A Multi-Frequency Approach to Microwave Imaging by Ultrasound Deformation

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Introduction

- Hybrid imaging techniques have been developed in recent years in medical imaging. Two different acquisition modalities are coupled simultaneously to obtain high contrast and high resolution images.
- The reconstruction of the parameters usually involves two steps:
  1. Internal functionals are constructed inside the domain of interest
  2. The parameters have to be found from the knowledge of these internal data
- In step 2. one often needs to construct solutions to the Helmholtz equation (or Maxwell's equations) satisfying certain non-zero properties

Outline of the talk

a. As a motivation: microwave imaging by ultrasound deformation

b. A multi-frequency approach to the boundary control of the Helmholtz equation, in order to obtain solutions satisfying particular non-zero properties

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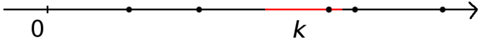
Microwave imaging by ultrasound deformation: Step 1

- $\Omega \subseteq \mathbb{R}^d$, $d = 2, 3$: $\mathcal{C}^2$ bounded domain
- $a \in \mathcal{C}^{0,\alpha}(\overline{\Omega})$: inverse of magnetic permeability
- $q \in L^\infty(\Omega)$: electric permittivity

$$c \leq a, q \leq C \text{ in } \Omega, \qquad a = q = 1 \text{ on } \partial\Omega.$$

- $k \in [K_{min}, K_{max}] \setminus \Sigma$: (frequency)^2



The electric field $u_k^\varphi \in \mathcal{C}^1(\overline{\Omega})$ satisfies

$$\begin{cases} -\mathrm{div}(a\,\nabla u_k^\varphi) - k\,q\,u_k^\varphi = 0 & \text{in } \Omega, \\ u_k^\varphi = \varphi & \text{on } \partial\Omega. \end{cases}$$

By locally perturbing the medium with ultrasounds and measuring the difference of the boundary data we obtain the internal functionals (Ammari et al., SIAP 2011)

$$E_k^{\varphi\psi} = a\,\nabla u_k^\varphi \cdot \nabla u_k^\psi, \qquad e_k^{\varphi\psi} = q\,u_k^\varphi u_k^\psi \quad \text{in } \Omega'.$$

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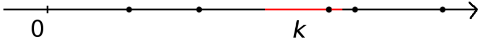

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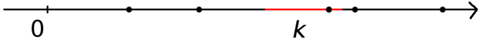

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- $K \subset [K_{min}, K_{max}] \setminus \Sigma$: finite set of frequencies
- φ_1, φ_2 and φ_3: boundary conditions
- $K \times \{\varphi_i\}$: set of measurements

$$\begin{cases} -\mathrm{div}(a\,\nabla u_k^i) - k\,q\,u_k^i = 0 & \text{in } \Omega, \\ u_k^i = \varphi_i & \text{on } \partial\Omega. \end{cases}$$

$$E_k^{ij} = a\,\nabla u_k^i \cdot \nabla u_k^j, \qquad e_k^{ij} = q\,u_k^i u_k^j \qquad \xrightarrow{?} \qquad a, q$$

- Exact formula for a/q (Ammari et al., SIAP 2011)

$$|\nabla(e_k/\mathrm{tr}(e_k))|_2^2\ \frac{a}{q} = 2\,\frac{\mathrm{tr}(e_k)\,\mathrm{tr}(E_k) - \mathrm{tr}(e_k E_k)}{\mathrm{tr}(e_k)^2}.$$

- Exact formula for q (GSA, 2013)

$$-\mathrm{div}\Big(\frac{a}{q}\,\mathrm{tr}(e)\,\nabla \log q\Big) = -\mathrm{div}\Big(\frac{a}{q}\,\nabla\,(\mathrm{tr}(e))\Big) + 2\sum_{k,i}\left(E_k^{ii} - k e_k^{ii}\right)$$

When are these formulae applicable?

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Complete Sets of Measurements

Given a set of measurements $K \times \{\varphi_1, \varphi_2, \varphi_3\}$, it turns out (Ammari et al., SIAP 2011 and GSA, 2013) that the reconstruction formulae are applicable if for every $x \in \Omega'$ there exists $\bar{k}(x) \in K$ such that:

1. $\left|u_{\bar{k}}^1\right|(x) \geq c_1 > 0$,
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This can be generalized to:

A set of measurements $K \times \{\varphi_i : i = 1, \ldots, d+1\}$ is *complete* if for every $x \in \Omega'$ there exists $\bar{k}(x) \in K$ such that:

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The usual conditions are with a fixed frequency and possibly more boundary conditions, but this formulation better suits the following developments.

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Similar conditions arise in various contexts:

- Microwaves and ultrasounds:
  - stability: need 1. (Triki, IP 2010)
  - reconstruction formulae: need 1., 2. and 3. (Ammari et al., SIAP 2011)
- Quantitative thermo-acoustics:
  - stability: need 1. (Bal et al., IP 2011)
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- General elliptic equations (quantitative photo-acoustics, elastography):
  - need 1., 2., 3. and further conditions (Bal and Uhlmann, CPAM 2013)

How can we construct complete sets of measurements?

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Complex geometric optics solutions (Sylvester and Uhlmann, AM 1987) are particular solutions to the Helmholtz equation in $\mathbb{R}^d$

$$u_{k_0}^{t}(x) = a^{-\frac{1}{2}} e^{tx_m} \left(\cos(tx_l) + i\sin(tx_l)\right)(1+\psi_t), \quad t \gg 0.$$

Theorem (Bal and Uhlmann, IP 2010)

If $t \gg 0$ then $u_{k_0}^{t}(x) \approx a^{-\frac{1}{2}} e^{tx_m} \left(\cos(tx_l) + i\sin(tx_l)\right)$ in $C^1(\overline{\Omega})$.

The traces on the boundary of these solutions give the required 1., 2. and 3..

Drawbacks:

- The result holds provided that a and q are smooth enough
- The construction of suitable illuminations depend on the parameters a and q
- Very oscillatory functions: numerically difficult to implement

Is there an alternative way to obtain these suitable illuminations?

Main idea: use different frequencies k.

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Multi-Frequency Approach: basic idea I

As an example, let us consider the 1D case with $a = q = 1$.

1. $\left|u_k^1(x)\right| > 0$: the zero sets of u_k^φ move when k changes...

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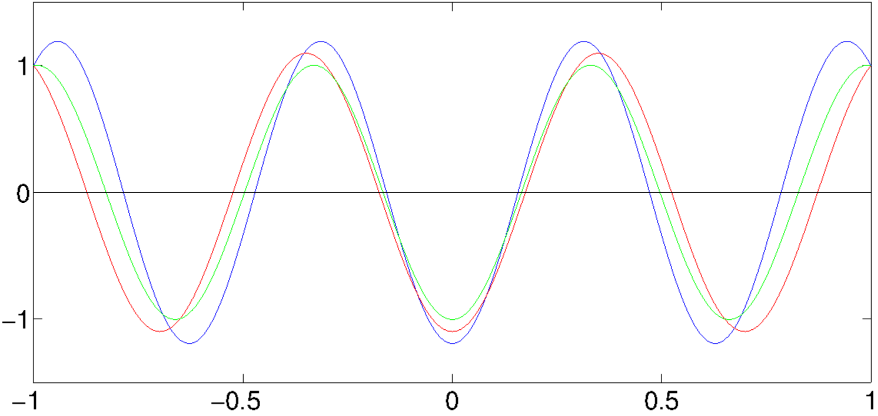
Boundary condition: $\varphi_1 = (1,1)$



Multi-Frequency Approach: basic idea II

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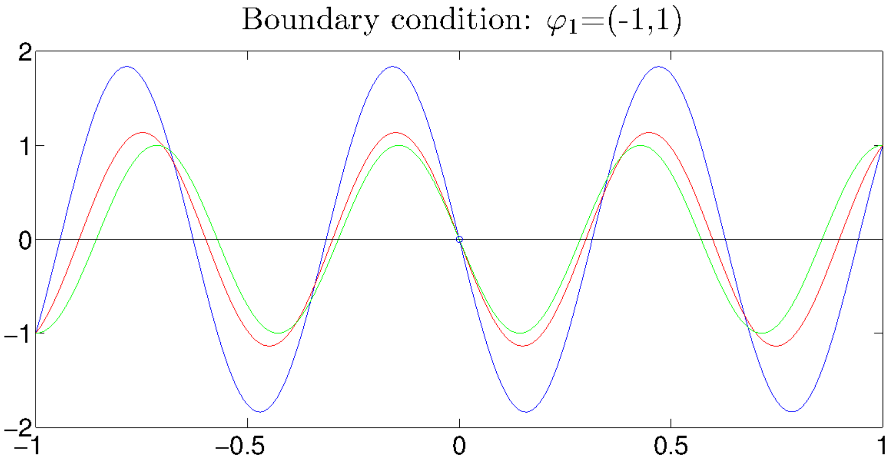
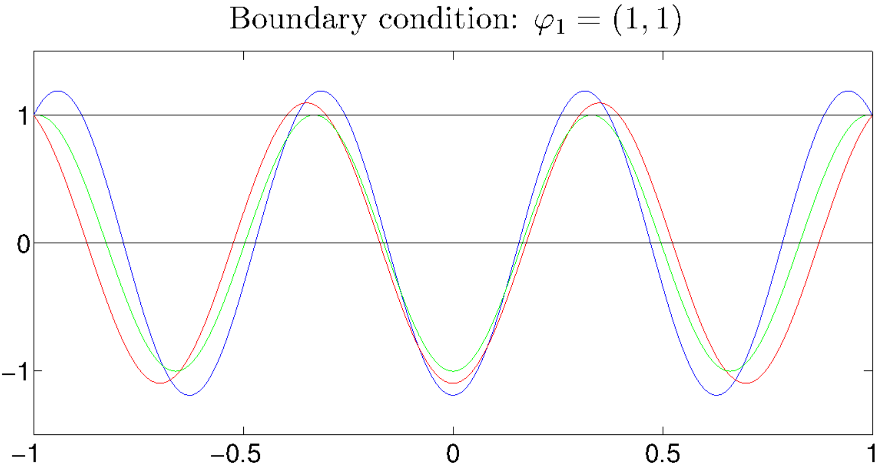


Multi-Frequency Approach: $k = 0$

1. $\left|u_k^1(x)\right| > 0$ everywhere for $k = 0$ $\Longrightarrow$ the zeros "move"



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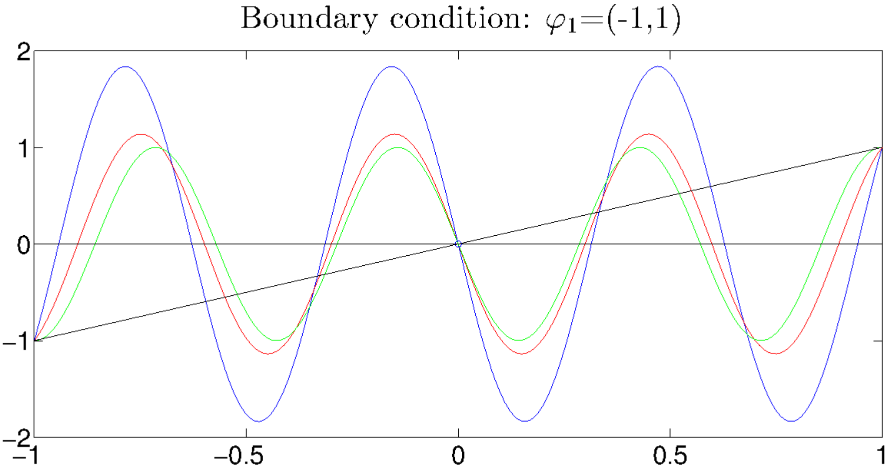

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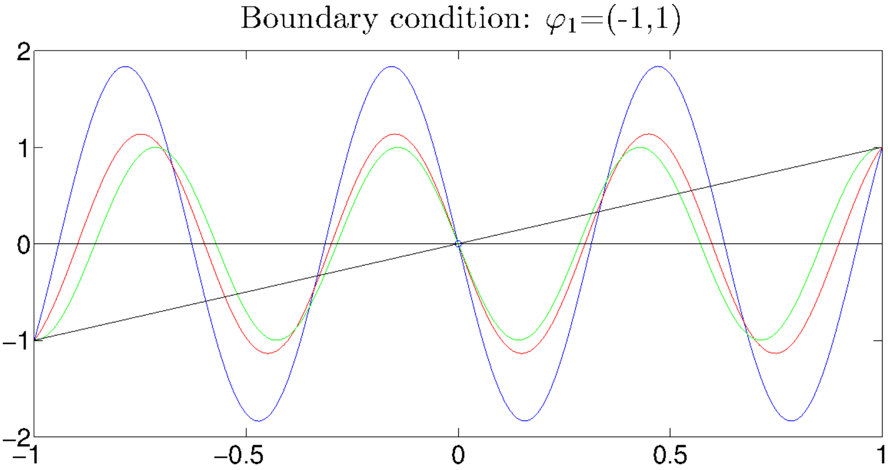

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What happens in $k = 0$?

$$\begin{cases} -\mathrm{div}(a\,\nabla u_k^i) - k\,q\,u_k^i = 0 & \text{in } \Omega, \\ u_k^i = \varphi_i & \text{on } \partial\Omega. \end{cases}$$

$K \times \{\varphi_i : i = 1, \ldots, d+1\}$ is *complete* if for all $x \in \Omega'$ there exists $\bar{k}(x) \in K$ s.t.:

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How to pass from 0 to k?

- Define $T\colon \mathbb{C}\setminus\Sigma \to \mathcal{C}^1(\overline{\Omega})$, $k \mapsto u_k^{\varphi}$.

Lemma

The map $T\colon \mathbb{C}\setminus\Sigma \longrightarrow \mathcal{C}^1(\overline{\Omega})$ is analytic.

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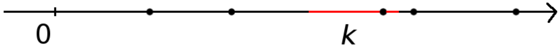

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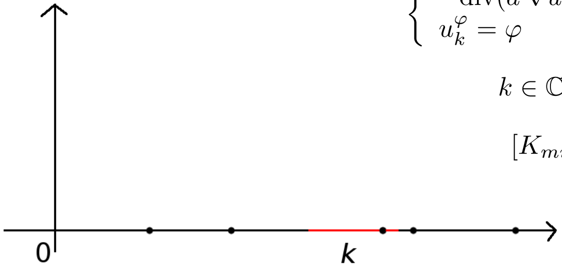

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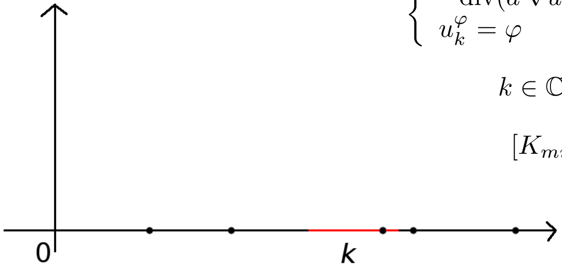

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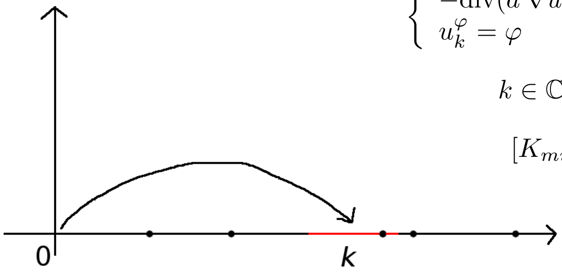

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Main results

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Let $\Omega \subseteq \mathbb{R}^2$ be convex. We can find a finite $K \subseteq [K_{min}, K_{max}]$ such that

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Proof.

- 1., 2. and 3. are satisfied in $k = 0$
- The analyticity of the map $k \mapsto u^{\varphi}_k$ "transfers" these properties to any range of frequencies.

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- The previous theorems give a constructive method to find the frequencies K. However, they do not provide an upper bound on $\#K$.
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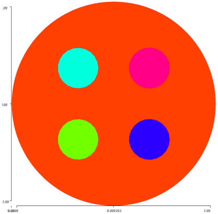

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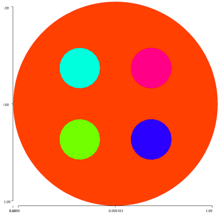

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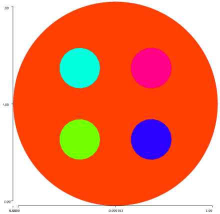

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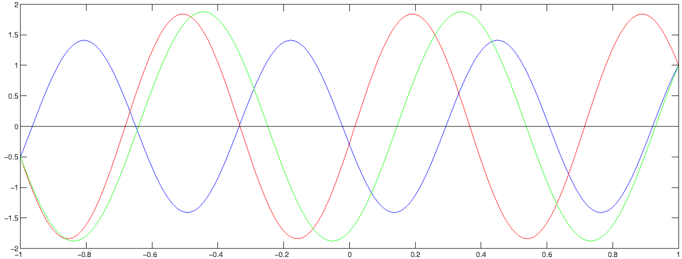

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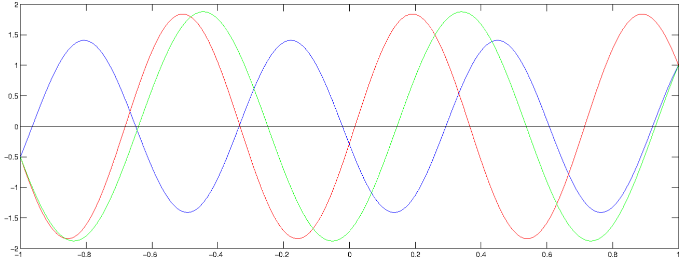
$$\sum_{k \in K} \left| \nabla u_k^{\varphi}(x) \right| \geq c > 0, \qquad \text{in } \Omega.$$

Various comments

- Everything works if a is an anisotropic tensor.
- The same theory applies to the full Maxwell equations.
- The assumption $a \approx 1$ in 3D seems necessary since the determinant of the gradients of solutions of the conductivity equation always vanishes (Briane et al., ARMA 2004). However, this is not needed for the theory to work:



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- In order to use the reconstruction algorithms for this and other hybrid techniques, we need to find illuminations such that the solutions of the Helmholtz equation satisfy some non-zero properties
- These are classically constructed with complex geometric optics solutions

Present

- We propose an alternative to the CGO by using a multi-frequency approach
- Some pros:
  - A priori conditions on the illuminations which do not depend on the coefficients
  - The coefficients do not have to be smooth
  - A few frequencies needed in numerical experiments
- Some cons:
  - No theoretical bounds on the number of frequencies
  - In 3D we currently have to assume $a \approx 1$

Future

- We conjecture that 3 frequencies are sufficient in 2D (with Y. Capdeboscq)
- $a \in \mathcal{C}^{0,\alpha}(\overline{\Omega}) \to a$ piecewise Hölder continuous (with L. Seppecher)
- In 3D, can we drop the assumption $a \approx 1$ by using genericity?

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Thank you for your attention!

Giovanni S Alberti, *On Multiple Frequency Power Density Measurements*, Tech. Report arXiv:1301.1508 (2013).