Non-uniqueness in Calderón's problem for anisotropic conductivities

Based on a Paper
by Kohn, Shen, Vogelius and Weinstein

Karthik Iyer

University of Washington

11th June 2015

- Suppose we have a conductor in a bounded region Ω with conductivity $\sigma(x)$ defined on $\bar{\Omega}$. We apply a voltage f on $\partial\Omega$ and measure the current that flows out of the region.
- Measuring the current flux through $\partial\Omega$ determines the quantity $\sigma(x)\nabla u(x) \cdot \eta(\hat{x})$.
- Application of Ohm's law leads us the BVP

$$\nabla \cdot (\sigma \nabla u) = 0 \quad \text{on} \quad \Omega \quad \text{with} \quad u = f \quad \text{on} \quad \partial\Omega \tag{1}$$

  PDE (1) determines a Dirichlet to Neumann map Λ_σ which takes boundary voltage to associated current flux

$$\Lambda_\sigma : u\big|_{\partial\Omega} \to \sigma \nabla u \cdot \nu\big|_{\partial\Omega} \tag{2}$$

- The inverse conductivity problem is to get information about σ from Λ_σ

- Different aspects of the inverse conductivity problem are uniqueness, stability and reconstruction.
- In the general case, we define the DN map weakly.
- Only concerned with uniqueness for this talk i.e does $\Lambda_{\sigma_1} = \Lambda_{\sigma_2} \implies \sigma_1 = \sigma_2$ on $\bar{\Omega}$.
- Yes if σ_I are scalar valued, positive and finite. No if σ_I is matrix valued, positive definite and finite.

- Let $y = F(x)$ be an invertible, orientation-preserving change of variables in Ω with $F(x) = x$ on $\partial\Omega$. Then

$$\int_\Omega \sum \sigma_{ij} \frac{\partial u}{\partial x_i} \frac{\partial u}{\partial x_j}\, dx = \int_\Omega \sum \sigma_{ij} \frac{\partial u}{\partial y_k} \frac{\partial y_k}{\partial x_i} \frac{\partial u}{\partial y_l} \frac{\partial y_l}{\partial x_j} det\left(\frac{\partial x}{\partial y}\right) dy.$$

- More compactly,

$$\int_\Omega (\sigma(x)\nabla_x u \cdot \nabla_x u) = \int_\Omega (F_*\sigma(y)\nabla_y u \cdot \nabla_y u)\, dy$$

where $F_*\sigma(y) = \frac{1}{det\ DF(x)} DF(x)\sigma(x)(DF(x))^T$ with $DF_{ij} = \frac{\partial y_i}{\partial x_j}$

- $\Lambda_\sigma = \Lambda_{F_*\sigma}$
- Is this the only impediment to uniqueness?

Cloaking

- Let $D \subset \Omega$ be fixed. Let $\sigma_c : \Omega \setminus D$ be a non-negative, matrix valued conductivity. We say σ_c cloaks D if its extensions across D,

$$\begin{aligned} \sigma_A(x) &= A(x) \quad \text{for } x \in D \\ &= \sigma_c(x) \quad \text{for } x \in \Omega \setminus D \end{aligned}$$

  produces the same boundary measurements as a uniform region with conductivity $\sigma = 1$, regardless of choice of $A(x)$.
- Appropriate name since cloaking renders D invisible with respect to boundary measurements.

Relation between uniqueness and cloaking

- No restricted uniqueness if cloaking is possible. We will show existence of cloak and hence non-uniqueness for the inverse conductivity problem.
- For simplicity, we will show existence of cloaks for $D = B_1$ with $\Omega = B_2$.
- Consider an F of the form F(x)

$$
\begin{aligned}
&= \frac{x}{\rho} \quad \text{for } |x| \leq \rho \\
&= \left(\frac{2-2\rho}{2-\rho} + \frac{1}{2-\rho}|x|\right)\frac{x}{|x|} \quad \text{for } \rho \leq |x| \leq 2
\end{aligned}
$$

- The associated near cloak is the push forward via F of $\sigma = 1$, F_*1, restricted to $B_2 \setminus B_1$.

- The boundary measurements of $\sigma_A(y)$

$$= A(y) \quad \text{for } y \in B_1 \tag{3}$$
$$= F_*1(y) \quad \text{for } y \in B_2 \setminus B_1 \tag{4}$$

  are identical, by change of variables principle, to $F_*^{-1}\sigma_A(x)$

$$= F_*^{-1}A(x) \quad \text{for } x \in B_\rho$$
$$= 1 \quad \text{for } x \in B_2 \setminus B_\rho$$

  where $F_*^{-1} = (F^{-1})_*$
- The boundary measurements associated with σ_A are the same as those of a uniform ball perturbed by a small inclusion at the center.
- If the radius ρ is small, we hope that the boundary measurements are close to that of a completely uniform ball thus almost cloaking the unit ball.

- By a result of **Kohn and Vogelius** (1989), we have the following bounds,

$$||\Lambda_{\sigma_A} - \Lambda_1|| \leq C\rho^d$$

- Thus the boundary measurements for σ_A are almost identical to those of a uniform ball with conductivity 1. This almost cloaks the unit ball.

Perfect cloak

- Pass to the limit $\rho \to 0$ to get $F(x) = (1 + \frac{|x|}{2})\frac{x}{|x|}$. We hope that the DN map for σ as in (3) and (4) is the same as that of uniform ball with conductivity 1.

- Have to be careful as F is singular at 0.

- Near the inner boundary of the shell, at least one eigenvalue of F_*1 tends to 0. We now show how to take care of this degeneracy. (Lack of uniform lower bound). Must define the DN map carefully

Existence and Uniqueness of solution

- Let v be the potential associated with Dirichlet data f: $\nabla \cdot (\sigma_A \nabla v) = 0$ in B_2 with $v = f$ on ∂B_2 where σ_A is as given in (3) and (4) using singular change of variables.
- Because of degeneracy of $F_* 1$ near $|y| = 1$, is existence and uniqueness of v now guaranteed?

- Consider the ansatz $v(y)$

$$\begin{aligned} &= u(x) \quad \text{for } y \in B_2 \setminus B_1 && (5)\\ &= u(0) \quad \text{for } y \in B_1 && (6) \end{aligned}$$

  where u is the harmonic function with the same Dirichlet data f and $x = F^{-1}(y)$

- We assume that $|v(y)| \leq C$ for $|y| \leq r$ for $1 < r \leq 2$ i.e v does not diverge as $|y| \to 1$.

- The idea of the proof is to show the following:
  - If v solves the BVP and satisfies the above assumption, then $v(y) = u(x)$ for $1 < |y| < 2$ with $x = F^{-1}(y)$.
  - Show that v is indeed a solution i.e $\sigma_A \nabla v$ is weakly divergence free.
  - v(y) = u(0) in B_1.

Potential outside cloaked region

Theorem 1

If v solves $\nabla \cdot (\sigma_A \nabla v) = 0$ in B_2 with $v = f$ on ∂B_2 and $|v(y)| \leq C$ for $|y| \leq r$ for $1 < r \leq 2$ then $v(y) = u(x)$ for $1 < |y| < 2$ where $x = F^{-1}(y)$ and u is the harmonic function on B_2 with the same Dirichlet data f.

Proof.

Change of variables + removal of isolated singularities for harmonic functions. □

Potential inside cloaked region

Theorem 2

Fix $f \in H^{\frac{1}{2}}(\partial B_2)$. Let v be as defined in (5) and (6). Then

- *(a) v is Lipschitz continuos away from ∂B_2 i.e $|\nabla v|$ is uniformly bounded in B_r for every $r < 2$.*
- *(b) $\sigma_A \nabla v$ is also uniformly bounded away from ∂B_2.*
- *(c) $(\sigma_A \nabla v) \cdot \nu \to 0$ uniformly as $|y|$ decreases to 1, where $\nu = \frac{y}{|y|}$ is the normal to ∂B_1.*
- *(d) $\sigma_A \nabla V$ is weakly divergence free in B_2.*

Potential inside the cloaked region

- To ensure that $v(y) = u(0)$ in B_1, we assume further that ∇v and $\sigma_A \nabla v$ are in $L^2(B_2)$.

- We have already shown that if v_1 is any solution, then $v_1 = u(x)$ on $1 < |y| < 2$.

- For $|y| = 1$,
  $v(y) = u(F^{-1}(y)) = u(0)$.
  $v_1(y) = v_1\Big|_{|y|\Downarrow 1} = u(x)\Big|_{|x|\Downarrow 0} = u(0)$.

- Both v and v_1 solve $\nabla \cdot (A\nabla g) = 0$ in B_1 and $g = u(0)$ on ∂B_1. Hence, $v_1 = v$.

Singular change of variables gives perfect cloaking

Theorem 3

Let σ_A be as given in (3) and (4) with singular F. Then $\Lambda_{\sigma_A} = \Lambda_1$

Proof.

- $\int_{B_2} (\sigma_A \nabla v \cdot \nabla v)\, dy = \int_{B_2 \setminus B_1} (\sigma_A \nabla v \cdot \nabla v)\, dy$
- Definition of σ_A and change of variables formula gives us

$$\int_{B_2 \setminus B_1} (\sigma_A \nabla v \cdot \nabla v) = \int_{B_2} |\nabla_x u|^2\, dx$$

  where u is harmonic with same Dirichlet data as v
- Thus $\Lambda_{\sigma_A}(f)(f) = \Lambda_1(f)(f)$

$\square$

Theorem 4

Let $G : B_2 \to \Omega$ be a bi-Lipschitz map and let $D = G(B_1)$. Then $H = G \circ F \circ G^{-1} : \Omega \to \Omega$ is identity on $\partial\Omega$ and blows up $z_0 = G(0)$ to D. Consider a conductivity σ_A of the form

$$\begin{aligned} \sigma_A(w) &= A(w) \quad \text{for } w \in D \\ &= H_*1(w) \quad \text{for } w \in \Omega \setminus D \end{aligned}$$

where A is symmetric and positive definite but otherwise arbitrary. Then we have $\Lambda_{\sigma_A} = \Lambda_1$.

Open questions

- Does Λ_σ determine σ uniquely? (in a restricted sense)
- If σ is only non-negative, then answer is no.
- The question is still open if σ is only supposed to be positive definite and finite in dimensions ≥ 3.