varphi \}_S (\boldsymbol{p}) - \varphi(\boldsymbol{p}) = \{ L\, v \}_S (\boldsymbol{p}) \qquad (\boldsymbol{p} \in \mathrm{E}) \tag{1a}$$

---

[1] Boundary Value Problems and Boundary Conditions
[2] Laplace Equation
[3] Representation of a line by flat panels
[4] Piecewise Polynomial Interpolation
[5] Solution of Fredholm Integral Equatins by Collocation
[6] Discretization of the Laplace Integral Operators
[7] Introduction to the Boundary Element Method
[8] Integral Equation Formulation of the Exterior Laplace Problem

$$\{\, M\varphi\}_S\,(\boldsymbol{p}) - \frac{1}{2}\,\varphi(\boldsymbol{p}) \;=\; \{\, L\, v\,\}_S\,(\boldsymbol{p}) \quad (\boldsymbol{p}\, \in\, S)\,. \tag{1b}$$

The application of the collocation method to equation (1b), that is applying the equation to every collocation point $\mathbf{p}$ on S gives the following linear system of equations:

$$(\mathrm{M}_{SS} - ½\mathrm{I})\underline{\hat{\varphi}}_S = \mathrm{L}_{SS}\underline{\hat{v}}_S \tag{2}$$

where L_{SS} and M_{SS} are square $n_S \times n_S$ matrices, I is the $n_S \times n_S$ identity matrix and $\underline{\hat{\varphi}}_S$ and $\underline{\hat{v}}_S$ list the approximate or representative values of φ and $\frac{\partial\varphi}{\partial n}$ at the collocation points. The matrices are defined and computed as integrals, for example the i, j^{th} component of the matrix L_{SS} is the integration with respect to the i^{th} collocation point over the j^{th} panel:

$$[\mathrm{L}_{SS}]_{ij} = \int_{\Delta\tilde{S}_j} G(\boldsymbol{p}_i, \boldsymbol{q})dS_q\,. \tag{3}$$

Except for the special case in which φ or $\frac{\partial\varphi}{\partial n}$ is specified at all collocation points, equation (2) respresents n_S equations with $2n_S$ unknowns. The further set of n_S equations to complete the problem are obtained from the discrete equivalent of the boundary condition (1b):

$$\alpha_{S_i}\,\hat{\varphi}_{S_i} + \beta_{S_i}\,\hat{v}_{S_i} = f_{S_i} \;\text{ for } i = 1 \ldots n\,, \tag{4}$$

Once the discrete equivalent (2) of the boundary integral equation (1b) is solved, solutions in the interior domain D can similarly be found from equation (1a), for example for the exterior point $\boldsymbol{p}_i$:

$$\varphi(\boldsymbol{p}_i) = \{\, M\varphi\}_S\,(\boldsymbol{p}_i) - \{\, L\, v\,\}_S\,(\boldsymbol{p}_i) \qquad (\boldsymbol{p}_i \in D) \tag{5}$$

Let $\underline{\hat{\varphi}}_E$ list the approximations to the solutions at the n_E exterior points $\boldsymbol{p}_i$ for $i = 1, .., n_E$, where the solution is sought, then the discrete equivalent of equation (5) is obtained:

$$\underline{\hat{\varphi}}_E = \mathrm{M}_{ES}\underline{\hat{\varphi}}_S - \mathrm{L}_{ES}\underline{\hat{v}}_S\,. \tag{6}$$

where the $n_E \times n_S$ matrices L_{ES} and M_{ES} are again the discrete forms of the Laplace integral operators.

Through differentiating the integral equation (1a) along the boundary normal, the following integral equation formulation of the interior Laplace equation is obtained[8]

$$\{\, N\varphi\}_S\,(\boldsymbol{p}; \boldsymbol{n}_p\,) - \frac{1}{2}\, v(\boldsymbol{p}) \;=\; \{\, M^t\, v\,\}_S\,\big(\boldsymbol{p}; \boldsymbol{n}_p\big) \quad (\boldsymbol{p}\, \in\, S)\,, \tag{7}$$

Using the same technique as discusssed above, this equation may be represented in discrete form, as follows:

$$\mathrm{N}_{SS}\underline{\hat{\varphi}}_S = (\mathrm{M}_{SS}^t + ½\mathrm{I})\underline{\hat{v}}_S. \tag{8}$$

Equation (8) along with the boundary condition is sufficient information to obtain the approximation to the solutions on the boundary $\underline{\hat{\varphi}}_S$ and $\underline{\hat{v}}_S$ and with these the solution in the domain can be found by equation (6).

With the two direct methods to choose from there is also the potential to use a hybrid of the two[9]:

$$\{(M - \frac{1}{2}I + \mu N)\varphi\}_S(\boldsymbol{p};\, \boldsymbol{n_p}) = \{(L + \mu\left(M^t + \frac{1}{2}I\right)v)\}_S(\boldsymbol{p};\, \boldsymbol{n_p}) \qquad (\boldsymbol{p}\, \in\, S), \tag{9}$$

in which the initial stage of the boundary element method, equation (2) or (8) are replaced by their hybrid

$$(\mathrm{M}_{SS} + ½\mathrm{I} + \mu\mathrm{N}_{SS})\underline{\hat{\varphi}}_S = \left(\mathrm{L}_{SS} + \mu(\mathrm{M}_{SS}^t - ½\mathrm{I})\right)\underline{\hat{v}}_S. \tag{10}$$

Indirect Boundary Element Method

The simplest indirect boundary element method can be obtained by representing the solutionas a simgle-layer potential, resulting in the following[8],

$$\varphi(\boldsymbol{p}) = \{L\sigma_0\}_S(\boldsymbol{p}) \qquad (\boldsymbol{p}\, \in\, E), \tag{11}$$

$$v(\boldsymbol{p}) = \{(M^t + ½I)\sigma_\rho\}_S(\boldsymbol{p}; \boldsymbol{n}_p) \quad (\boldsymbol{p}\, \in\, S)\,. \tag{12}$$

As in the earlier methods, the discrete equivalents of these formulations are as follows:

$$\underline{\hat{\varphi}}_S = \mathrm{L}_{SS}\underline{\widehat{\sigma_0}}_S, \tag{13}$$

$$\underline{\hat{v}}_S = (M_{SS}^t - ½I)\underline{\widehat{\sigma_0}}_S. \tag{14}$$

For the general Robin problem, the approximation to the single-layer potential σ_0 can be found through substitution the expressions for $\underline{\hat{\varphi}}_S$ and $\underline{\hat{v}}_S$ into (4), resulting in the following linear system of equations:

$$(D_\alpha \mathrm{L}_{SS} + D_\beta(\mathrm{M}_{SS}^t - ½\mathrm{I}))\underline{\widehat{\sigma_0}}_S = \underline{f}. \tag{15}$$

where D_α and D_β are diagonal matrices with $[D_\alpha]_{ii} = \alpha_i$ and $[D_\beta]_{ii} = \beta_i$. The discrete equivalent of equation (13) for points in the domain is then utilised in order to find the solution there after substituting the results from the solution of equation (15):

$$\underline{\hat{\varphi}}_E = \mathrm{L}_{ES}\underline{\widehat{\sigma_0}}_S. \tag{16}$$

An alternative method is obtained through representing the solution as a double-layer potential, returning the following boundary integral equations[8]

$$\varphi\,(\mathbf{p}) \;=\; \left\{(M + \frac{1}{2}\,I)\sigma_\infty\right\}_S(\boldsymbol{p}) \quad (\boldsymbol{p}\, \in\, S) \tag{17}$$

[9] Solution of exterior acoustic problems by the boundary element method

$$v(\boldsymbol{p}) \;=\; \{N\,\sigma_\infty\}_S(\boldsymbol{p};\boldsymbol{v}_p) \quad (\boldsymbol{p}\;\in\;S). \tag{18}$$

Following the usual method of collocation, the discrete equivalent of equations (17-18) equations are as follows:

$$\underline{\hat{\varphi}_S} = (\mathrm{M}_{SS}^t + ½\mathrm{I})\underline{\widehat{\sigma_\infty}_S}. \tag{19}$$

$$\underline{\hat{v}_S} = \mathrm{N}_{SS}\underline{\widehat{\sigma_\infty}_S}. \tag{20}$$

Again, substituting these expressions into the generalised boundary condition gives the following linear system:

$$(D_\alpha(\mathrm{M}_{SS}^t + ½\mathrm{I})\underline{\widehat{\sigma_\infty}_S} + D_\beta \mathrm{N}_{SS})\underline{\widehat{\sigma_\infty}_S} = \underline{f}. \tag{21}$$

Once the solution to equation (21) is found the solution for a set of points in the domain can be found from the discrete equivalent of equation[8]:

$$\varphi\,(\mathbf{p}) \;=\; \{M\,\sigma_\infty\}_S(\boldsymbol{p}) \quad (\boldsymbol{p}\;\in\;E) \tag{22}$$

and that is

$$\underline{\hat{\varphi}_E} = \mathrm{M}_{ES}\underline{\widehat{\sigma_\infty}_S}\,. \tag{23}$$

As with the direct method, with the indirect method we have found two solution methods. In addition, also as with the direct method, we also have the potential for hybrid methods[8]

$$\varphi(\boldsymbol{p}) = \{(L+\rho M)\sigma_\rho\}_S(\boldsymbol{p}) \quad (\boldsymbol{p}\;\in\;E)\,. \tag{24}$$

$$\varphi(\boldsymbol{p}) = \{(L+\rho(M+½I)\sigma_\rho)\}_S(\boldsymbol{p}) \quad (\boldsymbol{p}\;\in\;S)\,. \tag{25}$$

$$v(\boldsymbol{p}) = \{(M^t - ½I + \rho N\sigma_\rho)\}_S(\boldsymbol{p};\boldsymbol{n}_p) \quad (\boldsymbol{p}\;\in\;S)\,. \tag{26}$$

Field Modification

If there is an existing field $\varphi^i(\boldsymbol{p})$ then this is effectively superposed. So for example equation (6) is replaced by

$$\underline{\hat{\varphi}_E} = \underline{\varphi_E}^i + \mathrm{M}_{ES}\underline{\hat{\varphi}_S} - \mathrm{L}_{ES}\underline{\hat{v}_S}\,.$$

where $\underline{\varphi_E}^i$ represents the incident potential field at the domain points. The boundary integral equation (2) is replaced by

$$(\mathrm{M}_{SS} - ½\mathrm{I})\underline{\hat{\varphi}_S} = -\underline{\varphi_S}^i + \mathrm{L}_{SS}\underline{\hat{v}_S}$$

where $\underline{\varphi_S}^i$ is the incident field at the boundary collocation points.