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Tesi di Laurea

Reproducing Subgroups of the Symplectic Group

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Contents

Notation

Here we fix the notation used in this thesis:

$\mathcal{U}(\mathcal{H})$ denotes the group of unitary operators of a Hilbert space $\mathcal{H}$, equipped with the strong operator topology;

LCSC means locally compact and second countable;

Δ_G denotes the modular function of a LCSC group G;

$H < G$ means that H is a [Lie] subgroup of the [Lie] group G;

$\mathfrak{h} < \mathfrak{g}$ means that $\mathfrak{h}$ is a Lie subalgebra of the Lie algebra $\mathfrak{g}$;

$\mathcal{F} = \hat{}$ denote the Fourier transform on $L^2\left(\mathbb{R}^d\right)$;

$\mathcal{C}_c(G)$ denotes the space of compactly supported continuous functions on a topological group G;

$i_g(h) = ghg^{-1}$ denotes the conjugation of h by g, where g, h are elements of a group;

$I_d = \begin{bmatrix} 1 & & 0 \\ & \ddots & \\ 0 & & 1 \end{bmatrix} \in GL(d, \mathbb{R})$ denotes the identity matrix;

$R_\theta = \left[\begin{smallmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{smallmatrix}\right]$ is the anticlockwise rotation of an angle $\theta \in \mathbb{R}$;

$A_t = \left[\begin{smallmatrix} \cosh t & \sinh t \\ \sinh t & \cosh t \end{smallmatrix}\right]$ is the hyperbolic rotation with $t \in \mathbb{R}$;

$J = \left[\begin{smallmatrix} 0 & I_d \\ -I_d & 0 \end{smallmatrix}\right]$ denotes the standard skew-symmetric form.

Chapter 1

Introduction

Mathematics often deals with simplification, in order to make various objects easier to describe and to work with. For instance, consider a point in the plane and draw an arrow, starting from it. Describing directly and precisely that arrow may be not so easy. In 1637 Descartes and Fermat introduced the Cartesian coordinates, by which a vector v is split in its components v^i, along some principal directions e_i. In modern notation and generalizing to n-dimensional space we can write the initial arrow as $v = v^1 e_1 + \cdots + v^n e_n$, where $v^i = \langle v, e_i \rangle$ are called Fourier coefficients. For certain problems, working with the coordinates of a vector often is more efficient and simpler, especially because we have to deal with real numbers instead of vectors.

In the first years of the XX century Hilbert made an important extension of this construction, passing to infinite dimensional spaces, frequently function spaces. Also in this case we can write a very similar formula for a function f, that is

$$f = \sum_{i=0}^{\infty} \langle f, e_i \rangle \, e_i = \int_{\mathbb{N}} \langle f, e_i \rangle \, e_i \, d\nu(i)$$

where $\{e_i\}$ is an orthonormal basis of the considered space and ν denotes the counting measure. This kind of formulae, called Fourier expansions, may have some applicability difficulties, in case some Fourier coefficients are not known. In order to solve this problem, we could consider redundant systems instead of orthonormal bases in a Hilbert space $\mathcal{H}$: this idea was introduced for the first time in the last decades of the XX century. We obtain a generalization of the previous expansion, namely a weakly convergent integral

$$f = \int_G \langle f, \eta_g \rangle \, \eta_g \, dg,$$

where G is a locally compact group, dg its left Haar measure and $\eta_g \in \mathcal{H}$. The above equality is called *reproducing formula* and G is called *reproducing* if it holds for every $f \in \mathcal{H}$. This theory is mainly applied to signal analysis, with $\mathcal{H} = L^2\left(\mathbb{R}^d\right)$. A natural choice for the elements η_g is to fix a function $\eta \in L^2\left(\mathbb{R}^d\right)$, that is called *admissible vector*, and then setting $\eta_g = \pi(g)\eta$, where π is a unitary representation of G on $\mathcal{H}$. In this thesis we study a special kind of these formulae.

The restriction of the metaplectic representation of $Sp(d,\mathbb{R})$ to its closed subgroups produces a wealth of useful reproducing formulae for functions in $L^2(\mathbb{R}^d)$. This general fact has motivated both a general theory of "mock" metaplectic representations (and the abstract harmonic analysis thereof [DD11]), and a more applications-oriented approach, where the main focus is the actual study of these formulae in connection with the classical themes of signal analysis. In this thesis we study the class $\mathcal{E}$ of closed subgroups of $Sp(d,\mathbb{R})$ that we believe is the "right" class for signal analysis and we illustrate its relevance in $2D$-analysis by exhibiting the full list of reproducing formulae that it yields, up to the appropriate notion of equivalence.

As pointed out in [Cor+06a] and [Cor+06b], many known reproducing formulae (notably those associated to wavelets, shearlets and some of their variants) arise in this way, or are at least are equivalent to them via natural intertwining operators such as the Fourier transform, perhaps combined with geometric (affine) transformations of phase space. But much more is true: all the reproducing groups that we are aware of share the common structural feature that we use to define the class $\mathcal{E}$. Namely, they are block triangular semidirect products of a particular type. Written as $d \times d$ blocks, their elements have the form

$$\begin{bmatrix} h & 0 \\ \sigma h & {}^t h^{-1} \end{bmatrix}$$

where σ ranges in a non trivial vector space Σ of symmetric $d \times d$ matrices (the vector components) and h ranges, independently of σ, in a non trivial closed and connected Lie subgroup H of $GL(d,\mathbb{R})$ (the homogeneous component), that acts on Σ via

$$h^\dagger[\sigma] = {}^t h^{-1} \sigma h^{-1}.$$

Thus, a group in the class $\mathcal{E}$ is a semidirect product $G = \Sigma \rtimes H$.

In Chapter 2 we introduce all the theoretical topics we will need to develop the theory of reproducing formulae and to study the class $\mathcal{E}$. They are mainly the theory of Haar measure on locally compact groups, the theory of image measures and the basic notions of unitary representations.

In Chapter 3 we start by exposing the general setting of reproducing formulae. Successively, we introduce the class $\mathcal{E}$ and we study its basic properties, in relation with the metaplectic representation. We present the basic results of [DD11], that allow us to establish if a group in $\mathcal{E}$ is reproducing or not and to characterize the admissible vectors. We introduce an important notion of equivalence among the groups in the class $\mathcal{E}$, which arises from particular conjugations within $Sp(d,\mathbb{R})$. We explain a possible strategy to determine all the groups in $\mathcal{E}$ in the case $\dim\Sigma = 1$.

Finally, in Chapter 4 we study the case of two-dimensional signals. We classify all the possibilities of reproducing groups in $\mathcal{E}$, up to conjugation in $Sp(2,\mathbb{R})$, by using the theory developed in Chapter 3. In the case where $\dim\Sigma = 2$, we also give a full description of the admissible vectors.

Chapter 2

Preliminaries

In this chapter we introduce, often without proofs, the main topics we will need in the following chapters.

2.1 Locally Compact Groups

Here the main reference is [Fol95]. In this section we develop the theory of Haar measures on LCSC groups; indeed, this subject has a central importance for the following chapters. Let's see why we may need Haar measures to write reproducing formulae.

Recall from the introduction that we would like to express any function $f \in L^2\left(\mathbb{R}^d\right)$ as a combination of particular functions $\eta_g \in L^2\left(\mathbb{R}^d\right)$ indexed on a measure space (G, μ)

$$f = \int_G \langle f, \eta_g \rangle \eta_g \, d\mu\,(g)\,. \tag{2.1}$$

Such formulae often arise by considering a topological group G and a measure μ compatible with the group structure, as we will specify.

2.1.1 Haar Measures

Throughout this section, we fix a topological group G which we assume locally compact, Hausdorff and second countable.

Definition 2.1. A topological space X is called **LCSC** if it is locally compact, Hausdorff and second countable.

The last hypothesis is not necessary for most results that follow. However, we will deal almost only with LCSC spaces for two reasons. First, the second countability often simplifies the theory. Second, this is enough for what we develop.

Thus, for this entire section let G be a LCSC group. Set

$$\mathcal{C}_c^+(G) = \{ f \in \mathcal{C}_c(G) \mid f \geq 0 \,\text{and}\, f \neq 0 \} .$$

We denote by $L_h f$ and $R_h f$ the left and right translates of a function f on G through h, namely

$$L_h f(g) = f\left(h^{-1}g\right), \qquad R_h f(g) = f\left(gh\right), \qquad g \in G.$$

Now we recall the definition of Radon measure.

Definition 2.2. A **Radon measure** on a locally compact Hausdorff space X is a positive Borel measure that is finite on compact sets, outer regular on all Borel sets (that we denote by $\mathcal{B}(X)$) and inner regular on open sets.

Remark 2.3. Note that, if X is LCSC, the last two hypotheses are automatically satisfied, since the measure becomes regular. Hence, in particular, a Radon measure on X is simply a Borel measure that is finite on compact sets. Further, since X is also σ-compact, a Radon measure is σ-finite too.

Definition 2.4. A **left Haar measure** on G is a nonzero Radon measure μ on G such that

$$\mu(gE) = \mu(E) \qquad \text{for all} \quad g \in G,\, E \in \mathcal{B}(G). \tag{2.2}$$

Similarly, a **right Haar measure** must satisfy $\mu(Eg) = \mu(E)$.

In other words, a left (respectively right) Haar measure is invariant under left (resp. right) translations of the group G: this property represents the compatibility between group structure and measure.

The basic properties of Haar measures are collected in the following

Proposition 2.5. *Let λ be a Radon measure on G and define $\rho(E) = \lambda(E^{-1})$ for all $E \in \mathcal{B}(G)$. Then:*

1. *λ is a left Haar measure if and only if ρ is a right Haar measure. In this case*

$$\int_G f\left(g^{-1}\right)\, d\rho = \int_G f(g)\, d\lambda \qquad \textit{for all} \quad f \in \mathcal{C}_c(G);$$

2. *λ is a left Haar measure if and only if*

$$\int_G L_h f\, d\lambda = \int_G f\, d\lambda \qquad \textit{for all} \quad f \in \mathcal{C}_c^+(G),\, h \in G.$$

Thanks to part 1, we can focus on left Haar measures. Note that, if G is Abelian, then the two notions coincide. Part 2 of this Proposition expresses the left invariance of left Haar measures. As a consequence we have

$$\int_G L_h f \, d\lambda = \int_G f \, d\lambda, \qquad h \in G,\ f \in L^1(\lambda) \tag{2.3}$$

For example, $G = \mathbb{R}^n$ with the Euclidean topology and the sum is a LCSC group. Since the Lebesgue measure is invariant under translations, it is a left (and right) Haar measure on $\mathbb{R}^n$. Although in some simple cases the existence of Haar measures is easily established (like in the Euclidean space), the following general theorem holds, as a consequence of the Riesz representation theorem.

Theorem 2.6. *A LCSC group possesses a left Haar measure, that is unique up to a positive constant.*

Note that it is impossible to obtain a better uniqueness result: indeed, if λ is a left Haar measure on G then $c\,\lambda$ is a left Haar measure for all $c > 0$.

We saw that if G is Abelian and λ is a left Haar measure on G, then λ is also a right Haar measure. This is not true in general and so we want to study how far is a left Haar measure λ from being a right Haar measure. For $h \in G$ define $\lambda_h(E) = \lambda(Eh)$. Since

$$\lambda_h(gE) = \lambda\left((gE)h\right) = \lambda\left(g(Eh)\right) = \lambda(Eh) = \lambda_h(E),$$

λ_h is another left Haar measure on G. Hence, by the uniqueness in Theorem 2.6, there exists a number $\Delta(h) > 0$ such that

$$\lambda_h = \Delta(h)\,\lambda.$$

Note that, if $c > 0$ and $\mu = c\,\lambda$, then $\mu_h = c\,\lambda_h$. Therefore $\Delta(h)$ does not depend on the choice of the left Haar measure but only on the group G.

Definition 2.7. The map $\Delta \colon G \to (0, \infty)$ is called the **modular function** of G.

Let's investigate the algebraic properties of Δ. By Definition 2.4, there is $E \in \mathcal{B}(G)$ such that $\lambda(E) > 0$. Hence, if $h_1,\ h_2 \in G$, we have:

$$\Delta(h_1\, h_2)\lambda(E) = \lambda(E\, h_1\, h_2) = \Delta(h_2)\lambda(E\, h_1) = \Delta(h_2)\Delta(h_1)\lambda(E),$$

whence

$$\Delta(h_1\, h_2) = \Delta(h_1)\Delta(h_2), \qquad h_1,\ h_2 \in G.$$

Therefore $\Delta \colon G \to \mathbb{R}_+$ is a group homomorphism from G to the multiplicative group of positive real numbers $\mathbb{R}_+$.

Recall that the left invariance of a left Haar measure involves the relation (2.3). Since the modular function measures how far is a left Haar measure from being a right Haar measure, it allows to compare $\int R_h f \, d\lambda$ with $\int f \, d\lambda$.

Proposition 2.8. *Let λ be a left Haar measure on G and Δ the modular function of G. For every $f \in L^1(\lambda)$ and every $h \in G$*

$$\int_G R_h f \, d\lambda = \Delta\left(h^{-1}\right) \int_G f \, d\lambda, \qquad h \in G.$$

From this proposition and the continuity of the translations on L^1, we deduce the continuity of the modular function.

Corollary 2.9. *The modular function of a LCSC group is continuous.*

Via the modular function, we can introduce an important class of LCSC groups. This notion will be crucial in the following.

Definition 2.10. The group G is called **unimodular** if $\Delta = 1$.

Thus, a group G is unimodular if $\lambda_h = \lambda$ for all $h \in G$, that is, if $\lambda(Eh) = \lambda$ for all $h \in G$ or, equivalently, if left Haar measures on G are also right Haar measures. Actually, in this case, by Proposition 2.8 we have:

$$\int_G R_h f \, d\lambda = \int_G f \, d\lambda \qquad \text{for all} \quad h \in G.$$

Abelian groups are always unimodular, but they are not the only ones.

Proposition 2.11. *If G is compact then G is unimodular.*

Proof. Thanks to Corollary 2.9, $\Delta(G)$ is a compact subgroup of $\mathbb{R}_+$. Therefore $\Delta(G) = \{ 1 \}$, that is $\Delta = 1$. $\square$

2.1.2 Homogeneous Spaces

We need some further basic notions of measure theory.

Definition 2.12. Two Radon measures on a locally compact Hausdorff space are called **equivalent** if they are mutually absolutely continuous.

Now we give some equivalent conditions in the following proposition.

Proposition 2.13. *Let X be a locally compact Hausdorff space and μ, ν two Radon measures on X. Then the following facts are equivalent:*

1. *μ and ν are equivalent;*

2. *$\mu(E) = 0$ if and only if $\nu(E) = 0$ for all $E \in \mathcal{B}(X)$;*

3. *the Radon derivative $\frac{d\mu}{d\nu} \in L^1_{loc}$ and is strictly positive almost everywhere.*

The last condition expresses the equivalence of the measures through a property of their Radon derivatives. We can strengthen this hypothesis.

Definition 2.14. Two Radon measures μ and ν on a locally compact Hausdorff space are called **strongly equivalent** if the Radon derivative

$$\frac{d\mu}{d\nu} \colon X \to (0, \infty)$$

exists and is continuous.

By using the basic properties of absolutely continuous measures it is easy to prove that these two notions do induce equivalence relations among Radon measures on X. We are ready to introduce homogeneous spaces.

Definition 2.15. An **action** of G on a LCSC space X is a continuous map

$$\begin{aligned} G \times X &\longrightarrow X \\ (g, x) &\longmapsto g[x] \end{aligned}$$

such that:

1. the map $x \mapsto g[x]$ is a homeomorphism for all $g \in G$;
2. $(gh)[x] = g\,[h[x]]$ for all g, $h \in G$ and $x \in X$.

In this case, the space X is called G-**space**.

If $x_0 \in X$, the subset $G[x_0] = \{\, g[x_0] \mid g \in G \,\}$ of X is called the **orbit** of x_0. Define $G_{x_0} = \{\, g \in G \mid g[x_0] = x_0 \,\}$, a closed subgroup (easy verification) of G called the **stabilizer** of x_0. It consists of the elements of G whose action doesn't move the point x_0. The action is called **transitive** if $G[x_0] = X$ for all $x_0 \in X$ and in this case X is called a **homogeneous space**.

If H is a closed subgroup of G we denote by $G/H = \{\, gH : g \in G \,\}$ the set of left cosets modulo H equipped with the quotient topology.

Proposition 2.16. *Let X be a homogeneous space and $x_0 \in X$. The map*

$$\begin{aligned} \Phi \colon G/G_{x_0} &\longrightarrow X \\ g\,G_{x_0} &\longmapsto g[x_0] \end{aligned} \tag{2.4}$$

is a homeomorphism.

The inverse of Φ is determined by a map

$$\begin{aligned} X &\longrightarrow G \\ x &\longmapsto g_x \end{aligned} \tag{2.5}$$

such that $g_x[x_0] = x$ for all $x \in X$, whose existence is a consequence of the transitivity of the action considered. Although this map is not unique, Lemma 1.1 on [Mac52] allows to choose it measurably. Further,

In order to study the problem of the existence of a Radon measure on X that is invariant under the action of G, we begin with the following Definition.

Definition 2.17. Let μ be a nonzero Radon measure on X and for $g \in G$ denote by μ_g the measure defined by $\mu_g(E) = \mu(g[E])$, for $E \in \mathcal{B}(X)$. The measure μ is called **G-invariant** if

$$\mu_g = \mu \qquad \text{for all } g \in G. \tag{2.6}$$

In some situations, G-invariant measures exist. For instance, consider $X = G$ and define $g[x] = g\,x$, that is the action of G on itself determined by the left translations. In this case, we already know that an invariant measure exists: the left Haar measure. Hence, in this context, Haar measures are a particular case of G-invariant measures.

As we did in the Proposition 2.5 part b), we can rewrite (2.6) for a G-invariant measure μ in the following way:

$$\int_X \varphi\left(g^{-1}[x]\right)\, d\mu(x) = \int_X \varphi(x)\, d\mu(x), \qquad g \in G,\ \varphi \in \mathcal{C}_c(X). \tag{2.7}$$

Unfortunately, the situation usually is not as simple. Indeed, G-invariant measures often don't exist.

Example 2.18. Consider the group $G =$ "$ax + b$" acting on $X = \mathbb{R}$ by the natural action $(b,a)[x] = ax + b$ for all $a > 0$, $b \in \mathbb{R}$ and $x \in \mathbb{R}$. The only measure (up to a positive constant) on $\mathbb{R}$ that is invariant under the translations $x \mapsto x + b$ is the Lebesgue measure, that is not invariant under the dilatations $x \mapsto ax$. Hence this homogeneous space doesn't possess a G-invariant measure μ, that is such that $\mu_{(b,a)} = \mu$ for all $(b,a) \in$ "$ax + b$". Alternatively, denoting with $|\ |$ the Lebesgue measure on $\mathbb{R}$ we can compute, for $E \in \mathcal{B}(\mathbb{R})$

$$|E|_{(b,a)} = |(b,a)[E]| = |a\,E| = a\,|E|.$$

Although for $a \neq 1$ we have $|\ |_{(b,a)} \neq |\ |$, this two measures are connected: precisely, they are all (for $a > 0$) strongly equivalent to each other. Indeed, the map $x \mapsto a$ is continuous and strictly positive, and so the measures $|\ |$ and $|\ |_{(b,a)}$ satisfy the condition in Definition 2.14. Moreover, the map

$$\begin{aligned} \lambda \colon \text{“}ax + b\text{”} \times X &\longrightarrow (0, \infty) \\ ((b,a), x) &\longmapsto a \end{aligned}$$

is jointly continuous in (b,a) and in x. Therefore, although it is impossible to determine a G-invariant measure, the Lebesgue measure does a good job.

Generalizing the ideas of the previous example, we can give a weaker definition of invariance, with the hope that in this case we will not face non-existence problems.

Definition 2.19. A nonzero Radon measure μ on X is called **strongly quasi-invariant** if there is a continuous function $\lambda\colon G \times X \to (0, \infty)$ such that for all $g \in G$ and $x \in X$

$$d\mu_g(x) = \lambda(g, x)\, d\mu(x).$$

Thus we require all the measures μ_g to be strongly equivalent. Further, we need the Radon Nykodym derivatives $\frac{d\mu_g}{d\mu}(x) = \lambda(g, x)$ to be jointly continuous in g and in x. Recall Example 2.18: the Lebesgue measure is a strongly quasi-invariant measure on $\mathbb{R}$ under the action of the group "$ax + b$".

In this case, the formula (2.7) transforms in a natural way to the relation:

$$\int_X \varphi\left(g^{-1}[x]\right)\, d\mu(x) = \int_X \varphi\left(g^{-1}[g[x]]\right)\, d\mu_g(x) = \int_X \varphi(x)\lambda(g, x)\, d\mu(x) \tag{2.8}$$

for all $g \in G$ and $\varphi \in \mathcal{C}_c(X)$. By Proposition 2.8 the modular function quantifies how much a left Haar measure fails to be right invariant, and so it permits to find a relation between the two integrals. Now, the function λ does the same job: it measures the non-invariance of μ and so it appears in the formula if μ is not G-invariant.

For completeness, we present another concept of invariance even if it will not be studied here.

Definition 2.20. A nonzero Radon measure μ on X is called **quasi-invariant** if the measures μ_g are all equivalent.

Thus, we have defined three concepts of invariance for a Radon measure, starting from the strongest: invariant, strongly quasi-invariant and quasi-invariant measures. We know that the first one is too strict a request, but what about the other two? Recall the Latin quotation "in medio stat virtus": probably we should study the intermediate case. The rest of this section will be devoted to obtain a theorem of existence of a strongly quasi-invariant measure on a homogeneous space X and to determine its properties.

Thanks to Proposition 2.16, we suppose $X = G/H$ where H is a closed subgroup of G. Denote by dg and dh two left Haar measures, respectively, on G and on H and with Δ_G and Δ_H their modular functions. We define a map $P\colon \mathcal{C}_c(G) \longrightarrow \mathcal{C}_c\left(G/H\right)$ by

$$Pf(g\, H) = \int_H f(g\, h)\, dh \tag{2.9}$$

that is well-defined by the left invariance of the measure dh. The following Proposition is the first fundamental step to reach our purpose.

Proposition 2.21. *Let H be a closed subgroup of G. There is a continuous function $\rho\colon G \to (0,\infty)$ such that for all $g \in G$ and $h \in H$*

$$\rho(g\,h) = \frac{\Delta_H(h)}{\Delta_G(h)}\,\rho(g). \tag{2.10}$$

*Such a function is called **rho-function** for the pair (G,H).*

We will see that a strongly quasi-invariant measure on X depends on the choice of a rho-function. So, before determining the measure, we consider a rho-function ρ for the pair (G,H) and we define the function $\lambda_\rho\colon G \times G/H \longrightarrow (0,\infty)$ by

$$\lambda_\rho(g_1, g_2\,H) = \frac{\rho(g_1\,g_2)}{\rho(g_2)}. \tag{2.11}$$

for all g_1, $g_2 \in G$. Let us verify that it is well-defined. If $g_1, g_2 \in G$ and $h \in H$ the following equalities hold, thanks to (2.10):

$$\lambda_\rho(g_1, g_2 h\,H) = \frac{\rho(g_1\,g_2 h)}{\rho(g_2 h)} = \frac{\Delta_H(h)}{\Delta_G(h)}\,\rho(g_1\,g_2)\,\frac{\Delta_G(h)}{\Delta_H(h)}\,\frac{1}{\rho(g_2)} = \frac{\rho(g_1\,g_2)}{\rho(g_2)} = \lambda_\rho(g_1, g_2\,H).$$

Now we are ready to state the existence theorem.

Theorem 2.22. *Let H be a closed subgroup of G and ρ be a rho-function for the pair (G,H). There is a unique strongly quasi-invariant measure μ^ρ on G/H such that for all $f \in \mathcal{C}_c(G)$*

$$\int_{G/H} Pf\,d\mu^\rho = \int_G f(g)\rho(g)\,dg.$$

The measure μ^ρ satisfies

$$\frac{d\mu^\rho_{g_1}}{d\mu^\rho}(g_2\,H) = \lambda_\rho(g_1, g_2\,H). \tag{2.12}$$

There is also a sort of uniqueness theorem. Indeed, we may ask if all the strongly quasi-invariant measures on a homogeneous space arise from a rho-function. The answer is positive.

Theorem 2.23. *If μ is a strongly quasi-invariant measure on X then there is a rho-function ρ such that $\mu = \mu^\rho$.*

We have come to an end relatively to the study of strongly quasi-invariant measures on a homogeneous space X. What can we say now about the existence of a G-invariant measure? Remember the equation (2.10)

$$\rho(g\,h) = \frac{\Delta_H(h)}{\Delta_G(h)}\,\rho(g).$$

Suppose that $\Delta_H = \Delta_{G|H}$. In this case we can simply choose $\rho = 1$. Hence, by (2.11), $\lambda_\rho = 1$ holds. Therefore, by (2.12), the strongly quasi-invariant measure μ^ρ obtained by Theorem 2.22 actually satisfies $\mu^\rho_g = \mu^\rho$ for all $g \in G$, namely the measure μ^ρ is G-invariant. The following theorem explains that the condition $\Delta_H = \Delta_{G|H}$ is also necessary.

Theorem 2.24. *Let H be a closed subgroup of G. There is a G-invariant measure on G/H if and only if $\Delta_H = \Delta_{G|H}$.*

2.2 Image and Pseudo-Image Measures

Here the main references are [Sch93] for the first part and [Bou59] for the second part. We deal with a common problem in mathematics, namely the transfer of a particular structure of a space onto another space by means of a map. For example, we can pull back the metric from a manifold to another. In this case, we want to deal with measures and we would like to be able to *push forward* a measure.

Consider a σ-finite measure space $(X, \mathcal{S}, \mu)$, where $\mathcal{S}$ is a σ-algebra and μ a measure. Let $H \colon X \to Y$ be a measurable map into a measure space $(Y, \mathcal{T})$. Our goal is to transfer the measure μ on Y via the map T. Since we want to obtain a σ-finite measure on Y, we must suppose the existence of a family $\{ Y_n \in \mathcal{T} : n \in \mathbb{N} \}$ such that

$$Y = \bigcup_{n\in\mathbb{N}} Y_n \quad \text{and} \quad \mu\left(H^{-1}(Y_n)\right) < \infty \quad \text{for all } n \in \mathbb{N}.$$

Remark 2.25. If H is an injection and H^{-1} is measurable then the previous condition is trivially satisfied. Indeed, take $Y_n = H(X_{n-1})$ for $n > 0$ and $Y_0 = Y \setminus \operatorname{Im} H$, where $X_n \in \mathcal{S}$ are given by the hypothesis of σ-finiteness of μ. We have $Y = \bigcup_{n\in\mathbb{N}} Y_n$ and

$$\begin{aligned} &\mu\left(H^{-1}(Y_0)\right) = \mu(\emptyset) = 0 < \infty && \text{if } n = 0 \\ &\mu\left(H^{-1}(Y_n)\right) = \mu\left(H^{-1}(H(X_{n-1}))\right) = \mu\left(X_{n-1}\right) < \infty && \text{if } n > 1 \end{aligned}$$

since H is one-to-one.

In these hypotheses we can define

$$H\mu(B) = \mu\left(H^{-1}(B)\right) \quad \text{for all } B \in \mathcal{T} \tag{2.13}$$

and the following theorem holds.

Theorem 2.26. *The function $H\mu \colon \mathcal{T} \to [0, \infty]$ is a σ-finite measure on Y.*

Thanks to this result we reached our objective and we can give the following definition.

Definition 2.27. The measure $H\mu$ on Y defined by (2.13) is called the **image measure** (or **push forward measure**) of μ under H.

The following theorem presents the basic property of image measures.

Theorem 2.28. *If $f \in L^1(H\mu)$ then $f \circ H \in L^1(\mu)$ and*

$$\int_X f \circ H \, d\mu = \int_Y f \, dH\mu.$$

Finally, we give an interesting result that relates Haar measures with image measures. Since it is not in the cited books we prove it.

Proposition 2.29. *Let G_1, G_2 be two LCSC groups and let $\varphi\colon G_1 \to G_2$ a topological group isomorphism. Denote by μ_1 and μ_2, respectively, their left Haar measures. Then:*

1. *$\varphi\,\mu_1$ is a left Haar measure on G_2;*
2. *there exists $c > 0$ such that $\varphi\,\mu_1 = c\,\mu_2$;*
3. *if $f \in L^1(\mu_2, \mathbb{C})$ then $f \circ \varphi \in L^1(\mu_1, \mathbb{C})$ and there exists $c > 0$ such that*

$$\int_{G_1} f \circ \varphi \, d\mu_1 = c \int_{G_2} f \, d\mu_2.$$

Proof. Note that $\varphi\,\mu_1$ is well-defined because G_1 is a LCSC space, μ_1 a Radon measure and φ a homeomorphism (see Remarks 2.3 and 2.25). In order to prove 1, let $E \in \mathcal{B}(G_2)$ and $g \in G_2$. We have

$$\varphi\,\mu_1(g\,E) = \mu_1\left(\varphi^{-1}(g\,E)\right) = \mu_1\left(\varphi^{-1}(g)\,\varphi^{-1}(E)\right) = \mu_1\left(\varphi^{-1}(E)\right) = \varphi\,\mu_1(E)$$

whence $\varphi\,\mu_1$ is left invariant. We still need to prove that $\varphi\,\mu_1$ is a Radon measure: since G_2 is LCSC, by Remark 2.3, it is sufficient to show that it is finite on compact sets. If $K \in \mathcal{B}(G_2)$ is a compact set, then $\varphi^{-1}(K)$ is compact too since φ^{-1} is continuous. Hence

$$\varphi\,\mu_1(K) = \mu_1\left(\varphi^{-1}(K)\right) < \infty$$

because μ_1 is a Radon measure.

In order to prove 2 it is sufficient to apply Theorem 2.6. Finally, part 3 follows from part 2 and from Theorem 2.28. □

Finally, we want to weaken this notion.

Definition 2.30. Let X and Y be two locally compact spaces. Let μ a positive measure on X and $H\colon X \to Y$ a μ-measurable map. A positive Borel measure ν on Y is called **pseudo-image measure** of μ under H if for every $E \in \mathcal{B}(Y)$, $\nu(E) = 0$ if and only if $\mu\left(H^{-1}(E)\right) = 0$.

The following result proves the existence of pseudo-images measures.

Proposition 2.31. *Let X and Y be two locally compact spaces. Let μ a positive measure on X and $H\colon X \to Y$ a μ-measurable map. If X is σ-compact then there exists e pseudo-image measure on Y of μ under H.*

2.3 Unitary Representations

Here the main reference is [Fol95]. In this section we present the fundamental notions of the theory of unitary representations.

As we saw in section 2.1, we would like to express a function f in $L^2\left(\mathbb{R}^d\right)$ as a superposition of particular functions $\eta_g \in L^2$:

$$f = \int_G \langle f, \eta_g \rangle \eta_g \, d\mu\,(g) \tag{2.14}$$

where G is a LCSC group with its left Haar measure μ. Let's focus on the choice of the functions η_g and consider a possible way to define them. Let $\eta_g = \pi\,(g)\,\eta$, where $\pi\,(g)$ is a unitary operator on L^2 and η is a fixed element of L^2. In other words, the functions η_g are the images, through several unitary operators indexed on a group G, of a particular function $\eta \in L^2$. Thus, the determination of the functions η_g is based on a fixed vector η and a family of unitary operators.

Definition 2.32. Let G be a LCSC group, $\mathcal{H}$ a nonzero Hilbert space and denote by $\mathcal{U}\,(\mathcal{H})$ the group of unitary operators on $\mathcal{H}$, equipped with the strong operator topology. A continuous group homomorphism $\pi\colon G \to \mathcal{U}\,(\mathcal{H})$ is called a **unitary representation** of G on $\mathcal{H}$, that is called **representation space** of π.

Remark 2.33. By definition, a map $\pi\colon G \to \mathcal{U}\,(\mathcal{H})$ is a unitary representation if

$$\begin{aligned}
&\pi\,(g_1\, g_2) = \pi\,(g_1)\,\pi\,(g_2) && \text{for all} \quad g_1, g_2 \in G,\\
&\pi\left(g^{-1}\right) = \pi(g)^{-1} = \pi(g)^* && \text{for all} \quad g \in G,\\
&g \mapsto \pi(g)u && \text{is continuous for all} \quad u \in \mathcal{H}.
\end{aligned}$$

One can easily verify that this last condition is equivalent to

$$g \mapsto \langle \pi(g)u, v\rangle \qquad \text{is continuous for all} \quad u, v \in \mathcal{H},$$

which expresses the continuity respect to the weak operator topology on $\mathcal{U}\,(\mathcal{H})$. Thanks to the relations

$$\begin{aligned}\|\pi(g)u - \pi(g_0)u\| &= \|\pi(g_0)\pi(g_0^{-1}g)u - \pi(g_0)u\| \\ &= \|\pi(g_0)\left(\pi(g_0^{-1}g)u - u\right)\| \\ &= \|\pi(g_0^{-1}g)u - u\|,\end{aligned}$$

it is enough to verify the continuity of the representation in e.

Now we present a simple construction that will be useful. Consider two LCSC groups G_1 and G_2 and a continuous homomorphism $\varphi\colon G_1 \to G_2$. Trivially, by Definition 2.32, we have that if $\pi\colon G_2 \to \mathcal{U}(\mathcal{H})$ is a unitary representation of G_2, then $\pi \circ \varphi$ is a unitary representation of G_1. In particular, if $\varphi\colon H \hookrightarrow G$ is the immersion of a subgroup $H < G$ of G, we obtain $\pi \circ \varphi = \pi_{|H}$ that is simply the restriction of π to H. Further, we can consider another kind of restriction, described by the following

Definition 2.34. Let $\pi\colon G \to \mathcal{U}(\mathcal{H})$ be a unitary representation. A closed subspace $X \subseteq \mathcal{H}$ is called π-**invariant** if $\pi(G)X \subseteq X$. Then, if X is π-invariant, we have $\pi(g)_{|X} \in \mathcal{U}(\mathcal{H})$ for all $g \in G$. Thus we obtain a new representation

$$\begin{aligned}\pi_{|X}\colon G &\longrightarrow \mathcal{U}(\mathcal{H}) \\ g &\longmapsto \pi(g)_{|X}\end{aligned}$$

which is called **subrepresentation** of π.

Now we point out a first important result. As we will understand from the proof, this proposition doesn't hold for an arbitrary representation, for which $\pi(g) \in \mathcal{L}(\mathcal{H})$.

Proposition 2.35. *Let $\pi\colon G \to \mathcal{U}(\mathcal{H})$ be a unitary representation. If a subspace $X \subseteq \mathcal{H}$ is π-invariant, then $X^\perp$ is π-invariant.*

Proof. Let $y \in X^\perp$. For all $g \in G$ and $x \in X$ we have

$$\langle \pi(g)y, x\rangle = \langle y, \pi(g)^* x\rangle = \langle y, \pi(g)^{-1}x\rangle = \langle y, \pi(g^{-1})x\rangle = 0$$

because $\pi(g^{-1})x \in X$ and $y \in X^\perp$. Then $\pi(g)y \in X^\perp$. □

The spaces 0 and $\mathcal{H}$ always are π-invariant for a unitary representation $\pi\colon G \to \mathcal{U}(\mathcal{H})$, hence they are trivial invariant subspaces. There is an important class of unitary representations, for which there aren't other invariant subspaces.

Definition 2.36. A unitary representation $\pi\colon G \to \mathcal{U}(\mathcal{H})$ is called **irreducible** if it has no nontrivial π-invariant subspaces. Otherwise, it is called **reducible**. A vector $v \in \mathcal{H}$ is called **cyclic** if

$$\overline{\operatorname{span}\{\, \pi(g)v : g \in G \,\}} = \mathcal{H}.$$

Switching to the orthogonal, we obtain that v is cyclic if and only if

$$\{\, \pi(g)v : g \in G \,\}^{\perp} = \emptyset. \tag{2.15}$$

As we will see in detail, the notion of cyclic vector is fundamental. If $\eta \in L^2\left(\mathbb{R}^d\right)$ is a cyclic vector, then the functions $\eta_g = \pi(g)\eta$ span the entire space: they may be suitable to obtain the decomposition (2.14).
There is a strong connection between the two concepts introduced in the previous definition, the following proposition explains it.

Proposition 2.37. *Let $\pi\colon G \to \mathcal{U}(\mathcal{H})$ be a unitary representation. Then π is irreducible if and only if every $0 \neq v \in \mathcal{H}$ is cyclic.*

Now, consider a reducible unitary representation. A question naturally arises: can we reduce it in simpler components? What does it mean?

Definition 2.38. Let $\{\pi_i\}_{i\in I}$ be a family of unitary representations of a LCSC group G on the Hilbert spaces $\mathcal{H}_i$. Consider their direct sum $\mathcal{H} = \bigoplus\limits_{i\in I}\mathcal{H}_i$ and define the representation $\bigoplus\limits_{i\in I}\pi_i$ on $\mathcal{H}$ by

$$\bigoplus_{i\in I}\pi_i(g)\left(\sum_{i\in I} v_i\right) = \sum_{i\in I}\pi_i(g)v_i$$

for all $v_i \in \mathcal{H}_i$ such that $\sum_{i\in I}\|v_i\|^2 < \infty$. The representation $\bigoplus\limits_{i\in I}\pi_i$ is called the **direct sum** of the $\{\pi_i\}_{i\in I}$.

Note that, since $\|\pi(g)v_i\| = \|v_i\|$, $\bigoplus\limits_{i\in I}\pi_i$ is well-defined and unitary. The continuity follows from the Lebesgue convergence theorem, indeed

$$\|\sum_{i\in I}\pi_i(g)v_i - \sum_{i\in I}v_i\|^2_{\mathcal{H}} = \sum_{i\in I}\|\pi_i(g)v_i - v_i\|^2_{\mathcal{H}_i}$$
$$\|\pi_i(g)v_i - v_i\|_{\mathcal{H}_i} \xrightarrow[g\to e]{} 0 \quad \text{and} \quad \sum_{i\in I}\|\pi_i(g)v_i - v_i\|^2_{\mathcal{H}_i} \leq 4\sum_{i\in I}\|v_i\|^2_{\mathcal{H}_i} < \infty.$$

Corollary 2.39. *Let $\pi\colon G \to \mathcal{U}(\mathcal{H})$ be a unitary representation. If $X \subseteq \mathcal{H}$ is a non trivial invariant subspace, then $\pi = \pi_{|X} \oplus \pi_{|X^\perp}$.*

Thus, we are able to decompose a reducible representation into (at least) two subrepresentations: this explains the reason of the names in the definition 2.36.

Let's analyze a typical example of unitary representation, that will play a role in the following chapters.

Example 2.40. Let G be a LCSC group and λ a left Haar measure of G. Define $\mathcal{H} = L^2(\lambda)$ and consider the map $\pi_L \colon G \to \mathcal{U}\left(L^2(\lambda)\right)$ defined by

$$[\pi_L(g)f]\,(h) = f\left(g^{-1}h\right) \qquad \text{for all} \quad f \in L^2(\lambda) \quad g, h \in G.$$

Let's prove it is well-defined. By left invariance of the measure λ

$$\int_G |\pi_L(g)f|^2\, d\lambda = \int_G |f\left(g^{-1}h\right)|^2\, d\lambda(h) = \int_G |f|^2\, d\lambda.$$

Hence $\pi_L(g)$ is an isometry. It is an easy exercise to verify the homomorphism properties. Since $\pi_L(g^{-1}) = \pi_L(g)^{-1}$, $\pi_L(g)$ is unitary. The continuity of π follows from the continuity of the translations on $L^2(\lambda)$ and from Remark 2.33. Therefore π_L is a unitary representation on $L^2(\lambda)$, which is called the **left regular representation**. Similarly, if we consider $\mathcal{H} = L^2(\rho)$ where ρ is a right Haar measure on G and define $[\pi_R(g)f]\,(h) = f(h\,g)$, we obtain that π_R is a unitary representation on $L^2(\rho)$, called the **right regular representation**.

These two representations are substantially the same. Suppose $\rho(E) = \lambda\left(E^{-1}\right)$ for all Borel sets $E \subseteq G$ [1] and consider the operators

$$U \colon L^2(\lambda) \to L^2(\rho)\,, \qquad U f(g) = f\left(g^{-1}\right) \qquad\qquad U' \colon L^2(\rho) \to L^2(\lambda)\,. \qquad U' f(g) = f\left(g^{-1}\right)$$

We have

$$\|Uf\|^2_{L^2(\rho)} = \int_G |f\left(g^{-1}\right)|^2\, d\rho(g) = \int_G |f(g)|^2\, d\lambda(g) = \|f\|^2_{L^2(\lambda)}$$

because $d\lambda(g) = d\rho\left(g^{-1}\right)$. Then U is an isometry and since $U' = U^{-1}$ it is also a unitary operator.

Consider the following diagram

$$\begin{array}{ccc} L^2(\lambda) & \xrightarrow{\pi_L(g)} & L^2(\lambda) \\ \uparrow{\scriptstyle U^{-1}} & & \downarrow{\scriptstyle U} \\ L^2(\rho) & \xrightarrow{?} & L^2(\rho) \end{array}$$

and *tranfer* $\pi_L(g)$, via the operator U, on $L^2(\rho)$. What is the result to write on the lower arrow? Let's compute the composition: if $f \in L^2(\rho)$ and $h \in G$ we have

$$\left[U\,\pi_L(g)\,U^{-1}\,f\right](h) = \left[\pi_L(g)\,U^{-1}\,f\right]\left(h^{-1}\right) = U^{-1}\,f\left(g^{-1}\,h^{-1}\right) = f(h\,g) = [\pi_R(g)\,f]\,(h).$$

[1] This is always true, up to a constant.

Therefore, $U\,\pi_L(g)\,U^{-1} = \pi_R(g)$ holds or, equivalently,

$$U\,\pi_L(g) = \pi_R(g)\,U \qquad \text{for all} \quad g \in G.$$

This means that π_L and π_R are the *same* representation, modulo a unitary operator that identifies $L^2(\lambda)$ with $L^2(\rho)$ and transforms the action of G from one space to the other. We will say that these two representations are **equivalent**.

Now, we are ready to make these ideas more rigorous and more general, in order to introduce a notion of equivalence of representations.

Definition 2.41. Let π_1 and π_2 be two unitary representations of a LCSC group G with representation spaces $\mathcal{H}_1$ and $\mathcal{H}_2$ respectively. An **intertwining operator** for π_1 and π_2 is a continuous linear map $T\colon \mathcal{H}_1 \to \mathcal{H}_2$ such that

$$T\,\pi_1(g) = \pi_2(g)\,T \qquad \text{for all} \quad g \in G. \tag{2.16}$$

Denote by $\mathcal{C}\,(\pi_1, \pi_2)$ the set of all such operators and by $\mathcal{C}(\pi) := \mathcal{C}\,(\pi, \pi)$. If $\mathcal{C}\,(\pi_1, \pi_2) \cap \mathcal{U}\,(\mathcal{H}_1, \mathcal{H}_2) \neq \emptyset$ we say that π_1 and π_2 are **unitarily equivalent**.

It is easy to prove that this notion yields an equivalence relation. Going back to the Example 2.40, since U is a unitary operator, the representations π_L and π_R are unitarily equivalent. In analogy with the example, we may interpret (2.16) as the commutativity of the following diagram

$$\begin{array}{ccc} \mathcal{H}_1 & \xrightarrow{\pi_1(g)} & \mathcal{H}_1 \\ \Big\downarrow T & & \Big\downarrow T \\ \mathcal{H}_2 & \xrightarrow{\pi_2(g)} & \mathcal{H}_2 \end{array} \qquad \text{for all} \quad g \in G.$$

Finally, we give a fundamental theorem of this theory.

Theorem 2.42 (Schur's Lemma)**.** *Let π, π_1 and π_2 be unitary representations of a LCSC group G on three Hilbert spaces. One has*

1. *π is irreducible if and only if $\mathcal{C}(\pi) = \{c\,\mathrm{id} : c \in \mathbb{C}\}$;*
2. *if π_1, π_2 are irreducible and equivalent then $\dim \mathcal{C}\,(\pi_1, \pi_2) = 1$;*
3. *if π_1, π_2 are irreducible but not equivalent then $\mathcal{C}\,(\pi_1, \pi_2) = \{0\}$.*

Chapter 3

Reproducing Formulae

3.1 Basic Notions

In this section we present the notion of reproducing formula, which is our primary object of study. We will deal only with a continuous setting, even though the discrete counterpart has a considerable importance in the applications. The main reference is [**fÃijhr2005abstract**].

Fix a LCSC group G with a left Haar measure dg and a complex separable Hilbert space $\mathcal{H}$, that represents the space of signals we want to analyze. Usually, $\mathcal{H}$ is a subspace of $L^2\left(\mathbb{R}^d\right)$ where d is the dimension of the signals.

Definition 3.1. Let $\pi\colon G \to \mathcal{U}(\mathcal{H})$ be a unitary representation of G on $\mathcal{H}$ and $\eta \in \mathcal{H}$. Denote by $\eta_g = \pi(g)\,\eta$. If the equality

$$f = \int_G \langle f, \eta_g\rangle\, \eta_g\, dg \tag{3.1}$$

holds weakly for every $f \in \mathcal{H}$, the couple (π, η) is called **reproducing system**. Alternatively, G is called **reproducing group** and π **reproducing representation**. The element η is called **admissible vector** or **wavelet**.

The condition (3.1), which is called **reproducing formula**, by the definition of weak integral may be rewritten as

$$\langle f_1, f_2\rangle = \int_G \langle f_1, \eta_g\rangle\, \langle \eta_g, f_2\rangle\, dg \quad \text{for all } f_1, f_2 \in \mathcal{H}.$$

Thanks to the polarization formula, we obtain the following equivalent formulation:

$$\|f\|^2_{\mathcal{H}} = \int_G |\langle f, \eta_g\rangle|^2\, dg \quad \text{for all } f \in \mathcal{H}. \tag{3.2}$$

As we said in Section 2.3, the notion of cyclic vector (see condition (2.15)) is related with the concept of admissible vector. Precisely, the following proposition holds.

Proposition 3.2. *Let $\pi\colon G \to \mathcal{U}(\mathcal{H})$ be a unitary representation of G on $\mathcal{H}$ and $\eta \in \mathcal{H}$ an admissible vector for π. Then η is a cyclic vector.*

Proof. By contradiction, suppose there exists a nonzero $f \in \{\, \eta_g : g \in G \,\}^{\perp}$. Since η is admissible we have

$$0 < \|f\|^2_{\mathcal{H}} = \int_G |\langle f, \eta_g\rangle|^2 \, dg = 0,$$

a contradiction. □

Let (π, η) be a reproducing system and, for $f \in \mathcal{H}$ and $g \in G$, define $(Wf)(g) = \langle f, \eta_g\rangle$, which are the coefficients of the reproducing formula. Recalling the equation (3.2) we obtain an isometry $W\colon \mathcal{H} \to L^2(G)$ which is called **wavelet transform**. By strong continuity of the representation π

$$W(\mathcal{H}) \subseteq \mathcal{C}(G) \cap L^2(G).$$

Recall the definition of the left regular representation π_L of G on $L^2(G)$ introduced in Example 2.40. By the properties explained in the Remark 2.33 we have, for $h \in G$

$$\begin{aligned}
\pi_L(g)\, Wf(h) &= Wf\left(g^{-1}h\right) \\
&= \langle f, \eta_{g^{-1}h}\rangle \\
&= \langle f, \pi(g^{-1}h)\eta\rangle \\
&= \langle f, \pi(g^{-1})\pi(h)\eta\rangle \\
&= \langle \pi(g)f, \eta_h\rangle \\
&= W\pi(g)f(h)
\end{aligned}$$

whence $\pi_L(g)\, W = W\pi(g)$ for all $g \in G$. Since W is a unitary operator from $\mathcal{H}$ to $\operatorname{Im} W$, it becomes an intertwining operator between π and $\pi_{L|\operatorname{Im} W}$. Hence the original representation π is equivalent via W to a subrepresentation of the left regular representation π_L.

Example 3.3. Take $G = \mathbb{R}$ with the Lebesgue measure and $\mathcal{H} = L^2(\mathbb{R})$. Consider the left regular representation $\lambda\colon \mathbb{R} \to \mathcal{U}(\mathcal{H})$ defined for every $b \in \mathbb{R}$ by

$$\lambda_b f(x) = f(x - b)\,, \qquad f \in L^2(\mathbb{R}).$$

Now we show that λ is not reproducing. By contradiction, suppose that there exists an admissible vector $\eta \in L^2(\mathbb{R})$ and take $f \in L^2(\mathbb{R})$. Consider the Fourier transform

$\mathcal{F}\colon L^2(\mathbb{R}) \to L^2(\mathbb{R})$ defined by

$$(\mathcal{F}g)(\xi) = \int_{\mathbb{R}} g(x)\, e^{-2\pi i x\xi}\, dx$$

for every $g \in L^2(\mathbb{R}) \cap L^1(\mathbb{R})$ and $\xi \in \mathbb{R}$. By definition of wavelet transform and thanks to Plancharel's formula we have

$$Wf(b) = \langle f, \lambda_b \eta\rangle = \langle \hat{f}, \widehat{\lambda_b\eta}\rangle = \int_{\mathbb{R}} \hat{f}(\xi)\overline{e^{-2\pi i\xi b}\hat{\eta}(\xi)}\, d\xi = \mathcal{F}^{-1}(\hat{f}\bar{\hat{\eta}})(b)$$

whence

$$\|\hat{f}\|^2_{L^2(\mathbb{R})} = \|f\|^2_{L^2(\mathbb{R})} = \|Wf\|^2_{L^2(\mathbb{R})} = \|\hat{f}\hat{\eta}\|^2_{L^2(\mathbb{R})}.$$

By choosing suitably the function f we deduce that $|\hat{\eta}| = 1$, a contradiction.

Roughly speaking, the translations of a fixed function η are not sufficient to capture every detail of a one-dimensional signal. The following example shows how to solve the problem just presented. We need to add the dilatations, in order to be able to express every function in $L^2(\mathbb{R})$ that may be discontinuous.

Example 3.4 (Wavelets)**.** Take the group $G =$"$ax+b$" and the Hilbert space $\mathcal{H} = L^2(\mathbb{R})$. Let us consider the representation $U\colon$ "$ax+b$"$\to \mathcal{U}(\mathcal{H})$ defined by

$$U_{(b,a)}f(x) = a^{-\frac{1}{2}}\, f\Big(\frac{x-b}{a}\Big), \qquad f \in L^2(\mathbb{R})$$

for every $(b,a) \in G$. An easy computation shows that U is a unitary representation, that is called **wavelet representation**. A classical result states that U is reproducing and gives a characterization of the admissible vectors (see for example [Dau04]).

Now we want to write U in an equivalent form. Denote by $U'\colon G \to \mathcal{U}(\mathcal{H})$ the representation obtained by intertwining U by means of the Fourier transform $\mathcal{F}$, namely

$$U'_{(b,a)} = \mathcal{F}\, U_{(b,a)}\, \mathcal{F}^{-1}$$

for all $(b,a) \in G$. Now compute for $f \in L^2(\mathbb{R}) \cap L^1(\mathbb{R})$ and $\xi \in \mathbb{R}$

$$\begin{aligned} U'_{(b,a)}f(\xi) &= \int_{\mathbb{R}} U_{(b,a)}\, \mathcal{F}^{-1} f(x)\, e^{-2\pi i x\xi}\, dx \\ &= \int_{\mathbb{R}} a^{-\frac{1}{2}}\, \mathcal{F}^{-1} f\Big(\frac{x-b}{a}\Big)\, e^{-2\pi i x\xi}\, dx \\ &= a^{\frac{1}{2}} e^{-2\pi i\xi b} \int_{R} \mathcal{F}^{-1} f(x) e^{-2\pi i x a\xi}\, dx, \end{aligned}$$

whence we obtain

$$U'_{(b,a)}f(\xi) = \sqrt{a}\, e^{-2\pi i\xi b}\, f(a\xi) \tag{3.3}$$

for all $f \in L^2(\mathbb{R})$, $(b, a) \in G$ and $\xi \in \mathbb{R}$.

The next proposition relates the concepts of equivalent and reproducing representations.

Proposition 3.5. *Let $\mathcal{H}_1$, $\mathcal{H}_2$ be two separable complex Hilbert spaces and $\pi_i\colon G \to \mathcal{U}(\mathcal{H}_i)$ two unitary representations of G. If π_1 and π_2 are equivalent, then π_1 is reproducing if and only if π_2 is reproducing and the intertwining operator transforms the admissible vectors.*

Proof. Suppose that π_1 is reproducing and let's prove that π_2 is reproducing. The converse follows applying the same argument. Since the two representations are equivalent, there exists a unitary intertwining operator $U\colon \mathcal{H}_1 \to \mathcal{H}_2$ such that

$$U^{-1}\,\pi_2(g)\,U = \pi_1(g) \quad \text{for all } g \in G.$$

Let η be an admissible vector for π_1, we prove that $\eta' = U\,\eta$ is an admissible vector for π_2. Let $f' \in \mathcal{H}_2$ and $f \in \mathcal{H}_1$ such that $Uf = f'$. Since η is an admissible vector we have:

$$\begin{aligned}
\|f'\|^2_{\mathcal{H}_2} &= \|f\|^2_{\mathcal{H}_1} \\
&= \int_G |\langle f, \pi_1(g)\eta\rangle_1|^2\, dg \\
&= \int_G |\langle f, U^{-1}\,\pi_2(g)\,U\eta\rangle_1|^2\, dg \\
&= \int_G |\langle Uf, \pi_2(g)\,U\eta\rangle_2|^2\, dg \\
&= \int_G |\langle f', \pi_2(g)\eta'\rangle_2|^2\, dg,
\end{aligned}$$

where $\langle\ ,\ \rangle_i$ denotes the scalar product on $\mathcal{H}_i$. Hence η' is an admissible vector for π_2. □

Remark 3.6. The previous proposition still holds true if the intertwining operator between the two representations is such that

$$\|Uf\|_{\mathcal{H}_2} = c\,\|f\|_{\mathcal{H}_1} \quad \text{for all } f \in \mathcal{H}_1$$

with some $c > 0$. Indeed, it is enough to normalize correctly the admissible vector η'.

Proposition 3.7. *Let G_1, G_2 be two LCSC groups and let $\varphi\colon G_1 \to G_2$ be a topological group isomorphism. Let $\pi_2\colon G_2 \to \mathcal{U}(\mathcal{H})$ a unitary representation and define $\pi_1 = \pi_2 \circ \varphi\colon G_1 \to \mathcal{U}(\mathcal{H})$. Then π_1 is reproducing if and only if π_2 is reproducing and they have the same admissible vectors, up to a constant.*

Proof. Denote by μ_i the left Haar measures of G_i. Suppose π_2 to be reproducing and we prove that π_1 is reproducing. The converse follows applying the same argument because

$\pi_1 = \pi_2 \circ \varphi^{-1}$. By hypothesis, there exists an admissible vector $\eta \in \mathcal{H}$ for π_2. Thus, for $f \in \mathcal{H}$, we have:

$$\|f\|^2_{\mathcal{H}} = \int_{G_2} |\langle f, \pi_2(g_2)\eta\rangle|^2 \, d\mu_2(g_2).$$

Hence the function $\delta\colon g_2 \mapsto |\langle f, \pi_2(g_2)\eta\rangle|^2$ is in $L^1(\mu_2)$. By Proposition 2.29, $\delta \circ \varphi \in L^1(\mu_1)$ and there exists $c > 0$ such that

$$\begin{aligned}
\|f\|^2_{\mathcal{H}} &= \int_{G_2} \delta \, d\mu_2 \\
&= c \int_{G_1} \delta \circ \varphi \, d\mu_1 \\
&= c \int_{G_1} |\langle f, \pi_1(g_1)\eta\rangle|^2 \, d\mu_1(g_1) \\
&= \int_{G_1} |\langle f, \pi_1(g_1)(\sqrt{c}\,\eta)\rangle|^2 \, d\mu_1(g_1)
\end{aligned}$$

for all $f \in \mathcal{H}$. Therefore $\sqrt{c}\,\eta$ is an admissible vector for π_1, that is therefore reproducing. □

As a consequence of the previous two propositions we can give a fundamental result.

Theorem 3.8. *Let G_1, G_2 two LCSC subgroups of G. Suppose that G_1 and G_2 are conjugated, that is, there exists $g \in G$ such that $g\, G_1\, g^{-1} = G_2$. Suppose that $\pi\colon G \to \mathcal{U}(\mathcal{H})$ is a unitary representation and denote $\pi_i = \pi_{|G_i}$. Then π_1 is reproducing if and only if π_2 is reproducing and $\pi(g)$ transforms the admissible vectors, up to a constant.*

Proof. Write $i_g(g_1) = g\, g_1\, g^{-1}$. Then

$$\pi_2 \circ i_g(g_1) = \pi(g\, g_1\, g^{-1}) = \pi(g)\pi_1(g_1)\pi(g^{-1}) \quad \text{for all } g_1 \in G_1, \tag{3.4}$$

whence $\pi_2 \circ i_g$ and π_1 are equivalent. Thanks to Propositions 3.5 and 3.7 we are done, because i_g is a homeomorphism. □

3.2 The Class $\mathcal{E}$

From now on, we fix our signal space $\mathcal{H} = L^2\left(\mathbb{R}^d\right)$. In the last section we have seen general reproducing formulae, without specifying any characteristic of the group G or of the representation π involved. Now we focus on a special class of groups and a particular representation. The choice is motivated by the possibility of including many known formulae of wavelet theory in this context (see Example 3.14 and Section 5 in [Cor+06a]).

The main ingredients are the symplectic group defined by

$$Sp(d,\mathbb{R}) = \left\{ A \in GL(2d,\mathbb{R}) : {}^tAJA = J \right\}, \quad J = \begin{bmatrix} 0 & I_d \\ -I_d & 0 \end{bmatrix} \tag{3.5}$$

and the Euclidean space consisting of the real symmetric matrices of dimension d, denoted by $\mathrm{Sym}(d,\mathbb{R})$, equipped with the usual scalar product $\langle \sigma_1, \sigma_2 \rangle_S = \mathrm{tr}(\sigma_1 \sigma_2)$.

Regarding the representation, we consider the metaplectic representation μ of $Sp(d,\mathbb{R})$ (see [Fol89] for details). Rigorously, the metaplectic representation is a representation of the metaplectic group, which is a double cover of $Sp(d,\mathbb{R})$, and it is defined on $Sp(d,\mathbb{R})$ with a sign ambiguity. However, since the reproducing formula (3.1) is insensitive to phase factors, this ambiguity is irrelevant. Regarding the group G, we want to consider particular Lie subgroups of $Sp(d,\mathbb{R})$. In order to introduce this class of Lie subgroups, we need to make some observations on group actions.

Let V be a vector space of dimension k and G be a LCSC group. For $n \leq k$ denote by $Gr_n(V)$ the set of all n-dimensional vector subspaces of V. Consider a representation $\gamma\colon G \to GL(V)$. Since $\gamma(g) \in GL(V)$, if $W \in Gr_n(V)$ then $\bar{\gamma}(g)W := \{ \gamma(g)w : w \in W \} \in Gr_n(V)$. Hence we have just obtained an induced action

$$\bar{\gamma}\colon G \times Gr_n(V) \longrightarrow Gr_n(V) .$$

Now we want to study the stabilizers of $\bar{\gamma}$. If $W \in Gr_n(V)$ denote by $H^\gamma(W)$ the stabilizer of W, namely

$$H^\gamma(W) = \{ g \in G : \bar{\gamma}(g)W = W \} = \{ g \in G : \gamma(g)w \in W \text{ for all } w \in W \}$$

that is a closed subgroup of G. Recall the basic property of the stabilizers

$$H^\gamma(\bar{\gamma}(g)W) = g\, H^\gamma(W)\, g^{-1} =: i_g H^\gamma(W) \tag{3.6}$$

that relates the stabilizers of the elements of the same orbit.

Define the representation

$$\begin{aligned} \delta\colon GL(d,\mathbb{R}) &\longrightarrow GL(\mathrm{Sym}(d,\mathbb{R})) , \\ h &\longmapsto h[\quad] \end{aligned}$$

where $h[\sigma] = h\,\sigma\,{}^th$ for all $\sigma \in \mathrm{Sym}(d,\mathbb{R})$. If we put

$$h^\dagger[\sigma] = {}^th^{-1}\,\sigma\,h^{-1}$$

for all $h \in GL(d,\mathbb{R})$ and $\sigma \in \mathrm{Sym}(d,\mathbb{R})$, we obtain

$$\begin{aligned} \langle h^{-1}[\sigma_1], \sigma_2 \rangle_S &= \mathrm{tr}(h^{-1}\,\sigma_1\,{}^th^{-1}\,\sigma_2) \\ &= \mathrm{tr}(\sigma_1\,{}^th^{-1}\,\sigma_2\,h^{-1}) \\ &= \langle \sigma_1,\,{}^th^{-1}\,\sigma_2\,h^{-1} \rangle_S \\ &= \langle \sigma_1, h^\dagger[\sigma_2] \rangle_S \end{aligned} \tag{3.7}$$

for all $h \in GL(d,\mathbb{R})$ and $\sigma_1, \sigma_2 \in \mathrm{Sym}(d,\mathbb{R})$. Hence

$$\begin{aligned} \delta^\dagger \colon GL(d,\mathbb{R}) &\longrightarrow GL(\mathrm{Sym}(d,\mathbb{R})) \\ h &\longmapsto h^\dagger[\quad] \end{aligned}$$

is the contragradient representation of δ. Since $h^{-1}[\sigma] = h^{-1}\sigma\,{}^th^{-1} = {}^th^\dagger[\sigma]$, we have

$$\langle {}^th^\dagger[\sigma_1], \sigma_2 \rangle_S = \langle \sigma_1, h^\dagger[\sigma_2] \rangle_S \quad \text{for all } \sigma_1, \sigma_2 \in \mathrm{Sym}(d,\mathbb{R}) \text{ and } h \in GL(d,\mathbb{R}). \tag{3.8}$$

We are ready to introduce the groups of our interest. The standard maximal parabolic subgroup Q of the symplectic group that we are interested in is the closed Lie group corresponding to the parabolic algebra obtained by omitting the long simple negative restricted root (see for example [Kna02] for details), namely

$$Q = \left\{ g(\sigma, h) := \begin{bmatrix} h & 0 \\ \sigma h & {}^th^{-1} \end{bmatrix} : h \in GL(d,\mathbb{R}),\ \sigma \in \mathrm{Sym}(d,\mathbb{R}) \right\} < Sp(d,\mathbb{R}), \tag{3.9}$$

whose Langlands decomposition $Q = MAN$ is easily checked to be

$$\begin{aligned} M &= \left\{ \begin{bmatrix} g & 0 \\ 0 & {}^tg^{-1} \end{bmatrix} : \det g = \pm 1 \right\} \\ A &= \left\{ \begin{bmatrix} \lambda I_d & 0 \\ 0 & \lambda^{-1} I_d \end{bmatrix} : \lambda > 0 \right\} \\ N &= \left\{ \begin{bmatrix} I & 0 \\ \sigma & I \end{bmatrix} : \sigma \in \mathrm{Sym}(d,\mathbb{R}) \right\}. \end{aligned} \tag{3.10}$$

Thus, $MA \cong GL(d,\mathbb{R})$ is the homogeneous component, while $N \cong \mathrm{Sym}(d,\mathbb{R})$ is the vector component. Let's compute the product in Q. If $h_1, h_2 \in GL(d,\mathbb{R})$ and $\sigma_1, \sigma_2 \in$

$\mathrm{Sym}(d, \mathbb{R})$ we have

$$
\begin{aligned}
g(\sigma_1, h_1)g(\sigma_2, h_2) &= \begin{bmatrix} h_1 & 0 \\ \sigma_1 h_1 & {}^t h_1^{-1} \end{bmatrix} \begin{bmatrix} h_2 & 0 \\ \sigma_2 h_2 & {}^t h_2^{-1} \end{bmatrix} \\
&= \begin{bmatrix} h_1 h_2 & 0 \\ \sigma_1 h_1 h_2 + {}^t h_1^{-1}\, \sigma_2 h_2 & {}^t h_1^{-1}\, {}^t h_2^{-1} \end{bmatrix} \\
&= \begin{bmatrix} h_1 h_2 & 0 \\ (\sigma_1 + h_1^{\dagger}[\sigma_2]) h_1 h_2 & {}^t (h_1 h_2)^{-1} \end{bmatrix}
\end{aligned}
$$

whence we obtain

$$g(\sigma_1, h_1)\, g(\sigma_2, h_2) = g(\sigma_1 + h_1^{\dagger}[\sigma_2], h_1 h_2). \tag{3.11}$$

The inverse is given by

$$g(\sigma, h)^{-1} = g(-(h^{-1})^{\dagger}[\sigma], h^{-1}).$$

Therefore Q is a semidirect product, namely

$$Q = \mathrm{Sym}(d, \mathbb{R}) \rtimes_{\delta^{\dagger}} GL(d, \mathbb{R}).$$

The projection

$$\pi \colon Q \longrightarrow GL(d, \mathbb{R}), \quad g(\sigma, h) \longmapsto h \tag{3.12}$$

is a smooth and regular surjective group homomorphism (see equation (3.11)). If G is a Lie subgroup of Q, denote by π_G its restriction to G and by $H_G = \pi_G(G)$, a Lie subgroup of $GL(d, \mathbb{R})$. Now consider $\ker(\pi_G)$: it is a closed Lie subgroup of G. By definition of π we can write

$$\ker(\pi_G) = \{\, g(\sigma, I_d) : \sigma \in \Sigma_G \,\}$$

with Σ_G a closed Lie subgroup of $\mathrm{Sym}(d, \mathbb{R})$.

The following proposition characterizes the Lie subgroups of Q.

Proposition 3.9. *Let G be a Lie closed subgroup of Q. Then there exists a map $\varphi_G \colon H_G \to \mathrm{Sym}(d, \mathbb{R})$, unique up to elements in Σ_G, such that $g(\varphi_G(h), h) \in G$ for all $h \in H_G$. Moreover*

1. $\varphi_G(h) + h^{\dagger}[\varphi_G(h')] - \varphi_G(hh') \in \Sigma_G$ *for all* $h, h' \in H_G$*;*
2. $G = \{\, g(\sigma + \varphi_G(h), h) : \sigma \in \Sigma_G, h \in H_G \,\} = \Sigma_G \rtimes_{\varphi_G} H_G$

Further

a) $h^{\dagger}[\Sigma_G] \subseteq \Sigma_G$ *for all* $h \in H_G$*;*

b) *if G is connected then H_G is connected.*

Proof. If $h \in H_G$ there exists $g \in G$ such that $\pi(g) = h$, hence there exists $\varphi_G(h) \in \mathrm{Sym}(d, \mathbb{R})$ such that $g(\varphi_G(h), h) \in G$. For $\sigma, \sigma' \in \mathrm{Sym}(d, \mathbb{R})$ and $h \in H_G$ such that $g(\sigma, h), g(\sigma', h) \in G$ we have

$$g(\sigma, h)g(\sigma', h)^{-1} = g(\sigma, h)g(-(h^{-1})^{\dagger}[\sigma'], h^{-1}) = g(\sigma - \sigma', I_d) \in G.$$

Hence the map φ_G is defined uniquely up to elements in Σ_G. Furthermore

$$\pi^{-1}(h) = \{\, g(\sigma + \varphi_G(h), h) : \sigma \in \Sigma_G \,\} \tag{3.13}$$

for every $h \in H_G$.

For $h, h' \in H_G$ we have

$$g(\varphi_G(h), h)g(\varphi_G(h'), h') = g(\varphi_G(h) + h^{\dagger}[\varphi_G(h')], hh')$$

whence we obtain part 1 because φ_G is defined up to elements in Σ_G.

In order to show part 2 it is sufficient to observe that, by definition of H_G and by (3.13), we have

$$G = \pi^{-1}(H_G) = \bigcup_{h \in H_G} \pi^{-1}(h) = \{\, g(\sigma + \varphi_G(h), h) : \sigma \in \Sigma_G, h \in H_G \,\}.$$

Now we prove part **a**. For $h \in H_G$ and $\sigma \in \Sigma_G$ we have

$$\begin{aligned} g(\varphi_G(h), h)g(\sigma, I_d)g(\varphi_G(h), h)^{-1} &= g(\varphi_G(h) + h^{\dagger}[\sigma], h)g(-(h^{-1})^{\dagger}[\varphi_G(h)], h^{-1}) \\ &= g(h^{\dagger}[\sigma], I_d) \in G \end{aligned}$$

whence $h^{\dagger}[\sigma] \in \Sigma_G$.

Part **b** follows trivially by the continuity of π. □

Remark 3.10. The uniqueness of the map φ_G up to elements in Σ_G can be expressed also by stating the uniqueness of $\overline{\varphi_G} = p \circ \varphi_G$, where p is the canonical projection

$$p \colon \mathrm{Sym}(d, \mathbb{R}) \longrightarrow \mathrm{Sym}(d, \mathbb{R})/\Sigma_G.$$

Hence to every closed Lie subgroup G of Q the map $\overline{\varphi_G}$ is determined univocally.

To simplify the notation, denote by $\mathcal{S}_n(d) = Gr_n(\mathrm{Sym}(d, \mathbb{R}))$ and set

$$\mathcal{S}(d) = \bigcup_{0<n\leq d} \mathcal{S}_n(d).$$

The upper bound for n is motivated by Theorem 3.18. Take $\Sigma \in \mathcal{S}(d)$, H a closed Lie subgroup of $H^{\delta^\dagger}(\Sigma)$ and consider the set

$$G = \Sigma \rtimes_0 H = \{\, g(\sigma, h) \in GL(2d, \mathbb{R}) : h \in H,\, \sigma \in \Sigma \,\}\,.$$

Since $H \subseteq H^{\delta^\dagger}(\Sigma)$ and thanks to the product law (3.11), G is a Lie subgroup of Q. The Class $\mathcal{E}_d$ is the family of triangular Lie subgroups of the symplectic group of this type, namely

$$\mathcal{E}_d = \Big\{\, \Sigma \rtimes_0 H : \Sigma \in \mathcal{S}(d), \{I_d\} \neq H < H^{\delta^\dagger}(\Sigma) \text{ closed} \,\Big\}\,.$$

Hereafter we write simply $G = \Sigma \rtimes H$ and $H(\Sigma) = H^{\delta^\dagger}(\Sigma)$. Further we define

$$\mathcal{E} = \bigcup_{d \geq 1} \mathcal{E}_d.$$

Remark 3.11. It is natural to ask whether a subgroup G of Q is necessarily in the class $\mathcal{E}_d$ or not. The answer is negative, as the following example shows. Consider the matrices

$$\sigma_{x,n} = \begin{bmatrix} x & n \\ n & 0 \end{bmatrix}, \qquad h_n = \begin{bmatrix} 1 & 0 \\ n & 1 \end{bmatrix}$$

with $n, x \in \mathbb{R}$ and form the corresponding symplectic matrices

$$g_{x,n} = g(\sigma_{x,n}, h_n) = \begin{bmatrix} h_n & 0 \\ \sigma_{x,n} h_n & {}^t h_n^{-1} \end{bmatrix} \in Sp(2, \mathbb{R}).$$

It is easy to see that they form a subgroup

$$G_{\text{bad}} = \big\{ g_{x,n} : x, n \in \mathbb{R} \big\}$$

of Q. Indeed, first of all $g_{0,0}$ is the 4×4 identity matrix. Further,

$$-\,{}^t h_n \sigma_{x,n} h_n = \sigma_{-x-2n^2,-n}.$$

Therefore, since $h_n^{-1} = h_{-n}$, it follows that $g_{x,n}^{-1} = g_{-x-2n^2,-n} \in G_{\text{bad}}$. Finally, it is clear that $h_n h_m = h_{n+m}$, so that

$$h_n^\dagger[\sigma_{y,m}] = \sigma_{y-2nm,m}$$

shows that if $g_{x,n}, g_{y,m} \in G_{\text{bad}}$, then

$$g_{x,n} g_{y,m} = g_{x+y-2nm,n+m} \in G_{\text{bad}},$$

as desired. Thus, G_{bad} is a connected, Abelian two-dimensional Lie subgroup of Q. However, $G_{\text{bad}} \notin \mathcal{E}_2$. Roughly speaking, this is because σ and h are linked rather than varying independently. The connected subgroup $N = \{h_n : n \in \mathbb{R}\}$ of $GL(2, \mathbb{R})$ is of course the Iwasawa N-group of $SL(2,\mathbb{R})$. The set where the σ-components range is $\Sigma_2 = \left\{\sigma_{x,n} : x, n \in \mathbb{R}\right\}$, but $G_{\text{bad}} \neq \Sigma_2 \rtimes N$. Among other things, this follows from the dimension formula $\dim(\Sigma_2 \rtimes H) = \dim \Sigma_2 + \dim H = 3$ while $\dim G_{\text{bad}} = 2$.

The following proposition expresses the algebraic importance of the class $\mathcal{E}_d$, by using the map $\overline{\varphi_G}$ defined in Remark 3.10.

Proposition 3.12. *Let G be a closed Lie subgroup of Q such that $\Sigma_G \neq \{0\}$ and $H_G \neq \{I_d\}$. Then $G \in \mathcal{E}$ if and only if $\overline{\varphi_G} = 0$.*

Proof. It is an immediate consequence of Proposition 3.9. □

Next, we write the metaplectic representation for the groups of the class $\mathcal{E}_d$.

Proposition 3.13. *Let $\Sigma \in \mathcal{S}_n(d)$, H a closed subgroup of $H(\Sigma)$ and $G = \Sigma \rtimes H$. For $f \in L^2\left(\mathbb{R}^d\right)$, $\sigma \in \Sigma$ and $h \in H$ we have*

$$\mu_{g(\sigma,h)} f(x) = |\det h|^{-1/2}\, e^{\pi i \langle \sigma x, x\rangle} f\left(h^{-1}x\right) \qquad \textit{for all } x \in \mathbb{R}^d.$$

Proof. It is an easy consequence of Proposition 4.46 in [Fol89] because

$$g(\sigma, h) = g(\sigma, I) g(0, h) = \begin{bmatrix} I & 0 \\ \sigma & I \end{bmatrix} \begin{bmatrix} h & 0 \\ 0 & {}^t h^{-1} \end{bmatrix}.$$

□

The following example shows what happens if $d = 1$, where there is only one non trivial case. The goal of Chapter 4 will be to study all the possibilities with two-dimensional signals.

Example 3.14. We want to study a simple case with one-dimensional signals, namely with $d = 1$. Set $\Sigma = \mathbb{R}$, $H = H(\Sigma)_0 = \mathbb{R}_+$ and $G = \mathbb{R} \rtimes R_+ \in \mathcal{E}_1$. A straightforward verification shows that ζ: "$ax + b$"$\to G$ defined by

$$(b, a) \overset{\zeta}{\longmapsto} \begin{bmatrix} a^{-\frac{1}{2}} & 0 \\ b\, a^{-\frac{1}{2}} & a^{\frac{1}{2}} \end{bmatrix}$$

is a Lie group isomorphism. Now we write the metaplectic representation for the group G by exploiting the isomorphism ζ. By Proposition 3.13 we have

$$\mu_{\zeta(b,a)} f(x) = a^{\frac{1}{4}}\, e^{\pi i b x^2}\, f\left(a^{\frac{1}{2}} x\right) \tag{3.14}$$

for every $(b, a) \in$ "$ax + b$", $f \in L^2(\mathbb{R})$ and $x \in \mathbb{R}$. We want to find the relationship between this representation and the wavelet representation introduced in Example 3.4, in the form (3.3). The ideas are similar to those in Theorem 4.2 of [Cor+06b]. By comparing equations (3.3) and (3.14) we see that the main difference is in the exponential of e, because in the original form it is linear in x while now it is quadratic in x. In order to solve this problem, consider the diffeomorphisms $\Phi_\pm \colon \mathbb{R}_\pm \to \mathbb{R}_-$ defined by $\Phi_\pm(x) = -\frac{1}{2}x^2$ for every $x \in \mathbb{R}_\pm$. The link between the two representations is based on the unitary operators

$$\Psi_\pm \colon L^2(\mathbb{R}_-) \longrightarrow L^2(\mathbb{R}_\pm), \qquad f \longmapsto (f \circ \Phi_\pm)\,(J\Phi_\pm)^{\frac{1}{2}},$$

with inverse defined by $\Psi_\pm^{-1}(g) = (g \circ \Phi_\pm^{-1})\,(J\Phi_\pm^{-1})^{\frac{1}{2}}$ for every $g \in L^2(\mathbb{R}_\pm)$. Thanks to the unitary identification $L^2(\mathbb{R}) = L^2(\mathbb{R}_-) \oplus L^2(\mathbb{R}_+)$ into two invariant subspaces for $\mu \circ \zeta$ and for U', we can consider the unitary operator $\Psi \colon L^2(\mathbb{R}_-) \oplus L^2(\mathbb{R}_-) \to L^2(\mathbb{R})$ defined by $\Psi = \Psi_- \oplus \Psi_+$. To synthesize the situation we can consider the following diagram

$$\begin{tikzcd}
L^2(\mathbb{R}) \arrow[r, "\mu_{\zeta(b,a)}"] & L^2(\mathbb{R}) \arrow[d, "\Psi^{-1}"] \\
L^2(\mathbb{R}_-) \oplus L^2(\mathbb{R}_-) \arrow[u, "\Psi"] \arrow[r, "U'_{(b,a)} \oplus U'_{(b,a)}"] & L^2(\mathbb{R}_-) \oplus L^2(\mathbb{R}_-)
\end{tikzcd}$$

and we show that is commutative. Indeed we can write for f in the first copy of $L^2(\mathbb{R}_-)$ (the other case is similar) and $x \in \mathbb{R}_-$

$$\begin{aligned}
\Psi^{-1}\,\mu_{\zeta(b,a)}\,\Psi f(x) &= \Psi_-^{-1}\mu_{\zeta(b,a)}\,\Psi_- f(x) \\
&= \mu_{\zeta(b,a)}\,\Psi_-\, f\big(\Phi_-^{-1}(x)\big)(-2x)^{-\frac{1}{4}} \\
&= a^{\frac{1}{4}} e^{-2\pi ibx}\,\Psi_- f(-\sqrt{-2ax})(-2x)^{-\frac{1}{4}} \\
&= a^{\frac{1}{4}} e^{-2\pi ibx} f(ax) a^{\frac{1}{4}} \\
&= U'_{(b,a)} f(x).
\end{aligned}$$

Hence the metaplectic representation of G is equivalent to two copies of the wavelet representation restricted to $L^2(\mathbb{R}_-)$.

The group G is determined by the representation $\delta^\dagger \colon H \to GL(\Sigma)$. By an easy consequence of the relation (3.7), it is the contragradient representation of $\delta' \colon h \mapsto P_\Sigma \circ \delta(h)$, where P_Σ is the orthogonal projection onto Σ.

Denote by Δ_H and Δ_G the modular functions on H and G respectively. Further, consider the character $\alpha \colon H \to \mathbb{R}_+$ defined by

$$\alpha(h) = |\det(\delta^\dagger_{|\Sigma}(h))| = |\det(\sigma \mapsto h^\dagger[\sigma])| \quad \text{for all } h \in H.$$

By equation (3.7), we have $\alpha(h) = |\det(\delta'_{|\Sigma}(h))|^{-1}$. With the following proposition we study the Haar measure on $G \in \mathcal{E}_d$.

Proposition 3.15. *Let $\Sigma \in \mathcal{S}(d)$, H a closed subgroup of $H(\Sigma)$ and $G = \Sigma \rtimes H$. Denote by dh a Haar measure on H and by $d\sigma$ the Lebesgue measure on Σ. Then*

1. *$dg = \frac{1}{\alpha(h)} d\sigma dh$ is a left Haar measure on G;*
2. *$\Delta_G(\sigma, h) = \frac{\Delta_H(h)}{\alpha(h)}$ for all $\sigma \in \Sigma$ and $h \in H$.*

Proof. In order to prove part 1 we use Proposition 2.5. Let $f \in \mathcal{C}_c^+(G)$ and $g' = (\sigma', h') \in G$. By Fubini's theorem we have

$$\int_G L_{g'^{-1}} f(g)\, dg = \int_G f(g'\, g)\, dg = \int_H \frac{1}{\alpha(h)} \left(\int_\Sigma f\left(\sigma' + h'^{\dagger}[\sigma], h'h\right)\, d\sigma \right)\, dh.$$

Further, by the change of variables formula and since the Lebesgue measure $d\sigma$ is invariant under translations

$$\int_G L_{g'^{-1}} f(g)\, dg = \int_H \frac{1}{\alpha(h)\alpha(h')} \left(\int_\Sigma f\left(\sigma, h'h\right)\, d\sigma \right)\, dh.$$

Again, by Fubini's theorem and by Proposition 2.5 applied to (H, dh) we obtain

$$\begin{aligned} \int_G L_{g'^{-1}} f(g)\, dg &= \int_\Sigma \left(\int_H \frac{1}{\alpha(h'h)} f\left(\sigma, h'h\right)\, dh \right)\, d\sigma \\ &= \int_\Sigma \left(\int_H \frac{1}{\alpha(h)} f\left(\sigma, h\right)\, dh \right)\, d\sigma \\ &= \int_G f(g)\, dg. \end{aligned}$$

Hence, again by Proposition 2.5, we obtain our first claim.

Now we prove part 2. Let $f \in \mathcal{C}_c^+(G)$ such that $\int_G f\, dg > 0$ and $g' \in G$. As above, we have

$$\begin{aligned} \int_G R_{g'} f\, dg &= \int_G f(gg') dg \\ &= \int_H \frac{1}{\alpha(h)} \left(\int_\Sigma f\left(\sigma + h^{\dagger}[\sigma'], hh'\right)\, d\sigma \right)\, dh \\ &= \int_H \frac{1}{\alpha(h)} \left(\int_\Sigma f\left(\sigma, hh'\right)\, d\sigma \right)\, dh \\ &= \int_\Sigma \alpha(h') \int_H \frac{1}{\alpha(hh')} f\left(\sigma, hh'\right) d\sigma\ dh. \end{aligned}$$

By Proposition 2.8 applied to (H, dh) we obtain

$$\begin{aligned}\int_G R_{g'} f\, dg &= \int_\Sigma \alpha(h')\Delta_H(h'^{-1}) \int_H \frac{1}{\alpha(h)} f(\sigma, h)\, d\sigma\; dh \\ &= \alpha(h')\Delta_H(h'^{-1}) \int_\Sigma \int_H \frac{1}{\alpha(h)} f(\sigma, h)\, d\sigma\; dh \\ &= \alpha(h')\Delta_H(h'^{-1}) \int_G f\, dg.\end{aligned}$$

Thanks again to Proposition 2.8 we obtain

$$\Delta_G(\sigma', h')^{-1} = \alpha(h')\Delta_H(h'^{-1})$$

and so we are finally done. □

3.3 Main Results

In this section we present the main results of [DD11] that allow us to establish if a group $G \in \mathcal{E}$ is reproducing with respect to the restriction of the metaplectic representation. First, there is a general dimensional upper bound (for the proof, see Theorem 2.4 in [Cor+10]).

Theorem 3.16. *If $G \in \mathcal{E}_d$ is reproducing then* $\dim G \leq d^2 + 1$.

Since $\langle \sigma x, x\rangle$ is linear in σ, there exists $\Phi(x) \in \Sigma$ such that

$$\langle \Phi(x), \sigma\rangle_S = -\frac{1}{2}\langle \sigma x, x\rangle \quad \text{for all } x \in \mathbb{R}^d \text{ and } \sigma \in \Sigma. \tag{3.15}$$

Denote by Φ_{*x} its differential and by $J(\Phi)(x) = \sqrt{\det\left(\Phi_{*x} \cdot {}^t\Phi_{*x}\right)}$ the Jacobian of Φ at x. Denote by

$$\mathcal{R} = \left\{\, x \in \mathbb{R}^d : J(\Phi)(x) > 0 \,\right\}$$

the set of all regular points of Φ. The following Lemma is a consequence of Lemma 2 and Appendix B in [DD11]. It deals with the map Φ and the two actions of H, δ' and the natural action on $\mathbb{R}^d$ defined by $x \mapsto hx$, for $h \in H$ and $x \in \mathbb{R}^d$.

Lemma 3.17. *Take $\Sigma \in \mathcal{S}(d)$, H a closed Lie subgroup of $H(\Sigma)$, X an open $H-$invariant subset of $\mathcal{R}$ and put $Y = \Phi(X)$. The following properties hold:*

a) *$\mathcal{R}$ and Y are open;*

b) *$\mathcal{R}$ is H-invariant and Y is H-invariant;*

c) *if $n = d$, then $\Phi^{-1}(y)$ is a finite set for every $y \in \Phi(\mathcal{R})$.*

We are ready to give the first important result, that is a necessary condition on the n-dimensional subspace Σ of $\mathrm{Sym}(d, \mathbb{R})$. It is a consequence of Theorem 1 and Proposition 5 of [DD11].

Theorem 3.18. *Take $G \in \mathcal{E}_d$. If the system $(G, \mu_{|G})$ is reproducing then $n \leq d$, $\mathbb{R}^d \setminus \mathcal{R}$ has Lebesgue measure zero and G is non-unimodular.*

For $\sigma \in \Sigma$, denote by $H_\sigma = \{ h \in H : \delta'(h)\sigma = \sigma \}$ the stabilizers of δ'. We are ready to give the following important result, that is Theorem 10 of [DD11].

Theorem 3.19 (De Mari, De Vito)**.** *Let $\Sigma \in \mathcal{S}_d(d)$, H a closed Lie subgroup of $H(\Sigma)$, $G = \Sigma \rtimes H \in \mathcal{E}_d$ and suppose that the H-orbits of $\Phi(\mathbb{R}^d)$ are locally closed. The system $(G, \mu_{|G})$ is reproducing if and only if G is non-unimodular and the stabilizer H_σ is compact for almost every $\sigma \in \Phi(\mathbb{R}^d)$.*

The following statement is similar to the previous one, but it is applicable also when $n < d$. For the proof, see Theorem 9 in [DD11].

Theorem 3.20. *Let $\Sigma \in \mathcal{S}(d)$, H a closed Lie subgroup of $H(\Sigma)$, $G = \Sigma \rtimes H \in \mathcal{E}_d$ and suppose that the H-orbits of $\Phi(\mathbb{R}^d)$ are locally closed. If G is non-unimodular and the stabilizer H_σ is compact for almost every $\sigma \in \Phi(\mathbb{R}^d)$ then the system $(G, \mu_{|G})$ is reproducing.*

For the characterization of local closedness, see for example [Bou66] (Chapter I, §3, Proposition 5). The following result may be useful to verify the first hypothesis of the previous theorem when $\Phi(\mathbb{R}^d)$ is closed.

Proposition 3.21. *Let T be a topological space, $S \subseteq T$ a subspace and $C \subseteq T$ a closed subspace. If S is locally closed then $S \cap C$ is locally closed.*

Proof. Suppose there exist $G, F \subseteq T$, with G open and F closed, such that $S = G \cap F$. Hence
$$S \cap C = (G \cap F) \cap C = G \cap (F \cap C).$$ □

We want to give a result that characterizes the admissible vectors for $(G, \mu_{|G})$ if $n = d$. Let us fix $\Sigma \in \mathcal{S}_d(d)$, H a closed Lie subgroup of $H(\Sigma)$, $G = \Sigma \rtimes H \in \mathcal{E}_d$ and suppose that the H-orbits of $\Phi(\mathbb{R}^d)$ are locally closed. Further, let X and Y as above, with $\mathcal{R} \setminus X$ negligible. Thanks to what is proved in Section 3.3 of [DD11], there exists a LCSC space Z and a Borel measurable surjective map $\pi \colon Y \to Z$ such that $\pi(y) = \pi(y')$ if and only if y and y' belong to the same orbit. In other words, the points of Z label the orbits of the action δ' of H on Y. Indeed this construction makes sense since, by Lemma 3.17, Y is H-invariant. Since Y is open and consequently σ-compact, by Proposition 2.31 there exists a pseudo-image measure λ on Z of the Lebesgue measure on Y under the map π.

Recalling that for every $y \in Y$, $\Phi^{-1}(y)$ is a finite set, we can state the following theorem, that describes the admissible vectors. It is a consequence of Theorem 3 and of Corollary 4 of [DD11].

Theorem 3.22 (De Mari, De Vito)**.** *A function $\eta \in L^2\left(\mathbb{R}^d\right)$ is an admissible vector for $(G, \mu_{|G})$ if and only if for λ-almost every $z \in Z$ there exists $y \in \pi^{-1}(z)$ such that*

$$\int_H \eta(h^{-1}P_i)\overline{\eta(h^{-1}P_j)}\frac{dh}{|\det(h)|\alpha(h)} = J(\Phi)(P_i)\delta_{ij} \quad \text{for all } P_i, P_j \in \Phi^{-1}(y). \tag{3.16}$$

If the above equality is satisfied for a pair $P_i, P_j \in \Phi^{-1}(y)$, then it holds true for any pair $sP_i, sP_j \in \Phi^{-1}(y)$ with $s \in H_y$. If (3.16) *is satisfied for a point $y \in Y$ then it is satisfied for every point in $\delta'(H)y$.*

3.4 Classification of the Class $\mathcal{E}$

The first step in our study must be the determination of the groups in the Class $\mathcal{E}_d$. After that, we will have to establish which are reproducing relative to the restriction of the metaplectic representation. At a first glance, the problem of classifying the groups in the Class $\mathcal{E}_d$ may seem difficult. Let's proceed in this way: fix $\Sigma \in \mathcal{S}_n(d)$ and determine $H(\Sigma)$, then consider all the closed Lie subgroups $H < H(\Sigma)$ and finally the groups $\Sigma \rtimes H \in \mathcal{E}_d$. For simplicity, we will deal only with the connected Lie subgroups H. Further, we will not consider the trivial case $H = \{I\}$ because $\Sigma \rtimes \{I\} \cong \Sigma$ is unimodular and consequently, by Theorem 3.18, it is not reproducing.

3.4.1 Useful Relations

Our first task is to determine the stabilizer $H(\Sigma)$ for $\Sigma \in \mathcal{S}(d)$. In other words, which elements of $GL(d, \mathbb{R})$ do not move Σ under the action $\delta^\dagger$? Since we have to determine a stabilizer (for the action $\overline{\delta^\dagger}$), we shall use the useful property (3.6) and calculate $H(\Sigma)$ only for a particular Σ in every orbit. More precisely, suppose that $\Sigma, \Sigma' \in \mathcal{S}_n(d)$ and that there exists $q \in GL(d, \mathbb{R})$ such that $\Sigma' = \overline{\delta^\dagger}(q)\Sigma$, namely $\Sigma' = {}^t q^{-1} \Sigma\, q^{-1}$. Thanks to the equation (3.6) we have

$$H(\Sigma') = H(\overline{\delta^\dagger}(\Sigma)) = q\, H(\Sigma)\, q^{-1} \tag{3.17}$$

that allows us to determine $H(\Sigma')$ for every Σ' in the same orbit of Σ. Passing to the Lie subgroups of $H(\Sigma)$, if $H < H(\Sigma)$ we have the corresponding subgroup $H' = q\, H\, q^{-1}$ of $H(\Sigma')$. To summarize we can consider the following diagram:

$$\begin{array}{ccc} \Sigma & \longleftrightarrow & \Sigma' = {}^tq^{-1}\Sigma q \\ H(\Sigma) & \longleftrightarrow & H(\Sigma') = qH(\Sigma)\,q^{-1} \\ \vee & & \vee \\ H & \longleftrightarrow & H' = qH\,q^{-1} \\ G = \Sigma \rtimes H & \longleftrightarrow & G' = \Sigma' \rtimes H' \end{array}$$

With this formulation, for every subgroup $G \in \mathcal{E}_d$ associated to Σ there exists another subgroup $G' \in \mathcal{E}_d$ associated to Σ' by the relation written above. The following proposition synthesizes what we have just proved and shows other properties. It deals with the conjugations within the class $\mathcal{E}_d$ via elements in MA (see the Langlands decomposition (3.10)), that is

$$MA = \{\, g(0,q) : q \in GL(d,\mathbb{R}) \,\}\,.$$

Proposition 3.23. *Take $G = \Sigma \rtimes H \in \mathcal{E}_d$ and $q \in GL(d,\mathbb{R})$. Then $i_{g(0,q)}(G) \in \mathcal{E}_d$. More precisely, if $G' = \Sigma' \rtimes H' \in \mathcal{E}_d$, then the following are equivalent:*

1. $i_{g(0,q)}(G) = G'$
2. $q^\dagger[\Sigma] = \Sigma'$ *and* $i_q(H) = H'$.

In this case, conjugation by $g(0,q)$ establishes one to one correspondences between:

- *$\mathcal{E}$-subgroups of G and $\mathcal{E}$-subgroups of G';*
- *reproducing $\mathcal{E}$-subgroups of G and reproducing $\mathcal{E}$-subgroups of G'.*

Proof. First we show that 2 implies 1, the other implication follows by a similar argument. Let's prove that G and G' are conjugated in $Sp(d,\mathbb{R})$ by the matrix $g = g(0,q) = \left[\begin{smallmatrix} q & 0 \\ 0 & {}^tq^{-1} \end{smallmatrix}\right]$. If we take

$$\sigma' \in \Sigma' = {}^tq^{-1}\Sigma\, q^{-1} \text{ and } h' \in H' = qH\,q^{-1}$$

there exist $\sigma \in \Sigma$ and $h \in H$ such that

$$\sigma' = {}^tq^{-1}\sigma\, q^{-1} \text{ and } h' = qh\,q^{-1}$$

Hence an arbitrary element of G' can be written

$$\begin{bmatrix} h' & 0 \\ \sigma'h' & {}^th'^{-1} \end{bmatrix} = \begin{bmatrix} qhq^{-1} & 0 \\ {}^tq^{-1}\sigma hq^{-1} & {}^t(qhq^{-1})^{-1} \end{bmatrix} = \begin{bmatrix} q & 0 \\ 0 & {}^tq^{-1} \end{bmatrix} \begin{bmatrix} h & 0 \\ \sigma h & {}^th^{-1} \end{bmatrix} \begin{bmatrix} q^{-1} & 0 \\ 0 & {}^tq \end{bmatrix}$$

and so $g\,Gg^{-1} = G'$.

The reproducing condition follows trivially by Theorem 3.8 (with $\pi = \mu$ the metaplectic representation) because a closed subgroup of a LCSC group is LCSC. □

By Proposition 3.23, it is enough to calculate $H(\Sigma)$ and its Lie subgroups only for one Σ in every orbit. Moreover, the restricted representations satisfy (3.4).

There is a similar result that exploits the Euclidean structure $\langle\ ,\ \rangle_S$ on $\mathrm{Sym}(d,\mathbb{R})$. Before we have related $H(\overline{\delta^\dagger}(q)\Sigma)$ with $H(\Sigma)$: now we want to do the same with $H\left(\Sigma^\perp\right)$. Unfortunately, as we will see in Remark 4.22, we cannot infer that $\Sigma^\perp \rtimes {}^tH$ is reproducing if such is $\Sigma \rtimes H$.

Proposition 3.24. *Let $\Sigma, \Sigma' \in \mathcal{S}_n(d)$, $H < GL(d,\mathbb{R})$ and $q \in GL(d,\mathbb{R})$. Then*

1. $\Sigma' = \overline{\delta^\dagger}(q)\Sigma$ *if and only if* $\Sigma'^\perp = \overline{\delta^\dagger}\left({}^tq^{-1}\right)\Sigma^\perp$;
2. $H\left(\Sigma^\perp\right) = {}^tH(\Sigma)$;
3. $H < H(\Sigma^\perp)$ *if and only if* ${}^tH < H(\Sigma)$.

Proof. First we prove part 1. Suppose $\Sigma' = \overline{\delta^\dagger}(q)\Sigma$. We have

$$\begin{aligned}
\Sigma'^\perp &= \left\{\, q^\dagger[\sigma] : \sigma \in \Sigma \,\right\}^\perp \\
&= \left\{\, \sigma_0 \in \mathrm{Sym}(d,\mathbb{R}) : \langle \sigma_0, q^\dagger[\sigma]\rangle_S = 0 \text{ for all } \sigma \in \Sigma \,\right\} \\
&\overset{(3.8)}{=} \left\{\, \sigma_0 \in \mathrm{Sym}(d,\mathbb{R}) : \langle {}^tq^\dagger[\sigma_0], \sigma\rangle_S = 0 \text{ for all } \sigma \in \Sigma \,\right\} \\
&= \left\{\, \sigma_0 \in \mathrm{Sym}(d,\mathbb{R}) :\ {}^tq^\dagger[\sigma_0] \in \Sigma^\perp \,\right\} \\
&= \left({}^tq^{-1}\right)^\dagger [\Sigma^\perp] \\
&= \overline{\delta^\dagger}\left({}^tq^{-1}\right)\Sigma^\perp
\end{aligned}$$

The converse statement is obtained simply by the same argument because ${\Sigma^\perp}^\perp = \Sigma$ and ${\Sigma'^\perp}^\perp = \Sigma'$.

Now we prove part 2. The following are equivalent conditions: $h \in H\left(\Sigma^\perp\right)$ if and only if $\langle h^\dagger[\sigma'], \sigma\rangle = 0$ for all $\sigma \in \Sigma$ and $\sigma' \in \Sigma^\perp$ if and only if $\mathrm{tr}(\sigma' h^{-1} \sigma\, {}^th^{-1}) = 0$ for all $\sigma \in \Sigma$ and $\sigma' \in \Sigma^\perp$ if and only if $\langle \sigma', {}^th^\dagger[\sigma]\rangle = 0$ for all $\sigma \in \Sigma$ and $\sigma' \in \Sigma^\perp$ if and only if ${}^th^\dagger[\Sigma] = \left(\Sigma^\perp\right)^\perp = \Sigma$ if and only if ${}^th \in H(\Sigma)$ if and only if $h \in {}^tH(\Sigma)$. Therefore $H\left(\Sigma^\perp\right) = {}^tH(\Sigma)$.

Part 3 follows trivially from part 1. $\square$

Thus, if Σ and Σ' are on the same orbit, then $\Sigma^\perp$ and $\Sigma'^\perp$ are on the same orbit too. Let us see why this will be useful. Let $n, n' \in \mathbb{N}$ such that $n + n' = \dim(\mathrm{Sym}(d,\mathbb{R}))$ so that, if $\Sigma \in \mathcal{S}_n(d)$, then $\Sigma^\perp \in \mathcal{S}_{n'}(d)$. Suppose that, for some reasons, we know the orbits of $\overline{\delta^\dagger}$ on $\mathcal{S}_n(d)$ and also the maximal subgroups $H(\Sigma)$ for one Σ for every orbit. Hence we know also the orbits in $\mathcal{S}_{n'}(d)$. Further, thanks to part 2, we can also determine $H(\Sigma^\perp)$.

3.4.2 Case $n = 1$

We have just seen that the study of $H(\Sigma)$ leads to the determination of the orbits in $\mathcal{S}_n(d)$ of the action $\overline{\delta^\dagger}$. In general this is not a simple problem.

We start by studying the case where $n := \dim(\Sigma) = 1$, hence set $\Sigma = \mathrm{span}\{\sigma\}$ and $H(\sigma) := H(\Sigma)$. The rest of this subsection is devoted to the determination of $H(\sigma)$.

By definition of $H(\sigma)$, we have that Σ is an invariant subspace for the action $\delta^\dagger_{|H(\sigma)}$ that for simplicity we still denote by $\delta^\dagger$. Considering the canonical vector space isomorphism $\psi\colon \mathbb{R} \to \Sigma$ given by $a \mapsto a\sigma$, we can write the action $\delta^\dagger\colon H(\sigma) \to GL(\mathbb{R})$ as $h \to h^\dagger[\quad]$, where

$$h^\dagger[a] = \psi^{-1}\left(h^\dagger[\psi(a)]\right), \tag{3.18}$$

with a slight abuse of notation, because $h^\dagger[\quad]$ denotes both the action on matrices and real numbers. Identifying $GL(\mathbb{R})$ with $\mathbb{R}^*$ we obtain $\delta^\dagger(h) = h^\dagger[1]$ and so, by (3.18) with $a = 1$, we have

$${}^t h^{-1}\sigma h^{-1} = \delta^\dagger(h)\sigma \quad \text{for all } h \in H(\sigma). \tag{3.19}$$

Now, consider the subgroup $\{\,-1, 1\,\}$ of $\mathbb{R}^*$ and set $F(\sigma) := {\delta^\dagger}^{-1}\{-1,1\}$, that is a subgroup of $H(\sigma)$ characterized by

$$F(\sigma) = \left\{\, l \in GL(d,\mathbb{R}) : {}^t l^{-1}\sigma\, l^{-1} = \pm\sigma \,\right\}. \tag{3.20}$$

The following statement is the fundamental first step towards the determination of $H(\sigma)$.

Proposition 3.25. *If $\sigma \in \mathrm{Sym}(d,\mathbb{R})$ then $\varphi\colon \mathbb{R}_+ \times F(\sigma) \to H(\sigma)$ defined by*

$$(\,\lambda, l\,) \quad \longmapsto \frac{l}{\sqrt{\lambda}}$$

is a group isomorphism.

Proof. First, we prove that φ is well-defined, namely that $\frac{l}{\sqrt{\lambda}} \in H(\sigma)$ for all $l \in F(\sigma)$ and $\lambda \in \mathbb{R}_+$. One has

$${}^t\left(\frac{l}{\sqrt{\lambda}}\right)^{-1}\sigma\left(\frac{l}{\sqrt{\lambda}}\right)^{-1} = \lambda\, {}^t l^{-1}\sigma l^{-1} = \pm\lambda\,\sigma \in \Sigma$$

by definition of $F(\sigma)$. Hence $\frac{l}{\sqrt{\lambda}} \in H(\sigma)$.

Now we verify that φ is a group homomorphism: for $\lambda_1, \lambda_2 \in \mathbb{R}_+$ and $l_1, l_2 \in F(\sigma)$ we have

$$\varphi(\lambda_1, l_1)\varphi(\lambda_2, l_2) = \frac{l_1}{\sqrt{\lambda_1}}\frac{l_2}{\sqrt{\lambda_2}} = \frac{l_1 l_2}{\sqrt{\lambda_1\lambda_2}} = \varphi(\lambda_1\lambda_2, l_1 l_2) = \varphi\left((\lambda_1, l_1)(\lambda_2, l_2)\right).$$

Let us prove that φ is injective. Let $\lambda \in \mathbb{R}_+$ and $l \in F(\sigma)$ be such that $\varphi(\lambda, l) = I$.

Thus we obtain $l = \sqrt{\lambda}\, I \in F(\sigma)$ and by (3.20) we have $\frac{1}{\lambda}\sigma = \pm\sigma$. Hence $\lambda = 1$ and so $l = I$.

Now we must prove that φ is surjective, the crucial part of the proof. Take $h \in H(\sigma)$ and set $a = \delta^\dagger(h) \in \mathbb{R}^*$ which can be written as $a = \operatorname{sign}(a)|a|$. Hence we have ${}^t h^{-1}\sigma h^{-1} = \operatorname{sign}(a)|a|\sigma$. Therefore

$$ {}^t(\sqrt{|a|}\, h)^{-1}\sigma(\sqrt{|a|}\, h)^{-1} = \operatorname{sign}(a)\sigma $$

whence $\sqrt{|a|}\, h \in F(\sigma)$ and

$$ h = \frac{\sqrt{|a|}\, h}{\sqrt{|a|}} = \varphi(\sqrt{|a|}, \sqrt{|a|}\, h) \in \operatorname{Im}(\varphi). \qquad \square $$

Thanks to this proposition, the determination of $H(\sigma)$ amounts to the determination of its subgroup $F(\sigma)$. In general this may remain quite complicated.

We want to use Sylvester's law of inertia to diagonalize the matrix σ. For this purpose, if $p, q, r \in \mathbb{N}$ with $p + q + r = d$ denote by I_{pqr} the canonical form of the metric with signature (p, q, r), namely

$$ I_{pqr} = \begin{bmatrix} I_p & 0 & 0 \\ 0 & -I_q & 0 \\ 0 & 0 & 0 \end{bmatrix} \in M_d(\mathbb{R}). $$

Now, since $\sigma \in \operatorname{Sym}(d, \mathbb{R})$, we can apply Sylvester's law of inertia: there exists $q \in GL(d, \mathbb{R})$ and $p, q, r \in \mathbb{N}$ as above such that ${}^t q\sigma q = I_{pqr}$, namely $q^\dagger[I_{pqr}] = \sigma$. Further, we can suppose $p \geq q$, because $\operatorname{span}\{\,\sigma\,\} = \operatorname{span}\{\,-\sigma\,\}$. Hence we have that $q^\dagger\,[\operatorname{span}\{I_{pqr}\}] = \Sigma$. Thanks to (3.17) (and more in general to Proposition 3.23), we can suppose $\Sigma = \operatorname{span}\{I_{pqr}\}$ and $\sigma = I_{pqr}$. If we denote by $F(p, q, r) = L(I_{pqr})$, by Proposition 3.25 we have $H(I_{pqr}) \cong \mathbb{R}_+ \times F(p, q, r)$ and so

$$ H(I_{pqr}) = \{\, \lambda\, h : \lambda \in \mathbb{R}_+, h \in F(p, q, r) \,\}\,. $$

Denote by $O(p, q, r) < GL(d, \mathbb{R})$ the stability group of the metric I_{pqr}, namely

$$ O(p, q, r) = \left\{\, h \in GL(d, \mathbb{R}) : {}^t h I_{pqr} h = I_{pqr} \,\right\}. $$

Recalling that $F(p, q, r) = \{\, h \in GL(d, \mathbb{R}) : {}^t h^{-1} I_{pqr} h^{-1} = \pm I_{pqr} \,\}$ we set

$$ O^*(p, q, r) := \left\{\, h \in GL(d, \mathbb{R}) : {}^t h I_{pqr} h = -I_{pqr} \,\right\} $$

whence

$$F(p,q,r) = O(p,q,r) \cup O^*(p,q,r).$$

As a consequence of what we have just seen the following corollary holds.

Corollary 3.26. *If $p,q,r \in N$ and $p+q+r=d$, then*

$$H(I_{pqr}) = \{\, \lambda\, h : \lambda \in \mathbb{R}_+, h \in (O(p,q,r) \cup O^*(p,q,r))\, \}$$

In particular, if $p \neq q$ then $H(I_{pqr}) = \{\, \lambda\, h : \lambda \in \mathbb{R}_+, h \in O(p,q,r)\, \}$.

Proof. It remains to prove that if $p \neq q$ then $O^*(p,q,r) = \emptyset$. This is trivial because ${}^t h I_{pqr} h$ has signature (p,q,r) while $-I_{pqr}$ has signature (q,p,r). $\square$

Chapter 4

Two-Dimensional Signals

In this chapter we want to present the complete study for $d = 2$. First, we need to determine the groups in the class $\mathcal{E}_2$ up to conjugation (see Proposition 3.23). In order to do this, we have to exploit the results developed in Subsection 3.4.1. Since $\dim(\mathrm{Sym}(2,\mathbb{R})) = 3$, the interesting cases are only $n = 1$ and $n = 2$ because, to be reproducing, a group must have Σ with $n \leq d$. The case $n = 0$ will not be studied here. By Proposition 3.24 we can concentrate only on the case $n = 1$ and then obtain the solution for the problem with $n = 2$, indeed $1 + 2 = 3$. Thanks to what we saw in Subsection 3.4.2, it is sufficient to consider the spaces generated by the matrices

$$\sigma_1 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad \sigma_2 = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \quad \sigma_3 = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}.$$

For each of these three subspaces $\Sigma_i = \mathrm{span}\,\{\, \sigma_i \,\}$ we have to determine $H(\sigma_i)$. After that, all the groups of the form $\Sigma_i \rtimes H$, with $H < H(\sigma_i)$ will be in $\mathcal{E}_2$. We choose to focus our attention only on the Lie subgroups $H < H(\sigma_i)$ connected. We need another observation. Proposition 3.23 with $q \in H(\sigma_i)$ yields to $\Sigma_i' = \Sigma_i$ and so the groups $G = \Sigma_i \rtimes H$ and $G' = \Sigma_i \rtimes H'$ are conjugated, and associated to the same subspace Σ_i. Hence we can study all the connected Lie subgroups of $H(\sigma_i)$ up to conjugation by elements of $H(\sigma_i)$. Further, this construction respects the passage to $\Sigma^\perp$. Indeed, if H and H' are conjugated by q then ${}^tH$ and ${}^tH'$ are conjugated by ${}^tq^{-1}$ and $H\left(\Sigma^\perp\right) = {}^tH(\Sigma)$ by Proposition 3.24.

In order to do this, we can pass to the corresponding Lie algebras and to their subalgebras, up to conjugation by elements of $H(\sigma_i)$. Indeed, thanks to Corollary (a) of Theorem 3.19 of [War71], there is a bijection between the connected Lie subgroups of $H(\sigma_i)$ and the subalgebras of its Lie algebra $\mathfrak{h}(\sigma_i) = \mathrm{Lie}(H(\sigma_i))$. Moreover, by Proposition A.3, this correspondence respects conjugation. Precisely, we will proceed following these steps:

1. determination of $H(\sigma_i)$ (by means of Corollary 3.26);
2. calculation of $\mathfrak{h}(\sigma_i)$;
3. determination of all the subalgebras $\mathfrak{h}$ of $\mathfrak{h}(\sigma_i)$ up to conjugation by elements of $H(\sigma_i)$;
4. calculation of the corresponding connected Lie subgroups $H < H(\sigma)$ such that $\mathrm{Lie}(H) = \mathfrak{h}$.

4.1 Signature $(2, 0, 0)$

We start studying all the groups in $\mathcal{E}_2$ associated to Σ_1, up to conjugation by elements of $H(\sigma_1)$. Since in this case $p = 2 \neq 0 = q$, by Corollary 3.26 we obtain

$$H(\sigma_1) = \left\{ e^{\lambda} h : \lambda \in \mathbb{R}, h \in O(2) \right\},$$

where, as usual, we denote by $O(2) = O(2, 0, 0)$. Since we can also write $H(\sigma_1) = \mathbb{R}_+ \times O(2)$, its Lie algebra $\mathfrak{h}(\sigma_1)$ can be written as

$$\mathfrak{h}(\sigma_1) = \mathbb{R} \oplus \mathfrak{so}(2) = \{ \alpha J + \beta I : \alpha, \beta \in \mathbb{R} \}.$$

All the non trivial subalgebras are vector subspaces of $\mathfrak{h}(\sigma_1)$ of dimension 1, which are automatically Abelian subalgebras. Since $\dim(\mathfrak{h}(\sigma_1)) = 2$, its 1-dimensional vector subspaces are in bijection with $\mathbb{P}^1_{\mathbb{R}}$. Hence we must consider

$$\mathfrak{h}_\infty(\sigma_1) = \mathrm{span}\, \{ J \}$$

and, for $\alpha \in \mathbb{R}$,

$$\mathfrak{h}_\alpha(\sigma_1) = \mathrm{span}\, \{ \alpha J + I \}.$$

With the following proposition we complete step 3. by finding all the conjugations among these subalgebras.

Proposition 4.1. *If $\alpha_1, \alpha_2 \in \mathbb{R} \cup \{ \infty \}$ then $\mathfrak{h}_{\alpha_1}(\sigma_1)$ is conjugated with $\mathfrak{h}_{\alpha_2}(\sigma_1)$ by an element of $H(\sigma_1)$ if and only if $\alpha_1 = \pm\alpha_2$.*

Proof. We have to conjugate with a matrix $g = \lambda h \in H(\sigma_1)$. The factor $\lambda \in \mathbb{R}_+$ doesn't contribute to the conjugation because it commutes with every matrix. Hence we can suppose $g \in O(2)$. The following relations help us with the computation:

$$O(2) = SO(2) \cup \Lambda \cdot SO(2), \quad SO(2) = \{ R_\theta : \theta \in [0, 2\pi] \},$$

where R_θ denotes the rotation matrix $R_\theta = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$ and $\Lambda = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$.

Since $R_\theta J R_{-\theta} = J$ and $\Lambda J \Lambda = -J$ the matrix J is not modified by a conjugation, so the algebra $\mathfrak{h}_\infty(\sigma_1)$ has no other conjugate algebras. From the computations

$$R_\theta(\alpha J + I)R_{-\theta} = (\alpha J + I), \qquad \Lambda\, R_\theta\,(\alpha J + I)\,R_{-\theta}\,\Lambda = \Lambda\,(\alpha J + I)\,\Lambda = -\alpha J + I$$

we obtain our claim. □

Therefore the family of subalgebras of $\mathfrak{h}(\sigma_1)$ up to conjugation with elements in $H(\sigma_1)$ is

$$\{\, \mathfrak{h}_\alpha(\sigma_1) : \alpha \in [0,\infty] \,\}\,.$$

Now, we need to pass to step 4. We must determine, for $\alpha \in [0,\infty]$, the Lie subgroup $H_\alpha(\sigma_1)$ such that $\mathrm{Lie}(H_\alpha(\sigma_1)) = \mathfrak{h}_{-\alpha}(\sigma_1)$. We use Theorem 3.31 in [War71], by which the exponential map

$$\exp\colon \mathfrak{h}_\alpha(\sigma_1) \longrightarrow H_\alpha(\sigma_1)$$

is a diffeomorphism of a neighborhood of 0 in $\mathfrak{h}_\alpha(\sigma_1)$ onto a neighborhood of e in $H_\alpha(\sigma_1)$. Let $t \in (-\epsilon, \epsilon)$ small enough and compute

$$\exp(t\,J) = R_{-t}, \quad \exp\left(t(\alpha J + I)\right) = \exp(t\alpha J)\exp(tI) = e^t R_{-\alpha t} \quad \text{for all } \alpha \in \mathbb{R},$$

whence we obtain

$$H_\infty(\sigma_1) = \{\, R_\theta : \theta \in [0, 2\pi) \,\}$$

and

$$H_\alpha(\sigma_1) = \left\{\, e^t R_{\alpha t} : t \in \mathbb{R} \,\right\} \quad \text{for } \alpha \in \mathbb{R}.$$

Therefore the family of 1-dimensional connected Lie subgroups of $H(\sigma_1)$ up to conjugation with elements in $H(\sigma_1)$ is

$$\{\, H_\alpha(\sigma_1) : \alpha \in [0,\infty] \,\}\,.$$

Finally, thanks to Proposition 3.24 we can write all the groups in the class $\mathcal{E}_2$ associated to Σ_1 and to $\Sigma_1^\perp$, up to conjugation and only with H connected.

Proposition 4.2. *The following are all the groups in the class $\mathcal{E}_2$ associated to Σ_1 and $\Sigma_1^\perp$, up to conjugation with MA and with H connected:*

(1.i) $\Sigma_1 \rtimes H^0(\sigma_1)$

(1.ii) $\Sigma_1 \rtimes H_\alpha(\sigma_1)$, *with* $\alpha \in [0,\infty]$

(1.iii) $\Sigma_1^\perp \rtimes H^0(\sigma_1)$

(1.iv) $\Sigma_1^\perp \rtimes H_\alpha(\sigma_1)$, *with* $\alpha \in [0,\infty]$.

Proof. The first part follows from what we saw before. In order to prove the rest, it is sufficient to use Proposition 3.24, Proposition 4.1 and to note the following equalities for

$\alpha \in \mathbb{R}$:

$$ {}^tH^0(\sigma_1) = H^0(\sigma_1), \quad {}^tH_\infty(\sigma_1) = H_\infty(\sigma_1), \quad {}^tH_\alpha(\sigma_1) = H_{-\alpha}(\sigma_1). \qquad \square $$

4.1.1 Reproducing Groups, $n = 1$

In this subsection we want to use Theorem 3.20 to determine the reproducing Lie subgroups associated to Σ_1, listed in Proposition 4.2.

First, we need to determine the map Φ associated to Σ_1. We can write $\Sigma_1 = \{ \sigma_a = aI : a \in \mathbb{R} \}$ and define $\psi\colon \mathbb{R} \to \Sigma_1,\ a \mapsto \sigma_a$. This isomorphism satisfies the relation $\langle \psi(a), \psi(a') \rangle_S = 2\,aa'$ for every $a, a' \in \mathbb{R}$. By the equation (3.15) we have for $x \in \mathbb{R}^2$ and $a \in \mathbb{R}$

$$ \langle \Phi(x), \sigma_a \rangle_S = -\frac{1}{2}\langle \sigma_a x, x \rangle = -\frac{1}{2}\langle ax, x \rangle = -\frac{1}{2}|x|^2\, a = -\frac{1}{4}\langle \psi(|x|^2), \sigma_a \rangle_S $$

whence $\Phi(x) = -\frac{1}{4}\psi(|x|^2)$ and so $\operatorname{Im}\Phi = \{ aI : a \leq 0 \}$.

Proposition 4.3. *The groups $\Sigma_1 \rtimes H^0(\sigma_1)$ and $\Sigma_1 \rtimes H_\alpha(\sigma_1)$ with $\alpha \in [0, \infty)$ are reproducing.*

Proof. Since ${}^tH(\sigma_1) = H(\sigma_1)$ we have $\delta' = \delta$. Recall that

$$ H^0(\sigma_1) = \left\{ e^t\, R_\theta : t \in \mathbb{R}, \theta \in [0, 2\pi) \right\}. $$

The action δ for an element of $H^0(\sigma_1)$ is for $t, a \in \mathbb{R}$ and $\theta \in [0, 2\pi)$

$$ \delta(e^t\, R_\theta)(\sigma_a) = (e^t\, R_\theta)\sigma_a\, {}^t(e^t\, R_\theta) = e^{2t}\sigma_a. \tag{4.1} $$

Now we show that $\Sigma_1 \rtimes H^0(\sigma_1)$ is reproducing. There are two orbits of the action in $\operatorname{Im}\Phi$ and they are locally closed. We have $\alpha(e^t\, R_\theta) = e^{-2t}$. Since $H^0(\sigma_1)$ is Abelian and so unimodular, by Proposition 3.15 the group $\Sigma_1 \rtimes H^0(\sigma_1)$ is non-unimodular. Finally, by (4.1) the stabilizer is trivial for every $a < 0$. By Theorem 3.20 the group $\Sigma_1 \rtimes H^0(\sigma_1)$ is reproducing.

Now we show that $\Sigma_1 \rtimes H_\alpha(\sigma_1)$ is reproducing. Recall that $H_\alpha(\sigma_1) = \{ e^t\, R_{\alpha t} : t \in \mathbb{R} \}$. By the above relation the action δ is $\delta(e^t\, R_{\alpha t})(\sigma_a) = e^{2t}\sigma_a$. There are two orbits of the action in $\operatorname{Im}\Phi$ and they are locally closed. We have $\alpha(e^t\, R_{\alpha t}) = e^{-2t}$. Since $H_\alpha(\sigma_1)$ is Abelian and so unimodular, by Proposition 3.15 the group $\Sigma_1 \rtimes H_\alpha(\sigma_1)$ is non-unimodular. Finally, by the above equation the stabilizer is trivial for every $a < 0$. By Theorem 3.20 the group $\Sigma_1 \rtimes H_\alpha(\sigma_1)$ is reproducing. $\square$

Now we show that $\Sigma_1 \rtimes H_\infty(\sigma_1)$ is not reproducing. Recall that $H_\infty(\sigma_1) = \{ R_\theta : \theta \in [0, 2\pi) \}$. By the relation (4.1), $\alpha(R_\theta) = 1$. Since $H_\infty(\sigma_1)$ is Abelian and so unimodular, by Proposition 3.15 the group $\Sigma_1 \rtimes H_\infty(\sigma_1)$ is unimodular. Hence, by Theorem 3.18, it is not reproducing.

4.1.2 Reproducing Groups, $n = 2$

In this subsection we can use the results contained in section 3.3 to establish which of the groups associated to $\Sigma_1^\perp$ are reproducing and to describe the admissible vectors. First of all, by definition of $\langle\ ,\ \rangle_S$, we obtain that

$$\Sigma_1^\perp = \left\{ \sigma_{(a,b)} : a, b \in \mathbb{R} \right\}$$

where $\sigma_{(a,b)} = \begin{bmatrix} a & b \\ b & -a \end{bmatrix}$. Indeed $\langle I, \sigma\rangle_S = \operatorname{tr}(I\,\sigma) = \operatorname{tr}(\sigma)$ for all $\sigma \in \operatorname{Sym}(2,\mathbb{R})$. Now we want to determine the map Φ associated to the subspace $\Sigma_1^\perp$. Consider the canonical isomorphism $\psi_1 \colon \mathbb{R}^2 \to \Sigma_1^\perp$ defined by $q = (a,b) \mapsto \sigma_q$ that identifies $\Sigma_1^\perp = \mathbb{R}^2$. The map ψ_1 satisfies the relation $\langle \psi_1(q), \psi_1(q')\rangle_S = 2\,\langle q, q'\rangle$ for all $q, q' \in \mathbb{R}^2$. Hence by the equality (3.15) we obtain for $x = (x_1, x_2), q = (a,b) \in \mathbb{R}^2$

$$\begin{aligned}
\langle \Phi(x), \sigma_q\rangle_S &= -\frac{1}{2}\langle \sigma_q\, x, x\rangle \\
&= -\frac{1}{2}(ax_1^2 + 2bx_1x_2 - ax_2^2) \\
&= -\frac{1}{2}\langle (x_1^2 - x_2^2, 2x_1x_2), (a,b)\rangle \\
&= -\frac{1}{4}\langle \psi_1(x_1^2 - x_2^2, 2x_1x_2), \sigma_q\rangle_S
\end{aligned}$$

whence $\Phi(x_1,x_2) = -\frac{1}{4}\psi_1(x_1^2 - x_2^2, 2x_1x_2)$. Hence Φ is surjective. For the rest of this section we often identify $\Sigma_1^\perp = \mathbb{R}^2$ and consequently neglect the map ψ_1. The Jacobian of Φ is $J(\Phi)(x) = \frac{1}{4}(x_1^2 + x_2^2)$ for every $x \in \mathbb{R}^2$ and so we set $X = \mathbb{R}^2 \setminus \{\,0\,\}$ and $Y = \Sigma_1^\perp \setminus \{\,0\,\}$.

In order to study, with the following proposition, the case with $H^0(\sigma_1) = \mathbb{R}_+ \times SO(2)$ let us introduce the measure ν in $\mathbb{R}^2$ defined by $d\nu = \frac{dx}{|x|^2}$, the map $j(x) = -x$ for $x \in \mathbb{R}^2$ and the unit vector $e_\theta = (\cos\theta, \sin\theta)$ for $\theta \in \mathbb{R}$.

Proposition 4.4. *The group $\Sigma_1^\perp \rtimes H^0(\sigma_1)$ is reproducing for $\mu_{|\Sigma_1^\perp \rtimes H^0(\sigma_1)}$ and a function $\eta \in L^2(\mathbb{R}^2)$ is an admissible vector for the system $(\Sigma_1^\perp \rtimes H^0(\sigma_1), \mu_{|\Sigma_1^\perp \rtimes H^0(\sigma_1)})$ if and only if $\|\eta\|_{L^2(\nu)} = 1/2$ and $\langle \eta, \eta \circ j\rangle_{L^2(\nu)} = 0$.*

Proof. From an easy computation we obtain for $\theta \in \mathbb{R}$ and $q = (a,b) \in \mathbb{R}^2$

$$R_\theta\, \sigma_q\, R_{-\theta} = \begin{bmatrix} a\cos(2\theta) - b\sin(2\theta) & b\cos(2\theta) + a\sin(2\theta) \\ b\cos(2\theta) + a\sin(2\theta) & b\sin(2\theta) - a\cos(2\theta) \end{bmatrix} \in \Sigma_1^\perp$$

and so we may write

$$R_\theta\, \sigma_q\, R_{-\theta} = \psi\left(R_{2\theta}\, q\right). \tag{4.2}$$

Note that, since ${}^tH(\sigma_1) = H(\sigma_1)$, we have $\delta' = \delta$.

Now we prove that $\Sigma_1^\perp \rtimes H^0(\sigma_1)$ is reproducing. First, we need to verify that the $H^0(\sigma_1)$-orbits of $\Sigma_1^\perp$ with respect to the action $\delta' = \delta$ are locally closed. Recall that

$$H^0(\sigma_1) = \left\{ e^t\, R_\theta : t \in \mathbb{R}, \theta \in [0, 2\pi) \right\}.$$

Thus we can compute for $t \in \mathbb{R}$, $\theta \in [0, 2\pi)$ and $q \in \mathbb{R}^2 \setminus \{ 0 \}$

$$\delta(e^t\, R_\theta)\sigma_q = (e^t\, R_\theta)[\sigma_q] = e^t\, R_\theta\, \sigma_q\, {}^t(e^t\, R_\theta) = e^{2t}\, R_\theta\, \sigma_q\, R_{-\theta} = e^{2t}\, \psi\,(R_{2\theta}\, q)\,.$$

Hence the orbit for σ_q is $\{ e^t\, \psi\,(R_\theta\, q) : t \in \mathbb{R}, \theta \in [0, 2\pi) \} = Y$ that is open and consequently locally closed. Now we need to prove that $\Sigma_1^\perp \rtimes H^0(\sigma_1)$ is non-unimodular. Since

$$\delta^\dagger(e^t\, R_\theta)\sigma_q = {}^t(e^t\, R_\theta)^{-1}\, \sigma_q\, (e^t\, R_\theta)^{-1} = e^{-2t}\, R_\theta\, \sigma_q\, R_{-\theta} = e^{-2t}\, \psi\,(R_{2\theta}\, q)\,,$$

we have that $\alpha(e^t\, R_\theta) = \det(e^{-2t}\, R_{2\theta}) = e^{-4t}$. Hence, thanks to Proposition 3.15, since $H^0(\sigma_1)$ is Abelian and as a result unimodular we obtain that $\Sigma_1^\perp \rtimes H^0(\sigma_1)$ is non-unimodular. Now we find the stabilizer of σ_q with $q \in \mathbb{R}^2 \setminus \{ 0 \}$ for the action δ. The condition $\delta(e^t\, R_\theta)\sigma_q = \sigma_q$ is equivalent to $e^{2t}\, R_{2\theta}\, q = q$. Hence the stabilizer is $H_{\sigma_q} = \{ I, R_\pi \}$ and so compact for almost every $q \in \mathbb{R}^2$. Thanks to Theorem 3.19 we obtain that $\Sigma_1^\perp \rtimes H^0(\sigma_1)$ is reproducing.

Now we use Theorem 3.22 to determine the admissible vectors. We saw before that there is a unique δ-orbit in Y, and so we choose $Z = \{ 0 \}$ and $\lambda(Z) = 1$ is a pseudo-image measure of the Lebesgue measure on Y. Set $y_0 = -\frac{1}{4}(1, 0) \in Y$ and so $\Phi^{-1}(y_0) = \{ P_1, P_2 \}$ where $P_1 = (1, 0)$ and $P_2 = -P_1$. Since $R_\pi \in H_{y_0}$ and $R_\pi P_1 = P_2$ we have to consider equation (3.16) for the couples (P_1, P_1) and (P_1, P_2). The first condition yields

$$\begin{aligned}
\frac{1}{4} &= \int_{\mathbb{R}} \int_0^{2\pi} |\eta(e^{-t} R_{-\theta} P_1)|^2\, \frac{1}{e^{2t}\, e^{-4t}}\, d\theta dt \\
&= \int_{\mathbb{R}} \int_0^{2\pi} |\eta(e^{-t}(\cos\theta, -\sin\theta))|^2\, e^{2t}\, d\theta dt \\
&= \int_{\mathbb{R}} \int_0^{2\pi} |\eta(e^{-t}\, e_\theta)|^2\, e^{2t}\, d\theta dt \\
&= \int_{\mathbb{R}+} \int_0^{2\pi} |\eta(\rho\, e_\theta)|^2\, \frac{\rho}{\rho^4}\, d\theta d\rho \\
&= \int_{\mathbb{R}^2} |\eta|^2\, d\nu \\
&= \|\eta\|^2_{L^2(\nu)}.
\end{aligned}$$

The other condition follows analogously with the same change of variable, because $P_2 =$

$-P_1$. □

Now we show that the group $\Sigma_1^\perp \rtimes H_\infty(\sigma_1)$ is not reproducing. The orbits of the action δ are circumferences and so locally closed. Since $H_\infty(\sigma_1)$ is commutative and consequently unimodular, by Proposition 3.15 and by the equation (4.2) the group $\Sigma_1^\perp \rtimes H_\infty(\sigma_1)$ is unimodular. Hence, thanks to Theorem 3.19, it is not reproducing.

In order to study, with the following proposition, the case with $H_\alpha(\sigma_1)$ let us introduce the 1-dimensional subvarieties of $\mathbb{R}^2$ for $\nu \in [0, 2\pi)$

$$\mathcal{O}_\nu^\alpha = \left\{ e^t R_{\alpha t}\, e_\nu : t \in \mathbb{R} \right\}$$

that are spirals centered in the origin.

Proposition 4.5. *Let $\alpha \in [0, \infty)$. The group $\Sigma_1^\perp \rtimes H_\alpha(\sigma_1)$ is reproducing for $\mu_{|\Sigma_1^\perp \rtimes H_\alpha(\sigma_1)}$ and a function $\eta \in L^2(\mathbb{R}^2)$ is an admissible vector for the system $(\Sigma_1^\perp \rtimes H_\alpha(\sigma_1), \mu_{|\Sigma_1^\perp \rtimes H_\alpha(\sigma_1)})$ if and only if*

$$\int_{\mathcal{O}_\nu^\alpha} \frac{|\eta(x)|^2}{|x|^3}\, ds(x) = \frac{\sqrt{\alpha^2+1}}{4}, \qquad \int_{\mathcal{O}_\nu^\alpha} \frac{\eta(x)\, \overline{\eta \circ j(x)}}{|x|^3}\, ds(x) = 0$$

for almost every $\nu \in [0, 2\pi)$.

Proof. Now we show that $\Sigma_1^\perp \rtimes H_\alpha(\sigma_1)$ is reproducing. First, we need to verify that the $H_\alpha(\sigma_1)$-orbits of $\Sigma_1^\perp$ with respect to the action δ are locally closed. Recall that $H_\alpha(\sigma_1) = \{ e^t R_{\alpha t} : t \in \mathbb{R} \}$. Thus we can compute for $t \in \mathbb{R}$ and $q \in \mathbb{R}^2 \setminus \{ 0 \}$

$$\delta(e^t R_{\alpha t})\sigma_q = (e^t R_{\alpha t})[\sigma_q] = e^t R_{\alpha t}\, \sigma_q\, {}^t(e^t R_{\alpha t}) = e^{2t} R_{\alpha t}\, \sigma_q\, R_{-\alpha t} = e^{2t}\, \psi_1(R_{2\alpha t} q).$$

Hence the orbit for σ_q is $\{ e^t\, \psi_1(R_{\alpha t} q) : t \in \mathbb{R} \}$ and so homeomorphic to $\mathcal{O}_q = \{ e^t R_{\alpha t}\, q : t \in \mathbb{R} \}$ that is a spiral in $\mathbb{R}^2$. $\mathcal{O}_q$ is locally closed because $\mathcal{O}_q = (\mathbb{R}^2 \setminus \{ 0 \}) \cap \overline{\mathcal{O}_q}$. Now we need to prove that $\Sigma_1^\perp \rtimes H_\alpha(\sigma_1)$ is non-unimodular. Since

$$\delta^\dagger(e^t R_{\alpha t})\sigma_q = {}^t(e^t R_{\alpha t})^{-1}\, \sigma_q\, (e^t R_{\alpha t})^{-1} = e^{-2t} R_{\alpha t}\, \sigma_q\, R_{-\alpha t} = e^{-2t}\, \psi_1(R_{2\alpha t} q),$$

we have that $\alpha(e^t R_{\alpha t}) = \det(e^{-2t} R_{\alpha t}) = e^{-4t}$. Hence, thanks to Proposition 3.15, since $H(\sigma_1)_\alpha$ is Abelian and consequently unimodular we obtain that $\Sigma_1^\perp \rtimes H_\alpha(\sigma_1)$ is non-unimodular. Now we find the stabilizer of σ_q with $q \in \mathbb{R}^2 \setminus \{ 0 \}$ for the action δ. The condition $\delta(e^t R_{\alpha t})\sigma_q = \sigma_q$ is equivalent to $e^{2t} R_{2\alpha t}\, q = q$. Hence the stabilizer is $\{ I \}$ and so compact for almost every $q \in \mathbb{R}^2$. Thanks to Theorem 3.19 we deduce that $\Sigma_1^\perp \rtimes H_\alpha(\sigma_1)$ is reproducing.

Now we use Theorem 3.22 to determine the admissible vectors. We saw before that the orbits of δ in Y are $\mathcal{O}_q = \{ e^t R_{\alpha t}\, q : t \in \mathbb{R} \}$ and so every orbit passes for a unique point in the unit circle. Hence we set $Z = [0, 2\pi)$. As a consequence of Fubini's theorem, it is easy to verify that the Lebesgue measure λ on Z is a pseudo-image measure of the Lebesgue measure on Y. For almost every $\theta \in Z$ let $y_\theta = \frac{1}{4} e_\theta \in Y$ such that $\pi(y_\theta) = \theta$. Hence $\Phi^{-1}(y_\theta) = \{ e_{\nu_1}, e_{\nu_2} \}$ where $\nu_1 = \frac{\theta - \pi}{2}$ and $\nu_2 = \nu_1 + \pi$. Condition (3.16) for the couples (e_{ν_1}, e_{ν_1}) and (e_{ν_2}, e_{ν_2}) yields for almost every $\nu \in [0, 2\pi)$

$$\frac{1}{4} = \int_{\mathbb{R}} |\eta(e^{-t} R_{-\alpha t}\, e_\nu)|^2 \, \frac{1}{e^{-4t} e^{2t}} \, dt = \int_{\mathbb{R}} |\eta(e^{-t} R_{-\alpha t}\, e_\nu)|^2 \, e^{2t} \, dt.$$

We want to simplify this formula. Consider the map $\gamma_\nu \colon \mathbb{R} \to \mathbb{R}^2$ defined by $\gamma_\nu(t) = e^{-t} R_{-\alpha t}\, e_\nu$ for every $t \in \mathbb{R}$. Hence $|\gamma_\nu(t)| = e^{-t}$. Moreover we have

$$\gamma_\nu(t) = e^{-t} R_{-\alpha t}\, R_\nu\, e_0 = R_\nu\, e^{-t} R_{-\alpha t}\, e_0 = R_\nu\, \gamma_0(t) \quad \text{for all } t \in \mathbb{R}$$

and so

$$|\gamma_\nu'(t)| = |\gamma_0'(t)| = e^{-t} \sqrt{\alpha^2 + 1} \quad \text{for all } t \in \mathbb{R}$$

because $\gamma_\nu'(t) = R_\nu \gamma_0'(t)$ for every $t \in \mathbb{R}$. Hence the above equation may be rewritten

$$\frac{1}{4} = \frac{1}{\sqrt{\alpha^2 + 1}} \int_{\mathbb{R}} \frac{|\eta(\gamma_\nu(t))|^2}{|\gamma_\nu(t)|^3} \, |\gamma_\nu'(t)| \, dt = \frac{1}{\sqrt{\alpha^2 + 1}} \int_{\mathcal{O}_\nu^\alpha} \frac{|\eta(x)|^2}{|(x)|^3} \, ds$$

whence we obtain the first condition. The second one follows similarly. $\square$

4.2 Signature $(1, 1, 0)$

We study all the groups in $\mathcal{E}_2$ associated to the space Σ_2, up to conjugation by elements of $H(\sigma_2)$. Since in this case $p = 1 = q$, thanks to Corollary 3.26 we obtain

$$H(\sigma_2) = \{ \lambda\, h : \lambda \in \mathbb{R}_+, h \in (O(1,1) \cup O^*(1,1)) \}$$

where, as usual, we denote by $O(1,1) = O(1,1,0)$. Its Lie algebra $\mathfrak{h}(\sigma_1)$ can be written as

$$\mathfrak{h}(\sigma_2) = \mathbb{R} \oplus \mathfrak{so}(1,1) = \{ \alpha\Omega + \beta I : \alpha, \beta \in \mathbb{R} \}, \quad \text{where } \Omega = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}.$$

Now we have to deal with step 3. All the non trivial subalgebras are vector subspaces of $\mathfrak{h}(\sigma_2)$ of dimension 1, which are automatically Abelian subalgebras. Since $\dim(\mathfrak{h}(\sigma_2)) = 2$, its 1-dimensional vector subspaces are in bijection with $\mathbb{P}^1_{\mathbb{R}}$. Hence we must consider

$$\mathfrak{h}_\infty(\sigma_2) = \operatorname{span}\{ \Omega \}$$

and, for $\alpha \in \mathbb{R}$,

$$\mathfrak{h}_\alpha(\sigma_2) = \operatorname{span}\{\, \alpha\Omega + I \,\}\,.$$

With the following proposition we complete step 3. by finding all the conjugations among these subalgebras.

Proposition 4.6. *If $\alpha_1, \alpha_2 \in \mathbb{R} \cup \{\, \infty \,\}$ then $\mathfrak{h}_{\alpha_1}(\sigma_2)$ is conjugated with $\mathfrak{h}_{\alpha_2}(\sigma_2)$ by an element of $H(\sigma_2)$ if and only if $\alpha_1 = \pm\alpha_2$.*

Proof. We have to conjugate with a matrix $g = \lambda h \in H(\sigma_2)$. The factor $\lambda \in \mathbb{R}_+$ doesn't contribute to the conjugation because it commutes with every matrix. Hence we can suppose $g \in L(1, 1) := L(1, 1, 0)$. For $t \in \mathbb{R}$ denote by

$$A_t = \begin{bmatrix} \cosh t & \sinh t \\ \sinh t & \cosh t \end{bmatrix}.$$

The following relations are easily provable and help us with the computation:

$$O(1, 1) = \{\, \pm A_t, \pm\Lambda A_t : t \in \mathbb{R} \,\}\,, \quad L(1, 1) = O(1, 1) \cup \Omega \cdot O(1, 1).$$

Since $A_t\, \Omega\, A_t^{-1} = \Omega$ and $\Lambda\, \Omega\, \Lambda = -\Omega$, the algebra $\mathfrak{h}_\infty(\sigma_2)$ has no other conjugate algebras. For the same reason for which we have excluded the factors λ, we don't have to consider the signs "$-$". Finally, from the computations $A_t(\alpha\Omega + I)A_t^{-1} = (\alpha\Omega + I)$,

$$\Lambda A_t(\alpha\Omega + I)A_t^{-1}\Lambda = \Lambda(\alpha\Omega + I)\Lambda = -\alpha\Omega + I$$

and

$$\Omega(\alpha\Omega + I)\Omega = \alpha\Omega + I,$$

we obtain our claim. □

Therefore the family of subalgebras of $\mathfrak{h}(\sigma_2)$ up to conjugation with elements in $H(\sigma_2)$ is

$$\{\, \mathfrak{h}_\alpha(\sigma_2) : \alpha \in [0, \infty] \,\}\,.$$

Now, we need to pass to step 4. We must determine, for $\alpha \in [0, \infty]$, the corresponding Lie subgroup $H_\alpha(\sigma_2)$ such that $\operatorname{Lie}(H_\alpha(\sigma_2)) = \mathfrak{h}_\alpha(\sigma_2)$. As above, let $t \in (-\epsilon, \epsilon)$ small enough and compute

$$\exp(t\Omega) = A_t, \quad \exp\left(t(\alpha\Omega + I)\right) = \exp(t\alpha\Omega)\exp(tI) = e^t A_{\alpha t} \quad \text{for all } \alpha \in [0, \infty),$$

whence we obtain

$$H_\infty(\sigma_2) = \{\, A_t : t \in \mathbb{R} \,\}$$

and

$$H_\alpha(\sigma_2) = \left\{ e^t A_{\alpha t} : t \in \mathbb{R} \right\} \quad \text{for } \alpha \in [0,\infty).$$

Therefore the family of 1-dimensional Lie connected subgroups of $H(\sigma_2)$ up to conjugation with elements in $H(\sigma_2)$ is

$$\{ H_\alpha(\sigma_2) : \alpha \in [0,\infty] \}.$$

Finally, thanks to Proposition 3.24 we can write all the groups in the class $\mathcal{E}_2$ associated to Σ_2 and to $\Sigma_2^\perp$, up to conjugation and only with H connected.

Proposition 4.7. *The following are all the groups in the class $\mathcal{E}_2$ associated to Σ_2 and $\Sigma_2^\perp$, up to conjugation with MA and with H connected:*

(2.i) $\Sigma_2 \rtimes H^0(\sigma_2)$

(2.ii) $\Sigma_2 \rtimes H_\alpha(\sigma_2)$, *with* $\alpha \in [0,\infty]$

(2.iii) $\Sigma_2^\perp \rtimes H^0(\sigma_2)$

(2.iv) $\Sigma_2^\perp \rtimes H_\alpha(\sigma_2)$, *with* $\alpha \in [0,\infty]$.

Proof. The first part follows from what we saw before. In order to prove the rest, it is sufficient to use Proposition 3.24, Proposition 4.6 and to note the following equalities for $\alpha \in \mathbb{R}$:

$$^tH^0(\sigma_2) = H^0(\sigma_2), \quad ^tH_\infty(\sigma_2) = H_\infty(\sigma_2), \quad ^tH_\alpha(\sigma_2) = H_\alpha(\sigma_2). \qquad \square$$

4.2.1 Reproducing Groups, $n = 1$

In this subsection we want to use Theorem 3.20 to determine the reproducing Lie subgroups associated to Σ_2, listed in Proposition 4.7.

First, we need to determine the map Φ associated to Σ_2. We can write $\Sigma_2 = \{ \sigma_a = a\sigma_2 : a \in \mathbb{R} \}$ and define $\psi\colon \mathbb{R} \to \Sigma_1$, $a \mapsto \sigma_a$. This isomorphism satisfies the relation $\langle \psi(a), \psi(a') \rangle_S = 2\,aa'$ for every $a, a' \in \mathbb{R}$. By the equation (3.15) we have for $x \in \mathbb{R}^2$ and $a \in \mathbb{R}$

$$\langle \Phi(x), \sigma_a \rangle_S = -\frac{1}{2}\langle \sigma_a x, x\rangle = -\frac{1}{2}\langle a\sigma_2 x, x\rangle = -\frac{1}{2}(x_1^2 - x_2^2)\,a = -\frac{1}{4}\langle \psi(x_1^2 - x_2^2), \sigma_a \rangle_S$$

whence $\Phi(x) = \frac{1}{4}\psi(x_2^2 - x_1^2)$ and so $\operatorname{Im}\Phi = \Sigma_2$.

Proposition 4.8. *The groups $\Sigma_2 \rtimes H^0(\sigma_2)$ and $\Sigma_2 \rtimes H_\alpha(\sigma_2)$ with $\alpha \in [0,\infty)$ are reproducing.*

Proof. Since $^tH(\sigma_2) = H(\sigma_2)$ we have $\delta' = \delta$. Recall that

$$H^0(\sigma_2) = \mathbb{R}_+ \times O(1,1)_0 = \left\{ e^t A_s : t, s \in \mathbb{R} \right\}.$$

The action δ for an element of $H^0(\sigma_2)$ is for $t,s,a\in\mathbb{R}$

$$\delta(e^t\,A_s)(\sigma_a) = (e^t\,A_s)\sigma_a\,{}^t(e^t\,A_s) = e^{2t}\,A_s\,\sigma_a\,A_s = e^{2t}\,\sigma_a. \tag{4.3}$$

Now we show that $\Sigma_2\rtimes H^0(\sigma_2)$ is reproducing. There are three orbits of the action in $\operatorname{Im}\Phi$ and they are locally closed. We have $\alpha(e^t\,A_s) = e^{-2s}$. Since $H^0(\sigma_2)$ is Abelian and so unimodular, by Proposition 3.15 the group $\Sigma_2\rtimes H^0(\sigma_2)$ is non-unimodular. Finally, by the above equation the stabilizer is trivial for every $a\neq 0$. By Theorem 3.20 the group $\Sigma_2\rtimes H^0(\sigma_2)$ is reproducing.

Now we show that $\Sigma_2\rtimes H_\alpha(\sigma_2)$ is reproducing. Recall that $H_\alpha(\sigma_2) = \{\, e^t\,A_{\alpha t} : t\in\mathbb{R}\,\}$. By the above relation the action δ is $\delta(e^t\,A_{\alpha t})(\sigma_a) = e^{2t}\sigma_a$. There are three orbits of the action in $\operatorname{Im}\Phi$ and they are locally closed. We have $\alpha(e^t\,A_{\alpha t}) = e^{-2t}$. Since $H_\alpha(\sigma_2)$ is Abelian and so unimodular, by Proposition 3.15 the group $\Sigma_2\rtimes H_\alpha(\sigma_2)$ is non-unimodular. Finally, the stabilizer is trivial for every $a\neq 0$. By Theorem 3.20 the group $\Sigma_2\rtimes H_\alpha(\sigma_2)$ is reproducing. □

Now we show that $\Sigma_2\rtimes H_\infty(\sigma_2)$ is not reproducing. Recall that $H_\infty(\sigma_2) = \{\, A_t : t\in\mathbb{R}\,\}$. By the relation (4.3), $\alpha(A_t) = 1$. Since $H_\infty(\sigma_1)$ is Abelian and so unimodular, by Proposition 3.15 the group $\Sigma_2\rtimes H_\infty(\sigma_2)$ is unimodular. Hence, by Theorem 3.18, it is not reproducing.

4.2.2 Reproducing Groups, $n = 2$

In this subsection we use the results contained in section 3.3 to establish which of the groups associated to $\Sigma_2^\perp$ are reproducing and to describe the admissible vectors. First of all, by definition of $\langle\ ,\ \rangle_S$, we obtain that

$$\Sigma_2^\perp = \left\{\, \sigma_{(a,b)} : a,b\in\mathbb{R} \,\right\}$$

where $\sigma_{(a,b)} = \begin{bmatrix} a & b \\ b & a \end{bmatrix}$. Indeed $\langle\sigma_2,\sigma\rangle_S = \operatorname{tr}(\sigma_2\,\sigma) = \sigma_{1,1}-\sigma_{2,2}$ for all $\sigma\in\operatorname{Sym}(2,\mathbb{R})$. Now we want to determine the map Φ associated to the subspace $\Sigma_2^\perp$. Consider the canonical isomorphism $\psi_2\colon\mathbb{R}^2\to\Sigma_2^\perp$ defined by $q=(a,b)\mapsto\sigma_q$ that identifies $\Sigma_2^\perp = \mathbb{R}^2$. The map ψ_2 satisfies the relation $\langle\psi_2(q),\psi_2(q')\rangle_S = 2\,\langle q,q'\rangle$ for all $q,q'\in\mathbb{R}^2$. Hence by the equality (3.15) we obtain for $x=(x_1,x_2), q=(a,b)\in\mathbb{R}^2$

$$\begin{aligned}\langle \Phi(x),\sigma_q\rangle_S &= -\frac{1}{2}\langle \sigma_q\, x, x\rangle \\ &= -\frac{1}{2}(ax_1^2+2bx_1x_2+ax_2^2) \\ &= -\frac{1}{2}\langle (x_1^2+x_2^2, 2x_1x_2),(a,b)\rangle \\ &= -\frac{1}{4}\langle \psi_2(x_1^2+x_2^2,2x_1x_2),\sigma_q\rangle_S\end{aligned}$$

whence $\Phi(x_1,x_2)=-\frac{1}{4}\psi_2(x_1^2+x_2^2,2x_1x_2)$. Hence $\operatorname{Im}\Phi=\{\,(y_1,y_2): y_1\le 0, |y_2|\le -y_1\,\}$. For the rest of this section we often identify $\Sigma_2^\perp=\mathbb{R}^2$ and consequently neglect the map ψ_2. The Jacobian of Φ is $J(\Phi)(x)=\frac{1}{4}|x_1^2-x_2^2|$ for every $x\in\mathbb{R}^2$ and so we set $X=\{\,x\in\mathbb{R}^2 : x_1^2\neq x_2^2\,\}$ and $Y=\{\,(y_1,y_2): y_1<0, |y_2|<-y_1\,\}$.

With the following proposition we study the case with $H^0(\sigma_2)$. Let us introduce some useful objects. Set

$$M_0=I,\quad M_1=\Omega,\quad M_2=-I,\quad M_3=-\Omega\in GL(\mathbb{R}^2)$$

that must be seen as isomorphism of $\mathbb{R}^2$. Further, if $i\in\{\,0,1,2,3\,\}$ denote by V_i the cone

$$V_i=\left\{\,\rho e_\theta\in\mathbb{R}^2 : \rho\in\mathbb{R}_+, \theta\in(\frac{\pi}{2}i-\frac{\pi}{4},\frac{\pi}{2}i+\frac{\pi}{4})\,\right\}$$

and by ν_i the measure on V_i defined by $d\nu_i=\frac{dx_1dx_2}{(x_1^2-x_2^2)^2}$. Note that by definition we have

$$X=\bigcup_{i=0}^{3}V_i,\qquad Y=V_2,\qquad M_i\colon V_0\xrightarrow{\cong}V_i.$$

Proposition 4.9. *The group $\Sigma_2^\perp\rtimes H^0(\sigma_2)$ is reproducing for $\mu_{|\Sigma_2^\perp\rtimes H^0(\sigma_2)}$ and a function $\eta\in L^2(\mathbb{R}^2)$ is an admissible vector for the system $(\Sigma_2^\perp\rtimes H^0(\sigma_2),\mu_{|\Sigma_2^\perp\rtimes H^0(\sigma_2)})$ if and only if $\|\eta\|_{L^2(\nu_i)}=1/2$ for every $i=0,\dots,3$ and $\langle\eta\circ M_i,\eta\circ M_j\rangle_{L^2(\nu_0)}=0$ for every $0\le i<j\le 3$.*

Proof. From an easy computation we obtain for $s\in\mathbb{R}$ and $q=(a,b)\in\mathbb{R}^2$

$$A_s\,\sigma_q\,A_s=\begin{bmatrix} a\cosh(2s)+b\sinh(2s) & b\cosh(2s)+a\sinh(2s)\\ b\cosh(2s)+a\sinh(2s) & a\cosh(2s)+b\sinh(2s)\end{bmatrix}\in\Sigma_2^\perp$$

and so we may write

$$A_s\,\sigma_q\,A_s=\psi_3\left(A_{2s}\,q\right). \tag{4.4}$$

Note that, since ${}^tH(\sigma_2)=H(\sigma_2)$, we have $\delta'=\delta$.

Now we prove that $\Sigma_2^\perp \rtimes H^0(\sigma_2)$ is reproducing. First, we need to verify that the $H^0(\sigma_2)$-orbits of $\operatorname{Im}\Phi$ with respect to the action $\delta' = \delta$ are locally closed. We can compute for $t, s \in \mathbb{R}$ and $q \in \operatorname{Im}\Phi$

$$\delta(e^t A_s)\sigma_q = (e^t A_s)[\sigma_q] = e^t A_s \, \sigma_q \, {}^t(e^t A_s) = e^{2t} A_s \, \sigma_q \, A_s = e^{2t} \, \psi_2 \, (A_{2s} \, q) \, . \tag{4.5}$$

Hence the orbit for σ_q is $\{ \, e^t \, \psi_2 \, (A_s \, q) : t, s \in \mathbb{R} \, \}$. If $q \in Y$ the orbit is Y that is open and consequently locally closed. Otherwise the orbit is a ray that is locally closed. Now we need to prove that $\Sigma_2^\perp \rtimes H^0(\sigma_2)$ is non-unimodular. Since

$$\delta^\dagger(e^t A_s)\sigma_q = {}^t(e^t A_s)^{-1} \, \sigma_q \, (e^t A_s)^{-1} = e^{-2t} \, A_{-s} \, \sigma_q \, A_{-s} = e^{-2t} \, \psi_2 \, (A_{-2s} \, q) \, ,$$

we have that $\alpha(e^t A_s) = \det(e^{-2t} A_{-2s}) = e^{-4t}$. Hence, thanks to Proposition 3.15, since $H^0(\sigma_2)$ is Abelian and as a result unimodular we obtain that $\Sigma_2^\perp \rtimes H^0(\sigma_2)$ is non-unimodular. Now we find the stabilizer of σ_q with $q \in Y$ for the action δ. The condition $\delta(e^t A_s)\sigma_q = \sigma_q$ is equivalent to $e^{2t} A_{2s} \, q = q$. Taking $q = (-1, 0)$, recalling that the orbit is Y, we obtain that the stabilizer is $\{ \, I \, \}$ and so compact for almost every $q \in \operatorname{Im}\Phi$. Thanks to Theorem 3.19 we obtain that $\Sigma_2^\perp \rtimes H^0(\sigma_2)$ is reproducing.

Now we use Theorem 3.22 to determine the admissible vectors. We saw before that there is a unique δ-orbit in Y, and so we choose $Z = \{ \, 0 \, \}$ and $\lambda(Z) = 1$ is a pseudo-image measure of the Lebesgue measure on Y. Set $y_0 = -\frac{1}{4}(1,0) \in Y$ and so $\Phi^{-1}(y_0) = \left\{ \, P_i = e_{\frac{\pi}{2} i} : i = 0, \ldots, 3 \, \right\}$. Now we consider equation (3.16) for the couple (P_i, P_i) with $i \in \{ \, 0, 1, 2, 3 \, \}$:

$$\frac{1}{4} = \int_{\mathbb{R}} \int_{\mathbb{R}} |\eta(e^{-t} A_{-s} P_i)|^2 \, \frac{1}{e^{-4t} e^{2t}} \, dt \, ds = \int_{\mathbb{R}} \int_{\mathbb{R}} |\eta(e^{-t} A_{-s} P_i)|^2 \, e^{2t} \, dt \, ds.$$

In order to simplify this condition, consider the parametrization $\gamma^i \colon \mathbb{R}^2 \to V_i$ defined by $(t, s) \mapsto e^{-t} A_{-s} P_i$ for $t, s \in \mathbb{R}$. A simple computation shows that

$$|\gamma_1^i(t,s)^2 - \gamma_2^i(t,s)^2| = e^{-2t} = J(\gamma^i)(t,s) \quad \text{for all } t, s \in \mathbb{R}.$$

Therefore the above condition may be rewritten as

$$\frac{1}{4} = \int_{\mathbb{R}} \int_{\mathbb{R}} \frac{|\eta(\gamma^i(t,s))|^2}{(\gamma_1^i(t,s)^2 - \gamma_2^i(t,s)^2)^2} \, J(\gamma^i)(t,s) \, dt \, ds = \int_{V_i} \frac{|\eta(x_1, x_2)|^2}{(x_1^2 - x_2^2)^2} \, dx_1 dx_2.$$

Now we have to consider equation (3.16) for a mixed couple (P_i, P_j) with $i < j$. It easy to prove that $A_{-s}P_i = M_i A_{-s} P_0$ for every i. With the same change of variable we obtain

$$\begin{aligned} 0 &= \int_{\mathbb{R}} \int_{\mathbb{R}} \eta(e^{-t} A_{-s} P_i)\, \overline{\eta(e^{-t} A_{-s} P_j)}\, e^{2t}\, dt\, ds \\ &= \int_{\mathbb{R}} \int_{\mathbb{R}} \eta(M_i\, \gamma^0(t,s))\, \overline{\eta(M_j\, \gamma^0(t,s))}\, e^{2t}\, dt\, ds \\ &= \int_{V_0} \frac{\eta \circ M_i(x_1,x_2)\, \overline{\eta \circ M_j(x_1,x_2)}}{(x_1^2 - x_2^2)^2}\, dx_1 dx_2, \end{aligned}$$

and so we are done. □

Now we show that the group $\Sigma_2^{\perp} \rtimes H_\infty(\sigma_2)$ is not reproducing. The orbits of the action δ are rays or branches of hyperbola and so locally closed. Since $H_\infty(\sigma_2)$ is commutative and consequently unimodular, by Proposition 3.15 and by the equation (4.4) the group $\Sigma_2^{\perp} \rtimes H_\infty(\sigma_2)$ is unimodular. Hence, thanks to Theorem 3.19, it is not reproducing.

In order to study, with the following proposition, the case with $H_\alpha(\sigma_2)$ let us introduce the vectors $q_\xi = (-\cosh\xi, \sinh\xi)$ for $\xi \in \mathbb{R}$.

Proposition 4.10. *Let $\alpha \in [0,\infty)$. The group $\Sigma_2^{\perp} \rtimes H_\alpha(\sigma_2)$ is reproducing for $\mu_{|\Sigma_2^{\perp} \rtimes H_\alpha(\sigma_2)}$ and a function $\eta \in L^2\left(\mathbb{R}^d\right)$ is an admissible vector for the system $(\Sigma_2^{\perp} \rtimes H_\alpha(\sigma_2), \mu_{|\Sigma_2^{\perp} \rtimes H_\alpha(\sigma_2)})$ if and only if for almost every $\xi \in \mathbb{R}$*

$$\int_{\mathbb{R}} |\eta(e^{-t} A_{-\alpha t}\, M_i\, q_\xi)|^2\, e^{2t}\, dt = \frac{1}{4} \qquad \text{for every } i = 0, \ldots, 3$$

and

$$\int_{\mathbb{R}} \eta(e^{-t} A_{-\alpha t}\, M_i\, q_\xi))\, \overline{\eta(e^{-t} A_{-\alpha t}\, M_j\, q_\xi))}\, e^{2t}\, dt = 0 \qquad \text{for every } 0 \le i < j \le 3.$$

Proof. First of all, we need to study more deeply the action δ of $H_\alpha(\sigma_2) = \{\, e^t A_{\alpha t} : t \in \mathbb{R} \,\}$ on Y. We claim that every orbit has a unique point on the branch of the hyperbola $\{\, (a,b) \in Y : a^2 - b^2 = 1 \,\}$. By the equation (4.5) the action for an arbitrary point $q = (a,b) \in Y$ is

$$\delta(e^t A_{\alpha t})\sigma_q = e^{2t}\, \psi_2(A_{2\alpha t} q) \quad \text{for all } t \in \mathbb{R}. \tag{4.6}$$

If $\alpha = 0$ then $H_\alpha(\sigma_2)$ acts by dilation and so the claim is established. Suppose $\alpha > 0$. The orbit for $q \in Y$ is $\mathcal{O}_q = \{\, e^t A_{\alpha t} q : t \in \mathbb{R} \,\}$. One has that $(e^t A_{\alpha t} q)_2 = e^t\, (\sinh(\alpha t)\, a + \cosh(\alpha t)\, b)$ and so $(e^t A_{\alpha t} q)_2 = 0$ if and only if $\tanh(\alpha t) = -b/a$. Since $|-b/a| < 1$ there is a unique point of the form $(a,0)$ in $\mathcal{O}_q$. Hence we can suppose $q = (a,0)$ with $a < 0$. Finally, it is easy to prove that there exists a unique point in $\{\, (a,b) \in Y : a^2 - b^2 = 1 \,\}$ of the form $a\, e^t\, (\cosh(\alpha t), \sinh(\alpha t))$, whence the claim follows. Therefore every orbit can be written as $\mathcal{O}_\xi := \mathcal{O}_{q_\xi}$. The following relation is easily established

$$\mathcal{O}_\xi = \left\{ e^t A_{\alpha t}\, q_\xi = -e^t (\cosh(\alpha t - \xi), \sinh(\alpha t - \xi)) : t \in \mathbb{R} \right\},$$

and so it is locally closed.

Now we show that $\Sigma_2^\perp \rtimes H_\alpha(\sigma_2)$ is reproducing. First, we need to prove that $\Sigma_2^\perp \rtimes H_\alpha(\sigma_2)$ is non-unimodular. By equation (4.6) we have that $\alpha(e^t A_{\alpha t}) = \det(e^{2t} A_{2\alpha t})^{-1} = e^{-4t}$. Hence, thanks to Proposition 3.15, since $H_\alpha(\sigma_2)$ is Abelian and consequently unimodular we obtain that $\Sigma_2^\perp \rtimes H_\alpha(\sigma_2)$ is non-unimodular. Now we find the stabilizer of σ_q with $q \in Y$ for the action δ. Thanks to what we have proved above, we can suppose $q = q_\xi$ with $\xi \in \mathbb{R}$. The condition $\delta(e^t A_{\alpha t})\sigma_{q_\xi} = \sigma_{q_\xi}$ is equivalent to $e^{2t} A_{2\alpha t}\, q_\xi = q_\xi$, which yields to the couple of relations

$$-e^{2t} \cosh(2\alpha t - \xi) = -\cosh \xi, \quad -e^{2t} \sinh(2\alpha t - \xi) = \sinh \xi.$$

and so $e^{4t} = 1$. Hence the stabilizer is $\{ I \}$ and so compact for almost every $q \in \operatorname{Im} \Phi$. Thanks to Theorem 3.19 we deduce that $\Sigma_2^\perp \rtimes H_\alpha(\sigma_2)$ is reproducing.

Now we use Theorem 3.22 to determine the admissible vectors. Thanks to the claim previously proved, we can set $Z = \mathbb{R}$ with $\pi \colon Y \to Z$ defined by the condition $q \in \mathcal{O}_{-2\pi(q)}$. As a consequence of Fubini's theorem, it is easy to verify that the Lebesgue measure λ on Z is a pseudo-image measure of the Lebesgue measure on Y. For almost every $\xi \in Z$ let $y_\xi = q_{-2\xi} \in \pi^{-1}(\xi)$. Hence $\Phi^{-1}(y_\xi) = \{ P_i = 2M_i\, q_\xi : i = 0, \ldots, 3 \}$. Condition (3.16) for the couples (P_i, P_i) yields for almost every $\xi \in \mathbb{R}$

$$1 = \int_{\mathbb{R}} |\eta(e^{-t} A_{-\alpha t}\, P_i)|^2 \, \frac{1}{e^{-4t} e^{2t}}\, dt = \int_{\mathbb{R}} |\eta(e^{-t} A_{-\alpha t}\, P_i)|^2 \, e^{2t}\, dt.$$

Finally, if $i < j$, condition (3.16) for the couples (P_i, P_j) yields for almost every $\xi \in \mathbb{R}$

$$0 = \int_{\mathbb{R}} \eta(e^{-t} A_{-\alpha t}\, P_i)\, \overline{\eta(e^{-t} A_{-\alpha t}\, P_j)}\, \frac{1}{e^{-4t} e^{2t}}\, dt = \int_{\mathbb{R}} \eta(e^{-t} A_{-\alpha t}\, P_i)\, \overline{\eta(e^{-t} A_{-\alpha t}\, P_j)}\, e^{2t}\, dt.$$

□

4.3 Signature $(1,0,1)$

We study all the groups in $\mathcal{E}_2$ associated to the space Σ_3, up to conjugation by elements of $H(\sigma_3)$. Since in this case $p = 1 \neq 0 = q$, thanks to Corollary 3.26 we obtain

$$H(\sigma_3) = \{ \lambda\, h : \lambda \in \mathbb{R}_+, h \in O(1,0,1) \}. \tag{4.7}$$

In order to give a characterization of the group $H(\sigma_3)$, consider the following matrices for $v \in R$ and $u, w \neq 0$

$$B = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \quad A = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}, \quad l_{u,v,w} = \begin{bmatrix} u & 0 \\ v & w \end{bmatrix}.$$

Lemma 4.11. *The group $H(\sigma_3)$ and its Lie algebra $\mathfrak{h}(\sigma_3)$ can be written as*

$$H(\sigma_3) = \{ \, l_{u,v,w} : v \in \mathbb{R}, u, w \neq 0 \, \}, \quad \mathfrak{h}(\sigma_3) = \operatorname{span} \{ \, I, B, A \, \}.$$

Proof. Denoting by $l_{v,w} = \left[\begin{smallmatrix} 1 & 0 \\ v & w \end{smallmatrix}\right], l'_{v,w} = \left[\begin{smallmatrix} -1 & 0 \\ v & w \end{smallmatrix}\right]$, the group $O(1,0,1)$ and its Lie algebra $\mathfrak{so}(1,0,1)$ can be written as

$$O(1,0,1) = \left\{ \, l_{v,w}, l'_{v,w} : v \in \mathbb{R}, w \neq 0 \, \right\}, \quad \mathfrak{so}(1,0,1) = \operatorname{span} \{ \, B, A \, \}.$$

Indeed, for $u, z, v, w \in \mathbb{R}$ we have

$$\begin{bmatrix} u & v \\ z & w \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} u & z \\ v & w \end{bmatrix} = \begin{bmatrix} u^2 & uz \\ uz & z^2 \end{bmatrix}$$

whence we obtain our first claim. The characterization of the Lie algebra $\mathfrak{so}(1,0,1)$ follows trivially from the first part of the statement. The characterization of the group $H(\sigma_3)$ and of its Lie algebra $\mathfrak{h}(\sigma_3)$ follows from the relation (4.7). □

Now we have to deal with step 3. We start with the 1-dimensional subalgebras of $\mathfrak{h}(\sigma_3)$. Since $\dim(\mathfrak{h}(\sigma_3)) = 3$, its 1-dimensional vector subspaces are in bijection with $\mathbb{P}^2_{\mathbb{R}}$. Hence we must consider

$$\mathfrak{h}_\infty(\sigma_3) = \operatorname{span} \{ \, I \, \},$$

for $\alpha \in \mathbb{R}$ we have

$$\mathfrak{h}_\alpha(\sigma_3) = \operatorname{span} \{ \, \alpha I + B \, \}$$

and, for $\alpha, \beta \in \mathbb{R}$,

$$\mathfrak{h}_{\alpha,\beta}(\sigma_3) = \operatorname{span} \{ \, \alpha I + \beta B + A \, \}.$$

With the following proposition we complete step 3. for the 1-dimensional subalgebras by finding all the conjugations among these subalgebras.

Proposition 4.12. *Let $\alpha_1, \alpha_2, \beta_1, \beta_2 \in \mathbb{R}$. Hence:*

1. *$\mathfrak{h}_0(\sigma_3)$ and $\mathfrak{h}_\infty(\sigma_3)$ have no nontrivial subalgebras conjugated by an element of $H(\sigma_3)$;*

2. *if $\alpha \neq 0$ then $\mathfrak{h}_\alpha(\sigma_3)$ is conjugated to $\mathfrak{h}_1(\sigma_3)$ by an element of $H(\sigma_3)$;*

3. *$\mathfrak{h}_{\alpha_1,\beta_1}(\sigma_3)$ is not conjugated to $\mathfrak{h}_{\alpha_2}(\sigma_3)$ by an element of $H(\sigma_3)$;*

4. *$\mathfrak{h}_{\alpha_1,\beta_1}(\sigma_3)$ and $\mathfrak{h}_{\alpha_2,\beta_2}(\sigma_3)$ are conjugate by an element of $H(\sigma_3)$ if and only if $\alpha_1 = \alpha_2$.*

Proof. We have to conjugate with a matrix $g \in H(\sigma_3)$. From easy computations the following relations are established for $v \in \mathbb{R}$ and $u, w \neq 0$:

$$l_{u,v,w}\, B\, {l_{u,v,w}}^{-1} = \frac{w}{u}\, B, \tag{4.8a}$$

$$l_{u,v,w}\, A\, {l_{u,v,w}}^{-1} = -\frac{v}{u}\, B + A. \tag{4.8b}$$

Part 1 follows immediately from the relation (4.8a).

In order to prove part 2 it is sufficient to see that from the relation (4.8a) we have for $\alpha \neq 0$

$$l_{1,0,\alpha}\,(\alpha I + B)\, {l_{1,0,\alpha}}^{-1} = \alpha I + \alpha B = \alpha(I + B).$$

Part 3 follows from the relation (4.8a), indeed we have

$$l_{u,v,w}\,(\alpha_2 I + B)\, {l_{u,v,w}}^{-1} = \alpha_2 I + \frac{w}{u}\, B.$$

Part 4 follows from the relations (4.8), indeed we have

$$l_{u,v,w}\,(\alpha_1 I + \beta_1 B + A)\, {l_{u,v,w}}^{-1} = \alpha_1 I + \beta_1 \frac{w}{u}\, B - \frac{v}{u}\, B + A = \alpha_1 I + (\beta_1 \frac{w}{u} - \frac{v}{u}) B + A. \quad \square$$

Therefore the family of 1-dimensional subalgebras of $\mathfrak{h}(\sigma_3)$ up to conjugation with elements in $H(\sigma_3)$ is

$$\{\, \mathfrak{h}_0(\sigma_3), \mathfrak{h}_1(\sigma_3), \mathfrak{h}_\infty(\sigma_3) \,\} \cup \{\, \mathfrak{h}_{\alpha,0}(\sigma_3) : \alpha \in \mathbb{R} \,\}.$$

Now, we pass to step 4. We must determine all the connected Lie subgroups corresponding to these subalgebras. As above, the first one is

$$H_\infty(\sigma_3) = \left\{ e^t I : t \in \mathbb{R} \right\}.$$

Further, set $t \in (-\epsilon, \epsilon)$ small enough and compute

$$\exp\left(t(\alpha I + B)\right) = \exp(t\alpha I)\exp(t\,B) = e^{t\alpha}(I + t\,B) \quad \text{for all } \alpha \in \{\, 0, 1 \,\},$$

since $B^2 = 0$. Hence we obtain

$$H_0(\sigma_3) = \{\, I + t\,B : t \in \mathbb{R} \,\} = \left\{ \begin{bmatrix} 1 & 0 \\ t & 1 \end{bmatrix} : t \in \mathbb{R} \right\}$$

and

$$H_1(\sigma_3) = \left\{ e^t(I + t\,B) : t \in \mathbb{R} \right\} = \left\{ e^t \begin{bmatrix} 1 & 0 \\ t & 1 \end{bmatrix} : t \in \mathbb{R} \right\}.$$

Since A is diagonal we have

$$\exp\left(t(\alpha I + A)\right) = \exp(t\alpha I)\exp(t\,A) = e^{t\alpha}(I + (e^t - 1)\,A) \quad \text{for all } \alpha \in \mathbb{R}.$$

Hence we obtain the group

$$H_{\alpha,0}(\sigma_3) = \left\{ e^{t\alpha}(I + (e^t - 1)\,A) : t \in \mathbb{R} \right\} = \left\{ \begin{bmatrix} e^{\alpha t} & 0 \\ 0 & e^{(\alpha+1)t} \end{bmatrix} : t \in \mathbb{R} \right\}.$$

Therefore the family of 1-dimensional connected Lie subgroups of $H(\sigma_3)$ up to conjugation with elements in $H(\sigma_3)$ is

$$\{ H_\infty(\sigma_3), H_0(\sigma_3), H_1(\sigma_3) \} \cup \{ H_{\alpha,0}(\sigma_3) : \alpha \in \mathbb{R} \}.$$

Now we study step 3. for the 2-dimensional subalgebras of $\mathfrak{h}(\sigma_3)$.

Lemma 4.13. *The 2-dimensional subalgebras of $\mathfrak{h}(\sigma_3)$ are*

$$\mathfrak{k}_\infty(\sigma_3) = \operatorname{span}\{ I, B \}, \ \ \mathfrak{l}_\alpha(\sigma_3) = \operatorname{span}\{ B, \alpha I + A \}, \ \ \mathfrak{k}_\beta(\sigma_3) = \operatorname{span}\{ I, \beta B + A \},$$

for $\alpha, \beta \in \mathbb{R}$.

Proof. Computing the brackets inside $\mathfrak{h}(\sigma_3)$ we have:

$$[I, B] = [I, A] = 0, \quad [A, B] = B.$$

Therefore $\mathfrak{k}_\infty(\sigma_3)$, $\mathfrak{l}_\alpha(\sigma_3)$, $\mathfrak{k}_\beta(\sigma_3)$ are 2-dimensional subalgebras of $\mathfrak{h}(\sigma_3)$ for all $\alpha, \beta \in \mathbb{R}$. Conversely, if $\mathfrak{h}$ is a 2-dimensional subalgebra of $\mathfrak{h}(\sigma_3)$ then there exist $\alpha_i, \beta_i, \gamma_i \in \mathbb{R}$ such that

$$\mathfrak{h} = \operatorname{span}\{ \alpha_1 I + \beta_1 B + \gamma_1 A\,,\ \alpha_2 I + \beta_2 B + \gamma_2 A \}.$$

Since $\mathfrak{h}$ is a subalgebra, the element

$$[\alpha_1 I + \beta_1 B + \gamma_1 A\,,\ \alpha_2 I + \beta_2 B + \gamma_2 A] = [\beta_1 B + \gamma_1 A\,,\ \beta_2 B + \gamma_2 A] = (\gamma_1\,\beta_2 - \beta_1\,\gamma_2)\,B$$

must be in $\mathfrak{h}$. If $B \in \mathfrak{h}$ the last condition is satisfied and we obtain the first two types of subalgebras $\mathfrak{k}_\infty(\sigma_3)$, $\mathfrak{l}_\alpha(\sigma_3)$. If $B \notin \mathfrak{h}$ then $\gamma_1\,\beta_2 - \beta_1\,\gamma_2 = 0$ and so it is easy to prove that $I \in \mathfrak{h}$. Hence we obtain the remaining types $\mathfrak{k}_\beta(\sigma_3)$. $\square$

With the following proposition we complete step 3. for the 2-dimensional subalgebras by finding all the conjugations among these subalgebras.

Proposition 4.14. *Let $\alpha, \beta \in \mathbb{R}$. Hence:*

1. *$\mathfrak{k}_\infty(\sigma_3)$ have no nontrivial subalgebras conjugate by an element of $H(\sigma_3)$;*
2. *$\mathfrak{l}_\alpha(\sigma_3)$ have no nontrivial subalgebras conjugate by an element of $H(\sigma_3)$;*
3. *$\mathfrak{k}_\beta(\sigma_3)$ and $\mathfrak{k}_0(\sigma_3)$ are conjugated by an element of $H(\sigma_3)$.*

Proof. Part 1 follows immediately from the relation (4.8a).

Now we prove part 2. The subalgebra $\mathfrak{l}_\alpha(\sigma_3)$ can't be conjugated with $\mathfrak{k}_\beta(\sigma_3)$ because of the relation (4.8a) since $B \notin \mathfrak{k}_\beta(\sigma_3)$ for all $\beta \in \mathbb{R}$. Suppose that $\mathfrak{l}_\alpha(\sigma_3)$ is conjugated with $\mathfrak{l}_{\alpha_1}(\sigma_3)$, we have to prove that $\alpha = \alpha_1$. Let us assume that the conjugating matrix is of type $l_{u,v,w}$. Recall that

$$\mathfrak{l}_\alpha(\sigma_3) = \operatorname{span}\{\, B, \alpha I + A\,\}, \quad \mathfrak{l}_{\alpha_1}(\sigma_3) = \operatorname{span}\{\, B, \alpha_1 I + A\,\}.$$

By hypothesis, there exist $\lambda_1, \lambda_2 \in \mathbb{R}$ such that

$$\alpha_1 I + A = l_{u,v,w}\,(\lambda_1\, B + \lambda_2\,(\alpha I + A))l_{u,v,w}{}^{-1}.$$

Thanks to the relations (4.8) we have

$$\alpha_1 I + A = \lambda_1 \frac{w}{u} B + \lambda_2\, \alpha I + \lambda_2\,(-\frac{v}{u} B + A) = (\lambda_1 \frac{w}{u} - \lambda_2 \frac{v}{u})B + \lambda_2 \alpha I + \lambda_2\, A$$

whence $\lambda_2 = 1$ and so $\alpha = \alpha_1$.

Now we prove part 3. Recall that

$$\mathfrak{k}_\beta(\sigma_3) = \operatorname{span}\{\, I, \beta B + A\,\}, \quad \mathfrak{k}_0(\sigma_3) = \operatorname{span}\{\, I, A\,\}.$$

Since, thanks to the relations (4.8), we have

$$l_{1,\beta,1}(\beta\, B + A)l_{1,\beta,1}{}^{-1} = \beta\, B - \beta\, B + A = A, \quad l_{1,\beta,1}\, I\, l_{1,\beta,1}{}^{-1} = I,$$

we are done. □

Therefore the family of 2-dimensional subalgebras of $\mathfrak{h}(\sigma_3)$ up to conjugation with elements in $H(\sigma_3)$ is

$$\{\, \mathfrak{k}_\infty(\sigma_3), \mathfrak{k}_0(\sigma_3)\,\} \cup \{\, \mathfrak{l}_\alpha(\sigma_3) : \alpha \in \mathbb{R}\,\}.$$

Now, we pass to step 4. We must determine all the connected Lie subgroups corresponding to these subalgebras. As above, set $t, s \in (-\epsilon, \epsilon)$ small enough and compute

$$\exp\,(t\, I + s\, B) = \exp(t\, I)\exp(s\, B) = e^t(I + s\, B)$$

whence we obtain

$$K_\infty(\sigma_3) = \left\{\, e^t(I + s\,B) : t, s \in \mathbb{R} \,\right\} = \left\{ \begin{bmatrix} e^t & 0 \\ s & e^t \end{bmatrix} : t, s \in \mathbb{R} \right\}.$$

Analogously, from

$$\exp\,(t\,I + s\,A) = \exp(t\,I)\exp(s\,A) = e^t(I + (e^s - 1)A)$$

we obtain the group

$$K_0(\sigma_3) = \left\{\, e^t(I + (e^s - 1)A) : t, s \in \mathbb{R} \,\right\} = \left\{ \begin{bmatrix} e^t & 0 \\ 0 & e^s \end{bmatrix} : t, s \in \mathbb{R} \right\}.$$

Since $(s\,B + t\,A)^k = st^{k-1}B + t^k A$ for all $k \in \mathbb{N}^*$ we have $\exp(s\,B + t\,A) = I + \frac{s}{t}(e^t - 1)B + (e^t - 1)A$. Hence

$$\exp\,(s\,B + t\,(\alpha\,I + A)) = \exp(\alpha t\,I)\exp(s\,B + t\,A) = e^{\alpha t}\left(I + \frac{s}{t}(e^t - 1)B + (e^t - 1)A\right)$$

whence we obtain

$$L_\alpha(\sigma_3) = \left\{ e^{\alpha t}\left(I + \frac{s}{t}(e^t - 1)B + (e^t - 1)A\right) : t, s \in \mathbb{R} \right\} = \left\{ \begin{bmatrix} e^{\alpha t} & 0 \\ s & e^{(\alpha+1)t} \end{bmatrix} : t, s \in \mathbb{R} \right\}.$$

With a change of variables $\beta = \frac{\alpha}{\alpha+1}$, for $\beta \in \mathbb{R}$ we denote by

$$L'_\beta(\sigma_3) = \left\{ \begin{bmatrix} e^{\beta t} & 0 \\ s & e^t \end{bmatrix} : t, s \in \mathbb{R} \right\}.$$

Since $K_\infty(\sigma_3) = L'_1(\sigma_3)$, the family of 2-dimensional connected Lie subgroups of $H(\sigma_3)$ up to conjugation with elements in $H(\sigma_3)$ is

$$\{\, K_0(\sigma_3), L_{-1}(\sigma_3) \,\} \cup \left\{\, L'_\beta(\sigma_3) : \beta \in \mathbb{R} \,\right\}.$$

Finally, thanks to Proposition 3.24 we can write all the groups in the class $\mathcal{E}_2$ associated to Σ_3 and to $\Sigma_3^\perp$, up to conjugation and only with H connected.

Proposition 4.15. *The following are all the groups in the class $\mathcal{E}_2$ associated to Σ_3 and*

$\Sigma_3^\perp$, *up to conjugation with* MA *and with* H *connected:*

(3.i) $\Sigma_3 \rtimes H^0(\sigma_3)$	*(3.ix)* $\Sigma_3^\perp \rtimes {}^tH^0(\sigma_3)$
(3.ii) $\Sigma_3 \rtimes H_0(\sigma_3)$	*(3.x)* $\Sigma_3^\perp \rtimes {}^tH_0(\sigma_3)$
(3.iii) $\Sigma_3 \rtimes H_1(\sigma_3)$	*(3.xi)* $\Sigma_3^\perp \rtimes {}^tH_1(\sigma_3)$
(3.iv) $\Sigma_3 \rtimes H_\infty(\sigma_3)$	*(3.xii)* $\Sigma_3^\perp \rtimes H_\infty(\sigma_3)$
(3.v) $\Sigma_3 \rtimes H_{\alpha,0}(\sigma_3), \quad \alpha \in \mathbb{R}$	*(3.xiii)* $\Sigma_3^\perp \rtimes H_{\alpha,0}(\sigma_3), \quad \alpha \in \mathbb{R}$
(3.vi) $\Sigma_3 \rtimes K_0(\sigma_3)$	*(3.xiv)* $\Sigma_3^\perp \rtimes K_0(\sigma_3)$
(3.vii) $\Sigma_3 \rtimes L_{-1}(\sigma_3)$	*(3.xv)* $\Sigma_3^\perp \rtimes {}^tL_{-1}(\sigma_3)$
(3.viii) $\Sigma_3 \rtimes L'_\beta(\sigma_3), \quad \beta \in \mathbb{R}$	*(3.xvi)* $\Sigma_3^\perp \rtimes {}^tL'_\beta(\sigma_3), \quad \beta \in \mathbb{R}.$

4.3.1 Weyl Group Conjugations

With the previous proposition we have classified all the groups in the class $\mathcal{E}_2$ whose normal factor is equal or orthogonal to Σ_3, up to conjugation in $Sp(2,\mathbb{R})$ with elements in MA (see the Langlands decomposition (3.10) and Proposition 3.23). In this section we want to analyze another kind of conjugation, that is by means of elements in $W = W(Sp(2,\mathbb{R}))$, the Weyl group of $Sp(2,\mathbb{R})$. For a precise characterization of W see, for example, [De 87]. Here we only consider the two matrices

$$w_1 = \begin{bmatrix} s_1 & s_2 \\ -s_2 & s_1 \end{bmatrix}, \qquad w_2 = \begin{bmatrix} s_2 & s_1 \\ -s_1 & s_2 \end{bmatrix}, \tag{4.9}$$

where

$$s_1 = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \qquad s_2 = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}.$$

With the following proposition we study which conjugations arise thanks to these elements.

Proposition 4.16. *The following couples of subgroups of* $Sp(2,\mathbb{R})$ *are conjugated:*

1. $\Sigma_3 \rtimes H^0(\sigma_3)$ – $\Sigma_3^\perp \rtimes K_0(\sigma_3)$;
2. $\Sigma_3 \rtimes H_\infty(\sigma_3)$ – $\Sigma_3 \rtimes H_{-1/2,0}(\sigma_3)$;
3. $\Sigma_3 \rtimes H_{\alpha,0}(\sigma_3)$ – $\Sigma_3 \rtimes H_{-\frac{\alpha}{2\alpha+1},0}(\sigma_3)$ *for* $\alpha \neq -\frac{1}{2}$;
4. $\Sigma_3 \rtimes L_{-1}(\sigma_3)$ – $\Sigma_3^\perp \rtimes H_{-1,0}(\sigma_3)$;
5. $\Sigma_3 \rtimes L'_{-1}(\sigma_3)$ – $\Sigma_3^\perp \rtimes H_\infty(\sigma_3)$;
6. $\Sigma_3 \rtimes L'_\beta(\sigma_3)$ – $\Sigma_3^\perp \rtimes H_{-\frac{\beta}{\beta+1},0}(\sigma_3)$ *for* $\beta \neq -1$;
7. $\Sigma_3^\perp \rtimes {}^tL'_\beta(\sigma_3)$ – $\Sigma_3^\perp \rtimes {}^tL'_{-\beta}(\sigma_3)$.

Proof. 1. Recall that $H^0(\sigma_3) = \{\, l_{u,v,w} : v \in \mathbb{R}, u, w > 0 \,\}$. An easy computation shows that for $a, v \in \mathbb{R}$ and $u, w > 0$ we have

$$i_{w_1} g(a\,\sigma_3, l_{u,v,w}) = g\Big(\left[\begin{smallmatrix} a & -v/u \\ -v/u & 0 \end{smallmatrix}\right], \left[\begin{smallmatrix} u & 0 \\ 0 & 1/w \end{smallmatrix}\right]\Big)$$

and by conjugating with $g(0,\Omega)$ we obtain our claim.

2. Recall that $H_\infty(\sigma_3)=\{\,e^tI : t\in\mathbb{R}\,\}$. An easy computation shows that for $a,t\in\mathbb{R}$ we have

$$i_{w_1}g(a\,\sigma_3, e^tI)=g\big(a\,\sigma_3,\left[\begin{smallmatrix} e^t & 0\\ 0 & e^{-t}\end{smallmatrix}\right]\big)$$

whence we obtain our claim.

3. Recall that $H_{\alpha,0}(\sigma_3)=\left\{\,\left[\begin{smallmatrix} e^{\alpha t} & 0\\ 0 & e^{(\alpha+1)t}\end{smallmatrix}\right] : t\in\mathbb{R}\,\right\}$. An easy computation shows that for $a,t\in\mathbb{R}$ we have

$$i_{w_1}g(a\,\sigma_3,\left[\begin{smallmatrix} e^{\alpha t} & 0\\ 0 & e^{(\alpha+1)t}\end{smallmatrix}\right])=g\big(a\,\sigma_3,\left[\begin{smallmatrix} e^{\alpha t} & 0\\ 0 & e^{-(\alpha+1)t}\end{smallmatrix}\right]\big)$$

whence with a change of variable we obtain our claim.

4. Recall that $L_{-1}(\sigma_3)=\left\{\,\left[\begin{smallmatrix} e^{t} & 0\\ s & 1\end{smallmatrix}\right] : t,s\in\mathbb{R}\,\right\}$. An easy computation shows that for $a,t,s\in\mathbb{R}$ we have

$$i_{w_1}g(a\,\sigma_3,\left[\begin{smallmatrix} e^{t} & 0\\ s & e^{t}\end{smallmatrix}\right])=g\big(\left[\begin{smallmatrix} a & -se^{-t}\\ -se^{-t} & 0\end{smallmatrix}\right],\left[\begin{smallmatrix} e^{t} & 0\\ 0 & 1\end{smallmatrix}\right]\big)$$

whence we obtain our claim.

5, 6. Recall that $L'_\beta(\sigma_3)=\left\{\,\left[\begin{smallmatrix} e^{\beta t} & 0\\ s & e^{t}\end{smallmatrix}\right] : t,s\in\mathbb{R}\,\right\}$. An easy computation shows that for $a,t,s\in\mathbb{R}$ we have

$$i_{w_1}g(a\,\sigma_3,\left[\begin{smallmatrix} e^{\beta t} & 0\\ s & e^{t}\end{smallmatrix}\right])=g\big(\left[\begin{smallmatrix} a & -se^{-\beta t}\\ -se^{-\beta t} & 0\end{smallmatrix}\right],\left[\begin{smallmatrix} e^{\beta t} & 0\\ 0 & e^{-t}\end{smallmatrix}\right]\big)$$

whence with a change of variable we obtain our claim.

7. An easy computation shows that for $a,b,t,s\in\mathbb{R}$ we have

$$i_{w_2}g(\left[\begin{smallmatrix} 0 & b\\ b & a\end{smallmatrix}\right],\left[\begin{smallmatrix} e^{\beta t} & s\\ 0 & e^{t}\end{smallmatrix}\right])=g\big(\left[\begin{smallmatrix} 0 & -se^{-\beta t}\\ -se^{-\beta t} & a+2bse^{-t}\end{smallmatrix}\right],\left[\begin{smallmatrix} e^{-\beta t} & be^{t}\\ 0 & e^{t}\end{smallmatrix}\right]\big)$$

whence we obtain our claim. □

Thanks to these new relations we can refine the result of Proposition 4.15.

Corollary 4.17. *The following are all the groups in the class $\mathcal{E}_2$ associated to Σ_3 and $\Sigma_3^\perp$, up to conjugation with MA or with w_i and with H connected:*

(3.ii) $\Sigma_3 \rtimes H_0(\sigma_3)$
(3.iii) $\Sigma_3 \rtimes H_1(\sigma_3)$
(3.v) $\Sigma_3 \rtimes H_{\alpha,0}(\sigma_3)$, $\quad \alpha\in[-1,0]$
(3.vi) $\Sigma_3 \rtimes K_0(\sigma_3)$
(3.ix) $\Sigma_3^\perp \rtimes {}^tH^0(\sigma_3)$
(3.x) $\Sigma_3^\perp \rtimes {}^tH_0(\sigma_3)$
(3.xi) $\Sigma_3^\perp \rtimes {}^tH_1(\sigma_3)$
(3.xii) $\Sigma_3^\perp \rtimes H_\infty(\sigma_3)$
(3.xiii) $\Sigma_3^\perp \rtimes H_{\alpha,0}(\sigma_3)$, $\quad \alpha\in\mathbb{R}$
(3.xiv) $\Sigma_3^\perp \rtimes K_0(\sigma_3)$
(3.xv) $\Sigma_3^\perp \rtimes {}^tL_{-1}(\sigma_3)$
(3.xvi) $\Sigma_3^\perp \rtimes {}^tL'_\beta(\sigma_3)$, $\quad \beta\geq 0.$

4.3.2 Reproducing Groups, $n = 1$

In this subsection we want to use Theorem 3.20 to determine some reproducing Lie subgroups associated to Σ_3, listed in Corollary 4.17.

First, we need to determine the map Φ associated to Σ_3. We can write $\Sigma_3 = \{ \sigma_a = a\sigma_3 : a \in \mathbb{R} \}$ and define $\psi \colon \mathbb{R} \to \Sigma_1$, $a \mapsto \sigma_a$. This isomorphism satisfies the relation $\langle \psi(a), \psi(a') \rangle_S = aa'$ for every $a, a' \in \mathbb{R}$. By the equation (3.15) we have for $x \in \mathbb{R}^2$ and $a \in \mathbb{R}$

$$\langle \Phi(x), \sigma_a \rangle_S = -\frac{1}{2}\langle a\sigma_3 x, x\rangle = -\frac{1}{2}x_1^2\, a = -\frac{1}{2}\langle \psi(x_1^2), \sigma_a\rangle_S$$

whence $\Phi(x) = -\frac{1}{2}\psi(x_1^2)$ and so $\operatorname{Im}\Phi = \{ \sigma_a : a \leq 0 \}$.

Proposition 4.18. *The groups* $\Sigma_3 \rtimes H_1(\sigma_3)$, $\Sigma_3 \rtimes H_{\alpha,0}(\sigma_3)$ *with* $\alpha \in [-1, 0)$ *are reproducing.*

Proof. Recall that

$$H(\sigma_3) = \{ l_{u,v,w} : u, w \in \mathbb{R}^*, v \in \mathbb{R} \}.$$

The action δ for an element of $H(\sigma_3)$ is for $u, w \in \mathbb{R}^*, v \in \mathbb{R}$

$$\delta(l_{u,v,w})(\sigma_a) = l_{u,v,w}\, \sigma_a\, {}^t l_{u,v,w} = a \begin{bmatrix} u^2 & uv \\ uv & v^2 \end{bmatrix}$$

and so

$$\delta'(l_{u,v,w})(\sigma_a) = u^2\sigma_a \quad \text{for all } a \in \mathbb{R} \text{ and } u \neq 0. \tag{4.10}$$

Now we show that $\Sigma_3 \rtimes H_1(\sigma_3)$ is reproducing. Recall that $H_1(\sigma_3) = \{ l_{e^t, te^t, e^t} : t \in \mathbb{R} \}$. By the above equation the action is $\delta'(l_{e^t,te^t,e^t})\sigma_a = e^{2t}\sigma_a$. There are two orbits of the action in $\operatorname{Im}\Phi$ and they are locally closed. We have $\alpha(l_{e^t,te^t,e^t}) = e^{-2t}$. Since $H_1(\sigma_3)$ is Abelian and so unimodular, by Proposition 3.15 the group $\Sigma_3 \rtimes H_1(\sigma_3)$ is non-unimodular. Finally, the stabilizer is trivial for every $a < 0$. By Theorem 3.20 the group $\Sigma_3 \rtimes H_1(\sigma_3)$ is reproducing.

Now we show that $\Sigma_3 \rtimes H_{\alpha,0}(\sigma_3)$ is reproducing if $\alpha \neq 0$. Recall that $H_{\alpha,0}(\sigma_3) = \left\{ l_{e^{\alpha t},0,e^{(\alpha+1)t}} : t \in \mathbb{R} \right\}$. The action is $\delta'(l_{e^{\alpha t},0,e^{(\alpha+1)t}})\sigma_a = e^{2\alpha t}\sigma_a$. There are two orbits of the action in $\operatorname{Im}\Phi$ and they are locally closed. We have $\alpha(l_{e^{\alpha t},0,e^{(\alpha+1)t}}) = e^{-2\alpha t}$. Since $H_{\alpha,0}(\sigma_3)$ is Abelian and so unimodular, by Proposition 3.15 the group $\Sigma_3 \rtimes H_{\alpha,0}(\sigma_3)$ is non-unimodular. Finally, the stabilizer is trivial for every $a < 0$. By Theorem 3.20 the group $\Sigma_3 \rtimes H_{\alpha,0}(\sigma_3)$ is reproducing. □

Now we show that $\Sigma_3 \rtimes H_0(\sigma_3)$ is not reproducing. Recall that $H_0(\sigma_3) = \{ l_{1,t,1} : t \in \mathbb{R} \}$. By the relation (4.10), $\alpha(l_{1,t,1}) = 1$. Since $H_0(\sigma_1)$ is Abelian and so unimodular, by Proposition 3.15 the group $\Sigma_3 \rtimes H_0(\sigma_3)$ is unimodular. Hence, by Theorem 3.18, it is not reproducing.

Now we show that $\Sigma_3 \rtimes H_{0,0}(\sigma_3)$ is not reproducing. Recall that $H_{0,0}(\sigma_3) = \{ l_{1,0,e^t} : t \in \mathbb{R} \}$. By the relation (4.10), $\alpha(l_{1,0,e^t}) = 1$. Since $H_{0,0}(\sigma_1)$ is Abelian and so unimodular, by Proposition 3.15 the group $\Sigma_3 \rtimes H_{0,0}(\sigma_3)$ is unimodular. Hence, by Theorem 3.18, it is not reproducing.

4.3.3 Reproducing Groups, $n = 2$

In this subsection we use the results contained in section 3.3 to establish which of the groups associated to $\Sigma_3^\perp$ are reproducing and to describe the admissible vectors. Since $n = d$, we can give a complete classification. First of all, by definition of $\langle\ ,\ \rangle_S$, we obtain that

$$\Sigma_3^\perp = \left\{ \sigma_{(a,b)} : a, b \in \mathbb{R} \right\}$$

where $\sigma_{(a,b)} = \begin{bmatrix} 0 & b/\sqrt{2} \\ b/\sqrt{2} & a \end{bmatrix}$. Indeed $\langle \sigma_3, \sigma \rangle_S = \operatorname{tr}(\sigma_3\, \sigma) = \sigma_{1,1}$ for all $\sigma \in \operatorname{Sym}(2, \mathbb{R})$. Now we want to determine the map Φ associated to the subspace $\Sigma_3^\perp$. Consider the canonical isomorphism $\psi_3 \colon \mathbb{R}^2 \to \Sigma_3^\perp$ defined by $q = (a, b) \mapsto \sigma_q$ that identifies $\Sigma_3^\perp = \mathbb{R}^2$. The map ψ_3 satisfies the relation $\langle \psi_3(q), \psi_3(q') \rangle_S = \langle q, q' \rangle$ for all $q, q' \in \mathbb{R}^2$. Hence by the equality (3.15) we obtain for $x = (x_1, x_2), q = (a, b) \in \mathbb{R}^2$

$$\begin{aligned} \langle \Phi(x), \sigma_q \rangle_S &= -\frac{1}{2} \langle \sigma_q\, x, x \rangle \\ &= -\frac{1}{2} (a x_2^2 + \sqrt{2} b x_1 x_2) \\ &= -\frac{1}{2} \langle (x_2^2, \sqrt{2} x_1 x_2), (a, b) \rangle \\ &= -\frac{1}{2} \langle \psi_3(x_2^2, \sqrt{2} x_1 x_2), \sigma_q \rangle_S \end{aligned}$$

whence $\Phi(x_1, x_2) = -\frac{1}{2} \psi_3(x_2^2, \sqrt{2} x_1 x_2)$. Hence $\operatorname{Im}\Phi = \{ (y_1, y_2) : y_1 < 0, y_2 \in \mathbb{R} \} \cup \{ 0 \}$. For the rest of this section we often identify $\Sigma_3^\perp = \mathbb{R}^2$ and consequently neglect the map ψ_3. The Jacobian of Φ is $J(\Phi)(x) = \frac{x_2^2}{\sqrt{2}}$ for every $x \in \mathbb{R}^2$ and so we set $X = \{ x \in \mathbb{R}^2 : x_2 \neq 0 \}$ and $Y = \{ (y_1, y_2) : y_1 < 0, y_2 \in \mathbb{R} \}$.

Now we want to determine an explicit expression of the action δ' of ${}^tH(\sigma_3)$ on $\Sigma_3^\perp$. Recalling Lemma 4.11 we set $k_{u,v,w} = {}^t l_{u,v,w}$ and so we obtain

$${}^tH(\sigma_3) = \left\{ k_{u,v,w} = \begin{bmatrix} u & v \\ 0 & w \end{bmatrix} : u, w \neq 0, v \in \mathbb{R} \right\}.$$

Thanks to the form of $\Sigma_3^\perp$ an easy computation shows that

$$\delta'(k_{u,v,w}) \sigma_q = \psi_3 \left(w^2 a, w(\sqrt{2} a\, v + u\, b) \right) \tag{4.11}$$

for every $k_{u,v,w} \in {}^tH(\sigma_3)$ and $q=(a,b) \in \mathbb{R}^2$.

Now we prove that the group $\Sigma_3^\perp \rtimes {}^tH^0(\sigma_3)$ is not reproducing. Let us check that the δ'-orbits of $\operatorname{Im}\Phi$ are locally closed. By the relation (4.11), there are only the two orbits $\{\, (y_1,y_2) : y_1<0, y_2 \in \mathbb{R}\,\}$ and $\{\,0\,\}$ which are locally closed. If $q \in Y$ the set $\left\{\, k_{u,v,1} : v=(1-u)\,b/\sqrt{2}a,\ u \in \mathbb{R}\,\right\}$ is contained in the stabilizer of σ_q with as usual $q=(a,b)$. If $b \neq 0$ it is not compact, hence by Theorem 3.19 the group $\Sigma_3^\perp \rtimes {}^tH^0(\sigma_3)$ is not reproducing.

Now we prove that the group $\Sigma_3^\perp \rtimes {}^tH_0(\sigma_3)$ is not reproducing. Recall that ${}^tH_0(\sigma_3)=\{\, k_{1,t,1} : t \in \mathbb{R}\,\}$. By the equation (4.11) we have $\delta'(k_{1,t,1})\sigma_q = \psi_3\left(a, \sqrt{2}t\,a+b\right)$ and so the δ'-orbits of $\operatorname{Im}\Phi$ are locally closed but $\Sigma_3^\perp \rtimes {}^tH_0(\sigma_3)$ is unimodular because $\alpha(k_{1,t,1})=1$. Hence, by Theorem 3.19, the group $\Sigma_3^\perp \rtimes {}^tH_0(\sigma_3)$ is not reproducing.

Proposition 4.19. *The group $\Sigma_3^\perp \rtimes {}^tH_1(\sigma_3)$ is reproducing for $\mu_{|\Sigma_3^\perp \rtimes {}^tH_1(\sigma_3)}$ and a function $\eta \in L^2(\mathbb{R}^2)$ is an admissible vector for the system $(\Sigma_3^\perp \rtimes {}^tH_1(\sigma_3), \mu_{|\Sigma_3^\perp \rtimes {}^tH_1(\sigma_3)})$ if and only if for almost every $\xi \in \mathbb{R}$*

$$\int_{\mathbb{R}} |\eta(\pm e^{-t}(\xi-t,1))|^2\, e^{2t}\,dt = \frac{1}{\sqrt{2}}, \qquad \int_{\mathbb{R}} \eta(e^{-t}(\xi-t,1))\overline{\eta(e^{-t}(t-\xi,-1))}\, e^{2t}\,dt = 0.$$

Proof. Now we show that $\Sigma_3^\perp \rtimes {}^tH_1(\sigma_3)$ is reproducing. First, we need to verify that the ${}^tH_1(\sigma_3)$-orbits of $\operatorname{Im}\Phi$ with respect to the action δ' are locally closed. Recall that ${}^tH_1(\sigma_3)=\{\, e^tk_{1,t,1} : t \in \mathbb{R}\,\}$. Thus by equation (4.11) we have for $t \in \mathbb{R}$ and $q=(a,b)$ with $a<0$

$$\delta'(e^tk_{1,t,1})\sigma_q = e^{2t}\,\psi_3\left(a, \sqrt{2}a\,t+b\right) \tag{4.12}$$

whence every orbit has a unique point on the line $\left\{\, q_\xi = -\frac{1}{2}(1,\sqrt{2}\xi) : \xi \in \mathbb{R}\,\right\}$. Hence the orbit for σ_{q_ξ} is $\left\{\, -\frac{e^{2t}}{2}\,\psi_3\left(1,\sqrt{2}(t+\xi)\right) : t \in \mathbb{R}\,\right\}$. With the usual identification $\mathbb{R}^2 \overset{\psi_3}{=} \Sigma_3^\perp$ it becomes $O_\xi = \left\{\, -\frac{e^{2t}}{2}\left(1,\sqrt{2}(t+\xi)\right) : t \in \mathbb{R}\,\right\}$ that is locally closed because $\mathcal{O}_\xi = (\mathbb{R}^2 \setminus \{\,0\,\}) \cap \overline{\mathcal{O}_\xi}$. Now we need to prove that $\Sigma_3^\perp \rtimes {}^tH_1(\sigma_3)$ is non-unimodular. By the equation (4.12) we have that $\alpha(e^tk_{1,t,1}) = \det(e^{-2t}l_{1,-\sqrt{2}t,1}) = e^{-4t}$. Hence, thanks to Proposition 3.15, since ${}^tH_1(\sigma_3)$ is Abelian and consequently unimodular we obtain that $\Sigma_3^\perp \rtimes {}^tH_1(\sigma_3)$ is non-unimodular. Now we find the stabilizer of σ_{q_ξ} with $\xi \in \mathbb{R}$ for the action δ'. Since $O_\xi = \left\{\, -\frac{e^{2t}}{2}\left(1,\sqrt{2}(t+\xi)\right) : t \in \mathbb{R}\,\right\}$ the stabilizer is $\{\,I\,\}$ and so compact. Thanks to Theorem 3.19 we deduce that $\Sigma_3^\perp \rtimes {}^tH_1(\sigma_3)$ is reproducing.

Now we use Theorem 3.22 to determine the admissible vectors. We saw before that every orbit of δ' in Y passes for a unique point q_ξ. Hence we set $Z=\mathbb{R}$. As a consequence of Fubini's theorem, it is easy to verify that the Lebesgue measure λ on Z is a pseudo-image measure of the Lebesgue measure on Y. For almost every $\xi \in Z$ we need to compute $\Phi^{-1}(q_\xi) = \left\{\, P_1^\xi, P_2^\xi\,\right\}$ where $P_1^\xi = (\xi,1)$ and $P_2^\xi = -P_1^\xi$. Condition (3.16) for the couples (P_i^ξ, P_i^ξ) for almost every $\xi \in Z$ yields

$$\frac{1}{\sqrt{2}} = \int_{\mathbb{R}} |\eta(e^{-t}\, k_{1,-t,1}\, P_i^\xi)|^2 \frac{1}{e^{2t}e^{-4t}}\, dt = \int_{\mathbb{R}} |\eta(\pm e^{-t}(\xi - t, 1))|^2\, e^{2t}\, dt.$$

The other condition follows by applying the relation (3.16) to the couple (P_1^ξ, P_2^ξ) for almost every $\xi \in \mathbb{R}$. □

Proposition 4.20. *The group $\Sigma_3^\perp \rtimes {}^tH_\infty(\sigma_3)$ is reproducing for $\mu_{|\Sigma_3^\perp \rtimes {}^tH_\infty(\sigma_3)}$ and a function $\eta \in L^2(\mathbb{R}^2)$ is an admissible vector for the system $(\Sigma_3^\perp \rtimes {}^tH_\infty(\sigma_3), \mu_{|\Sigma_3^\perp \rtimes {}^tH_\infty(\sigma_3)})$ if and only if for almost every $\xi \in \mathbb{R}$*

$$\int_{\mathbb{R}} |\eta(\pm e^{-t}(\xi, 1))|^2\, e^{2t}\, dt = \frac{1}{\sqrt{2}}, \qquad \int_{\mathbb{R}} \eta(e^{-t}(\xi, 1))\overline{\eta(-e^{-t}(\xi, 1))}\, e^{2t}\, dt = 0.$$

Proof. Now we show that $\Sigma_3^\perp \rtimes {}^tH_\infty(\sigma_3)$ is reproducing. First, we need to verify that the ${}^tH_\infty(\sigma_3)$-orbits of $\operatorname{Im}\Phi$ with respect to the action δ' are locally closed. Recall that ${}^tH_\infty(\sigma_3) = \{\, e^t I : t \in \mathbb{R} \,\}$. Thus we have for $t \in \mathbb{R}$ and $q = (a, b)$ with $a < 0$

$$\delta'(e^t I)\sigma_q = e^{2t}\,\sigma_q. \tag{4.13}$$

Hence the orbit for σ_q is $\{\, e^t\,\sigma_q : t \in \mathbb{R} \,\}$ that is locally closed. Now we need to prove that $\Sigma_3^\perp \rtimes {}^tH_\infty(\sigma_3)$ is non-unimodular. By the equation (4.13) we have that $\alpha(e^t I) = \det(e^{-2t} I) = e^{-4t}$. Hence, thanks to Proposition 3.15, since ${}^tH_\infty(\sigma_3)$ is Abelian and consequently unimodular we obtain that $\Sigma_3^\perp \rtimes {}^tH_\infty(\sigma_3)$ is non-unimodular. Finally, by equation (4.13) the stabilizer of σ_q with $q \neq 0$ for the action δ' is $\{\, I \,\}$ and so compact. Thanks to Theorem 3.19 we deduce that $\Sigma_3^\perp \rtimes {}^tH_\infty(\sigma_3)$ is reproducing.

Now we use Theorem 3.22 to determine the admissible vectors. Every orbit of δ' in Y passes for a unique point $q_\xi = (-1/2, -\sqrt{2}/2\xi)$ for $\xi \in \mathbb{R}$. Hence we set $Z = \mathbb{R}$. As a consequence of Fubini's theorem, it is easy to verify that the Lebesgue measure λ on Z is a pseudo-image measure of the Lebesgue measure on Y. For almost every $\xi \in Z$ we need to compute $\Phi^{-1}(q_\xi) = \left\{\, P_1^\xi, P_2^\xi \,\right\}$ where $P_1^\xi = (\xi, 1)$ and $P_2^\xi = -P_1^\xi$. Condition (3.16) for the couples (P_i^ξ, P_i^ξ) yields for almost every $\xi \in Z$

$$\frac{1}{\sqrt{2}} = \int_{\mathbb{R}} |\eta(e^{-t}\, P_i^\xi)|^2 \frac{1}{e^{2t}e^{-4t}}\, dt = \int_{\mathbb{R}} |\eta(\pm e^{-t}(\xi, 1))|^2\, e^{2t}\, dt.$$

The other condition follows by applying the relation (3.16) to the couple (P_1^ξ, P_2^ξ) for almost every $\xi \in \mathbb{R}$. □

Proposition 4.21. *The group $\Sigma_3^\perp \rtimes H_{\alpha,0}(\sigma_3)$ is reproducing for $\mu_{|\Sigma_3^\perp \rtimes H_{\alpha,0}(\sigma_3)}$ if and only if $\alpha \neq -3/4$. A function $\eta \in L^2(\mathbb{R}^2)$ is an admissible vector for the system $(\Sigma_3^\perp \rtimes$*

$H_{\alpha,0}(\sigma_3), \mu_{|\Sigma_3^\perp \rtimes H_{\alpha,0}(\sigma_3)})$ *if and only if for almost every* $\xi \in \mathbb{R}$

$$\int_{\mathbb{R}} |\eta\left(e^{-\alpha t}(\xi, -e^{-t}\sqrt{2}|\xi|)\right)|^2\, e^{2(\alpha+1)t}\, dt = \sqrt{2}\,\xi^2,$$
$$\int_{\mathbb{R}} \eta\left(e^{-\alpha t}(\xi, -e^{-t}\sqrt{2}|\xi|)\right) \overline{\eta\left(-e^{-\alpha t}(\xi, -e^{-t}\sqrt{2}|\xi|)\right)}\, e^{2(\alpha+1)t}\, dt = 0.$$

Proof. In order to simplify the proof, we set $X = \{\, x \in \mathbb{R}^2 : x_1\, x_2 \neq 0 \,\}$ whence $Y = \{\, y \in \mathbb{R}^2 : y_1 < 0, y_2 \neq 0 \,\}$.

Now we show that $\Sigma_3^\perp \rtimes H_{\alpha,0}(\sigma_3)$ is reproducing if and only if $\alpha \neq -3/4$. First, we need to verify that the ${}^tH_{\alpha,0}(\sigma_3)$-orbits of $\operatorname{Im}\Phi$ with respect to the action δ' are locally closed. Recall that ${}^tH_{\alpha,0}(\sigma_3) = \{\, e^{\alpha t}\, k_{1,0,e^t} : t \in \mathbb{R} \,\}$. Thus we have for $t \in \mathbb{R}$ and $q = (a, b)$ with $a < 0$

$$\delta'(e^{\alpha t}\, k_{1,0,e^t})\sigma_q = e^{2\alpha t}\psi_3\left(e^{2t}a, e^t b\right) = e^{(2\alpha+1)t}\psi_3\left(e^t a, b\right). \tag{4.14}$$

The orbit for (a, b) is $\left\{\, (e^{(2\alpha+2)t}a, e^{(2\alpha+1)t}b) : t \in \mathbb{R} \,\right\}$ that is locally closed and the stabilizer is $\{\, I \,\}$ if $a, b \neq 0$ and so compact for almost every $q \in \operatorname{Im}\Phi$. By the equation (4.14) we have that $\alpha(e^{\alpha t}\, k_{1,0,e^t}) = e^{-(4\alpha+3)t}$. Hence, thanks to Proposition 3.15, since ${}^tH_{\alpha,0}(\sigma_3)$ is Abelian and consequently unimodular we obtain that $\Sigma_3^\perp \rtimes H_{\alpha,0}(\sigma_3)$ is unimodular if and only if $\alpha = -3/4$. Finally, thanks to Theorem 3.19, we obtain our claim.

Now we use Theorem 3.22 to determine the admissible vectors. An easy computation shows that every orbit of δ' in Y passes for a unique point $q_\xi = (-|\xi|, \xi)$ for every $\xi \in \mathbb{R}^*$. Hence we set $Z = \mathbb{R}^*$. As a consequence of Fubini's theorem, it is easy to verify that the Lebesgue measure λ on Z is a pseudo-image measure of the Lebesgue measure on Y. For almost every $\xi \in Z$ we need to compute $\Phi^{-1}(q_\xi) = \left\{\, P_1^\xi, P_2^\xi \,\right\}$ where $P_1^\xi = (\xi/\sqrt{|\xi|}, -\sqrt{2|\xi|})$ and $P_2^\xi = -P_1^\xi$. Condition (3.16) for the couples (P_i^ξ, P_i^ξ) yields for almost every $\xi \in Z$

$$\begin{aligned}\sqrt{2}|\xi| &= \int_{\mathbb{R}} |\eta(e^{-\alpha t}\, k_{1,0,e^{-t}}\, P_i^\xi)|^2 \frac{1}{e^{(2\alpha+1)t}e^{-(4\alpha+3)t}}\, dt = \\ &= \int_{\mathbb{R}} |\eta\left(e^{-\alpha t}(\xi/\sqrt{|\xi|}, -e^{-t}\sqrt{2|\xi|})\right)|^2\, e^{2(\alpha+1)t}\, dt.\end{aligned}$$

The other condition follows by applying the relation (3.16) to the couple (P_1^ξ, P_2^ξ) for almost every $\xi \in Z$. With a change of variables we are done. □

Remark 4.22. Thanks to Propositions 4.18 and 4.21 we know that $\Sigma_3 \rtimes H_{-3/4,0}(\sigma_3)$ is reproducing while $\Sigma_3^\perp \rtimes H_{-3/4,0}(\sigma_3)$ is not. This sheds light on what we said before Proposition 3.24.

Now we study the groups $\Sigma_3^\perp \rtimes H$ with H two-dimensional. Denote by W_i the open

ith quadrant of the Cartesian plane, namely $W_1 = \{\, x \in \mathbb{R}^2 : x_1, x_2 > 0 \,\}$ and $W_i = R_{\frac{\pi}{2}(i-1)} W_1$. Moreover, we introduce the measures ν_i on W_i defined by $d\nu_i = \frac{1}{|x_1\, x_2^3|} dx_1 dx_2$.

Proposition 4.23. *The group $\Sigma_3^\perp \rtimes K_0(\sigma_3)$ is reproducing for $\mu_{|\Sigma_3^\perp \rtimes K_0(\sigma_3)}$ and a function $\eta \in L^2(\mathbb{R}^2)$ is an admissible vector for the system $(\Sigma_3^\perp \rtimes K_0(\sigma_3), \mu_{|\Sigma_3^\perp \rtimes K_0(\sigma_3)})$ if and only if $\|\eta\|^2_{L^2(\nu_k)} = \frac{1}{\sqrt{2}}$ for every $k = 1, \ldots, 4$ and $\langle \eta, \eta \circ j \rangle_{L^2(\nu_k)} = 0$ for $k = 1, 2$.*

Proof. In order to simplify the proof, we set $X = \{\, x \in \mathbb{R}^2 : x_1\, x_2 \neq 0 \,\}$ whence $Y = \{\, y \in \mathbb{R}^2 : y_1 < 0, y_2 \neq 0 \,\}$.

Recall that $K_0(\sigma_3) = \{\, k_{e^t,0,e^s} : t, s \in \mathbb{R} \,\}$. Now we show that $\Sigma_3^\perp \rtimes K_0(\sigma_3)$ is reproducing. First, we need to verify that the $K_0(\sigma_3)$-orbits of $\operatorname{Im}\Phi$ with respect to the action δ' are locally closed. By the relation (4.11) we have for $t, s \in \mathbb{R}$ and $q = (a, b)$ with $a < 0$

$$\delta'(k_{e^t,0,e^s})\sigma_q = \psi_3\left(e^{2s}\, a, e^{s+t}\, b\right). \tag{4.15}$$

Hence there are four orbits: $\{\, 0 \,\}$, $\{\, (y_1, 0) \in \mathbb{R}^2 : y_1 < 0 \,\}$, $\{\, (y_1, y_2) \in \mathbb{R}^2 : y_1 < 0,\, y_2 < 0 \,\}$ and $\{\, (y_1, y_2) \in \mathbb{R}^2 : y_1 < 0, y_2 > 0 \,\}$. They are all locally closed. By the equation (4.15) we have that $\alpha(k_{e^t,0,e^s}) = e^{-3s-t}$. Hence, thanks to Proposition 3.15, since $K_0(\sigma_3)$ is Abelian and consequently unimodular we obtain that $\Sigma_3^\perp \rtimes K_0(\sigma_3)$ is non-unimodular. Further, by the equation (4.15), the stabilizer of σ_q is $\{\, I \,\}$ for every $q \in Y$ and so compact for almost every $\sigma_q \in \operatorname{Im}\Phi$. Therefore, by theorem 3.19 we obtain that the group $\Sigma_3^\perp \rtimes K_0(\sigma_3)$ is reproducing.

Now we use Theorem 3.22 to determine the admissible vectors. The action δ' has two orbits in Y and so we set $Z = \{\, 0, 1 \,\}$ and the counting measure λ on Z is a pseudo-image measure of the Lebesgue measure on Y. Let us set $y_1 = -1/2(1, -\sqrt{2})$ and $y_2 = -1/2(1, \sqrt{2})$. Hence $\Phi^{-1}(y_i) = \{\, P_1^i, P_2^i \,\}$ where $P_1^1 = (1, -1)$, $P_2^1 = -P_1^1$ and $P_1^2 = (1, 1)$, $P_2^2 = -P_1^2$. Condition (3.16) for the couples (P_i^1, P_i^1) yields

$$\frac{1}{\sqrt{2}} = \int_{\mathbb{R}^2} |\eta(\pm\, k_{e^{-t},0,e^{-s}}\, P_1^1)|^2 \frac{1}{e^{t+s} e^{-3s-t}}\, dtds = \int_{\mathbb{R}^2} |\eta\left(\pm\, (e^{-t}, -e^{-s})\right)|^2\, e^{2s}\, dtds.$$

To simplify this condition, we introduce the maps $\gamma^+ \colon \mathbb{R}^2 \to W_4$ and $\gamma^- \colon \mathbb{R}^2 \to W_2$ defined by $\gamma^\pm(t, s) = \pm\, (e^{-t}, -e^{-s})$ for every $(t, s) \in \mathbb{R}^2$. An easy computation shows that $J(\gamma^\pm)(t, s) = e^{-t-s}$ for every $(t, s) \in \mathbb{R}^2$. Hence the above relation can be rewritten as

$$\frac{1}{\sqrt{2}} = \int_{\mathbb{R}^2} \frac{|\eta\,(\gamma^\pm(t, s))|^2}{|\gamma_1^\pm(t, s)\, \gamma_2^\pm(t, s)^3|}\, J(\gamma^\pm)(t, s)\, dtds = \int_{W_k} \frac{|\eta\,(x_1, x_2)|^2}{|x_1\, x_2^3|}\, dx_1 dx_2 = \|\eta\|^2_{L^2(\nu_k)}.$$

with $k = 3 \pm 1$. The other conditions follows by applying the relation (3.16) with similar arguments to the couples (P_i^2, P_i^2) and (P_1^k, P_2^k). $\square$

Now we prove that the group $\Sigma_3^\perp \rtimes {}^tL_{-1}(\sigma_3)$ is not reproducing. Recall that

${}^{t}L_{-1}(\sigma_3) = \{\, k_{t,s,1} : t \in \mathbb{R}_+, s \in \mathbb{R} \,\} \cong$ "$ax + b$" and so $\Delta(k_{t,s,1}) = t^{-1}$. By the equation (4.11) we have $\delta'(k_{t,s,1})\sigma_q = \psi_3\left(a, \sqrt{2}s\, a + t\, b\right)$ and so the δ'-orbits of $\operatorname{Im}\Phi$ are locally closed and $\alpha(k_{t,s,1}) = t^{-1}$. Hence, by Proposition 3.15, the group $\Sigma_3^{\perp} \rtimes {}^{t}L_{-1}(\sigma_3)$ is unimodular. Therefore, by Theorem 3.19, the group $\Sigma_3^{\perp} \rtimes {}^{t}L_{-1}(\sigma_3)$ is not reproducing.

In order to study the group $\Sigma_3^{\perp} \rtimes {}^{t}L'_{\beta}(\sigma_3)$ we need the following lemma regarding

$$ {}^{t}L'_{\beta}(\sigma_3) = \left\{ \begin{bmatrix} e^{\beta t} & s \\ 0 & e^{t} \end{bmatrix} : t, s \in \mathbb{R} \right\}. $$

Lemma 4.24. *The measure $dh = e^{-\beta t} dt ds$ is a left Haar measure on ${}^{t}L'_{\beta}(\sigma_3)$ and $\Delta_{{}^{t}L'_{\beta}(\sigma_3)}(t, s) = e^{(1-\beta)t}$ for every $t, s \in \mathbb{R}$.*

Proof. Since

$$ \begin{bmatrix} e^{\beta t_1} & s_1 \\ 0 & e^{t_1} \end{bmatrix} \begin{bmatrix} e^{\beta t_2} & s_2 \\ 0 & e^{t_2} \end{bmatrix} = \begin{bmatrix} e^{\beta(t_1+t_2)} & e^{\beta t_1} s_2 + e^{t_2} s_1 \\ 0 & e^{t_1+t_2} \end{bmatrix}, $$

the group law can be expressed as $(t_1, s_1)\,(t_2, s_2) = (t_1 + t_2, e^{\beta t_1} s_2 + e^{t_2} s_1)$ for every $t_1, t_2, s_1, s_2 \in \mathbb{R}$. For every $f \in \mathcal{C}_C^{+}(H)$ and $h' = (t_1, s_1)^{-1} \in H$ we have

$$ \begin{aligned} \int_H L_{h'} f \, dh &= \int_{\mathbb{R}} \int_{\mathbb{R}} L_{h'} f\,(t_2, s_2)\; e^{-\beta t_2} dt_2 ds_2 \\ &= \int_{\mathbb{R}} \int_{\mathbb{R}} f\left((t_1, s_1)\,(t_2, s_2)\right)\; e^{-\beta t_2} dt_2 ds_2 \\ &= \int_{\mathbb{R}} \left(\int_{\mathbb{R}} f(t_1 + t_2, e^{\beta t_1} s_2 + e^{t_2} s_1)\, e^{-\beta t_2} ds_2 \right) dt_2 \\ &= \int_{\mathbb{R}} \left(\int_{\mathbb{R}} f(t_1 + t_2, s_2)\, e^{-\beta(t_1+t_2)} dt_2 \right) ds_2 \\ &= \int_{\mathbb{R}} \int_{\mathbb{R}} f(t_2, s_2)\, e^{-\beta t_2} dt_2 ds_2 \\ &= \int_H f \, dh, \end{aligned} $$

whence dh is a left Haar measure on H. Now, let us fix $f \in \mathcal{C}_C^{+}(H)$ such that $\int_H f\, dh > 0$.

For $(t_1, s_1) = h' \in H$ we have

$$\begin{aligned}
\int_H R_{h'} f\, dh &= \int_{\mathbb{R}} \int_{\mathbb{R}} R_{h'} f\,(t_2, s_2)\; e^{-\beta t_2} dt_2 ds_2 \\
&= \int_{\mathbb{R}} \int_{\mathbb{R}} f\,((t_2, s_2)\,(t_1, s_1))\; e^{-\beta t_2} dt_2 ds_2 \\
&= \int_{\mathbb{R}} \left(\int_{\mathbb{R}} f(t_1 + t_2, e^{\beta t_2} s_1 + e^{t_1} s_2)\, e^{-\beta t_2} ds_2 \right) dt_2 \\
&= e^{(\beta-1)t_1} \int_{\mathbb{R}} \left(\int_{\mathbb{R}} f(t_1 + t_2, s_2)\, e^{-\beta(t_1+t_2)} dt_2 \right) ds_2 \\
&= e^{(\beta-1)t_1} \int_{\mathbb{R}} \int_{\mathbb{R}} f(t_2, s_2)\, e^{-\beta t_2} dt_2 ds_2 \\
&= e^{(\beta-1)t_1} \int_H f\, dh,
\end{aligned}$$

whence $\Delta_H(t_1, s_1) = e^{(1-\beta)t_1}$. □

Let us denote by $\mathbb{R}^2_+$ and $\mathbb{R}^2_-$ the two half-planes $\mathbb{R}^2_\pm = \{\, x \in \mathbb{R}^2 : \pm x_2 > 0 \,\}$ provided with the measures ν_+ and ν_- defined by $d\nu_\pm = \frac{1}{x_2^4} dx_1 dx_2$.

Proposition 4.25. *Let $\beta \in \mathbb{R}$. The group $\Sigma_3^\perp \rtimes {}^t L'_\beta(\sigma_3)\,, \beta \in \mathbb{R}$ is reproducing for $\mu_{|\Sigma_3^\perp \rtimes {}^t L'_\beta(\sigma_3)}$ and a function $\eta \in L^2(\mathbb{R}^2)$ is an admissible vector for the system $(\Sigma_3^\perp \rtimes {}^t L'_\beta(\sigma_3), \mu_{|\Sigma_3^\perp \rtimes {}^t L'_\beta(\sigma_3)})$ if and only if $\|\eta\|^2_{L^2(\nu_\pm)} = \frac{1}{\sqrt{2}}$ and $\langle \eta, \eta \circ j \rangle_{L^2(\nu_+)} = 0$.*

Proof. Now we show that $\Sigma_3^\perp \rtimes {}^t L'_\beta(\sigma_3)$ is reproducing. First, we need to verify that the ${}^t L'_\beta(\sigma_3)$-orbits of $\operatorname{Im}\Phi$ with respect to the action δ' are locally closed. By the relation (4.11) we have for $t, s \in \mathbb{R}$ and $q = (a, b)$ with $a < 0$

$$\delta'(k_{e^{\beta t}, s, e^t})\sigma_q = \psi_3\left(e^{2t}\, a, e^t(\sqrt{2} a\, s + e^{\beta t}\, b)\right). \tag{4.16}$$

Hence there are the two orbits Y and $\{\, 0 \,\}$ that are locally closed. By the equation (4.16) we have that $\alpha(k_{e^{\beta t}, s, e^t}) = e^{-(3+\beta)t}$. Thanks to Proposition 3.15 and to Lemma 4.24 we obtain that $\Sigma_3^\perp \rtimes {}^t L'_\beta(\sigma_3)$ is non-unimodular. Further, by the equation (4.16), the stabilizer of σ_q is $\{\, I \,\}$ for every $q \in Y$ and so compact for almost every $\sigma_q \in \operatorname{Im}\Phi$. Hence, by theorem 3.19 we obtain that the group $\Sigma_3^\perp \rtimes {}^t L'_\beta(\sigma_3)$ is reproducing.

Now we use Theorem 3.22 to determine the admissible vectors. The action δ' is transitive on Y and so we set $Z = \{\, 0 \,\}$ and the measure λ defined by $\lambda(Z) = 1$ is a pseudo-image measure of the Lebesgue measure on Y. Let us set $y = (-1/2, 0)$ and so $\Phi^{-1}(y) = \{\, P_1, P_2 \,\}$ where $P_1 = (0, 1)$ and $P_2 = -P_1$. Since $dh = e^{-\beta t} dtds$ is a left Haar measure on ${}^t L'_\beta(\sigma_3)$, condition (3.16) for the couples (P_i, P_i) yields

$$\frac{1}{\sqrt{2}} = \int_{\mathbb{R}^2} |\eta(\pm\, k^{-1}_{e^{\beta t}, s, e^t}\, P_1)|^2\, e^{(2-\beta)t}\, dtds = \int_{\mathbb{R}^2} |\eta(\pm\,(-s e^{-(\beta+1)t}, e^{-t}))|^2\, e^{(2-\beta)t}\, dtds.$$

To simplify this condition, we introduce the maps $\gamma^{\pm}\colon \mathbb{R}^2 \to \mathbb{R}^2_{\pm}$ defined by $\gamma^{\pm}(t,s) = \pm\left(-se^{-(\beta+1)t}, e^{-t}\right)$ for every $(t,s) \in \mathbb{R}^2$. An easy computation shows that $J(\gamma^{\pm})(t,s) = e^{-(\beta+2)t}$ for every $(t,s) \in \mathbb{R}^2$. Hence the above relation can be rewritten as

$$\frac{1}{\sqrt{2}} = \int_{\mathbb{R}^2} \frac{|\eta(\gamma^{\pm}(t,s))|^2}{\gamma_2^{\pm}(t,s)^4} J(\gamma^{\pm})(t,s)\,dtds = \int_{R^2_{\pm}} \frac{|\eta(x_1,x_2)|^2}{x_2^4}\,dx_1dx_2 = \|\eta\|^2_{L^2(\nu_{\pm})}.$$

The other condition follows by applying the relation (3.16) to the couple (P_1, P_2) with the same argument. □

4.4 Last Conjugations

Here we want to prove that there are not other conjugations within $Sp(d,\mathbb{R})$ among the groups studied up to now. In order to do that, we introduce the class

$$\mathcal{E}'_d = \left\{\, i_{g(\tau,I)}\, G : \tau \in \mathrm{Sym}(d,\mathbb{R}),\, G \in \mathcal{E}_d \,\right\}$$

and the full collection of these groups is

$$\mathcal{E}' = \bigcup_{d\geq 1} \mathcal{E}'_d.$$

In order to describe the groups in $\mathcal{E}'$ we state the following

Lemma 4.26. *Take $G = \Sigma \rtimes H \in \mathcal{E}_d$, $\tau \in \mathrm{Sym}(d,\mathbb{R})$ and set $G' = i_{g(\tau,I)}G \in \mathcal{E}'_d$. Then*

1. *$\Sigma_{G'} = \Sigma$ and $H_{G'} = H$;*
2. *$\overline{\varphi_{G'}}(h) = \tau - h^{\dagger}[\tau] + \Sigma$ for every $h \in H$.*

Conversely, if $F < Q$ is a subgroup such that Σ_F is a vector subspace of $\mathrm{Sym}(d,\mathbb{R})$ *and $\overline{\varphi_F}(h) = \tau - h^{\dagger}[\tau] + \Sigma_F$ for every $h \in H_F$ then $F \in \mathcal{E}'_d$.*

Proof. For $\sigma \in \Sigma$ and $h \in H$ we have

$$i_{g(\tau,I)}\, g(\sigma,h) = g(\tau,I)\, g(\sigma,h)\, g(-\tau,h) = \begin{bmatrix} h & 0 \\ \sigma h + \tau h - {}^t h^{-1}\tau & {}^t h^{-1} \end{bmatrix} = g(\sigma + \tau - h^{\dagger}[\tau], h),$$

whence $G' = \left\{\, g(\sigma + \tau - h^{\dagger}[\tau], h) : \sigma \in \Sigma, h \in H \,\right\}$. Hence $H_{G'} = H$ and $\Sigma_{G'} = \Sigma$ because $I^{\dagger}[\;\,]$ is the identity action. Part 2 follows trivially by Proposition 3.9 and Remark 3.10 thanks to the above equality. □

As a consequence of part 1 of the previous lemma we have that for $G' \in \mathcal{E}'_d$ we have

$$\#\left(\left\{\, i_{g(\tau,I)}\, G' : \tau \in \mathrm{Sym}(d,\mathbb{R}) \,\right\} \cap \mathcal{E}_d\right) = 1. \tag{4.17}$$

Recalling the matrices described in (4.9), we set $W_0 = \{ I, J, w_1 \} \subseteq Sp(2,\mathbb{R})$. The following corollary is a consequence of the Bruhat decomposition of semisimple groups.

Corollary 4.27. *If $g \in Sp(2,\mathbb{R})$ then there exist $q_1, q_2 \in Q$ and $w_0 \in W_0$ such that $g = q_1\, w_0\, q_2$.*

Proof. Denote by $W = W(Sp(2,\mathbb{R}))$ the Weyl group of the symplectic group. By the Bruhat decomposition (for details, see [Hel78], Chapter IX, Theorem 1.4) every element $g \in Sp(2,\mathbb{R})$ may be decomposed as

$$g = q_1\, w\, q_2 \qquad q_i \in Q,\ w \in W. \tag{4.18}$$

The Weyl group may be described as follows (for details, see [De 87]). Set $W' = \{ I, J, w_1, w_2 \}$. Denote by P the subgroup of $Sp(2,\mathbb{R})$ formed by the elements of the form $g(0,p)$, with $p \in O(2)$ a permutation matrix. We have

$$W = \{ p\, w' : p \in P, w' \in W' \}.$$

Since $i_{g(0,\Omega)}\, w_1 = w_2$ our claim follows by (4.18). □

Now we want to study all the possible conjugations among the groups in $\mathcal{E}_2$, namely the family $\{ i_g\, G : g \in Sp(2,\mathbb{R}) \} \cap \mathcal{E}_2$ for $G \in \mathcal{E}_2$. By Lemma 4.27 we can study the conjugations of G by $q_1\, w_0\, q_2$ with $q_i \in Q$ and $w_0 \in W_0$. We have to ask ourselves whenever

$$i_{q_1\, w_0\, q_2}\, G \in \mathcal{E}_2.$$

Because we have already done a complete classification for the conjugations via elements in MA, by equation (4.17) we can study the simpler but equivalent problem

$$i_{w_0\, q}\, G \in \mathcal{E}_2'$$

for $q \in Q$ and $w_0 \in W_0$. There are three possibilities for w_0. If $w_0 = J$ an easy computation shows that $i_{J\, q}\, G \notin \mathcal{E}_2'$ for every $q \in Q$. By (4.17), if $w_0 = I$ the problem has already been solved in the previous sections.

Now suppose $w_0 = w_1$. By writing $q = g(\tau, k) = g(\tau, I)\, g(0,k)$ we have that $i_{g(0,k)}G$ is surely a group in $\mathcal{E}_2$ and so we can directly suppose $k = I$. Suppose $G = \Sigma \rtimes H$ where

$$\Sigma = \left\{ \sigma = \begin{bmatrix} a(\sigma) & b(\sigma) \\ b(\sigma) & c(\sigma) \end{bmatrix} : \sigma \in \Sigma \right\} < \mathrm{Sym}(2,\mathbb{R}),$$
$$H = \left\{ h = \begin{bmatrix} \alpha(h) & \gamma(h) \\ \beta(h) & \delta(h) \end{bmatrix} : h \in H \right\} < GL(2,\mathbb{R}).$$

Suppose $\tau = \tau_{t,s,z} = \left[\begin{smallmatrix} t & s \\ s & z \end{smallmatrix}\right]$ and denote by $\tau_z = \tau_{0,0,z}$ for $z \in \mathbb{R}$.

As said above, we ask ourselves when

$$\hat{G} = i_{w_1\, g(\tau, I)}\, G \in \mathcal{E}'.$$

Remark 4.28. The problem can be studied for all $G \in \mathcal{E}_2$ up to conjugation by elements of the form $g(0, \left[\begin{smallmatrix} \alpha & 0 \\ \beta & \delta \end{smallmatrix}\right])$ since $w_1\, g(0, \left[\begin{smallmatrix} \alpha & 0 \\ \beta & \delta \end{smallmatrix}\right])\, w_1^{-1} \in Q$.

Since $g(\tau_{t,s,z}, I) = g(\tau_{t,s,0}, I)\, g(\tau_z, I)$ and $i_{w_1}\, g(\tau_{t,s,0}, I) \in Q$ the previous question is equivalent to

$$\hat{G} = i_{w_1\, g(\tau_z, 0)}\, G \in \mathcal{E}'.$$

For $\sigma \in \Sigma$ and $h \in H$ we have

$$i_{w_1\, g(\tau_z, 0)}\, g(\sigma, h) = \begin{bmatrix} \cdots & 0 & -\gamma(h) \\ \cdots & -\gamma/\det h & z\alpha(h)/\det h - b(\sigma)\gamma(h) - z\delta(h) - c(\sigma)\delta(h) \\ \iddots & \vdots & \vdots \end{bmatrix}$$

whence necessary conditions are $\gamma(h) = 0$, $c(\sigma) = 0$ (by taking $h = I$) and $z(\delta(h) - 1) = 0$. These conditions are also sufficient to obtain $\hat{G} < Q$. We set

$$\Sigma_0 = \left\{ \begin{bmatrix} a & b \\ b & 0 \end{bmatrix} : a, b \in \mathbb{R} \right\}, \qquad H_0 = \left\{ \begin{bmatrix} \alpha & 0 \\ \alpha\beta & \delta \end{bmatrix} : \alpha, \delta > 0, \beta \in \mathbb{R} \right\}.$$

and so we can write $\Sigma < \Sigma_0$ and $H < H_0$. Note that $H_0 < H(\Sigma_0)$. For $h \in H_0$ let us change the parametrization $h = \left[\begin{smallmatrix} \alpha(h) & 0 \\ \alpha(h)\beta(h) & \delta(h) \end{smallmatrix}\right]$. An easy computation shows that for $\sigma \in \Sigma_0$ and $h \in H_0$ such that $z(\delta(h) - 1) = 0$ we have

$$\begin{aligned} i_{w_1\, g(\tau_z, 0)}\, g(\sigma, h) &= g(\left[\begin{smallmatrix} z\,\beta(h)^2 + 2b(\sigma)\,\beta(h) + a(\sigma) & -\beta(h) \\ -\beta(h) & 0 \end{smallmatrix}\right], \left[\begin{smallmatrix} \alpha(h) & 0 \\ \alpha(h)\,(b(\sigma) + \beta(h)\, z) & \delta(h)^{-1} \end{smallmatrix}\right]) \\ &= g(\hat{\sigma}(\sigma, h), \hat{h}(\sigma, h)) \end{aligned} \tag{4.19}$$

whence

$$\begin{aligned} &\hat{G} = \left\{ g(\hat{\sigma}(\sigma, h), \hat{h}(\sigma, h)) : \sigma \in \Sigma, h \in H \right\}, \\ &\hat{H} := H_{\hat{G}} = \left\{ \begin{bmatrix} \alpha(h) & 0 \\ \alpha(h)\,(b(\sigma) + \beta(h)\, z) & \delta(h)^{-1} \end{bmatrix} : \sigma \in \Sigma, h \in H \right\}, \\ &\hat{\Sigma} := \Sigma_{\hat{G}} = \left\{ \left[\begin{smallmatrix} b(\sigma)\,\beta(h) + a(\sigma) & -\beta(h) \\ -\beta(h) & 0 \end{smallmatrix}\right] : \sigma \in \Sigma, h \in H, \alpha(h) = \delta(h) = 1, b(\sigma) + z\beta(h) = 0 \right\}. \end{aligned}$$

By Lemma 4.26, $\hat{G} \in \mathcal{E}'$ if and only if

$$\exists\, \tau_0 \in \mathrm{Sym}(2, \mathbb{R})\, \forall \sigma \in \Sigma, h \in H \quad \hat{\sigma}(\sigma, h) - (\tau_0 - \hat{h}(\sigma, h)^\dagger[\tau_0]) \in \hat{\Sigma} \tag{4.20}$$

By Remark 4.28 an easy computation shows that we can suppose Σ to be one of the following

$$\Sigma_0, \qquad \Sigma_1 = \left\{ \begin{bmatrix} a & 0 \\ 0 & 0 \end{bmatrix} : a \in \mathbb{R} \right\}, \qquad \Sigma_2 = \left\{ \begin{bmatrix} 0 & b \\ b & 0 \end{bmatrix} : b \in \mathbb{R} \right\}.$$

Case Σ_2 In this case it never happens that $\hat{G} \in \mathcal{E}'$. Indeed the condition $H < H(\Sigma_2)$ implies that $\beta(h) = 0$ for every $h \in H$. Hence $\hat{\Sigma} = 0$ and so $\hat{G} \notin \mathcal{E}'$.

Cases Σ_0 and Σ_1 By Remark 4.28 it is sufficient to study the groups listed in Proposition 4.15 since $\Sigma_0 = \Omega^\dagger[\Sigma_3^\perp]$. Since in these case $\Sigma_0 \subseteq \hat{\Sigma}$, with an easy computation condition (4.20) may be rewritten

$$\begin{gathered} \exists v, w \in \mathbb{R}\, \forall \sigma \in \Sigma, h \in H\, \exists \sigma' \in \Sigma, h' \in H \;\; (\alpha(h') = \delta(h') = 1, b(\sigma') + z\beta(h') = 0) \; : \\ \begin{cases} v(\delta(h)/\alpha(h) - 1) - \delta(h)^2 w(b(\sigma) + z\beta(h)) - \beta(h) = -\beta(h') \\ w(\delta(h) - 1) = 0 \end{cases} . \end{gathered} \tag{4.21}$$

(3.i) Here $z = 0$ ($\delta \neq 1$) and we can set $v, w = 0$. See point 1 in Proposition 4.16.

(3.ii) If $z = 0$ then $\hat{H} = \{I\}$ and so $\hat{G} \notin \mathcal{E}'$. If $z \neq 0$ then $\hat{\Sigma} = \Sigma$ and $\hat{H} = H$ and so we are not interested in this case.

(3.iii) Here $z = 0$ ($\delta \neq 1$) and (4.4) is not satisfied ($w = 0$ and $\alpha(h) = \delta(h)$).

(3.iv) Here $z = 0$ ($\delta \neq 1$) and we can set $v, w = 0$. See point 2 in Proposition 4.16.

(3.v) Independently on z we can set $v, w = 0$. See points 2-3 in Proposition 4.16.

(3.vi) Here $z = 0$ ($\delta \neq 1$) and $\hat{\Sigma} = \Sigma$, $\hat{H} = H$, so we are not interested in this case.

(3.vii) If $z \neq 0$ then $\hat{\Sigma} = \Sigma$, $\hat{H} = H$ and so we are not interested in this case. If $z = 0$ we can set $v, w = 0$. See point 4 in Proposition 4.16.

(3.viii) Here $z = 0$ ($\delta \neq 1$) and we can set $v, w = 0$. See points 5-6 in Proposition 4.16.

(3.ix) Here $z = 0$ ($\delta \neq 1$) and $\hat{\Sigma} = \Sigma$, $\hat{H} = H$, so we are not interested in this case.

(3.x) Independently on $z \in \mathbb{R}$ we have $\hat{\Sigma} = \Sigma$, $\hat{H} = H$ and so we are not interested in this case.

(3.xi) Here $z = 0$ ($\delta \neq 1$) and (4.4) is not satisfied ($w = 0$ and $\alpha(h) = \delta(h)$).

(3.xii) Here $z = 0$ ($\delta \neq 1$) and we can set $v, w = 0$. See point 5 in Proposition 4.16.

(3.xiii) Independently on z we can set $v, w = 0$. See points 4-6 in Proposition 4.16.

(3.xiv) Here $z = 0$ ($\delta \neq 1$) and we can set $v, w = 0$. See point 1 in Proposition 4.16.

(3.xv) Independently on $z \in \mathbb{R}$ we have $\hat{\Sigma} = \Sigma$, $\hat{H} = H$ and so we are not interested in this case.

(3.xvi) Here $z = 0$ ($\delta \neq 1$) and we can set $v, w = 0$. See point 7 in Proposition 4.16.

4.5 Summary List

In this section we simply collect the results of this chapter, in a more concise way. Recall the three matrices σ_i introduced in Chapter 4 on page 40 and, as above, set $\Sigma_i = \operatorname{span}\{\sigma_i\}$. Since we gave a necessary and sufficient condition only in the case $n = d$, the following list is inclusive only if $\dim \Sigma = 2$, always up to conjugation in $Sp(d, \mathbb{R})$. Recall the notation

$$R_t = \begin{bmatrix} \cos t & -\sin t \\ \sin t & \cos t \end{bmatrix}, \qquad A_t = \begin{bmatrix} \cosh t & \sinh t \\ \sinh t & \cosh t \end{bmatrix}, \qquad t \in \mathbb{R}.$$

Theorem 4.29. *The following groups are reproducing:*

1. $\Sigma_1 \rtimes \left\{ e^t R_{\alpha t} : t \in \mathbb{R} \right\}$ *for* $\alpha \geq 0$;
2. $\Sigma_2 \rtimes \left\{ e^t A_{\alpha t} : t \in \mathbb{R} \right\}$ *for* $\alpha \geq 0$;
3. $\Sigma_3 \rtimes \left\{ e^t \left[\begin{smallmatrix} 1 & 0 \\ t & 1 \end{smallmatrix}\right] : t \in \mathbb{R} \right\}$;
4. $\Sigma_3 \rtimes \left\{ \left[\begin{smallmatrix} e^{\alpha t} & 0 \\ 0 & e^{(\alpha+1)t} \end{smallmatrix}\right] : t \in \mathbb{R} \right\}$ *for* $\alpha \in [-1, 0)$;
5. $\Sigma_1 \rtimes \left\{ e^t R_\theta : t \in \mathbb{R}, \theta \in [0, 2\pi) \right\}$;
6. $\Sigma_2 \rtimes \left\{ e^t A_s : t, s \in \mathbb{R} \right\}$;
   and
7. $\Sigma_1^\perp \rtimes \left\{ e^t R_{\alpha t} : t \in \mathbb{R} \right\}$ *for* $\alpha \geq 0$;
8. $\Sigma_2^\perp \rtimes \left\{ e^t A_{\alpha t} : t \in \mathbb{R} \right\}$ *for* $\alpha \geq 0$;
9. $\Sigma_3^\perp \rtimes \left\{ e^t I : t \in \mathbb{R} \right\}$;
10. $\Sigma_3^\perp \rtimes \left\{ e^t \left[\begin{smallmatrix} 1 & t \\ 0 & 1 \end{smallmatrix}\right] : t \in \mathbb{R} \right\}$;

11. $\Sigma_3^{\perp} \rtimes \left\{\left[\begin{smallmatrix} e^{\alpha t} & 0 \\ 0 & e^{(\alpha+1)t} \end{smallmatrix}\right] : t \in \mathbb{R}\right\}$ *for* $\alpha \neq -3/4$;
12. $\Sigma_1^{\perp} \rtimes \left\{e^t \, R_\theta : t \in \mathbb{R}, \theta \in [0, 2\pi)\right\}$;
13. $\Sigma_2^{\perp} \rtimes \left\{e^t \, A_s : t, s \in \mathbb{R}\right\}$;
14. $\Sigma_3^{\perp} \rtimes \left\{\left[\begin{smallmatrix} e^t & 0 \\ 0 & e^s \end{smallmatrix}\right] : t, s \in \mathbb{R}\right\}$;
15. $\Sigma_3^{\perp} \rtimes \left\{\left[\begin{smallmatrix} e^{\beta t} & s \\ 0 & e^t \end{smallmatrix}\right] : t, s \in \mathbb{R}\right\}$ *for* $\beta \in \mathbb{R}$.

Now we present a synthesis of all the main properties of these reproducing Lie subgroups. Note that every group H in the list above is solvable and obviously every subspace Σ is Abelian. Hence their semidirect product $\Sigma \rtimes H$ is always solvable.

1

$\Sigma = \left\{\left[\begin{smallmatrix} a & 0 \\ 0 & a \end{smallmatrix}\right] : a \in \mathbb{R}\right\}$

$H = \left\{e^t \, R_{\alpha t} : t \in \mathbb{R}\right\} \cong \mathbb{R}, \quad \alpha \in \mathbb{R}$

$\dim \Sigma \rtimes H = 2$

$dh = dt, \quad \Delta_H(t) = 1$

$dg = e^{2t} da \, dt, \quad \Delta_G(a, t) = e^{2t}$

$t^{\dagger}[a] = e^{-2t} \, a$

$\Phi(x) = -\frac{1}{4}|x|^2, \quad x \in \mathbb{R}^2$

2

$\Sigma = \left\{\left[\begin{smallmatrix} a & 0 \\ 0 & -a \end{smallmatrix}\right] : a \in \mathbb{R}\right\}$

$H = \left\{e^t \, A_{\alpha t} : t \in \mathbb{R}\right\} \cong \mathbb{R}, \quad \alpha \in \mathbb{R}$

$\dim \Sigma \rtimes H = 2$

$dh = dt, \quad \Delta_H(t) = 1$

$dg = e^{2t} da \, dt, \quad \Delta_G(a, t) = e^{2t}$

$t^{\dagger}[a] = e^{-2t} \, a$

$\Phi(x) = \frac{1}{4}(x_2^2 - x_1^2), \quad x \in \mathbb{R}^2$

3

$\Sigma = \left\{\left[\begin{smallmatrix} a & 0 \\ 0 & 0 \end{smallmatrix}\right] : a \in \mathbb{R}\right\}$

$H = \left\{e^t \left[\begin{smallmatrix} 1 & 0 \\ t & 1 \end{smallmatrix}\right] : t \in \mathbb{R}\right\} \cong \mathbb{R}$

$\dim \Sigma \rtimes H = 2$

$dh = dt, \quad \Delta_H(t) = 1$

$dg = e^{2t} da\, dt, \quad \Delta_G(a,t) = e^{2t}$

$t^\dagger[a] = e^{-2t}\, a$

$\Phi(x) = -\frac{1}{2}x_1^2, \quad x \in \mathbb{R}^2$

4

$\Sigma = \left\{\left[\begin{smallmatrix} a & 0 \\ 0 & 0 \end{smallmatrix}\right] : a \in \mathbb{R}\right\}$

$H = \left\{\left[\begin{smallmatrix} e^{\alpha t} & 0 \\ 0 & e^{(\alpha+1)t} \end{smallmatrix}\right] : t \in \mathbb{R}\right\} \cong \mathbb{R}, \quad \alpha \neq 0$

$\dim \Sigma \rtimes H = 2$

$dh = dt, \quad \Delta_H(t) = 1$

$dg = e^{2\alpha t} da\, dt, \quad \Delta_G(a,t) = e^{2\alpha t}$

$t^\dagger[a] = e^{-2\alpha t}\, a$

$\Phi(x) = -\frac{1}{2}x_1^2, \quad x \in \mathbb{R}^2$

5

$\Sigma = \left\{\left[\begin{smallmatrix} a & 0 \\ 0 & a \end{smallmatrix}\right] : a \in \mathbb{R}\right\}$

$H = \left\{e^t\, R_\theta : t \in \mathbb{R}, \theta \in [0, 2\pi)\right\} \cong \mathbb{R} \times \mathbb{T}$

$\dim \Sigma \rtimes H = 3$

$dh = dt\, d\theta, \quad \Delta_H(t,\theta) = 1$

$dg = e^{2t} da\, dt\, d\theta, \quad \Delta_G(a,t,\theta) = e^{2t}$

$(t,\theta)^\dagger[a] = e^{-2t}\, a$

$\Phi(x) = -\frac{1}{4}|x|^2, \quad x \in \mathbb{R}$

6

$\Sigma = \left\{\left[\begin{smallmatrix} a & 0 \\ 0 & -a \end{smallmatrix}\right] : a \in \mathbb{R}\right\}$

$H = \left\{e^t\, A_s : t, s \in \mathbb{R}\right\} \cong \mathbb{R}^2$

$\dim \Sigma \rtimes H = 3$

$dh = dt\, ds, \quad \Delta_H(t,s) = 1$

$dg = e^{2t} da\, dt\, ds, \quad \Delta_G(a,t,s) = e^{2t}$

$(t,s)^\dagger[a] = e^{-2t}\, a$

$\Phi(x) = \frac{1}{4}(x_2^2 - x_1^2), \quad x \in \mathbb{R}$

7

$\Sigma = \left\{ \left[\begin{smallmatrix} a & b \\ b & -a \end{smallmatrix} \right] : a, b \in \mathbb{R} \right\}$

$H = \left\{ e^t \, R_{\alpha t} : t \in \mathbb{R} \right\} \cong \mathbb{R}, \quad \alpha \geq 0$

$\dim \Sigma \rtimes H = 3$

$dh = dt, \quad \Delta_H(t) = 1$

$dg = e^{4t} da \, db \, dt, \quad \Delta_G(a, b, t) = e^{4t}$

$t^\dagger[a, b] = e^{-2t} R_{2\alpha t} \left[\begin{smallmatrix} a \\ b \end{smallmatrix} \right]$

$\Phi(x) = -\frac{1}{4}(x_1^2 - x_2^2, 2x_1 x_2), \quad x \in \mathbb{R}$

$\int_{\mathcal{O}_\nu^\alpha} \frac{|\eta(x)|^2}{|x|^3} \, ds(x) = \frac{\sqrt{\alpha^2+1}}{4}, \qquad \int_{\mathcal{O}_\nu^\alpha} \frac{\eta(x) \, \overline{\eta \circ j(x)}}{|x|^3} \, ds(x) = 0$ for almost every $\nu \in [0, 2\pi)$,

$\mathcal{O}_\nu^\alpha = \left\{ \, e^t R_{\alpha t} \, e_\nu : t \in \mathbb{R} \, \right\}$

8

$\Sigma = \left\{ \left[\begin{smallmatrix} a & b \\ b & a \end{smallmatrix} \right] : a, b \in \mathbb{R} \right\}$

$H = \left\{ e^t \, A_{\alpha t} : t \in \mathbb{R} \right\} \cong \mathbb{R}, \quad \alpha \geq 0$

$\dim \Sigma \rtimes H = 3$

$dh = dt, \quad \Delta_H(t) = 1$

$dg = e^{4t} da \, db \, dt, \quad \Delta_G(a, b, t) = e^{4t}$

$t^\dagger[a, b] = e^{-2t} A_{-2\alpha t} \left[\begin{smallmatrix} a \\ b \end{smallmatrix} \right]$

$\Phi(x) = -\frac{1}{4}(x_1^2 + x_2^2, 2x_1 x_2), \quad x \in \mathbb{R}$

$\int_{\mathbb{R}} |\eta(e^{-t} A_{-\alpha t} \, M_i \, q_\xi)|^2 \, e^{2t} \, dt = \frac{1}{4}, \quad i = 0, \ldots, 3,$ for a.e. $\xi \in \mathbb{R}$

$\int_{\mathbb{R}} \eta(e^{-t} A_{-\alpha t} \, M_i \, q_\xi)) \, \overline{\eta(e^{-t} A_{-\alpha t} \, M_j \, q_\xi))} \, e^{2t} \, dt = 0, \quad 0 \leq i < j \leq 3,$ for a.e. $\xi \in \mathbb{R}$

$M_0 = I, \quad M_1 = \left[\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix} \right], \quad M_2 = -I, \quad M_3 = -\left[\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix} \right], \quad q_\xi = (-\cosh \xi, \sinh \xi)$

9

$\Sigma = \left\{ \left[\begin{smallmatrix} 0 & b \\ b & a \end{smallmatrix} \right] : a, b \in \mathbb{R} \right\}$

$H = \left\{ e^t : t \in \mathbb{R} \right\} \cong \mathbb{R}$

$\dim \Sigma \rtimes H = 3$

$dh = dt, \quad \Delta_H(t) = 1$

$dg = e^{4t} da \, db \, dt, \quad \Delta_G(a, b, t) = e^{4t}$

$t^\dagger[a, b] = e^{-2t}(a, b)$

$\Phi(x) = -\frac{1}{2}(x_2^2, x_1 x_2), \quad x \in \mathbb{R}$

$\int_{\mathbb{R}} |\eta(\pm e^t(\xi, 1))|^2 \, e^{-2t} \, dt = \frac{1}{\sqrt{2}}, \quad \int_{\mathbb{R}} \eta(e^t(\xi, 1)) \overline{\eta(-e^t(\xi, 1))} \, e^{-2t} \, dt = 0$ for a.e. $\xi \in \mathbb{R}$

10

$\Sigma = \left\{ \left[\begin{smallmatrix} 0 & b \\ b & a \end{smallmatrix}\right] : a, b \in \mathbb{R} \right\}$

$H = \left\{ e^t \left[\begin{smallmatrix} 1 & t \\ 0 & 1 \end{smallmatrix}\right] : t \in \mathbb{R} \right\} \cong \mathbb{R}$

$\dim \Sigma \rtimes H = 3$

$dh = dt, \quad \Delta_H(t) = 1$

$dg = e^{4t} da\, db\, dt, \quad \Delta_G(a, b, t) = e^{4t}$

$t^\dagger[a, b] = e^{-2t}(a - 2tb, b)$

$\Phi(x) = -\frac{1}{2}(x_2^2, x_1 x_2), \quad x \in \mathbb{R}$

$\int_{\mathbb{R}} |\eta(\pm e^t(\xi + t, 1))|^2 \, e^{-2t} \, dt = \frac{1}{\sqrt{2}}, \quad \int_{\mathbb{R}} \eta(e^t(\xi + t, 1)) \overline{\eta(-e^t(\xi + t, 1))} \, e^{-2t} \, dt = 0$ for a.e. $\xi \in \mathbb{R}$

11

$\Sigma = \left\{ \left[\begin{smallmatrix} 0 & b \\ b & a \end{smallmatrix}\right] : a, b \in \mathbb{R} \right\}$

$H = \left\{ \left[\begin{smallmatrix} e^{\alpha t} & 0 \\ 0 & e^{(\alpha+1)t} \end{smallmatrix}\right] : t \in \mathbb{R} \right\} \cong \mathbb{R}, \alpha \neq -3/4$

$\dim \Sigma \rtimes H = 3$

$dh = dt, \quad \Delta_H(t) = 1$

$dg = e^{(4\alpha+3)t} da\, db\, dt, \quad \Delta_G(a, b, t) = e^{(4\alpha+3)t}$

$t^\dagger[a, b] = (e^{-2(\alpha+1)t} a, e^{-(2\alpha+1)t} b)$

$\Phi(x) = -\frac{1}{2}(x_2^2, x_1 x_2), \quad x \in \mathbb{R}$

$\int_{\mathbb{R}} |\eta\left(e^{-\alpha t}(\xi, -e^{-t}\sqrt{2}|\xi|)\right)|^2 \, e^{2(\alpha+1)t} \, dt = \sqrt{2}\, \xi^2$ for a.e. $\xi \in \mathbb{R}$,

$\int_{\mathbb{R}} \eta\left(e^{-\alpha t}(\xi, -e^{-t}\sqrt{2}|\xi|)\right) \overline{\eta\left(-e^{-\alpha t}(\xi, -e^{-t}\sqrt{2}|\xi|)\right)} \, e^{2(\alpha+1)t} \, dt = 0$ for a.e. $\xi \in \mathbb{R}$

12

$\Sigma = \left\{ \left[\begin{smallmatrix} a & b \\ b & -a \end{smallmatrix}\right] : a, b \in \mathbb{R} \right\}$

$H = \left\{ e^t R_\theta : t \in \mathbb{R}, \theta \in [0, 2\pi) \right\} \cong \mathbb{R} \times \mathbb{T}$

$\dim \Sigma \rtimes H = 4$

$dh = dt\, d\theta, \quad \Delta_H(t, \theta) = 1$

$dg = e^{4t} da\, db\, dt\, d\theta, \quad \Delta_G(a, b, t, \theta) = e^{4t}$

$(t, \theta)^\dagger[a, b] = e^{-2t} R_{2\theta} \left[\begin{smallmatrix} a \\ b \end{smallmatrix}\right]$

$\Phi(x) = -\frac{1}{4}(x_1^2 - x_2^2, 2x_1 x_2), \quad x \in \mathbb{R}$

$\|\eta\|_{L^2(\nu)} = 1/2$, $\langle \eta, \eta \circ j \rangle_{L^2(\nu)} = 0$, $d\nu = |x|^{-2} \, dx$

13

$\Sigma = \left\{ \left[\begin{smallmatrix} a & b \\ b & a \end{smallmatrix}\right] : a, b \in \mathbb{R} \right\}$
$H = \left\{ e^t A_s : t, s \in \mathbb{R} \right\} \cong \mathbb{R}^2$
$\dim \Sigma \rtimes H = 4$
$dh = dt\, ds, \quad \Delta_H(t,s) = 1$
$dg = e^{4t} da\, db\, dt\, ds, \quad \Delta_G(a,b,t,s) = e^{4t}$
$(t,s)^\dagger[a,b] = e^{-2t} A_{-2s} \left[\begin{smallmatrix} a \\ b \end{smallmatrix}\right]$
$\Phi(x) = -\frac{1}{4}(x_1^2 - x_2^2, 2x_1x_2), \quad x \in \mathbb{R}$
$\|\eta\|_{L^2(\nu_i)} = 1/2$ for every $i = 0, \ldots, 3$, $\langle \eta \circ M_i, \eta \circ M_j \rangle_{L^2(\nu_0)} = 0$ for every $0 \le i < j \le 3$
$d\nu_i = \frac{dx_1 dx_2}{(x_1^2 - x_2^2)^2}$ on $V_i = \{ \rho e_\theta \in \mathbb{R}^2 : \rho \in \mathbb{R}_+, \theta \in (\frac{\pi}{2} i - \frac{\pi}{4}, \frac{\pi}{2} i + \frac{\pi}{4}) \}$
$M_0 = I, \quad M_1 = \left[\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right], \quad M_2 = -I, \quad M_3 = -\left[\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right]$

14

$\Sigma = \left\{ \left[\begin{smallmatrix} 0 & b \\ b & a \end{smallmatrix}\right] : a, b \in \mathbb{R} \right\}$
$H = \left\{ \left[\begin{smallmatrix} e^t & 0 \\ 0 & e^s \end{smallmatrix}\right] : t, s \in \mathbb{R} \right\} \cong \mathbb{R}^2$
$\dim \Sigma \rtimes H = 4$
$dh = dt\, ds, \quad \Delta_H(t,s) = 1$
$dg = e^{t+3s} da\, db\, dt\, ds, \quad \Delta_G(a,b,t,s) = e^{t+3s}$
$(t,s)^\dagger[a,b] = (e^{-2s} a, e^{-t-s} b)$
$\Phi(x) = -\frac{1}{2}(x_2^2, x_1x_2), \quad x \in \mathbb{R}$
$\|\eta\|^2_{L^2(\nu_k)} = \frac{1}{\sqrt{2}}$ for every $k = 1, \ldots, 4$, $\langle \eta, \eta \circ j \rangle_{L^2(\nu_k)} = 0$ for every $k = 1, 2$
$d\nu_i = \frac{1}{|x_1\, x_2^3|} dx_1 dx_2$ on the i-th quadrant

15

$\Sigma = \left\{ \left[\begin{smallmatrix} 0 & b \\ b & a \end{smallmatrix}\right] : a, b \in \mathbb{R} \right\}$
$H = \left\{ \left[\begin{smallmatrix} 1 & s \\ 0 & 1 \end{smallmatrix}\right] \left[\begin{smallmatrix} e^{\beta t} & 0 \\ 0 & e^t \end{smallmatrix}\right] : s, t \in \mathbb{R} \right\} \cong \mathbb{R} \rtimes \mathbb{R}$
$\dim \Sigma \rtimes H = 4$
$dh = e^{(1-\beta)t} ds\, dt, \quad \Delta_H(s,t) = e^{(1-\beta)t}$
$dg = e^{4t} da\, db\, ds\, dt, \quad \Delta_G(a,b,t,s) = e^{4t}$
$(s,t)^\dagger[a,b] = (a e^{-2t} - 2bs e^{-(\beta+1)t}, b e^{-(\beta+1)t})$
$\Phi(x) = -\frac{1}{2}(x_2^2, x_1x_2), \quad x \in \mathbb{R}$
$\|\eta\|^2_{L^2(\nu_\pm)} = \frac{1}{\sqrt{2}}$, $\langle \eta, \eta \circ j \rangle_{L^2(\nu_+)} = 0$, $d\nu_\pm = \frac{1}{x_2^2} dx_1 dx_2$ on $\mathbb{R}^2_\pm = \{ x \in \mathbb{R}^2 : \pm x_2 > 0 \}$

Appendix A

Group and Algebra Conjugation

In this Appendix we recall a useful statement that deals with conjugation within Lie groups and algebras. The main reference is [War71]. For the rest of this Appendix, let G be a Lie group, denote by $\mathfrak{g} = \mathrm{Lie}(G)$ its Lie algebra and fix an element $g \in G$. Consider the group isomorphism

$$\begin{aligned} i_g \colon G &\longrightarrow G \\ h &\longmapsto ghg^{-1} \end{aligned}$$

and its differential

$$Ad(g) \colon \mathfrak{g} \longrightarrow \mathfrak{g}\,.$$

If G is a matrix group, then $Ad(g)X = gXg^{-1}$ for all $X \in \mathfrak{g}$.

Definition A.1. Two Lie subgroups $H_1, H_2 < G$ are called **conjugate** by g if $i_g(H_1) = H_2$. In this case we write $gH_1g^{-1} = H_2$.

There is a similar definition for the subalgebras of $\mathfrak{g}$.

Definition A.2. Two Lie subalgebras $\mathfrak{h}_1, \mathfrak{h}_2 < \mathfrak{h}$ are called **conjugate** by g if $Ad(g)(\mathfrak{h}_1) = \mathfrak{h}_2$. In this case we write $g\mathfrak{h}_1g^{-1} = \mathfrak{h}_2$.

The following result relates these two notions.

Proposition A.3. *Let $H_1, H_2 < G$ be two connected Lie subgroups of G and denote by $\mathfrak{h}_i = \mathrm{Lie}(H_i) < \mathfrak{g}$ their Lie algebras. Then H_1 and H_2 are conjugate by g if and only if $\mathfrak{h}_1$ and $\mathfrak{h}_2$ are conjugate by g.*

Proof. It is a simple consequence of Theorem 3.32 of [War71]. $\square$

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