Calderón's problem for quasilinear conductivities

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- Let $\sigma(x)$ be a symmetric uniformly elliptic matrix function in $L^{\infty}(\Omega)$.
- Consider the BVP

$$\nabla \cdot (\sigma \nabla u) = 0 \quad \text{on} \quad \Omega \quad \text{with} \quad u = f \quad \text{on} \quad \partial\Omega$$

Define the Dirichlet to Neumann map Λ_{σ}

$$\Lambda_{\sigma} : u\big|_{\partial\Omega} \to \sigma \nabla u \cdot \nu\big|_{\partial\Omega}$$

- The inverse conductivity problem is to get information about σ from Λ_{σ}

Different aspects of the inverse problem:

- Uniqueness
- Stability
- Reconstruction
- Numerical Implementation
- Partial Data

- We can extend the definition of Dirichlet-to-Neumann (DN) map $\Lambda_\sigma : H^{\frac{1}{2}}(\partial\Omega) \to H^{-\frac{1}{2}}(\partial\Omega)$

- $\langle \Lambda_\sigma(f), g \rangle = \int_\Omega \sigma(x) \nabla u \cdot \nabla v^g \, dx$ where u is the unique solution to conductivity equation with boundary data f and $v^g\big|_{\partial\Omega} = g$

- Does $\Lambda_{\sigma_1} = \Lambda_{\sigma_2} \implies \sigma_1 = \sigma_2$ on Ω ?

- Yes if σ_l are scalar valued. No if σ_l are matrix valued for $l = 1, 2$

Non-uniqueness in the anisotropic case

- Let $y = F(x)$ be smooth diffeomorphism $F : \bar{\Omega} \to \bar{\Omega}$ with $F(x) = x$ on $\partial\Omega$. Then

$$\int\limits_{\Omega} \sum \sigma_{ij} \frac{\partial u}{\partial x_i} \frac{\partial u}{\partial x_j}\, dx = \int\limits_{\Omega} \sum \sigma_{ij} \frac{\partial u}{\partial y_k} \frac{\partial y_k}{\partial x_i} \frac{\partial u}{\partial y_l} \frac{\partial y_l}{\partial x_j} det(\frac{\partial x}{\partial y})\, dy$$

- More compactly,

$$\int\limits_{\Omega} (\sigma(x) \nabla_x u \cdot \nabla_x u)\, dx = \int\limits_{\Omega} (F_* \sigma(y) \nabla_y u \cdot \nabla_y u)\, dy$$

where $F_*\sigma(y) = \frac{1}{det\ DF(x)} DF(x)^T \sigma(x)(DF(x))$ with $DF_{ij} = \frac{\partial y_i}{\partial x_j}$

- $\Lambda_\sigma = \Lambda_{F_*\sigma}$

Anisotropic uniqueness results for $n = 2$

Let $\Lambda_{\sigma_1} = \Lambda_{\sigma_2}$. Then

- $\sigma_2 = \Phi_* \sigma_1$, $\sigma_i \in C^{2,\alpha}(\bar{\Omega})$, Sylvester 1993 (with a smallness assumption)
- $\sigma_2 = \Phi_* \sigma_1$, $\sigma_i \in W^{1,p}(\Omega)$, $p > 2$ Sun and Uhlmann 2003
- $\sigma_2 = \Phi_* \sigma_1$, $\sigma_i \in L^{\infty}(\Omega)$ Astala, Päivärinta, Lassas 2006

Sun and Uhlmann anisotropic 2003 result

Theorem 1

Let $\Omega \subset \mathbb{R}^2$ be a bounded domain with C^1 boundary. Let σ_1 and σ_2 be two anisotropic conductivities in $W^{1,p}(\Omega)$ for $p > 2$ with $\sigma_1 - \sigma_2 \in W_0^{1,p}(\Omega)$ and $\Lambda_{\sigma_1} = \Lambda_{\sigma_2}$. Then $\exists$ a diffeomorphism $\Phi : \Omega \to \Omega$ in $W^{2,p}$ class with $\Phi\big|_{\partial\Omega} = I$ such that $\sigma_2 = \Phi_* \sigma_1$.

Conjecture 1

Let $\Omega \subset \mathbb{R}^2$ be a bounded domain with C^1 boundary. Let $\sigma_1(x,t)$ and $\sigma_2(x,t)$ be two anisotropic conductivities. Assume that $\sigma_1 - \sigma_2 \in W_0^{1,p}(\Omega)$ for each t with $p > 2$ and $\Lambda_{\sigma_1} = \Lambda_{\sigma_2}$. Then $\exists$ a diffeomorphism $\Phi : \Omega \to \Omega$ in $W^{2,p}$ class with $\Phi\big|_{\partial\Omega} = I$ such that $\sigma_2 = \Phi_* \sigma_1$.

Assumptions

- $\sigma(x,t)$ is a symmetric matrix function defined on $\bar{\Omega} \times \mathbb{R}$ where Ω is a bounded domain in $\mathbb{R}^2$ with C^1 boundary
- [Ellipticity] $\exists \lambda > 0$ so that $\frac{1}{\lambda} I \preceq \sigma(x,t) \preceq \lambda I$ for any $(x,t) \in \bar{\Omega} \times \mathbb{R}$
- [Smoothness in t] $\sigma(x,t)$ is C^1 w.r.t t such that $\frac{\partial \sigma^{ij}(x,t)}{\partial t}$ is in $L^\infty(\Omega)$ for any t with an $L > 0$ so that $||\frac{\partial \sigma^{ij}(x,t)}{\partial t}||_{L^\infty(\Omega)} \leq L$ for any t.
- [Smoothness in x] For each fixed t, $\sigma(x,t) \in W^{1,p}(\Omega)$ with $p > 2$

Known results for quasi-linear case

Let $\Lambda_{\sigma_1} = \Lambda_{\sigma_2}$. Then

- $\sigma_2 = \sigma_1$, $\sigma_i = \gamma_i I$, $\gamma_i \in C^{1,\alpha}(\bar{\Omega} \times \mathbb{R})$, Sun 1991
- $\sigma_2 = \Phi_* \sigma_1$, $\sigma_i \in C^{2,\alpha}(\bar{\Omega} \times \mathbb{R})$, Sun and Uhlmann 1997

- Conjecture makes sense as Φ preserves class.
- Strategy: Define the quasi-linear inverse problem, reduce to the case of linear inverse problem, use results from linear inverse problems, make reductions.
- Technique adapted from Sun and Uhlmann, 1997 who get uniqueness for anisotropic $\sigma \in C^{2,\alpha}(\bar{\Omega} \times \mathbb{R})$

DN map

- Consider

$$\nabla \cdot (\sigma(x,u)\nabla u) = 0 \quad \text{in} \quad \Omega \quad \textit{with} \quad u\big|_{\partial\Omega} = f$$

where $f \in H^{1/2}(\partial\Omega)$. $\exists!$ solution $u \in H^1(\Omega)$.

- Define the DN map $\Lambda_\sigma : H^{1/2}(\partial\Omega) \to H^{-1/2}(\partial\Omega)$ as $\langle \Lambda_\sigma(f), g\rangle = \int\limits_\Omega \sigma(x,u)\nabla u \cdot \nabla v^g \, dx$
- If Φ is a $W^{2,p}$ diffeomorphism of Ω which is identity on the boundary, we still have $\Lambda_\sigma = \Lambda_{\Phi^*\sigma}$

First Linearization

- Fix $t \in \mathbb{R}$ and $f \in H^{1/2}(\partial\Omega)$. Denote $\sigma^t(x)$ as the linear anisotropic conductivity obtained from $\sigma(x,t)$ by freezing t.

- $\lim\limits_{s \to 0} \| \frac{1}{s}\Lambda_\sigma(t + sf) - \Lambda_{\sigma^t}(f) \|_{H^{-1/2}(\partial\Omega)} = 0$

- $\Lambda_{\sigma_1} = \Lambda_{\sigma_2} \implies \Lambda_{\sigma_1^t} = \Lambda_{\sigma_2^t} \;\; \forall t \in \mathbb{R}.$

- Obtain family of diffeomorhphisms $\phi^t \in W^{2,p}(\Omega)$ and $\phi^t\big|_{\partial\Omega} = Id$ so that $\phi^t_* \sigma_1^t = \sigma_2^t$ for all $t \in \mathbb{R}$.

- Define $\Phi(x,t) = \phi^t(x)$. Φ is well defined.
- Need to show that for any t_0, $\frac{\partial \Phi}{\partial t}\Big|_{t=t_0} = 0$ in $\bar{\Omega}$.
- First show smoothness of Φ with respect to t.

Construction and properties of Φ

- For each t, construct a unique $W^{2,p}$ diffeomorphism of the plane so that $F_l^t : \Omega \to \Omega_l^t$ so that $(F_l^t)_* \sigma_l^t$ is isotropic for $l = 1, 2$.
- F_l^t is constructed solving a homogeneous Beltrami equation with a compactly supported dilation factor $\mu_l^t \in W^{1,p}(\mathbb{C})$.
- Since μ_l^t is C^1 in t, F_l^t will be C^1 in t
- $F_1^t = F_2^t$ on $\mathbb{R}^2 \setminus \Omega$. $\phi^t = (F_2^t)^{-1} \circ F_1^t$ is a $W^{2,p}$ diffeomorphism which is identity on the boundary.

- F_1^t and F_2^t are C^1 w.r.t t, $\phi^t = (F_2^t)^{-1} \circ F_1^t$ also inherits the same smoothness w.r.t to t.
- In fact it is possible to show that $\frac{\partial \Phi}{\partial t}\Big|_{t=t_0} \in W^{1,p}(\Omega)$ for any $t_0 \in \mathbb{R}$
- The task is thus reduced to showing $\frac{\partial \Phi}{\partial t}\Big|_{t=0} = 0$. The same argument works for $t \neq 0$.
- By a transformation one may assume that $\Phi^0 = Id$ so that $\sigma_1(x,0) = \sigma_2(x,0) = \sigma(x)$

Some reductions

- Fix $f \in W^{2-\frac{1}{p},p}(\partial\Omega)$. For $l = 1, 2$ and each t, solve

$$\nabla \cdot \sigma_l^t \nabla u_l^t = 0 \text{ in } \Omega \text{ with } u_l^t\big|_{\partial\Omega} = f$$

- Consider the unique $H_0^1(\Omega)$ solution v_l^t to

$$\nabla \cdot \sigma_l^t \nabla v_l^t = -\nabla \cdot \left[\frac{\partial \sigma_l^t}{\partial t} \nabla u_l^t\right]$$

- $t \to u_l(x, t)$ is differentiable in the H^1 topology with derivative v_l^t.

- $$\nabla \cdot \left[\left(\frac{\partial \sigma_1}{\partial t} - \frac{\partial \sigma_2}{\partial t} \right) \bigg|_{t=0} \nabla u \right] + \nabla \cdot \sigma \nabla (v_1^0 - v_2^0) = 0$$

  where $u = u_1(x, 0) = u_2(x, 0)$
- Let $X = \frac{\partial \Phi^t}{\partial t}\Big|_{t=0}$. For each $t \in \mathbb{R}$

  $$u_1^t(x) = u_2^t(\Phi^t(x))$$
- Differentiating in t at $t = 0$ gives

  $$v_1^0 - v_2^0 - X \cdot \nabla u = 0 \text{ in } \Omega$$

Need for second linearization

Lemma 2

For every f in $H^{1/2}(\partial\Omega) \cap L^{\infty}(\partial\Omega)$ and $t \in \mathbb{R}$

$$\lim_{s \to 0} \left\| \frac{1}{s} \left[\frac{\Lambda_{\sigma}(t + sf)}{s} - \Lambda_{\sigma^t}(f) \right] - K_{\sigma,t}(f) \right\|_{H^{-1/2}(\partial\Omega)} = 0$$

$K_{\sigma,t} : H^{1/2}(\partial\Omega) \cap L^{\infty}(\partial\Omega) \to H^{-1/2}(\partial\Omega)$ *is defined implicitly as*

$$\langle K_{\sigma,t}(f), g \rangle = \int_{\Omega} \nabla v^{g} \cdot \sigma_t(x,t) v^0 \nabla(v^0)\, dx$$

where v^g, v^0 are unique solutions to $\nabla \cdot \sigma(x,t)\nabla u = 0$ with trace g and f respectively.

- Second linearization implies

$$\int_\Omega \nabla u_1 \cdot \frac{\partial \sigma_1}{\partial t}\bigg|_{t=0} \nabla u_2^2 \, dx = \int_\Omega \nabla u_1 \cdot \frac{\partial \sigma_2}{\partial t}\bigg|_{t=0} \nabla u_2^2 \, dx$$

where u, u_1, u_2 are $W^{2,p}$ solutions to the linear equation $\nabla \cdot \sigma(x)\nabla u = 0$ in Ω

- Let $B(x) = \left(\frac{\partial \sigma_1}{\partial t} - \frac{\partial \sigma_2}{\partial t}\right)\Big|_{t=0}$. Above can be re-written as

$$\int_\Omega B(x)\nabla u \cdot \nabla(u_1 u_2) \, dx = 0$$

What needs to be done

- Need to show $\nabla \cdot B(x)\nabla u = 0$ for any $u \in W^{2,p}$ solution to $\nabla \cdot \sigma \nabla u = 0$.
- Need to show that $X \cdot \nabla u = 0$ for any $u \in W^{2,p}$ solution to $\nabla \cdot \sigma \nabla u = 0$ implies $X = 0$ in $\bar{\Omega}$.
- Can we improve the result even further to get the quasi-linear analog of the optimal regularity for the conductivities i.e remove the smoothness assumption in x?
- Stability issues for the quasi-linear case?