Determining differential equation from its spectral function

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Introduction

- Consider a second order ODE $y" + q(x)y = 0$ on $(0, \infty)$ with boundary conditions at one boundary, $y(0) = 1$ and $y'(0) = h$. The forward problem is to look at the eigenvalues and eigenfunctions of this ODE i.e solutions to $y" + (q(x) - \lambda)y = 0$ with the same boundary conditions and derive their properties. It turns out that one can find a function $\rho(\lambda)$ so that the eigenfunctions are orthonormal with respect to the measure $d\rho(\lambda)$.
- This enables us to write a Parseval type of identity which is analogous to the usual Parseval identity encountered in Fourier Transform, but this time instead of of $e^{ix\lambda}$, we choose eigenfunctions of a second order ODE. (The case of boundary conditions at both ends of a finite interval is also treated in the paper)

Forward Problem

- Such a function ρ is called the spectral function of the ODE $y'' + (q(x) - \lambda)y = 0$ with boundary conditions $y(0) = 1$ and $y'(0) = h$. For each such ODE, it is known that one can construct a spectral function ρ so that Parseval identity holds.
- The authors investigate the relation between $\phi(x, \lambda)$ and $\cos(\sqrt{\lambda}x)$. They assume a relation of the form $\phi(x, \lambda) = \cos(\sqrt{\lambda}x) + \int_0^x K(x, t) \cos(\sqrt{\lambda}t)\, dt$ for $h < \infty$.
- A similar such form is considered for $h = \infty$

- The idea for such a specific form for $\phi(x, \lambda)$ can be justified by a perturbation argument.
- Forcing $\phi(x, \lambda)$ to solve the eigenvalue problem (along with appropriate boundary values) leads to a PDE for the kernel $K(x, t)$ of the form $\frac{\partial^2 K(x,t)}{\partial x^2} - q(x)K(x, t) = \frac{\partial^2 K(x,t)}{\partial t^2}$ with some boundary conditions. The solution to such PDE exists i.e existence of the Kernel $K(x, t)$ is thus justified.
- Thus the eigen-functions can be constructed and as a result one knows $\lambda - q$.
- This observation is key to the inverse problem.

Inverse Problem

- Assume that now one is given a function $\rho(\lambda)$ from which one wants to construct a continuous q for which ρ becomes a spectral function for the the ODE $y" + (q - \lambda)y = 0$ with $y(0) = 1$ and $y'(0) = h$.
- Sufficient conditions to guarantee this is to assume ρ satisfies certain growth conditions viz that for each x, $\int\limits_{-\infty}^{0} e^{\sqrt{|\lambda|}x} d\rho(\lambda)$ exists, $\rho(\lambda)$ is a perturbation for the spectral function for the case $q = 0$ with the perturbation behaving 'nicely'. These conditions are also necessary if one wants to have $q \in C^1$
- These conditions are actually needed to find $K(x, t)$. The fundamental idea in the paper is to get not a differential equation for $K(x, t)$ but rather an integral equation for $K(x, t)$ where all the parameters can be constructed from the knowledge of ρ.

- More specifically, the imposed conditions on ρ allow us to get the parameter in the integral equation for K. This parameter is what the authors denote by $f(x, y)$ in the paper.
- To start off the proof, the first step is to find out an integral equation that the Kernel $K(x, t)$ satisfies. The form for the integral equation is gotten from the forward problem by using the relation $\phi(x, \lambda) = \cos(\sqrt{\lambda} x) + \int\limits_0^x K(x, t) \cos(\sqrt{\lambda} t)\, dt$ for $h < \infty$ and the validity of Parseval's identity.
- The authors then show that any such Kernel necessarily satisfies the integral equation $f(x, y) + \int\limits_0^x K(x, s) f(s, y)\, ds + K(x, y) = 0$ where $f = \int\limits_{-\infty}^{+\infty} \cos(\sqrt{\lambda} x) \cos(\sqrt{\lambda} y) d\sigma(\lambda)$ assuming the validity of Parseval's identity.

- Note that the integral equation for K depends only on ρ. The key idea in the inverse problem solution is to show that the integral equation for K actually has a unique solution.
- *Thus, starting from ρ, we construct f and show that the integral equation has a unique solution K*
- The authors then show that such a solution K must be sufficiently smooth in its variables.
- Now, *define* functions of the form $\phi(x,\lambda) = \cos(\sqrt{\lambda}x) + \int_0^x K(x,t)cos(\sqrt{\lambda}t)\,dt$.
- We need to show Parseval's identity holds and that the functions $\phi(x,\lambda)$ solve $y" + (\lambda - q(x))y = 0$ for some $q(x)$ with the requisite boundary conditions. (The arguments for finite and infinite h are slightly different)

Parseval's identity

- To show Parseval's identity, the authors first show that $\int_a^x \phi(t,\lambda)\,dt$ and $\int_b^y \phi(t,\lambda)\,dt$ are orthogonal with respect to $\rho(\lambda)$ whenever the intervals (a,x) and (b,y) do not intersect.
- Using this, the Parseval's identity for characteristic function of any finite interval (a,b) is shown.
- From here, it is fairly straightforward to show Parseval's equation for an arbitrary $f \in L^2(0,\infty)$.

Differential Equation for $\phi(x, \lambda)$

- Note that $\phi(x, \lambda) = \cos(\sqrt{\lambda}x) + \int\limits_0^x K(x,t) cos(\sqrt{\lambda}t)\, dt$.
  Multiply by $\cos(\sqrt{\lambda t})$ and use the fact that
  $\cos(\sqrt{\lambda}x) = \phi(x, \lambda) - \int\limits_0^x K_1(x,t)\phi(x,t)\, dt$ for some kernel K_1
  where K_1 can be found out from the equation for $\phi(x, \lambda)$ by solving it as a Volterra equation for $\cos(\sqrt{\lambda}t)$.
- It is then shown that $\phi(x, \lambda)$ satisfies
  $\phi" + (\lambda - q(x))\phi(x, \lambda) = 0$ with $\phi(0, \lambda) = 1$ and $\phi'(0, \lambda) = h$
  where $q = \lim_{t \to 0} \frac{-2}{t^2} \int\limits_{x-t}^{x+t} W(x,t,s)\, ds$ where $W(x,t,s)$ is a certain function constructed from $K(x,t)$ and $K_1(x,t)$.
- Similar argument goes through for $h = \infty$