# Problem Set 4 Solutions

2024

1. Estimate a VAR(4) for the growth rates of labor productivity and hours worked.
2. Identify a technology shock using the restriction that the shock is the only one driving labor productivity (in the log-level) in the long run.
3. Plot the impulse response functions of (log) labor productivity and (log) hours (point estimates and the 68% confidence bands computed with the bootstrap method) for level responses. Comment on your results.
4. Report the percentage of forecast error variance of (log) labor productivity and (log) hours explained at horizon 0, 4, 8, 20 and 40. Comment on your results.
5. BONUS: Compute the historical decomposition of the two series into the technology and the non-technology shock contributions over time.

# Solution Exercise 1:

In the script you can use the series from John Fernald or the ones from FRED. You can change the code easily to work with hours in per-capita terms, but this does not have much effect on the results. The plot below is for the growth rates computed by John Fernald (here in annualized terms):


Figure 1: *Time series for the VAR.*

We estimate a bivariate VAR(4) given by:

$$\begin{bmatrix}\Delta LP_t\\ \Delta H_t\end{bmatrix} = \begin{bmatrix}\mu_1\\ \mu_2\end{bmatrix} + A_1\begin{bmatrix}\Delta LP_{t-1}\\ \Delta H_{t-1}\end{bmatrix} + A_2\begin{bmatrix}\Delta LP_{t-2}\\ \Delta H_{t-2}\end{bmatrix} + A_3\begin{bmatrix}\Delta LP_{t-3}\\ \Delta H_{t-3}\end{bmatrix} + A_4\begin{bmatrix}\Delta LP_{t-4}\\ \Delta H_{t-4}\end{bmatrix} + \begin{bmatrix}\epsilon_{1t}\\ \epsilon_{2t}\end{bmatrix}$$

We can estimate the VAR(4) with a constant using standard OLS techniques. In this case the dimension of $Y$ will be $(T-p)\times 2$ and $X$ will be $(T-p)\times(1+Np)$. Hence, the matrix of parameter estimates, let's call it $\hat{\Pi} = (X'X)^{-1}X'Y$, will have dimensions $(1+Np)\times 2$. To get the estimation results, we take the transpose of $\hat{\Pi}$ and the first column will be the constant estimates and then the following $N\times N$ blocks are the matrices $\hat{A}_1,\ldots,\hat{A}_4$.

```
%% Load the data

% Fernald:
data = readtable('quarterly_tfp.xlsx', 'Sheet', 'quarterly');
```

```
hours = data.dhours(2:308);  % change in hours
prod = data.dLP(2:308);    % change in labor productivity
dates = datetime("1947-04-01"):calmonths(3):datetime("2023-10-01");

% FRED:
% data1 = readtable('techdata2.csv');
% hours_pc = diff(log(data1.HOANBS./data1.CNP16OV))*100;
% hours = diff(log(data1.HOANBS))*100;
% prod = diff(log(data1.OPHNFB))*100;
% dates = data1.DATE(2:end);

y = [prod, hours]; % Data from Q2:1947 to Q4:2023

figure;
subplot(2,1,1)
plot(dates, prod, 'LineWidth', 1.5)
recessionplot
axis tight
title('Labor productivity growth', 'FontSize', 14, ...
    'Interpreter','Latex')

subplot(2,1,2)
plot(dates, hours, 'LineWidth', 1.5)
recessionplot
axis tight
title('Hours worked growth', 'FontSize', 14, 'Interpreter','Latex')


%% Estimate the VAR
% Determine the lag length for the VAR using AIC
p = aicbic(y,4,1);
c = 1;

% Estimate the VAR yt = c + A1y_t-1 + ... Apyt-p + et
[beta, residuals] = VAR(y,p,c);

% Compute the variance-covariance matrix of the reduced form errors et
[T, N] = size(y);
Sigma = (residuals' * residuals)./(T-1-p-N*p);
```

# Solution Exercise 2:

The identification strategy requires long run restrictions. The technology shock is the only one driving (log) labor productivity in the long run. First, order the VAR as above $y_t = [\Delta LP_t, \Delta H_t]'$. Then derive the Wold representation coefficients.

$$y_t = C(L)\epsilon_t$$

Start again from this VMA representation, we need to find a linear combination $K$ which imposes our identification assumption and orthogonalizes the shocks.

$$y_t = C(L)KK^{-1}\epsilon_t \tag{1}$$

What is the long run response of the variables? Recall that to get the level response we have to sum the IRFs across all horizons. The long run response is the cumulated effect over all periods given by:

$$LR: \quad C(1)K \tag{2}$$

The long run responses are given by

$$C(1)K = D(1) = \begin{bmatrix} d_{11}^{LR} & 0 \\ d_{21}^{LR} & d_{22}^{LR} \end{bmatrix}$$

Then the second shock of the system has no impact in the long-run on the first variable and we interpret it as the non-technology shock. How do we construct $D(1)$ in this way and at the same time ensure that $Var(K^{-1}\epsilon_t) = I$ for the orthogonalization? The matrix $D(1)$ is lower traiangular. This could be the Cholesky factor of some unknown matrix $M$. Hence,

$$D(1)D(1)' = M \leftrightarrow C(1)KK'C(1)' = M \leftrightarrow KK' = C(1)^{-1}MC(1)^{-1'}$$

Furthermore, from the orthogonality condition we know that:

$$\begin{aligned} Var(K^{-1}\epsilon_t) &= K^{-1}E(\epsilon_t\epsilon_t')K^{-1'} = I \\ &= K^{-1}\Sigma K^{-1'} = I \end{aligned}$$

The equation is satisfied if $\Sigma = KK'$, so we equalize the two equations for $KK'$ and obtain

$$\Sigma = C(1)^{-1}MC(1)^{-1'}$$

Rearranging to isolate $M$ we get

$$M = C(1)\Sigma C(1)^{'}$$

and we know that $D(1)$ is the lower triangular Cholesky factor of M, hence we have identified the matrix K as

$$D(1) = chol(C(1)\Sigma C(1)^{'}) \leftrightarrow C(1)K = chol(C(1)\Sigma C(1)^{'}) \leftrightarrow K = C(1)^{-1}chol(C(1)\Sigma C(1)^{'})$$

Let's put this into MATLAB.

```
% Compute the Wold IRFs by inverting (I-A(L))
horizon = 40;
wold = woldirf(beta, c, p, horizon);

%% Compute the LR responses and identify technology shocks
% Gali assumes that only technology shocks can
% have a permanent effect on the level of technology. Second shock ...
    is the
% non-technology shock then.

C1 = sum(wold,3);
D1 = chol(C1*Sigma*C1', 'lower');
K = C1\D1;
% disp(K)

% Compute the LR responses
irf_bq = bqIRF(wold, K);
```

where the function **bqIRF** just calculates the structural IRFs.

```
function [bqirf] = bqIRF(wold, K, scaling)

% Function to compute the point estimate of the IRF of a VAR identified
% using BQ long-run restrictions

% Inputs:   wold    = (N x N x horizon +1) array of Wold IRFs
%           K       = N x N lower triangular matrix Cholesky factor ...
    of LR
%           scaling = 2 x 1 vector where the first argument is the ...
    variable
%                     to be used for scaleing and the second is the ...
    shock size

% Outputs:  bqirf = N x N x horizon + 1 array of LR identified IRFs

[N,¬, horizon] = size(wold);
bqirf = zeros(N,N,horizon);

for h=1:horizon

    bqirf(:,:,h) = wold(:,:,h) * K;

end

if nargin > 2
    bqirf = bqirf ./ (bqirf(scaling(1),scaling(1),1) * 1/scaling(2));
end

end
```

# Solution Exercise 3:

The only slight complication is that the question asks us for log-level responses. This means we have to cumulate the responses. In MATLAB we do the following:

```
%% Bootstrap confidence bands
nboot1 = 1000;
nboot2 = 2000;
prc = 68;
cumulate = [1,2];
[bootbq, upper, lower, boot_beta] = ...
    bootstrapBQ_corrected(y,p,c,beta,residuals,nboot1,horizon,prc,cumulate,nboot2);

%% Plot the IRFs
varnames = {'LABPROD', 'HOURS'};
shocknames = {'Technology', 'Non-Technology'} ;
shocks = [1,2];
plotLR(irf_bq, varnames, shocknames, cumulate, shocks, upper, lower, ...
    prc)
```

Here the function `bootstrapBQ_corrected` implements the bias correction for the IRFs and cumulates the responses where necessary (in our case both variables enter in growth rates, so both must be cumulated). The function `plotLR` returns a plot with the responses of the variables in the VAR to the desired shocks.

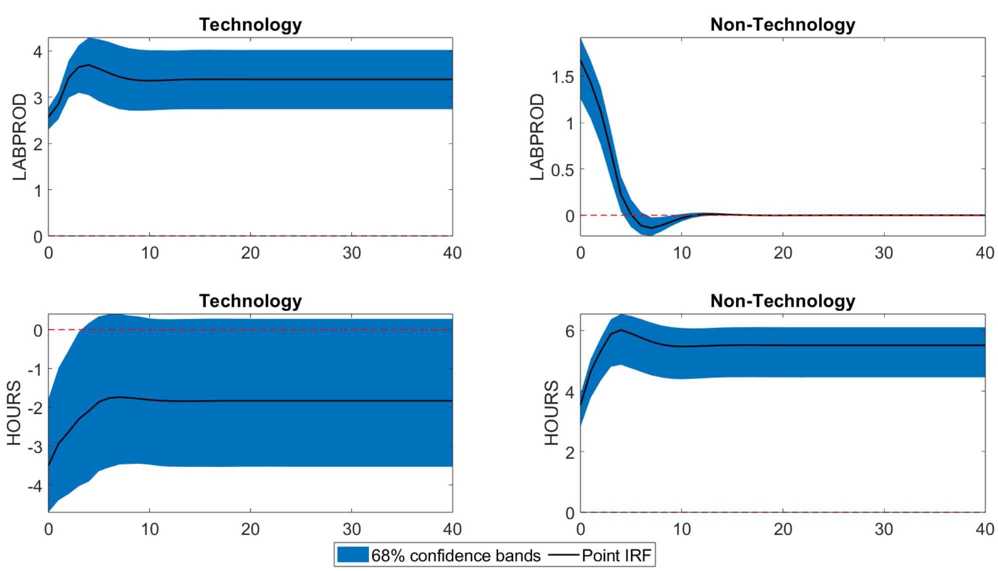


Figure 2: *IRFs to technology and non-technology shocks, identified using LR restrictions.*

We see that technology shocks lead to some creative destruction. Hours worked decrease as labor is replaced by capital or less labor is required to produce the same output.

The non-technology shock leads to a short run improvement in labor productivity and increases hours worked. While this is interpreted as monetary policy shocks in the original paper, it is likely a combination of undetermined shocks.

## Solution Exercise 4:

Here we compute the forecast error variance decomposition (FEVD) of the two variables due to the news shock and the other shock. Since we are asked for the level FEVD we use the cumulated IRFs in

$$FEVD_j^n = \frac{\sum_{h=0}^{H} IRF_{n,j,h}^2}{\sum_{k=1}^{K}\sum_{h=0}^{H} IRF_{n,k,h}^2}$$

This gives us the FEVD of the $n^{th}$ variable for the $j^{th}$ shock up to horizon $H$.

```
irf_cumu = cumsum(irf_bq,3);
fevd = variance_decomp(irf_cumu);
plot_vardec(fevd, varnames, shocknames)
```

where the function `variance_decomp` performs the decomposition under the assumption that all shocks in the system are orthogonal and have unit variance, as is the case for the identification strategy we have employed.

```
function fevd = variance_decomp(irf, shock)

% Function to compute forecast error variance decomposition. If only one
% shock is of interest, then input its position.

% Inputs:   irf = (N x N x horizon) array of impulse responses
%           shock = if a specific shock is required

% Output:  fevd = N x N x horizon array of forecast error shares

[N,¬,horizon] = size(irf);

% Square the IRFs
irf = irf.^2;

% Compute the total variation for each variable by summing over shocks
% (dimension 2) first and then summing over horizons (dimension 3).
% We use the cumsum() command to get the total variation at each horizon
total_variation = squeeze(cumsum(sum(irf,2),3));

% Now compute the variation of each individual shock over the ...
    horizon and
% divide it by the total variation
```

```

if nargin > 1 % For a specific shock
    shock_variation = cumsum(squeeze(irf(:,shock,:)),2);
    fevd = shock_variation ./ total_variation;
else % For all shocks
    fevd = zeros(N,N,horizon);
    for s=1:N
        shock_variation = cumsum(squeeze(irf(:,s,:)),2);
        fevd(:,s,:) = shock_variation ./ total_variation;
    end

end
```

The requested values at horizons 0,4,8,20, 40 are

| Horizon | LP | | Hours | |
|---|---|---|---|---|
| | Tech | Other | Tech | Other |
| 0 | 0.7784 | 0.2216 | 0.1357 | 0.8643 |
| 4 | 0.7794 | 0.2206 | 0.0711 | 0.9289 |
| 8 | 0.8337 | 0.1663 | 0.0623 | 0.9377 |
| 20 | 0.9201 | 0.0799 | 0.0588 | 0.9412 |
| 40 | 0.9592 | 0.0408 | 0.0580 | 0.9420 |

We can see that the technology shock is the main driver of labor productivity over all horizons and that hours worked fluctuate only mildly because of technological change in the long run. We can also plot the FEVD over time:

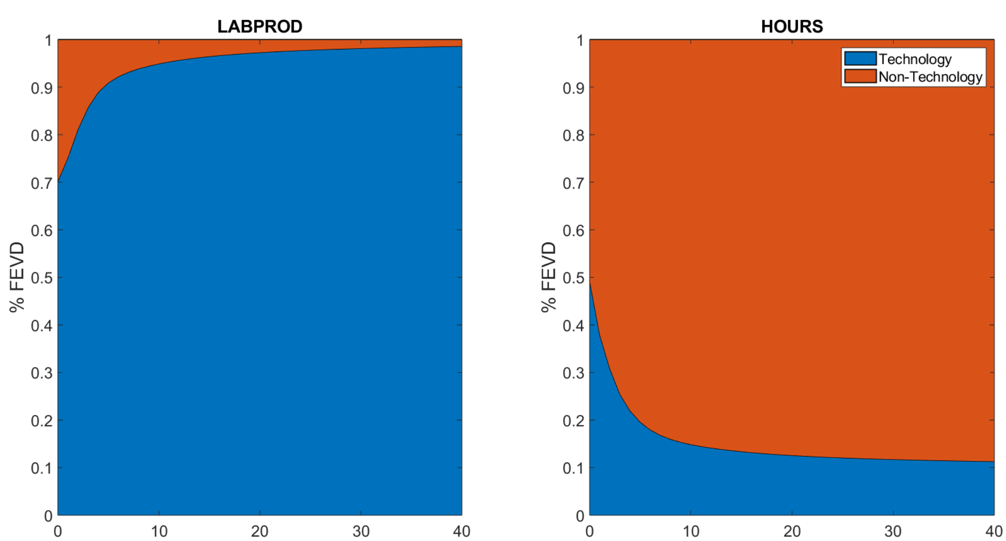


Figure 3: *FEVD to technology and non-technology shocks, identified using LR restrictions.*

# Solution Exercise 5:

Another tool that is frequently used in SVAR analysis is called historical decomposition. The idea is that given our VMA representation $y_t = \mu + D(L)u_t$, we can in principle compute the contribution of each shock to the variables in $y_t$ over time. From this we can analyze specific historic episodes and see what the economy would have looked like without the shocks. This is possible in stationary systems. The atarting point is the VMA representation:

$$y_t = D(L)u_t \tag{3}$$

We can subtract the deterministic component and express $y_t^*$ as a function of past shocks weighted by the MA coefficients at each period $t$. Hence,

$$\widehat{y_t^*} = \sum_{s=0}^{t-1} D_s u_{t-s} \tag{4}$$

For example, at $t$ the contribution of the first shock to the first variable will be equal to $D_{11}^0 u_{1t}$, at $t+1$ it will be the cumulative effect, i.e. $D_{11}^0 u_{1t} + D_{11}^1 u_{1t+1}$ and so on. We can compute this in MATLAB in the following way for a time series of interest:

```
%% Compute the historical decomposition
histedec_all = zeros(T-p,N,N);
ystar_all = zeros(T-p,N);

for series=1:N
    [histdec, ystar] = hist_decmp(y, beta, residuals, c, p, K, series);
    histedec_all(:,:,series) = histdec;
    ystar_all(:,series) = ystar;
end

%% Plot HD
dates_final = dates(p+1:end);

series = 1:1:N;
varnames = varnames(series);
plothistdec(histedec_all,ystar_all,shocks,series,dates_final,shocknames,varnames)
```

The actual decomposition is performed by the function `hist_decmp` for each series.

```
function [histdec, ystar] = hist_decmp(y, beta, residuals, c, p, K, ...
    series)

% Function to compute historical decomposition of series of choice

% Inputs:   y           = TxN matrix of original data
%           p           = integer VAR lag order
```

```
%            c            = 1 if constant required
%            beta         = (Np+1 x N) matrix of estimated ...
    coefficients (Np x N) if
%                           no constant is included
%            residuals    = (T-p) x N matrix of OLS residuals from VAR(p)
%                           estimation

% Outputs:  histdec      = T-p x N matrix of contributions of shocks ...
    to series
%            ystar        = demeaned series

[T, N] = size(y);

% Strucutral shocks
struct_shock = (K\residuals')';

% Demean the data
ystar = y(p+1:end,series) - mean(y(p+1:end,series),1);

% Compute the MA coefficients of the entire sample
wold_long = woldirf(beta, c, p, T-p);
ma_coeff = zeros(N,N,T-p);

for i=1:T-p
    ma_coeff(:,:,i) = wold_long(:,:,i) * K;
end

% Compute the historical decomposition for the series of choice
histdec = zeros(T-p, N);

for t=1:T-p % for each time period
    for j=1:N % for each shock
        histdec(t,j) =  squeeze(ma_coeff(series,j,1:t))' * ...
            flipud(struct_shock(1:t,j));
    end
end

end
```

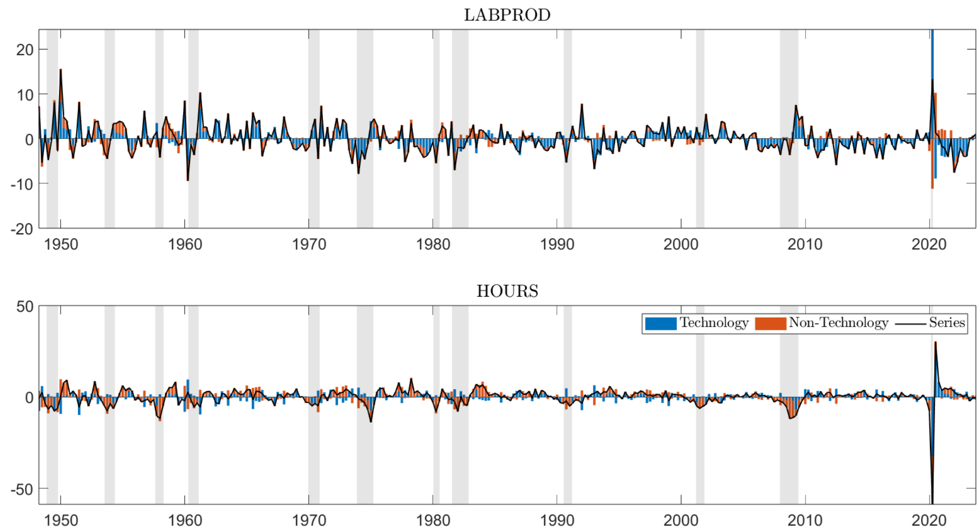


Figure 4: *HD to technology and non-technology shocks, identified using LR restrictions.*

As we can see, the historical contributions of technology shocks to labor productivity are much larger, and the reverse is true for the non-technology shocks and hours worked.

# Conditional Correlations

A final exercise could be to compute the conditional correlations after each of the two shocks. The idea is that the RBC model wants a strong positive correlation between technology and hours worked as your MPL increases and thus your wage is driven up, giving you more incentive to work. This is not what we get in the data. To do this, we proceed similar to computing the FEVD, just the ratio will be different:

$$\rho\left(\Delta LP_t, \Delta H_t \mid i\right) = \frac{\sum_{j=0}^{\infty} D_j^{1i} D_j^{2i}}{\sqrt{\operatorname{var}\left(\Delta LP_t \mid i\right) \operatorname{var}\left(\Delta H_t \mid i\right)}}$$

for $i = [$ tech, non-tech $]$, where $\operatorname{var}\left(\Delta LP_t \mid i\right) = \sum_{j=0}^{\infty}\left(D_j^{1i}\right)^2$ and $\operatorname{var}\left(\Delta H_t \mid i\right) = \sum_{j=0}^{\infty}\left(D_j^{2i}\right)^2$ are conditional variances of labor productivity and hours growth.

```
%% Compute the conditional correlations of Gali (1999)
[N,¬,horizon] = size(irf_bq);

% Compute the numerator for each shock (the product of the MA ...
    coefficients)
numerator = zeros(N,1);
```

```
for i=1:N

    temp = irf_bq(1,i,:) .* irf_bq(2,i,:);
    numerator(i) = sum(temp,3);

end

% Compute the denominators
irf_squared = irf_bq.^2;

denominator = zeros(N,1);
for i=1:N
    v1 = sum(irf_squared(1,i,:),3);
    v2 = sum(irf_squared(2,i,:),3);
    denominator(i) = sqrt(v1 * v2);
end

% Conditional correlations
cond_corr = numerator ./ denominator;

% Unconditional correlations
uncond_corr = corr(y);
```

We find the unconditional correlation between hours and productivity growth to be nearly zero (0.0035) and the conditional correlations to be -0.8280 (technology shock) and 0.4290 (non-technology shock), giving support to Gali's idea that the RBC model fails to account for this negative co-movement.