Study of the Wilson-Cowan model

The dynamics of the Wilson-Cowan model is governed by the following system of equations

$$\begin{cases} \tau_E \dfrac{dE}{dt}(t) & = & -E(t) + \mathcal{S}_E\left\{c_1 E(t) - c_2 I(t) + P_E(t)\right\} \\ \\ \tau_I \dfrac{dI}{dt}(t) & = & -I(t) + \mathcal{S}_I\left\{c_3 E(t) - c_4 I(t) + P_I(t)\right\} \end{cases} \tag{1}$$

where the nonlinear functions $\mathcal{S}_E$ and $\mathcal{S}_I$ are given by

$$\mathcal{S}_E(x) = \frac{1}{1 + exp(1.2 * (x - 2.8))} - \frac{1}{1 + exp(3.36)}$$

$$\mathcal{S}_I(x) = \frac{1}{1 + exp(1 * (x - 4))} - \frac{1}{1 + exp(4)}$$

and the time constants are $\tau_E = 1\ ms$, $\tau_I = 2\ ms$. The connection strengths parameters c_1, c_2, c_3, c_4 are given below.

Study the dynamics of the Wilson-Cowan model, using two or three of the following tasks as a starting point. Write a jupyter notebook with the code and a brief document commenting your results.
You can use your own implementation of the Wilson-Cowan model, writing the code from scratch, or you can employ the model available in the jupyter notebook at the following github repository:
`https://github.com/alessiomarta/mean_field_wilson_cowan_lecture`.

Task 1 - Find the steady states

Set $c_1 = 9$, $c_2 = 4$, $c_3 = 13$ and $c_4 = 11$ in Equation (1). One way to find the critical points of a dynamical system is to look for the intersections of the two nullcline curves, corresponding to the zeroes of the right hand sides. In our case the two nullcline curves are obtained setting $\frac{dE}{dt} = 0$ and $\frac{dI}{dt} = 0$.

Plot the nullcline curves of the Wilson-Cowan model as per Equation (1) and find an estimate for the points of intersection using a numerical root-finding algorithm.

Task 2 - Classification of the critical points

Critical points can be attractive of repulsive. Classify the nature of the critical point(s) found in **Task 1** using the Jacobian matrix of the dynamical system.

Task 3 - Change the nonlinearities

What happens to the critical points if you change the five parameters (the numbers appearing in the exponentials) of the nonlinear functions $\mathcal{S}_E$ and $\mathcal{S}_I$? Simulate and plot the activity of the population with different choices.

Task 4 - Change the connection strengths parameters

What happens to the critical points if you change the connection strengths parameters c_1, c_2, c_3 and c_4 assigned in **Task 1**? Simulate and plot the activity of the population with different choices of the parameters.

Task 5 - Add an external stimulus

What happens to the critical points if you add an external stimulus? Simulate and plot the activity of the population with different choices of P_E and P_I.

Task 6 - Oscillations

Some populations of neurons (for example the excitatory unipolar brush cells and the inhibitory Golgi cells in the cerebellar cortex) may show an oscillatory behavior, with the spiking rates of the two populations cyclically varying between a maximum and a minimum.

Set $c_1 = 6.4$, $c_2 = 4.8$, $c_3 = 6.0$ and $c_4 = 1.2$ in Equation (1) and suppose to apply an external stimulus P_E to the excitatory subpopulation, leaving the inhibitory subpopulation with no external input ($P_I = 0$).

Find which external stimuli $0.1 \leq P_E \leq 1.2$ give rise to an oscillatory behavior. Consider the initial conditions $E(0) = 0.32$ and $I(0) = 0.15$. In the case there is a limit cycle, determine if it is attractive or repulsive. *Suggestion: Use a phase portrait, employing the function 'quiver' from matplotlib.pyplot to see the direction of the trajectories.*