A 5 minute introduction to Inverse Problems

Karthik Iyer

Department of Mathematics
University of Washington

10th March, 2015

What constitutes an inverse problem

- Consider mappings between objects of interest, called *parameters*, and acquired information about these objects called *measurements*. The *forward problem* is called the *measurement operator* (MO) denoted by $\mathcal{M}$.

- The MO maps parameters in a Banach space $\mathcal{X}$ to data, typically in another Banach space $\mathcal{Y}$. We write

$$y = \mathcal{M}(x) \quad \text{for } x \in \mathcal{X} \text{ and } y \in \mathcal{Y} \tag{1}$$

- Solving the inverse problem amounts to finding point(s) $x \in \mathcal{X}$ so that (1) or an approximation to (1) holds or in other words reconstructing parameters from measurements.

- It is the choice of the spaces $\mathcal{X}$ and $\mathcal{Y}$ that very often proves to be difficult.

Injectivity and stability of MO

- The first question to ask about the MO is whether we have acquired enough data to uniquely reconstruct the parameters i.e whether $\mathcal{M}$ is injective.

- When $\mathcal{M}$ is injective, we can construct an inverse $\mathcal{M}^{-1}$. The main features of the inverse operator are captured by what are referred to as *stability estimates*.

- Stability estimates typically have the following form

$$||x_1 - x_2||_{\mathcal{X}} \leq \omega(||\mathcal{M}(x_1) - \mathcal{M}(x_2)||_{\mathcal{Y}}), \tag{2}$$

  where $\omega : \mathbb{R}_+ \to \mathbb{R}_+$ is an increasing function with $\omega(0) = 0$ which quantifies the modulus of continuity of the inverse operator $\mathcal{M}^{-1}$.

- Ideally we would like ω to go to 0 as fast as possible so that error in data acquisition does not translate to magnification of error in reconstructing parameters. This is what happens in an *well posed* problem.

Examples of MO and Inverse Problems

- $\mathcal{X} = C[0,1] = \mathcal{Y}$. Define

$$\mathcal{M}f(x) = \int_0^x f(y)\,dy \tag{3}$$

- $\mathcal{X} = C_C(\mathbb{R}^n)$, $Y = C(\mathbb{R} \times S^{n-1})$. Define

$$\mathcal{M}f(s,\omega) = \int_{x\cdot\omega=s} f(x)\,dH \tag{4}$$

  The operator $\mathcal{M}$ is called the n-dimensional Radon Transform.

- $\mathcal{M}$ is injective and there is an explicit formula for inversion of the Radon Transform.

Calderón's Inverse problem

- Let Ω be a bounded domain with Lipschitz boundary. Let $\gamma(x)$ be a positive function which is bounded above and below. Consider the following PDE

$$\nabla \cdot (\gamma \nabla u) = 0 \text{ in } \Omega.$$

- Given f in $H^{\frac{1}{2}}(\partial\Omega)$, there exists a unique u in $H^1(\Omega)$ such that

$$\begin{aligned} \nabla \cdot (\gamma \nabla u) &= 0 \\ u &= f \text{ on } \partial\Omega. \end{aligned}$$

- The Dirichlet to Neumann map Λ_γ is defined weakly as $(\Lambda_\gamma f, g) = \int\limits_\Omega \gamma \nabla u \cdot \nabla v$ where u solves the conductivity equation with boundary data f and v is any H^1 function such that $v\big|_{\partial\Omega} = g$. If $\gamma \in L^\infty(\bar{\Omega})$, then Λ_γ is a well defined map from $H^{\frac{1}{2}}(\partial\Omega)$ to $H^{\frac{-1}{2}}(\partial\Omega)$.

- The measurement operator maps $\gamma \to \Lambda\gamma$.

Practical applications

- Radon Transform in 2 dimensions i.e integrating a two dimensional function along all possible lines in the plane and its inversion form the mathematical backbone of *Computerized Tomography* (CT), one of the most successful medical imaging techiniques to date.
- A specific example of a weighted 2D Radon Transform, the attenuated Radon Transform finds an application in Single Photon Emission Computerized Tomography (SPECT).
- The Calderón problem has applications in two medical imaging methods, electrical impedance tomography (EIT) and optical tomography (OT).