UNIQUENESS IN CALDERÓN PROBLEM - HABERMAN AND TATARU'S PROOF

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Abstract

The problem of injectivity of Dirichlet to Neumann map for the conductivity equation has seen extensive progress over the years starting with [3] *which solved the problem for C^2 conductivities. Reducing the regularity for conductivity from C^2 has proven to be a formidable challenge. Haberman and Tataru in 2013* [7] *used a novel technique to reduce the regularity to C^1 conductivities, and Lipschitz conductivities close to the identity in a suitable sense. This note is my attempt to understand their paper. I would like to thank Gunther Uhlmann for introducing me to the field of Inverse problems, Francois Monard for the very many helpful discussions on this topic and enthusiastic support and Rusell Brown for e-mail clarification of doubts in* [1].

I hereby declare that this is not an original work but merely my attempt to understand the techniques in [7]**.** *All errors are entirely mine. Please let me know if you spot any. Comments, suggestions and ideas are always welcome.*

1. Introduction

In 1980, Calderón introduced an inverse boundary problem which considered whether one can determine the electrical conductivity of a medium by making voltage and current measurements on the boundary of the medium. This inverse method is known as Electrical Impedance Tomography (EIT).

Let us precisely describe the mathematical problem. Let Ω be a bounded domain with smooth boundary. The electrical conductivity of Ω is represented by a bounded and positive function $\gamma(x)$. In the absence of sinks or sources of current the equation of potential is given by

$$\nabla \cdot (\gamma \nabla u) = 0 \ \text{ in } \ \Omega.$$

since, by Ohm's law, $\gamma \nabla u$ represents the current flux. Given a potential f in $H^{\frac{1}{2}}(\partial\Omega)$ on the boundary, the induced potential u in $H^1(\Omega)$ solves the Dirichlet problem

$$\begin{aligned} \nabla \cdot (\gamma \nabla u) &= 0 \text{ in } \Omega \\ u &= f \text{ on } \partial\Omega \end{aligned} \tag{1.1}$$

The Dirichlet to Neumann map, or voltage to current map, is given by

$$\Lambda_\gamma(f) = (\gamma \frac{\partial u}{\partial \nu})\Big|_{\partial\Omega} \tag{1.2}$$

where ν denotes the unit outer normal on $\partial\Omega$. If $\gamma \in \mathrm{Lip}(\overline{\Omega})$, then Λ_γ is a well defined map from $H^{\frac{1}{2}}(\partial\Omega)$ to $H^{\frac{-1}{2}}(\partial\Omega)$. The inverse problem is to reconstruct γ from Λ_γ. An obvious condition for this to be necessary is that the map $\gamma \to \Lambda_\gamma$ be injective.

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The earliest progress in this direction was made by Calderón himself who considered a linearized version of the problem [2]. Calderón made the following observation.

Using the divergence theorem we have

$$Q_\gamma(f) := \int_\Omega \gamma |\nabla u|^2 \, dx = \int_{\partial\Omega} \Lambda_\gamma(f)(f) \, dS \tag{1.3}$$

where dS denotes surface measure and u is the solution to the given boundary value problem. In other words $Q_\gamma(f)$ is the quadratic form associated to the linear map $\Lambda_\gamma(f)$ and knowing the linear map $\Lambda_\gamma(f)$ is equivalent to knowing the quadratic form $Q_\gamma(f)$ for all f. Calderón's approach was to look at $Q_\gamma(f)$ and then come up with enough solutions to the conductivity equation with given f at the boundary to find γ in the interior. Since the DN map Λ_γ (or equivalently Q_γ) depends non-linearly on γ, Calderón considered the *linearized problem* at constant conductivity. A crucial ingredient in his approach was the use of complex exponential harmonic functions.

Though Calderón solved the inverse problem for constant conductivity, the problem for non-constant conductivity remained open until Sylvester and Uhlmann in 1987 [3] in a breakthrough result proved the following injectivity result for Λ_γ in dimensions $d \geq 3$.

Theorem 1.1 (Sylvester and Uhlmann's result)**.** *Let $\gamma_i \in C^2(\overline{\Omega})$ be strictly positive for $i = 1, 2$. If $\Lambda_{\gamma_1} = \Lambda_{\gamma_2}$ then $\gamma_1 = \gamma_2$ in $\overline{\Omega}$.*

A key step in their proof is the construction of 'enough' solutions to the conductivity PDE on the whole space $\mathbb{R}^d$ satisfying some growth conditions in appropriate normed spaces. To achieve this, a standard trick is to convert the conductivity equation in to a Schródinger type equation with potential.

To wit, if $\gamma \in C^2(\overline{\Omega})$ and $u \in H^1(\Omega)$ then

$$-\nabla \cdot (\gamma \nabla (\gamma^{\frac{-1}{2}} u)) = \gamma^{\frac{1}{2}}(-\Delta + q) \tag{1.4}$$

where

$$q = \frac{\Delta\sqrt{\gamma}}{\sqrt{\gamma}} \tag{1.5}$$

Therefore to construct solutions of $L_\gamma(u) = -\nabla \cdot (\gamma \nabla u) = 0$ in $\mathbb{R}^d$ it is sufficient to construct solutions to $(-\Delta + q)u = 0$ in $\mathbb{R}^d$ with q of the form (1.5)

As a consequence, Theorem 1.1 follows from the following more general result.

Theorem 1.2. *Let $q_i \in L^\infty(\Omega)$, $i = 1, 2$. If $\Lambda_{q_1} = \Lambda_{q_2}$ then $q_1 = q_2$ in Ω*

where $\Lambda_q(f) = \frac{\partial u}{\partial \nu}\big|_{\partial\Omega}$ and u solves

$$\begin{aligned} (-\Delta + q)(u) &= 0 \text{ in } \Omega \\ u\big|_{\partial\Omega} &= f. \end{aligned} \tag{1.6}$$

Let us quickly explain why. The proof of Theorem 1.1 proceeds in the following steps:

We first use the boundary identifiability result for Lipschitz conductivities proved by Alessandrini [4] and obtain *boundary* uniqueness. Using boundary uniqueness, we show that $\Lambda_{\gamma_1} = \Lambda_{\gamma_2} \implies \Lambda_{q_1} = \Lambda_{q_2}$. We then use Theorem 1.2 and show $q_1 = q_2$ in Ω. A simple argument (sketched at the end of Section 3) then shows $\gamma_1 = \gamma_2$ in Ω.

Hence it suffices to assume that $\Lambda_{q_1} = \Lambda_{q_2}$. The next step is to obtain an integral identity. If $u \in H^1(\Omega)$ satisfies $(-\Delta + q)u = 0$, then for any $v \in H^1(\Omega)$ we have

$$0 = \int_\Omega (-\Delta + q)uv\,dx = \int_\Omega (\nabla u \cdot \nabla v + quv)\,dx - \int_{\partial\Omega} \frac{\partial u}{\partial \nu} v\,d\sigma.$$

Thus if $u\big|_{\partial\Omega} = f$ and $v\big|_{\partial\Omega} = g$, we have

$$(\Lambda_q f, g)_{L^2(\partial\Omega)} = \int_{\partial\Omega} \frac{\partial u}{\partial \nu} v\,d\sigma = \int_\Omega (\nabla u \cdot \nabla v + quv)\,dx.$$

In particular if $\Lambda_{q_1} = \Lambda_{q_2}$ and $(-\Delta + q_i)u_i = 0$, then a simple calculation shows that

$$\int_\Omega (q_1 - q_2)u_1 u_2\,dx = 0$$

Using boundary determination [4] it is possible to extend the γ_i to functions in $W^{1,\infty}(\mathbb{R}^d)$ or $C^1(\mathbb{R}^d)$ so that $\gamma_1 = \gamma_2$ on $\mathbb{R}^d \setminus \Omega$. Given this, we can extend the domain of integration to all of $\mathbb{R}^d$ and eventually obtain

$$\int_{\mathbb{R}^d} (q_1 - q_2)u_1 u_2\,dx = 0. \tag{1.7}$$

We can now see why the derivation of the integral identity was helpful. *To show $q_1 = q_2$ in Ω, we need to produce enough solutions to the corresponding Schródinger equations so that their product is dense in $L^2(\Omega)$.* This step allows us to connect the knowledge of the DN map (a boundary measurement) to an integral identity in the *interior*.

Motivated by Calderón's exponential harmonic solutions, Sylvester and Uhlmann constructed in dimensions $d \geq 2$ complex geometrical optics solutions (CGO solutions) i.e complex exponentials with a linear phase, for C^2 conductivities that behave like Calderón's exponential harmonics for large frequencies. The proof of Theorem 1.2 involves constructing CGO solutions of the form $u_i = e^{x\cdot\zeta_i}(1+\psi)$. Here the $\zeta_i \in \mathbb{C}^d$ are chosen so that $\zeta_i \cdot \zeta_i = 0$ (so that $e^{x\cdot\zeta_i}$ is harmonic) and so that $\zeta_1 + \zeta_2 = ik$ for some fixed frequency $k \in \mathbb{R}^d$. In three or more dimensions, these conditions give sufficient freedom that it is possible to chose an infinite family of pairs ζ_1, ζ_2 with $|\zeta_i| \to \infty$. This in turn ensures that the perturbations of the harmonic exponential i.e the remainders ψ_i, decay to zero in an appropriate norm as $|\zeta_i| \to \infty$, so that the product $u_1 u_2$ converges to $e^{ix\cdot k}$. From (1.7) we can see that injectivity of Λ_q follows from Fourier inversion.

To construct these CGO solutions, fix $\zeta \in \mathbb{C}^d$ such that $\zeta \cdot \zeta = 0$, and note that $e^{-x\cdot\zeta}\Delta(e^{x\cdot\zeta}\psi) = (\Delta + 2\zeta \cdot \Delta)\psi$ in a weak sense. Thus $u = e^{x\cdot\zeta}(1+\psi)$ solves $\Delta u = qu$ in $\mathbb{R}^d$ if

$$\Delta_\zeta \psi = \Delta\psi + 2\zeta \cdot \nabla\psi = q(1+\psi). \tag{1.8}$$

We let m_q be the map sending q to $q\psi$. We (1.8) perturbatively, by viewing $\Delta_\zeta - m_q$ as a perturbation of Δ_ζ. The operator Δ_ζ has a right inverse defined by

$$\widehat{\Delta_\zeta^{-1} f}(\xi) = p_\zeta(\xi)^{-1}\hat{f}(\xi)$$

where

$$p_\zeta(\xi) = -|\xi|^2 + 2i\zeta \cdot \xi \tag{1.9}$$

is the symbol for the operator Δ_ζ.

Solutions to (1.8) are constructed using a fixed point argument for which we need to bound the operators Δ_ζ^{-1} and m_q in some iteration spaces. Sylvester and Uhlmann showed that

$$\|\Delta_\zeta^{-1}\|_{L^2_{\delta+1} \to L^2_\delta} \lesssim |\zeta|^{-1} \tag{1.10}$$

where $-1 < \delta < 0$ and

$$\|u\|^2_{L^2_\delta} = \int_{\mathbb{R}^d} (1+|x|^2)^\delta |f(x)|^2 \, dx.$$

For $\gamma \in C^2(\overline{\Omega})$, $q \in L^\infty(\mathbb{R}^d)$, and the bound $\|m_q\|_{L^2_\delta \to L^2_{\delta+1}} \lesssim 1$ is trivial. Combining this with (1.10) completes the iteration argument, showing CGO solutions exist and the remainder ψ goes to zero in L^2_δ as $|\zeta| \to \infty$ and as a result from (1.7), we get $q_1 = q_2$ in Ω which implies $\gamma_1 = \gamma_2$ in Ω.

If γ does not have two derivatives, then we can still use this approach by viewing q as having negative regularity. Brown in [5] used this observation and derived a bound for Δ_ζ^{-1} in certain weighted Besov spaces of negative order. Combined with a corresponding estimate for m_q, Brown proved injectivity for Λ_γ for γ having $\frac{3}{2}+\epsilon$ derivatives. ($\gamma \in W^{\frac{3}{2}+\epsilon,\infty}(\Omega)$) Uniqueness for $W^{\frac{3}{2},\infty}$ conductivities was shown in [**?**] later in 2003.

In 2013, Haberman and Tataru [7] proved an analog of Theorem 1.1 by relaxing the regularity assumption of to C^1 conductivities and Lipschitz conductivities close to the identity in a suitable sense. The goal of this note is to understand the key techniques used in their proof and give a detailed explanation thereof. Their main result is the following theorem.

Theorem 1.3 (Haberman and Tataru). *Let $\Omega \subset \mathbb{R}^d$ with $d \geq 3$ be a bounded domain with Lipschitz boundary. For $i = 1, 2$, let $\gamma_i \in W^{1,\infty}(\bar{\Omega})$ be real valued functions, and assume that there is some c such that $\gamma_i > c > 0$. The there exists a constant $\epsilon_{d,\Omega}$ such that if each γ_i satisfies either $\|\nabla \log \gamma_i\|_{L^\infty(\Omega)} \leq \epsilon_{d,\Omega}$ or $\gamma_i \in C^1(\bar{\Omega})$, then $\Lambda_{\gamma_1} = \Lambda_{\gamma_2}$ implies $\gamma_1 = \gamma_2$.*

Haberman and Tataru's proof uses two key new ideas. First is the use of spaces adapted to the operator Δ_ζ and second is the use of averaged estimates. We now try and understand these two ideas in detail.

2. Spaces adapted to Δ_ζ and Averaged Estimates

2.1. **Some properties of Bourgain spaces.** Let p_ζ be as defined in (1.9). We define the space $\dot{X}_{\zeta^b}$ to be the closure of the Schwartz space with the norm

$$\|f\|^2_{\dot{X}_{\zeta^b}} = \int_{\mathbb{R}^d} |\hat{f}(\xi)|^2 |p_\zeta(\xi)|^{2b}\, d\xi. \tag{2.1}$$

We will also need the inhomogenenous version of this space, X^b_ζ which has the norm

$$\|f|^2_{X_\zeta b} = \int_{\mathbb{R}^d} |\hat{f}(\xi)|^2 (|p_\zeta(\xi)| + |\zeta|)^{2b}\, d\xi. \tag{2.2}$$

We will only need $b = \pm\frac{1}{2}$. It is easily seen that the operator $\Delta_\zeta : \dot{X}_{\zeta \frac{1}{2}} \to \dot{X}_{\zeta \frac{1}{2}}$ is an isomorphism with norm 1. We are thus led to look at the behavior of m_q in these spaces. Since q is now a distribution, we need to define m_q carefully which we do at the beginning of section 2.2.

Our first result gives regularity of the functions in space $\dot{X}_{\zeta \frac{1}{2}}$. Let w is a non-negative Borel measurable function, we let $L^2(w)$ denote the set of all Borel measurable functions for which the norm

$$\|f\|^2_{L^2(w)} = \int_{\mathbb{R}^d} |\hat{f}(\xi)|^2 w(\xi)\, d\xi$$

is finite. We need the following result on maps between weighted L^2 spaces.

Lemma 2.1. *Let ϕ be in $L^1(\mathbb{R}^d)$ and let v and w be weights on $\mathbb{R}^d$. Define $Tf(\xi) = \phi * f(\xi) = \int_{\mathbb{R}^d} \phi(\xi - \eta) f(\eta)\, d\eta$. We have*

$$\|Tf\|_{L^2(w)} \leq A \|\phi\|^{\frac{1}{2}}_{L^1} \|f\|_{L^2(v)}$$

where $A = \min(A_1, A_2)$ and A_1 and A_2 are given by

$$A_1^2 = \sup_{\xi \in \mathbb{R}^d} \int |\phi(\xi - \eta)| \frac{w(\xi)}{v(\eta)}\, d\eta$$

$$A_2^2 = \sup_{\eta \in \mathbb{R}^d} \int |\phi(\xi - \eta) \frac{w(\xi)}{v(\eta)}\, d\xi \tag{2.3}$$

Proof. We first note that the boundedness of $T : L^2(w) \to L^2(v)$ is equivalent to showing the boundedness of $S : L^2 \to L^2$ where S is the map defined by

$$Sf(\xi) = \int w(\xi)^{\frac{1}{2}} \phi(\xi - \eta) v(\eta)^{\frac{-1}{2}} f(\eta)\, d\eta$$

By Cauchy-Schwarz,

$$|Sf(\xi)^2| \leq \int |f(\eta)|^2\, d\eta \int |\phi(\xi - \eta)|^2 \frac{w(\xi)}{v(\eta)}\, d\eta$$

And by Tonelli's theorem, $\|Sf\|_2^2 \leq \|f\|_2^2 \int\int |\phi(\xi-\eta)|^2 \frac{w(\xi)}{v(\eta)}\, d\eta d\xi \leq A_1^2 \|f\|_2^2 \|\phi\|_1$. A similar application of Tonelli's theorem gives us $\|Sf\|_2^2 \leq A_2^2 \|f\|_2^2 \|\phi\|_1$. The lemma is thus proved. ∎

Let us now introduce operators H and L which correspond to the low and high frequency parts for an $L^2(\mathbb{R}^d)$ function.

Let ϕ be a smooth function which is 1 for $\xi < 1$, 0 for $|\xi| > 2$. Given ζ we define $\hat{Lf}(\xi) = \phi(\frac{\xi}{16|\zeta|})\hat{f}(\xi)$ and $\hat{Hf}(\xi) = \hat{f}(\xi) - \hat{Lf}(\xi)$. Unwinding the definition of L shows that

$$\|\frac{\partial^\alpha}{\partial x^\alpha} Lf\|_2 \leq C|\zeta|^{|\alpha|}\|f\|_2 \tag{2.4}$$

The next lemma shows that functions in the space $\dot{X}_\zeta^{\frac{1}{2}}$ are locally in L^2.

Lemma 2.2. *Let ψ be a Schwartz function and suppose $\zeta \cdot \zeta = 0$, with $|\zeta| > 1$, then we have*

$$\|\psi u\|_{\dot{X}_\zeta^{\frac{-1}{2}}} \leq C\|u\|_{X_\zeta^{\frac{-1}{2}}} \tag{2.5}$$

$$\|\psi u\|_{X_\zeta^{\frac{1}{2}}} \leq C\|u\|_{\dot{X}_\zeta^{\frac{1}{2}}} \tag{2.6}$$

$$\|\psi u\|_2 \leq C|\zeta|^{\frac{-1}{2}}\|u\|_{\dot{X}_\zeta^{\frac{1}{2}}} \tag{2.7}$$

$$\|\nabla H(\psi u)\|_2 \leq C\|u\|_{\dot{X}_\zeta^{\frac{1}{2}}} \tag{2.8}$$

$$\|H(\psi u)\|_2 \leq |\zeta|^{-1}\|u\|_{\dot{X}_\zeta^{\frac{1}{2}}} \tag{2.9}$$

Proof. If we take the Fourier transform of $u\psi$, we obtain $\hat{u} * \hat{\psi}$. Thus $\|\psi u\|_{\dot{X}_\zeta^{\frac{-1}{2}}} = \|T(\hat{u})\|_{L^2(w)}$ with $w(\xi) = |p_\zeta(\xi)|^{-1}$. We can use Lemma 2.1 to get (2.5) with $v(\eta) = \frac{1}{|p_\zeta(\eta)|+|\zeta|}$ if we can show that

$$\int_{\mathbb{R}^d} |\hat{\psi}(\xi-\eta)| \frac{|p_\zeta(\eta)| + |\zeta|}{|p_\zeta(\xi)|}\, d\xi \leq C \text{ for some real } C. \tag{2.10}$$

To establish (2.10), we write $p_\zeta(\eta) = -|\eta-\xi+\xi|^2 + 2i(\eta-\xi+\xi)\cdot\zeta = -|\eta-\xi|^2 - 2\xi\cdot(\eta-\xi) + 2i\zeta\cdot(\eta-\xi) + p_\zeta(\xi)$. Hence LHS of (2.10) is controlled by

$$\int_{\mathbb{R}^d} |\hat{\psi}(\xi-\eta)| \left(\frac{|\zeta| + |\xi-\eta|^2 + |\zeta||\xi-\eta|}{|p_{\zeta(\xi)|}} + \frac{|\xi||\xi-\eta|}{|p_\zeta(\xi)|} + 1 \right) d\xi \tag{2.11}$$

Let us now fix η and first prove the estimate

$$\int_{\mathbb{R}^d} \frac{|\hat{\psi}(\xi-\eta)||\zeta|}{|p_\zeta(\xi)|}\, d\xi \leq C \tag{2.12}$$

To establish (2.12), we let $B_k = \xi : |\xi - \eta| < 2^k$ and let $A_k = B_k \setminus B_{k-1}$. Using the Lemma in Appendix A and the fact that ψ is in Schwartz class (so that it decays to any desired order in $|\xi - \eta|$), we obtain for any N

$$\int_{\mathbb{R}^d} \frac{|\hat{\psi}(\xi-\eta)||\zeta|}{|p_\zeta(\xi)|}\,d\xi \le \int_{B_0} \frac{|\hat{\psi}(\xi-\eta)||\zeta|}{|p_\zeta(\xi)|}\,d\xi + \sum \int_{A_k} \frac{|\hat{\psi}(\xi-\eta)||\zeta|}{|p_\zeta(\xi)|}\,d\xi \le C \sum 2^{k(d-1-N)} \tag{2.13}$$

This will be finite if we choose N large enough, for instance $N \ge d$ will do. The other terms in (2.11) are handled in the same manner using the Lemma in Appendix A. (2.5) is hence established.

We now prove the remaining estimates. Estimate (2.6) is simply the adjoint of (2.5). The adjoint of the map $Tu = \psi u$ is $T^*u = \bar{\psi} u$. By (2.5) since we know that $T^* : X_\zeta^{\frac{-1}{2}} \to \dot{X}_\zeta^{\frac{-1}{2}}$, it follows that $T : \dot{X}_\zeta^{\frac{-1}{2}} \to X_\zeta^{\frac{-1}{2}}$.

Estimate (2.7) follows from (2.6) since

$$|\zeta|^{\frac{1}{2}} \le (|\zeta| + |p_\zeta(\xi)|)^{\frac{1}{2}}.$$

(Note that (2.7) shows that $\dot{X}_\zeta^{\frac{1}{2}} \subset L^2_{loc}$ for any fixed non- zero ζ.)

Estimates (2.8) and (2.9) now follow easily if we recall the definition of H and observe that $|p_\zeta(\xi)| \ge \frac{1}{2}|\xi|^2$ if $|\xi| > 4|\zeta|$. ■

2.2. **Estimates on q with negative regularity.** Given a conductivity γ which is continuously differentiable and one near infinity, we may define a map $m_q : S(\mathbb{R}^d) \to S'(\mathbb{R}^d)$ by

$$(m_q(u), v) = -\int_{\mathbb{R}^d} \nabla\sqrt{\gamma} \cdot \nabla \frac{uv}{\sqrt{\gamma}}\,dx.$$

(Recall that $q = \frac{\Delta\sqrt{\gamma}}{\sqrt{\gamma}}$.) When γ has two derivatives, it is easy to see by integration by parts that $m_q(u)$ is the function qu. By density, it is clear that the expression for $m_q(u)(v)$ can be extended to functions u and v which lie in the Sobolev space H^1_{loc}.

The next step is to study the expression $m_q(u)(v)$ when u and v lie in $\dot{X}_\zeta^{\frac{1}{2}}$. Since we have assumed that γ is one near infinity, it follows that q is a compactly supported distribution. Thus if ψ is a smooth function which is one on neighborhood of support of q, we have $m_q(u)(v) = m_q(\psi u)(\psi v)$. This observation allows us to apply estimates of Lemma 2.2. (The hypothesis that $\gamma = 1$ near infinity is justified by Theorem 0.7 in [7].)

The key step in Haberman and Tataru's proof is the following theorem which proves that $\|m_q\|_{\dot{X}_\zeta^{\frac{1}{2}} \to \dot{X}_\zeta^{\frac{-1}{2}}}$ is small. (We use without proof the simple fact that $\dot{X}_\zeta^{\frac{-1}{2}}$ is isomorphic to dual of $\dot{X}_\zeta^{\frac{1}{2}}$.)

Theorem 2.3. *Let $\zeta_j \in \mathbb{C}^d$ be such that $\zeta_j \cdot \zeta_j = 0$ for $j = 1,2$ and $|\zeta_1| = |\zeta_2|$ and let $\theta \in [0,1]$. If $\gamma \in W^{1+\theta,\infty}(\mathbb{R}^d), \nabla\gamma$ is compactly supported, and $\gamma \geq c$ for some positive c then we have*

$$|m_q(u)(v)| \leq C\omega(\|\log\gamma\|_{W^{1+\theta,\infty}(\mathbb{R}^d)})|\zeta_1|^{-\theta}\|u\|_{\dot{X}^{\frac{1}{2}}_{\zeta_1}}\|v\|_{\dot{X}^{\frac{1}{2}}_{\zeta_2}}.$$

where the function ω satisfies $\lim_{t\to 0^+}\omega(t) = 0$. In addition if $\theta = 0$, we have

$$|m_q(u)(v)| \leq C\omega(\gamma,|\zeta|)\|u\|_{\dot{X}^{\frac{1}{2}}_{\zeta_1}}\|v\|_{\dot{X}^{\frac{1}{2}}_{\zeta_2}}.$$

where $\lim_{s\to\infty}\omega(\gamma,s) \leq \lim_{\epsilon\to 0^+}\|\log\gamma - (\log\gamma)_\epsilon\|_{W^{1,\infty}(\mathbb{R}^d)}$.

We first prove an auxillary lemma to prove Theorem 2.3.

Lemma 2.4. *Let ζ_1 and ζ_2 be as defined in Theorem 2.3 Let f be a vector valued function in $W^{\theta,\infty}(\mathbb{R}^d)$. Let ψ be a Schwartz function and put $u_\psi = u\psi$. Then we have*

$$\int f\cdot\nabla(u_\psi v_\psi) \leq C\|f\|_{L^\infty(\mathbb{R}^d)}\|u\|_{\dot{X}^{\frac{1}{2}}_{\zeta_1}}\|v\|_{\dot{X}^{\frac{1}{2}}_{\zeta_2}} \tag{2.14}$$

$$\int f\cdot\nabla(u_\psi v_\psi) \leq Cs^{-1}\|\nabla f\|_{L^\infty(\mathbb{R}^d)}\|u\|_{\dot{X}^{\frac{1}{2}}_{\zeta_1}}\|v\|_{\dot{X}^{\frac{1}{2}}_{\zeta_2}} \tag{2.15}$$

$$\int f\cdot\nabla(u_\psi v_\psi) \leq Cs^{-\theta}\|f\|_{W^{\theta,\infty}(\mathbb{R}^d)}\|u\|_{\dot{X}^{\frac{1}{2}}_{\zeta_1}}\|v\|_{\dot{X}^{\frac{1}{2}}_{\zeta_2}} \tag{2.16}$$

The constant C depends on d, θ and ψ and $\sqrt{2}s = |\zeta_j|$.

Proof of Lemma 2.4. To prove (2.15) we integrate by parts and use estimate (2.7).

To prove (2.14), we write $u_\psi = Hu_\psi + Lu_\psi$ and $v_\psi = Hv_\psi + Lv_\psi$. We have

$$\int_{\mathbb{R}^d} f\cdot\nabla(u_\psi v_\psi)\,dy = \int f\cdot\nabla(Hu_\psi Hv_\psi + Hu_\psi Lv_\psi + Lu_\psi Hv_\psi + Lu_\psi Lv_\psi)\,dy$$

We have four terms to consider. For the high- high term we use Leibnitz rule , Cauchy Schwartz inequality and estimates (2.8) and (2.9) to obtain

$$|\int_{\mathbb{R}^d} f\cdot\nabla(Hu_\psi Hv_\psi)\,dy| \leq Cs^{-1}\|f\|_{L^\infty(\mathbb{R}^d)}\|u\|_{\dot{X}^{\frac{1}{2}}_{\zeta_1}}\|v\|_{\dot{X}^{\frac{1}{2}}_{\zeta_2}}$$

To estimate high- low term involving $Hu_\psi Lv_\psi$, we use Leibnitz rule and Cauchy Schwartz inequality to obtain

$$|\int_{\mathbb{R}^d} f\cdot Lv_\psi\nabla(Hu_\psi)\,dy| \leq C\|f\|_{L^\infty(\mathbb{R}^d)}\|\nabla(Hu_\psi)\|_{L^2(\mathbb{R}^d)}\|Lv_\psi\|_{L^2(\mathbb{R}^d)}$$

We now use the estimates (2.4) and (2.8) to get

$$C\|f\|_{L^\infty(\mathbb{R}^d)}\|\nabla(Hu_\psi)\|_{L^2(\mathbb{R}^d)}\|Lv_\psi\|_{L^2(\mathbb{R}^d)} \leq C\|f\|_{L^\infty(\mathbb{R}^d)}\|u\|_{\dot{X}^{\frac{1}{2}}_{\zeta_1}}\|v_\psi\|_{L^2(\mathbb{R}^d)}$$

Using the trivial bound $\|v_\psi\|_{L^2(\mathbb{R}^d)} \le C\|v_\psi\|_{X^{\frac{1}{2}}_{\zeta_2}}$ and (2.6) gives us

$$\|u\|_{\dot{X}^{\frac{1}{2}}_{\zeta_1}} \|v_\psi\|_{L^2(\mathbb{R}^d)} \le \|u\|_{\dot{X}^{\frac{1}{2}}_{\zeta_1}} \|v\|_{\dot{X}^{\frac{1}{2}}_{\zeta_2}}.$$

Thus we obtain

$$|\int_{\mathbb{R}^d} f \cdot Lv_\psi \nabla(Hu_\psi)\, dy| \le C\|f\|_{L^\infty(\mathbb{R}^d)} \|u\|_{\dot{X}^{\frac{1}{2}}_{\zeta_1}} \|v\|_{\dot{X}^{\frac{1}{2}}_{\zeta_2}}$$

To estimate the other term obtained by Leibnitz rule, $\int_{\mathbb{R}^d} f \cdot Hu_\psi \nabla(Lv_\psi)\, dy$ we make use of (2.4), (2.6) and (2.9) to get

$$|\int_{\mathbb{R}^d} f \cdot Hu_\psi \nabla(Lv_\psi)\, dy \le C\|f\|_{L^\infty(\mathbb{R}^d)} \|u\|_{\dot{X}^{\frac{1}{2}}_{\zeta_1}} \|v\|_{\dot{X}^{\frac{1}{2}}_{\zeta_2}}$$

The mixed term involving $Lu_\psi Hv_\psi$ can be handled by interchanging u and v.

Finally for the low-low term, we use Leibnitz rule to obtain

$$\|\nabla(Lu_\psi Lv_\psi\|_{L^1(\mathbb{R}^d)} \le C|\zeta| \|Lu_\psi\|_{L^2(\mathbb{R}^d)} \|Lv_\psi\|_{L^2(\mathbb{R}^d)}$$

Now we use estimate (2.4) to get

$$\|\nabla(Lu_\psi Lv_\psi\|_{L^1(\mathbb{R}^d)} \le C|\zeta| \|Lu_\psi\|_{L^2(\mathbb{R}^d)} \|Lv_\psi\|_{L^2(\mathbb{R}^d)} \le C\|u\|_{\dot{X}^{\frac{1}{2}}_{\zeta_1}} \|v\|_{\dot{X}^{\frac{1}{2}}_{\zeta_2}}$$

Estimate (2.14) is thus proved.

We now prove the estimate (2.16). We will use an interpolation argument. We let ϕ_ϵ be a standard mollifier and set $f_\epsilon = \phi_\epsilon * f$. We observe that for $0 \le \theta \le 1$, we have $\|f - f_\epsilon\|_{L^\infty(\mathbb{R}^d)} \le C\epsilon^\theta \|f\|_{W^{\theta,\infty}(\mathbb{R}^d)}$ and $\|\nabla f_\epsilon\| \le C\epsilon^{\theta-1}\|f\|_{W^{\theta,\infty}(\mathbb{R}^d)}$. Thus, using estimates (2.14) and (2.15), we have

$$\begin{aligned} |\int f \cdot \nabla(u_\psi v_\psi)\, dy| &\le |\int (f - f_\epsilon) \cdot \nabla(u_\psi v_\psi)\, dy| + |\int f_\epsilon \cdot \nabla(u_\psi v_\psi)\, dy| \\ &\le C(\epsilon^\theta + |\zeta|^{-1}\epsilon^{-\theta-1}) \|f\|_{W^{\theta,\infty}(\mathbb{R}^d)} \|u\|_{\dot{X}^{\frac{1}{2}}_{\zeta_1}} \|v\|_{\dot{X}^{\frac{1}{2}}_{\zeta_2}}. \end{aligned}$$

Choosing $\epsilon = \frac{1}{s}$ gives us (2.16). ■

We now use this lemma to prove the estimates of Theorem 2.3.

Proof of Theorem 2.3. Choose a function ψ which is smooth and compactly supported on $\mathbb{R}^d$ with $\psi = 1$ in a neighborhood of support of $\nabla\gamma$. With u_ψ, we have by Leibnitz rule

$$m_q(u)(v) = -\int_{\mathbb{R}^d} \frac{\nabla\sqrt{\gamma}}{\sqrt{\gamma}} \cdot \nabla(u_\psi v_\psi) - \frac{|\nabla\sqrt{\gamma}|^2}{\gamma} u_\psi v_\psi\, dy$$

From the estimate (2.7), it is easy to obtain

$$|\int \frac{|\nabla\sqrt{\gamma}|^2}{\gamma} u_\psi v_\psi\, dy| \le C|\zeta|^{-1} \|\log\sqrt{\gamma}\|^2_{W^{1,\infty}(\mathbb{R}^d)} \|u\|_{\dot{X}^{\frac{1}{2}}_{\zeta_1}} \|v\|_{\dot{X}^{\frac{1}{2}}_{\zeta_2}}.$$

which is stronger than the desired estimate. For the other term in $m_q(u)(v)$, we use the estimate (2.16) the estimate

$$|\int_{\mathbb{R}^d} \nabla \log\sqrt{\gamma}\cdot\nabla(u_\psi v_\psi)\,dy| \leq C|\zeta|^{-\theta}\|\log\sqrt{\gamma}\|_{W^{1+\theta,\infty}(\mathbb{R}^d)}\|u\|_{\dot{X}^{\frac{1}{2}}_{\zeta_1}}\|v\|_{\dot{X}^{\frac{1}{2}}_{\zeta_2}}.$$

We thereby obtain the first estimate in Theorem 2.3.

Finally when $\theta = 0$, so that $\log\sqrt{\gamma}$ is only Lipschitz, we let $(\log\sqrt{\gamma})_\epsilon$ denote a standard mollification of $\log\sqrt{\gamma}$. We add and subtact the smoothed version and use integration by parts in the second term and the fact that $\gamma = 1$ near infinity to get

$$|\int_{\mathbb{R}^d} \nabla \log\sqrt{\gamma}\cdot(u_\psi v_\psi\,dy| \leq |\int_{\mathbb{R}^d} \nabla(\log\sqrt{\gamma}-(\log\sqrt{\gamma})_\epsilon)\cdot\nabla(u_\psi v_\psi)\,dy|$$

$$+|\int_{\mathbb{R}^d} \Delta(\log\sqrt{\gamma})u_\psi v_\psi\,dy|$$

Using the estimate (2.15), Cauchy Schwartz inequality and estimate (2.7) we obtain

$$|\int_{\mathbb{R}^d} \nabla(\log\sqrt{\gamma}-(\log\sqrt{\gamma})_\epsilon)\cdot\nabla(u_\psi v_\psi)\,dy| + |\int_{\mathbb{R}^d} \Delta(\log\sqrt{\gamma})u_\psi v_\psi\,dy|$$

$$\leq C[\|\nabla(\log\sqrt{\gamma}-(\log\sqrt{\gamma})_\epsilon)\|_{L^\infty(\mathbb{R}^d)} + \epsilon^{-1}|\zeta|^{-1}\|\Delta(\log\sqrt{\gamma})_\epsilon\|_{L^\infty(\mathbb{R}^d)}]\|u\|_{\dot{X}^{\frac{1}{2}}_{\zeta_1}}\|v\|_{\dot{X}^{\frac{1}{2}}_{\zeta_2}}.$$

We choose $\epsilon = |\zeta|^{\frac{-1}{2}}$ to obtain the second estimate of Theorem 2.3 ∎

2.3. **An averaged estimate.** The second innovation in Haberman and Tataru's paper is the use of an *averaging* argument used to show that $\|q\|_{\dot{X}^{\frac{-1}{2}}_{\zeta}} \to 0$ as $|\zeta| \to \infty$. We describe this method of averaged estimates below.

Need for averaged estimates

We wish to construct solutions of the Schrödinger equation in the form $e^{x\cdot\zeta}(1+\psi)$ where $\zeta\cdot\zeta = 0$. The function ψ will satisfy $\psi - G_\zeta(q\psi) = G_\zeta(q)$ with $G_\zeta = (\Delta + 2\zeta\cdot\nabla)^{-1}$ and $q\psi$ is interpreted as $m_q(\psi)$. Unwinding the definition of the norms, it is clear that $G_\zeta : \dot{X}^{\frac{-1}{2}}_{\zeta} \to \dot{X}^{\frac{1}{2}}_{\zeta}$ and Theorem 2.3 tells us that $\|m_q\|_{\dot{X}^{\frac{1}{2}}_{\zeta}\to\dot{X}^{\frac{-1}{2}}_{\zeta}} \leq 1$.

If $\gamma \in C^1$ we can prove thus using the first estimate in Theorem 2.3 and if $\gamma \in W^{1,\infty}$ we can prove this using the second estimate of Theorem 2.3 assuming that $\|\nabla\log\gamma\|_{L^\infty(\mathbb{R}^d)}$ is small.

It remains to study the right hand side $G_\zeta(q)$. In order to obtain control of the solutions ψ to the equation $(\Delta_\zeta - m_q)\psi = q$ in $\dot{X}_{\zeta^{\frac{1}{2}}}$ we need to estimate $\|q\|_{\dot{X}_{\zeta^{\frac{-1}{2}}}}$. Note that we need to make $\|q\|_{\dot{X}^{\frac{-1}{2}}_{\zeta}}$ small to be able to use fixed point theorem. The worst part of q looks like $\Delta\log(\sqrt{\gamma})$ which leads us to bound expressions of the form $\|\nabla h\|_{\dot{X}^{\frac{-1}{2}}_{\zeta}}$ where h is a continuous function with compact support. Now, since h is compactly supported, by (2.5) the $\dot{X}_\zeta^{\frac{-1}{2}}$ norm is bounded above by $X_\zeta^{\frac{-1}{2}}$ norm.

To compute the $X_\zeta^{\frac{-1}{2}}$ norm, we again split h using the projection operators H and L previously defined and note that at large $|\xi|$, $|p_\zeta(\xi)|^{-1} \approx |\xi|^{-2}$, so $\|H\nabla h\|_{X_\zeta^{\frac{-1}{2}}} \leq \|h\|_2$. The problem occurs at low frequencies where all we can say is $\|L\nabla h\|_{X_\zeta^{\frac{-1}{2}}} \lesssim s^{\frac{-1}{2}}\|L\nabla h\|_2 \lesssim s^{\frac{1}{2}}\|h\|_2$ where the last inequality follows by finite band property.

Unfortunately, for large $|s|$, the factor $s^{\frac{1}{2}}$ will dominate the estimate

$$\|m_q\|_{\dot{X}_\zeta^{\frac{1}{2}} \to \dot{X}_\zeta^{\frac{-1}{2}}} \lesssim s^{-\theta}$$

(which we obtained in Theorem 2.3) if $\theta < \frac{1}{2}$. Thus if $\theta < \frac{1}{2}$ we can no longer ensure that for large $|s|$, $\|m_q\|_{\dot{X}_\zeta^{\frac{1}{2}} \to \dot{X}_\zeta^{\frac{-1}{2}}}$ will be small. This is why we need averaged estimates.

2.3.1. **Construction of averaged estimates**. Let us introduce some notation. Fix a vector $k \in \mathbb{R}^d$ and let P be any 2 plane through the origin orthogonal to k (P may be arbitrary if $k = 0$.) Let e_1 and e_2 be an orthonormal basis for P. We identify $P \cap S^{d-1}$ with S^1 so that $e_i \in S^1$ for $i = 1, 2$.

The idea now is that by averaging over all possible choices of $e_1 \in S^1$ and $|\zeta|$ in a dyadic region $[\lambda, 2\lambda]$, we will get better estimates for the Schrödinger potential in $\dot{X}_{\zeta_i}^{\frac{-1}{2}}$.

Let us define

$$e_1(\theta) = e_1\cos(\theta) - e_2\sin(\theta)$$

$$e_1(\theta) = e_1\sin(\theta) + e_2\cos(\theta)$$

For $s > |k|$, we let

$$\zeta_1(s,\theta) = se_1(\theta) + i\left(\frac{k}{2} + \sqrt{s^2 - \frac{|k|^2}{4}}\right)e_2(\theta)$$

$$\zeta_2(s,\theta) = -se_1(\theta) + i\left(\frac{k}{2} - \sqrt{s^2 - \frac{|k|^2}{4}}\right)e_2(\theta)$$

Lemma 2.5. *Let $\lambda > |k|$ and fix ϕ a Schwartz function in $\mathbb{R}^d$. If $f \in H^\theta(\mathbb{R}^d)$, $0 \leq \theta \leq 1$, then for $j = 1$ or 2 we have*

$$\frac{1}{\lambda}\int_0^{2\pi}\int_\lambda^{2\lambda} \|\phi\nabla f\|^2_{\dot{X}_{\zeta_j}^{\frac{-1}{2}}}\, ds\, d\theta \leq C(\phi,k)\lambda^{-\theta}\|f\|^2_{H^\theta} \tag{2.17}$$

If $f \in L^2(\mathbb{R}^d)$, then

$$\frac{1}{\lambda}\int_0^{2\pi}\int_\lambda^{2\lambda} \|\phi\nabla f\|^2_{\dot{X}_{\zeta_j}^{\frac{-1}{2}}}\, ds\, d\theta = o(1) \qquad as\ \lambda \to \infty$$

Proof of Lemma 2.5. We let $\zeta_1 = \zeta$ and give the proof in detail for ζ. The proof for ζ_2 can be carried by replacing the basis (e_1, e_2) by $(-e_1, -e_2)$. We prove the estimate (2.17) by interpolating the estimates

$$\int_0^{2\pi}\int_\lambda^{2\lambda} \|\phi\nabla f\|^2_{\dot{X}_{\zeta}^{\frac{-1}{2}}}\, ds\, d\theta \le C\|\nabla f\|^2_{L^2} \tag{2.18}$$

$$\frac{1}{\lambda}\int_0^{2\pi}\int_\lambda^{2\lambda} \|\phi\nabla f\|^2_{\dot{X}_{\zeta}^{\frac{-1}{2}}}\, ds\, d\theta \le C\|f\|^2_{L^2} \tag{2.19}$$

To establish (2.18), we use (2.3) and the elementary estimate $\|\nabla f\|^2_{X_\zeta^{\frac{-1}{2}}} \le |\zeta|^{-1}\|\nabla f\|^2_{L^2}$ and then integrate in s and θ.

The estimate (2.19) is more difficult because it is not proved by integrating a pointwise estimate. To establish (2.19), we first use (2.5) and replace the homogeneous space $\dot{X}_\zeta^{\frac{-1}{2}}$ by the inhomogeneous space $X_\zeta^{\frac{-1}{2}}$, then we use Plancharel's theorem and divide the integral on $\mathbb{R}^d$ in to 3 regions,

$$\begin{aligned}
\frac{1}{\lambda}\int_0^{2\pi}\int_\lambda^{2\lambda} \|\phi\nabla f\|^2_{\dot{X}_{\zeta}^{\frac{-1}{2}}}\, ds\, d\theta &\le C\frac{1}{\lambda}\int_0^{2\pi}\int_\lambda^{2\lambda} \|\nabla f\|^2_{X_{\zeta}^{\frac{-1}{2}}}\, ds\, d\theta \\
&\le \frac{C}{\lambda}\int_0^{2\pi}\int_\lambda^{2\lambda}\int_{|\xi|<16|k|} \frac{|\xi|^2}{|p_\zeta(\xi)+|\zeta|}|\hat{f}(\xi)|^2\, d\xi\, ds\, d\theta \\
&+ \frac{C}{\lambda}\int_0^{2\pi}\int_\lambda^{2\lambda}\int_{|\xi|^2<16\lambda|\xi^\perp|} \frac{|\xi|^2}{|p_\zeta(\xi)+|\zeta|}|\hat{f}(\xi)|^2\, d\xi\, ds\, d\theta \\
&+ \frac{C}{\lambda}\int_0^{2\pi}\int_\lambda^{2\lambda}\int_{|\xi|^2\ge 16\lambda|\xi^\perp|, |\xi|\ge 16|k|} \frac{|\xi|^2}{|p_\zeta(\xi)+|\zeta|}|\hat{f}(\xi)|^2\, d\xi\, ds\, d\theta. \\
&= (I) + (II) + (III)
\end{aligned} \tag{2.20}$$

We use $|\xi^\perp|$ to denote the projection of ξ onto the plane P. Note that $|\xi^\perp|^2 = |e_1(\theta\cdot\xi|^2 + |e_2(\theta)\cdot\xi|^2$.

In the region $|\xi| < 16|k|$, we have

$$I \leq C\frac{|k|^2}{|\zeta|}\|f\|_{L^2}^2$$

which is sufficient since our constants are allowed to depend on k.

(II) is harder to bound. We change variables in (s,θ) integral by setting

$t_1 = \mathrm{Re}\ p_\zeta(\xi) = -|\xi|^2 - k\cdot\xi - \sqrt{s^2 - \frac{|k|^2}{4}}e_2(\theta)\cdot\xi$ and $t_2 = \mathrm{Im}\ p_\zeta(\xi) = 2se_1(\theta)\cdot\xi$.

The Jacobian of the map $(s,\theta)\to(t_1,t_2)$ is given by $J = \begin{vmatrix} \frac{2s}{\sqrt{s^2-\frac{|k|^2}{4}}}e_2(\theta)\cdot\xi & 2e_1(\theta)\cdot\xi \\ 2\sqrt{s^2-\frac{|k|^2}{4}}e_1(\theta)\cdot\xi & -2se_2(\theta) \end{vmatrix}$

$= 4\sqrt{s^2-\frac{|k|^2}{4}}|e_1(\theta)\cdot\xi|^2 + \frac{4s^2}{\sqrt{s^2-\frac{|k|^2}{4}}}|e_2(\theta)\cdot|\xi|^2.$

Under the assumption that $s \geq \lambda \geq |k|$, we have $2s\sqrt{3}|\xi^\perp|^2 \leq J \leq (\frac{8s}{\sqrt{3}})|\xi^\perp|^2$. We also have that the image of $(\lambda, 2\lambda)\times(0,2\pi)$ under the map $(s,\theta)\to(t_1,t_2)$ lies in a disk of radius $4\lambda|\xi^\perp|$. We thus have

$$(II) \leq C\int_{|\xi|^2<16\lambda|\xi^\perp|}|\hat{f}(\xi)|^2\frac{|\xi|^2}{\lambda^2|\xi^\perp|^2}\int_{|t|<4\lambda|\xi^\perp|}\frac{1}{|t|}\,dt\,d\xi \leq C\|f\|_{L^2(\mathbb{R}^d)}^2$$

Finally, when $|\xi| > 16|k|$ and $|\xi|^2 > 16\lambda|\xi^\perp|$ we have $|p_\zeta(\xi)| \geq \frac{1}{2}|\xi|^2 + |\xi|(\frac{1}{4}|\xi| - |k|) + (\frac{1}{4}|\xi|^2 - 4\lambda|\xi|^\perp \geq \frac{1}{2}|\xi|^2$ and thus it is easy to see that $(III) \leq C\|f\|_{L^2(\mathbb{R}^d)}^2$. This proves (2.19). As mentioned before, estimate (2.17) follows by a standard interpolation between estimates (2.18) and (2.19).

To prove the second estimate in Lemma 2.5, we write $f = (f - f_\epsilon) + f_\epsilon$ where f_ϵ is a standard mollification of f. Estimates (2.18) and (2.19) then imply

$$\frac{1}{\lambda}\int_0^{2\pi}\int_\lambda^{2\lambda}\|\phi\nabla f\|_{\dot{X}_\zeta^{\frac{-1}{2}}}^2\,ds\,d\theta \leq C(\lambda^{-1}\|\nabla f_\epsilon\|_{L^2(\mathbb{R}^d)}^2 + \|f - f_\epsilon\|_{L^2(\mathbb{R}^d)}^2).$$

We let $\epsilon = \lambda^{\frac{-1}{2}}$ to see that the right hand side above goes to zero. ∎

Theorem 2.6. *Let γ satisfy $a \leq \gamma(x) \leq a^{-1}$ for some positive constant a, suppose that $\gamma = 1$ outside some compact set and $\nabla\gamma \in L^2(\mathbb{R}^d)$. Let k and $\zeta_j(s,\theta)$ be as in Lemma 2.5. Then we have*

$$\lim_{\lambda\to\infty}\frac{1}{\lambda}\int_0^{2\pi}\int_{\eta_1\in S^1}\|\frac{\Delta\sqrt{\gamma}}{\sqrt{\gamma}}\|_{\dot{X}_{\zeta_j}^{\frac{-1}{2}}}\,ds\,d\theta = 0$$

Proof of Theorem 2.6. We may write $\frac{\Delta\sqrt{\gamma}}{\sqrt{\gamma}}$ as $\nabla\cdot\nabla\log\sqrt{\gamma} + |\nabla\log\sqrt{\gamma}|^2$. We may estimate the average of $\dot{X}_{\zeta_j}^{\frac{-1}{2}}$ norm for each of these terms by Lemma 2.5 with $f = |\nabla\log\gamma|$ for the average of $\dot{X}_{\zeta_j}^{\frac{-1}{2}}$ norm for the first term $\nabla\cdot\nabla\log\sqrt{\gamma}$ and $f = \log\sqrt{\gamma}$ for the average of $\dot{X}_{\zeta_j}^{\frac{-1}{2}}$ norm for second term $|\nabla\log\sqrt{\gamma}|^2$. ∎

Finally we use the above results to show, in the next section, that it is possible to find a sequence $|\zeta^j| \to \infty$ so that

$$\lim_{j\to\infty} \|q\|_{\dot{X}^{\frac{-1}{2}}_{\zeta^j}} = 0 \tag{2.21}$$

3. Construction of CGO solutions

We use the averaged estimates proved in the previous section to globally solve the equation $\Delta_\zeta \psi = q(1+\psi)$, which can be re-written as

$$\psi = \Delta_\zeta^{-1} q\psi + \Delta_\zeta^{-1} q. \tag{3.1}$$

To solve equation (3.1), we make use of the following lemma.

Lemma 3.1. *Let $d \geq 3$, $k \in \mathbb{R}^d$, and for $i = 1,2$ suppose γ_i is strictly positive, bounded above and below with $\nabla\gamma \in L^2(\mathbb{R}^d)$ and $\gamma_1 = \gamma_2 = 1$ outside some compact set. Then we may find sequences ζ_i^j, $i = 1,2$ so that $\zeta_1^j + \zeta_2^j = ik$ and*

$$\lim_{j\to\infty} \|q_i\|_{\dot{X}^{\frac{-1}{2}}_{\zeta_i^j}} = 0.$$

Proof of lemma 3.1. By Theorem 2.6, we have

$$\frac{1}{\lambda}\int_0^{2\pi}\int_\lambda^{2\lambda} \left\|\frac{\Delta\sqrt{\gamma_i}}{\sqrt{\gamma_i}}\right\|_{\dot{X}^{\frac{-1}{2}}_{\zeta_j}} ds\, d\theta \to 0 \quad \text{as} \quad \lambda \to \infty$$

Thus there exist $\zeta = \zeta_j$ with $|\zeta_j| \to \infty$ such that $\sum_{i,j} \|q_i\|_{\dot{X}^{\frac{-1}{2}}_{\zeta_j}} \to 0$. Since $\zeta_i^j + \zeta_2^j = ik$ for all j, we can see that the conclusion of Lemma 3.1 holds. ∎

We now use Lemma 3.1 to show that solution ψ to $\psi = \Delta_\zeta^{-1} q\psi + \Delta_\zeta^{-1} q$ is small in $\dot{X}^{\frac{1}{2}}_{\zeta_j}$ for large j.

Theorem 3.2. *Let $\gamma_i \in C(\mathbb{R}^d)$ or $W^{1,\infty}(\mathbb{R}^d)$, γ_i is bounded above and below by fixed positive constants and $\gamma_i = 1$ outside a compact set. Let ζ_i^j be as in lemma 3.0.3. Then we may find $\psi_i^j \in \dot{X}^{\frac{1}{2}}_{\zeta_i^j}$ so that $v_i = e^{x\cdot\zeta_i^j}(1+\psi_i^j)$ satisfies $(\Delta v_i - m_{qi})v_i = 0$ weakly and*

$$\lim_{j\to\infty} \|\psi_i^j\|_{\dot{X}^{\frac{1}{2}}_{\zeta_i^j}} = 0 \tag{3.2}$$

where $i = 1,2$.

Proof of Theorem 3.2. From the definition of $\dot{X}_{\zeta^b}$ norms, we have $\|\Delta_\zeta^{-1} q\|_{\dot{X}_{\zeta^{\frac{1}{2}}}} = \|q\|_{\dot{X}^{\frac{-1}{2}}_{\zeta}}$ and by Theorem 2.3 we have

$$\lim_{\zeta\to\infty} \|m_q\|_{L_{(\dot{X}^{\frac{1}{2}}_{\zeta}, \dot{X}^{\frac{-1}{2}}_{\zeta})}} = 0. \tag{3.3}$$

Thus we may find an R such that $\|\Delta_\zeta \circ m_q\| \leq \frac{1}{2}$ if $|\zeta| \geq R$. For such ζ, we have that $(I-\Delta_\zeta \circ m_q)^{-1}$ exists and is given by the series $\sum(\Delta_\zeta \circ m_q)^j$. This sum converges absolutely in the operator norm on $L(\dot{X}_\zeta^{\frac{1}{2}})$ and we have $\|(I-\Delta_\zeta \circ m_{q_i})^{-1}\|_{L(\dot{X}_\zeta^{\frac{1}{2}})} \leq 2$.

We let ζ_j^i be the sequence from Lemma 3.1. For j sufficiently large, we set

$$\psi_i^j = (I-\Delta_{\zeta_i^j} \circ m_{q_i})^{-1}(\Delta_{\zeta_i^j} q_i). \tag{3.4}$$

By the observations of the previous paragraph, $\lim_{j\to\infty} \|\psi_i^j\|_{\dot{X}^{\frac{1}{2}}_{\zeta_i^j}} \leq 2 \lim_{j\to\infty} \|q_i\|_{\dot{X}^{\frac{-1}{2}}_{\zeta_i^j}} = 0$. It is now easy to see that the corresponding v_i is a solution of $(\Delta - q_i)v_i = 0$, in sense of distributions. ■

Remark. *By (2.4) and (2.8) it follows that for $i=1$ and 2, $\nabla\psi_i^j$ and hence ∇v_i are in L^2_{loc}. By (2.7) ψ_i^j and thus v_i are in L^2_{loc}. Hence $v_i \in H^1_{loc}(\mathbb{R}^d)$*

For the proof of Theorem 1.3, we first reduce to the case where γ_i are globally defined and satisfy properties as required in Theorem 3.2. We can do this by Theorem 0.7 in [**7**]. After having the desired extension, we first prove the following lemma.

Lemma 3.3. *Let $\gamma_1, \gamma_2 \in W^{1,\infty}(\bar{\Omega})$ be Lipschitz or $C^1(\bar{\Omega})$ such that $\gamma_1 = \gamma_2$ outside Ω and $\gamma_1 = \gamma_2 = 1$ outside a compact set and assume that $\Lambda_{\gamma_1} = \Lambda_{\gamma_2}$. If $v_1 \in H^1_{loc}(\mathbb{R}^d)$, $i = 1,2$, are weak solutions of $(\Delta - q_i)v_1 = 0$ in $\mathbb{R}^d$, then we have*

$$(m_{q_1}(v_1), v_2) = (m_{q_2}(v_2), v_1)$$

Proof of Lemma 3.3. Unwinding the definition of m_q from the beginning of section 2.2, we want to prove

$$\int_{\mathbb{R}^d} \nabla\sqrt{\gamma_1}\cdot\nabla\frac{v_1v_2}{\sqrt{\gamma_1}}\,dx = \int_{\mathbb{R}^d} \nabla\sqrt{\gamma_2}\cdot\nabla\frac{v_1v_2}{\sqrt{\gamma_2}}\,dx$$

Let $\gamma \in W^{1,\infty}(\mathbb{R}^d)$ or $C^1(\mathbb{R}^d)$ with γ bounded above and below by fixed positive constants. Let q denote the potential $\frac{\Delta\sqrt{\gamma}}{\sqrt{\gamma}}$. We first observe that if v is a weak solution of $(\Delta - q)v = 0$, then $u = \frac{v}{\sqrt{\gamma}}$ is a solution of $\nabla\cdot\gamma\nabla u = 0$.

To see this, we let φ be a test function on $\mathbb{R}^d$ and let $v = \sqrt{\gamma}u$ and $\phi = \sqrt{\gamma}\varphi$ and use product rule to obtain

$$\begin{aligned}
&\int_{\mathbb{R}^d} \nabla(\sqrt{\gamma}u)\cdot\nabla(\sqrt{\gamma}\varphi) - \nabla(\sqrt{\gamma})\cdot\nabla(\sqrt{\gamma}u\varphi)\,dy \\
&= \int_{\mathbb{R}^d} \gamma\nabla u\cdot\nabla\varphi + \sqrt{\gamma}\varphi\nabla\sqrt{\gamma}\cdot\nabla u\,dy \\
&+ u\nabla\sqrt{\gamma}\cdot\nabla(\sqrt{\gamma}\varphi) - \nabla(\sqrt{\gamma})\cdot\nabla(\sqrt{\gamma}u\varphi)\,dy.
\end{aligned}$$

Since the map $\psi \to \psi\sqrt{\gamma}$ is invertible on $C^1(\mathbb{R}^d)$, we have

$$\int \nabla v \cdot \nabla\phi - \nabla\sqrt{\gamma}\cdot\nabla(\frac{v\phi}{\sqrt{\gamma}})\,dy = 0, \qquad \phi \in C_1^c(\mathbb{R}^d)$$

iff

$$\int_{\mathbb{R}^d} \gamma\nabla u\cdot\nabla\phi\,dy = 0 \qquad \phi \in C_1^c(\mathbb{R}^d)$$

Using this observation, let us now prove $(m_{q_1}(v_1), v_2) = (m_{q_2}(v_2), v_1)$.

Since $\gamma_1 = \gamma_2$ outside Ω by hypothesis, it is clear that

$$\int_{\mathbb{R}^d\setminus\bar{\Omega}} \nabla\sqrt{\gamma_1}\cdot\nabla(\frac{v_1v_2}{\sqrt{\gamma_1}})\,dy = \int_{\mathbb{R}^d\setminus\bar{\Omega}} \nabla\sqrt{\gamma_2}\cdot\nabla(\frac{v_1v_2}{\sqrt{\gamma_2}})\,dy. \tag{3.5}$$

We now obtain a corresponding result over Ω. We claim that

$$\Lambda_{\gamma_1}(u_1)(u_2) = \int_\Omega \nabla v_1\nabla v_2 - \nabla\sqrt{\gamma_1}\cdot\nabla(\frac{v_1v_2}{\sqrt{\gamma_1}})\,dy \tag{3.6}$$

Let us now prove (3.6). Put $\tilde{u_2} = \frac{v_2}{\gamma_1}$. Then by product rule we can easily see that

$$\int_\Omega \nabla v_1\cdot\nabla v_2 - \nabla\sqrt{\gamma_1}\cdot\nabla(\frac{v_1v_2}{\sqrt{\gamma_1}})\,dy = \int_\Omega \gamma_1\nabla u_1\cdot\nabla\tilde{u_2}\,dy. \tag{3.7}$$

Now we note that $u_2 - \tilde{u_2} = v_2(\frac{1}{\sqrt{\gamma_2}} - \frac{1}{\sqrt{\gamma_2}}) = Bv_2$ (Say).

Then by hypothesis of Lemma 3.3, we have that $B \in C^1(\bar{\Omega})$ and $B = 0$ on $tial\Omega$. By Sobolev space theory, $u_2 - \tilde{u_2} \in H_0^1(\Omega)$. Since u_1 weakly solves div $\gamma_1\nabla u_1 = 0$, we have

$$\int_\Omega \gamma_1\nabla u_1\cdot\nabla\tilde{u_2}\,dy = \int_\Omega \gamma_1\nabla u_1\cdot\nabla u_2\,dy \tag{3.8}$$

(3.7) and (3.8) together imply (3.6).

Using the hypothesis $\Lambda_{\gamma_1} = \Lambda_{\gamma_2}$ so that in particular $\Lambda_{\gamma_2}(u_1)(u_2) = \Lambda_{\gamma_2}(u_1)(u_2)$ and (3.6), we can conclude that

$$\int_\Omega \nabla\sqrt{\gamma_1}\cdot\nabla(\frac{v_1v_2}{\sqrt{\gamma_1}})\,dy = \int_\Omega \nabla\sqrt{\gamma_2}\cdot\nabla(\frac{v_1v_2}{\sqrt{\gamma_2}})\,dy \tag{3.9}$$

Adding (3.5) and (3.9) gives us the desired conclusion ■

We now all the tools to prove that $\Lambda_{\gamma_1} = \Lambda_{\gamma_2}$ implies $q_1 = q_2$ where $q_i = \frac{\Delta\sqrt{\gamma_i}}{\sqrt{\gamma_i}}$.

Theorem 3.4. *Let Ω be a C^1 or Lipschitz domain. Let γ_1 and γ_2 be two strictly positive conductivities in $C^1(\bar{\Omega})$ or $W^{1,\infty}(\bar{\Omega})$. Then $\Lambda_{\gamma_1} = \Lambda_{\gamma_2}$ implies $q_1 = q_2$.*

Proof of Theorem 3.4. By boundary deterination of Alessandrini [4], we have $\gamma_1 = \gamma_2$ and $\nabla\gamma_1 = \nabla\gamma_2$ on $\partial\Omega$. Thus, we may extend γ_1 and γ_2 to be functions in $C^1(\mathbb{R}^d)$ or $W^{1,\infty}(\mathbb{R}^d)$ which are bounded above and below such that $\gamma_1 = \gamma_2$ in $\mathbb{R}^d \setminus \bar{\Omega}$, and $\gamma_1 = \gamma_2$ outside a compact set. (Note that the boundary determination result is needed only to identify γ_1 and γ_2 at the boundary for which the Lipschitz hypothesis is sufficient.)

Since the distributions $q_i = \frac{\Delta\sqrt{\gamma_i}}{\sqrt{\gamma_i}}$ are compactly supported, the Fourier transforms $\hat{q}_i$ are smooth functions. We fix $k \in \mathbb{R}^d$. From Theorem 3.2, we can find sequences ζ_i^j and ψ_i^j such that $\zeta_i^j \cdot \zeta_i^j = 0$ for $i = 1, 2$ and $j \in \mathbb{N}$ so that $\zeta_1^j + \zeta_2^j = -ik$, $v_i^j = e^{x\cdot\zeta_i^j}(1 + \psi_i^j)$ are solutions of $\Delta v - qv = 0$ and $\lim_{j\to\infty} \|\psi_i^j\|_{\dot{X}^{\frac{1}{2}}_{\zeta_i^j}} = 0$.

It hence follows that

$$\lim_{j\to\infty} m_{q_i}(v_1^j)(v_2^j) = \hat{q}_i(k)$$

Lemma 3.3 implies that $\hat{q_1}(k) = \hat{q_2}(k)$. Since k is arbitrary we get $q_1 = q_2$ (as distributions). ∎

We are now finally (!) ready to prove Theorem 1.3.

Proof of Theorem 1.3. By Theorem 3.4, $\Lambda_{\gamma_1} = \Lambda_{\gamma_2}$ implies $q_1 = q_2$ as distributions. We can thus test q_i with a $C^1(\Omega)$ function ϕ to get

$$\int_\Omega \nabla\sqrt{\gamma_1} \cdot \nabla\left(\frac{\phi}{\sqrt{\gamma_1}}\right) - \nabla\sqrt{\gamma_2} \cdot \nabla\left(\frac{\phi}{\sqrt{\gamma_2}}\right) dx = 0$$

We now let $\varphi = \dfrac{\phi}{\sqrt{\gamma_1}\sqrt{\gamma_2}}$, then we have

$$\int_\Omega \sqrt{\gamma_1\gamma_2}\nabla(\log\sqrt{\gamma_1} - \log\sqrt{\gamma_2}) \cdot \nabla\varphi\, dx = 0.$$

If $\gamma_1 = \gamma_2$ on the boundary and Ω is C^1 or Lipschitz, then we have $\log(\frac{\gamma_1}{\gamma_2})$ in $H_0^1(\Omega)$. Thus $\log(\frac{\gamma_1}{\gamma_2})$ in $H_0^1(\Omega)$ and solves the differential equation div $\sqrt{\gamma_1\gamma_2}\ \nabla u = 0$. And thus by linear elliptic partial differential equation theory, $\log(\frac{\gamma_1}{\gamma_2}) = 0$ in Ω and hence $\gamma_1 = \gamma_2$ in Ω. ∎

Appendix A. Appendix A

Lemma A.1. *If $\eta \in \mathbb{R}^d$, $0 \le s \le 1$ and $r > 0$ then there exists a constant C depending only on the dimension d so that*

$$\int_{B_r(\eta)} \frac{|\xi|^s}{|-|\xi|^2 + 2i\zeta\cdot\xi|}\, d\xi \le \frac{Cr^{d-1}}{|\zeta|^{1-s}}$$

Proof. Without loss of generality we can assume that $\zeta = \frac{(e_1+ie_2)}{\sqrt{2}}$. Let Σ_ζ be the zero set of $p_\zeta(\xi)$.

We first consider the case $B_r(\eta)$ satisfies $r < \frac{1}{100}$, $\mathrm{dist}(\eta, \Sigma_\zeta) < 2r$. Since $|\zeta|^s$ is bounded by some constant on $B_r(\eta)$, it suffices to consider the case $s = 0$. We make a change of variables. We let $x_1 = \mathrm{Re}\ \zeta\cdot\xi$, $x_2 = |\,Im\ \zeta|^2 - |\,Im\ \zeta + \xi|^2$ and $x_j = \xi_j$ for $j = 3$ *to* d. We compute the Jacobian and see that it is bounded below on $B_r(\eta)$. This gives the bound

$$\int\limits_{B_{3r}(0)} \frac{1}{|-|\xi|^2 + 2i\zeta\cdot\xi|}\,d\xi \le C \int\limits_{B_{3r}(0)} \frac{1}{|x_1 + ix_2|}\,dx = C\, r^{d-1}$$

We now consider the case $B_r(\eta)$ satisfies $r \ge \frac{1}{100}$, $\mathrm{dist}(\eta, \Sigma_\zeta) < 2r$. Here we have,

$$\sup_{\xi\in B_r(\eta)} \frac{|\xi|^s|}{|-|\xi|^2 + 2i\zeta\cdot\xi|} \le \frac{C}{r}.$$

If $\mathrm{dist}(\eta, \Sigma_\zeta) < 8$, then the above inequality is true since $||-|\xi|^2 + 2i\zeta\cdot\xi|$ is comparable to to $\mathrm{dist}(\eta, \Sigma_\zeta)$ on $B_r(\eta)$. If $\mathrm{dist}(\eta, \Sigma_\zeta) \ge 8$, then $|-|\xi|^2 + 2i\zeta\cdot\xi|$ is comparable to $|\xi|^2$ for $\xi \in B_r(\eta)$. The lemma thus follows.

We now consider the case $r \ge \frac{1}{100}$ and $\mathrm{dist}(\eta, \Sigma_\zeta) < 2r$. In this case, let $B_r = B_0 \cup B_1$ where $B_0 = B_r \cap B_4(0)$ and $B_1 = B_r \setminus B_4(0)$. By case 1 and 2,

$$\int\limits_{B_0} \frac{|\xi|^s}{|-|\xi|^2 + 2i\zeta\cdot\xi|}\,d\xi \le C$$

And since $\Sigma_\zeta \subset B_4(0)$, we have on B_1

$$\frac{|\xi|^s}{|-|\xi|^2 + 2i\zeta\cdot\xi|} \le \frac{C}{|\xi|^{2-s}}$$

Integrating this estimate gives

$$\int\limits_{B_1} \frac{|\xi|^s}{|-|\xi|^2 + 2i\zeta\cdot\xi|} \le \frac{C}{|\xi|^{2-s}} \le C r^{n-2+s}$$

Since $r > \frac{1}{100}$, the estimates on B_1 and B_0 imply the estimates of lemma in this case ■

As a consequence of this lemma, we can define the operator $G_\zeta : S(\mathbb{R}^d) \to S'(\mathbb{R}^d)$ by

$$\widehat{G_\zeta(f)} = \frac{\hat{f}(\xi)}{|-|\xi|^2 + 2i\zeta\cdot\xi|}$$

since by the above lemma, the symbol of G_ζ satisfies a growth condition and is hence in $S'(\mathbb{R}^d)$.

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