Fall 2015: MATH 324F **Advanced Multi-variable Calculus**

Midterm Examination **50 minutes**

Name:

Student ID:

Instructions.

1. This exam is closed book. You may use one "8.5×11" sheet of handwritten notes. Do not share notes.
2. Show all the steps of your work clearly in order to receive credit.
3. Place a $\boxed{\text{box}}$ around your answer to each question.
4. Raise your hand if you have a question.
5. Do not cheat.

Question	Points	Your Score
Q1	10	
Q2	8	
Q3	12	
Q4	10	
Q5	10	
TOTAL	50	

Q1 a) [4 points]

Assume that all given functions are differentiable. If $z = f(x-y)$, show that $\frac{\partial z}{\partial x} + \frac{\partial z}{\partial y} = 0$.

> $z = f(u(x,y))$ where $u(x,y) = x - y$. By chain rule, $\frac{\partial z}{\partial x} = f^{'}(u)\frac{\partial u}{\partial x} = f^{'}(u)$. Similarly, $\frac{\partial z}{\partial x} = -f^{'}(u)$. Thus $\frac{\partial z}{\partial x} + \frac{\partial z}{\partial y} = 0$.

Q1 b) [6 points]

Use chain rule to find out $\frac{\partial z}{\partial u}$ for $z = x^4 + x^2 y$, $x = s + 2t - u$, $y = stu^2$; when $s = 4$, $t = 2$ and $u = 1$.

> For $s = 4$, $t = 2$, $u = 1$, $x = 7$ and $y = 8$. Using chain rule gives us $\frac{\partial z}{\partial u} = [(4x^3 + 2xy) \times -1] + [x^2 \times (2stu)]$. Plugging in the given values of s, t, u and the computed values of x and y, we get $\frac{\partial z}{\partial u} = -700$.

Q2) [8 points]

Find the direction in which $f(x,y,z) = ze^{xy}$ increases most rapidly at the point $(0,1,2)$. What is the maximum rate of increase?

> $\nabla f(x,y,z) = (yze^{xy}, zxe^{xy}, e^{xy})$. $\nabla f(0,1,2) = (2,0,1)$. The direction of most rapid change for f at the point $(0,1,2)$ is along $\nabla f(0,1,2)$ i.e along $(2,0,1)$. We can normalize to get a unit vector but that is not needed in this question as we are just asked to find the direction of the most rapid change.
>
> The maximum rate of increase of f at $(0,1,2)$ is $|\nabla f(0,1,2)| = |(2,0,1)| = \sqrt{5}$.

Q3 a) [4 points] Express the triple integral of the function $x^2 - y + z$ over the region bounded by the planes $x = 0$, $y = 0$, $z = 0$ and $2x + y + z = 8$. Do not evaluate the integral.

> We will write down the integral in the order $dz\,dy\,dx$. The desired expression is:
>
> $$\int_0^{8-2x-y} \int_0^{8-2x} \int_0^{4} (x^2 - y + z)\,dz\,dy\,dx.$$

Q3 b) [8 points]

Evaluate the integral $\iiint\limits_V x^2 z\,dV$ where V is the region bounded by the spheres $x^2 + y^2 + z^2 = 4$ and $x^2 + y^2 + z^2 = 1$.

> The region of integration is symmetric about all the axes. The function to be integrated is odd. The integral is hence 0.

Q4) [10 points]

Consider the iterated integral $\int\limits_0^1 \int\limits_0^{1-x^2} \int\limits_0^{1-x} f(x,y,z)\,dy\,dz\,dx$.

(a) [**5 points**] Rewrite this integral as an equivalent iterated integral in the order $dy\,dx\,dz$.

> $\int\limits_0^1 \int\limits_0^{\sqrt{1-z}} \int\limits_0^{1-x} f(x,y,z)\,dy\,dx\,dz.$

(b) [**5 points**] Rewrite the integral in the order $dz\,dx\,dy$.

> $\int\limits_0^1 \int\limits_0^{1-y} \int\limits_0^{1-x^2} f(x,y,z)\,dz\,dx\,dy.$

Q5 [10 points]

Compute $\iint\limits_R \frac{(x+y)^4}{(2-x+y)^4}\,dx\,dy$, where R is the square with vertices at $(1,0)$, $(0,1)$, $(-1,0)$ and $(0,-1)$.

[**Hint:** Use change of variables to make the integrand simpler. Compute the Jacobian, write down the expression for the integral to be evaluated in the uv-plane and then compute this new integral.]

> Since the region is bounded by the lines $x+y=\pm 1$ and $x-y=\pm-1$, we make a change of variables $u=x+y$ and $v=x-y$.
>
> The Jacobian $\frac{\partial(x,y)}{\partial(u,v)} = -1/2$. Thus, $dx\,dy = \frac{1}{2}\,du\,dv$. Hence we get $\iint\limits_R \frac{(x+y)^4}{(2-x+y)^4}\,dx\,dy = \int\limits_{-1}^1 \int\limits_{-1}^1 \frac{u^4}{(2-v)^4}\,\frac{1}{2}\,du\,dv = \int\limits_{-1}^1 \frac{1}{5(2-v)^4}\,dv = \frac{26}{405}$.