Determining rough first order perturbations of the polyharmonic operator

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What constitutes an inverse problem

- Consider mappings between objects of interest, called *parameters*, and acquired information about these objects called *measurements*. The *forward problem* is called the *measurement operator* (MO) denoted by $\mathcal{M}$.

- The MO maps parameters in a Banach space $\mathcal{X}$ to data, typically in another Banach space $\mathcal{Y}$. We write

$$y = \mathcal{M}(x) \quad \text{for } x \in \mathcal{X} \text{ and } y \in \mathcal{Y} \tag{1.1}$$

- Solving the inverse problem amounts to finding point(s) $x \in \mathcal{X}$ so that (1.1) or an approximation to (1.1) holds.

Preliminaries

- Let $\Omega \subset \mathbb{R}^n$, $n \geq 3$ be a bounded open set with C^∞ boundary.
- Consider the operator $(-\Delta)^m$ where $m \geq 1$ is an integer. $(-\Delta)^m$ is positive and self-adjoint on $L^2(\Omega)$ with domain $H^{2m}(\Omega) \cap H_0^m(\Omega)$, where $H_0^m(\Omega) = \{u \in H^m(\Omega) : \gamma u = 0\}$.
- $\gamma : H^{2m}(\Omega) \to \prod_{j=0}^{m-1} H^{2m-j-1/2}(\partial\Omega), \quad \gamma u = (u\big|_{\partial\Omega}, \partial_\nu u\big|_{\partial\Omega}, ..., \partial_\nu^{m-1} u\big|_{\partial\Omega})$,
- First consider $\mathcal{L}_{A,q} = (-\Delta)^m + A \cdot D + q$ where A and q are sufficiently smooth and $D = -i\nabla$.

- For $f = (f_0, f_1, ..., f_{m-1}) \in \prod_{j=0}^{m-1} H^{2m-j-1/2}(\partial\Omega)$, consider

$$\mathcal{L}_{A,q} u = 0 \text{ in } \Omega \quad \text{and} \quad \gamma u = f \text{ on } \partial\Omega. \tag{BVP}$$

  If 0 is not in the spectrum of $\mathcal{L}_{A,q}$, (BVP) has a unique solution.

- We define the Dirichlet-to-Neumann (DN) $\mathcal{N}_{A,q}$ map as

$$\begin{aligned} \mathcal{N}_{A,q} f &= (\partial_\nu^m u\big|_{\partial\Omega}, ..., \partial_\nu^{2m-1} u\big|_{\partial\Omega}) \\ &= \tilde{\gamma} u \in \prod_{j=m}^{2m-1} H^{2m-j-1/2}(\partial\Omega). \end{aligned}$$

- The inverse problem for the perturbed polyharmonic operator $\mathcal{L}_{A,q}$ is to determine A and q in Ω (parameters) from the knowledge of the Dirichlet to Neumann map $\mathcal{N}_{A,q}$ (measurements).

Motivation

- Higher order polyharmonic operators ($m \geq 2$) occur in areas of physics and geometry such as
  - Kirchoff plate equation in the theory of elasticity.
  - Paneitz-Branson operator in conformal geometry.
- The biharmonic equation models the displacements of a thin plate clamped near its boundary, the stresses in an elastic body and the stream function in creeping flow of a viscous incompressible fluid.

Set up

Let first order perturbation A be in $W^{-\frac{m}{2}+1,p'}(\mathbb{R}^n) \cap \mathcal{E}'(\bar{\Omega})$, where

$$\begin{cases} p' \in [2n/m, \infty) & \text{if} \quad m < n, \\ p' \in (2, \infty) & \text{if} \quad m = n \quad \text{or} \quad m = n+2, \\ p' \in [2, \infty) & \text{otherwise.} \end{cases} \tag{A}$$

For a fixed δ with $0 < \delta < \frac{1}{2}$, let the zeroth order perturbation q be in $W^{-\frac{m}{2}+\delta,r'}(\mathbb{R}^n) \cap \mathcal{E}'(\bar{\Omega})$, where

$$\begin{cases} r' \in [2n/(m-2\delta), \infty), & \text{if} \quad m < n, \\ r' \in [2n/(m-2\delta), \infty), & \text{if} \quad m = n, \\ r' \in [2, \infty), & \text{if} \quad m \geq n+1. \end{cases} \tag{q}$$

Solving the forward problem

- Define bi-linear forms B_A and b_q on $H^m(\Omega)$ as

$$B_A(u,v) := B_A^{\mathbb{R}^n}(\tilde{u},\tilde{v}) := \langle A, \tilde{v} D\tilde{u}\rangle$$
$$b_q(u,v) := b_q^{\mathbb{R}^n}(\tilde{u},\tilde{v}) := \langle q, \tilde{u}\tilde{v}\rangle$$

  for all $u, v \in H^m(\Omega)$, where $\langle \cdot, \cdot \rangle$ denotes the distributional duality on $\mathbb{R}^n$ and $\tilde{u}, \tilde{v} \in H^m(\mathbb{R}^n)$ are any extensions of u and v, respectively.

- Define $D_A(u)$ and $m_q(u)$ as

$$\langle D_A(u), \psi\rangle_\Omega = B_A(u,\psi)$$
$$\langle m_q(u), \psi\rangle_\Omega = b_q(u,\psi), \quad \psi \in C_0^\infty(\Omega),$$

- $D_A(u) \sim A \cdot Du,\ m_q(u) \sim qu.$

- We first show that D_A and m_q are bounded $H^m(\Omega) \to H^{-m}(\Omega)$ and that

$$\begin{aligned}\mathcal{L}_{A,q} = (-\Delta)^m + D_A + m_q : H^m(\Omega) \to & H^{-m}(\Omega) \\ = & (H_0^m(\Omega))'\end{aligned}$$

  is a Fredholm operator with index 0.

- For $f = (f_0, f_1, ..., f_{m-1}) \in \prod_{j=0}^{m-1} H^{m-j-1/2}(\partial\Omega)$, consider (BVP). If 0 is not in the spectrum of $\mathcal{L}_{A,q}$, we show (BVP) has a unique solution $u \in H^m(\Omega)$.

DN map

- Define the Dirichlet-to-Neumann map $\mathcal{N}_{A,q}$ weakly as

$$\langle N_{A,q} f, \bar{h} \rangle_{\partial\Omega} = \sum_{|\alpha|=m} \frac{m!}{\alpha!} (D^\alpha u, D^\alpha v_h)_{L^2(\Omega)} + B_A(u, \bar{v}_h) + b_q(u, \bar{v}_h),$$

  where $h = (h_0, h_1, \ldots, h_{m-1}) \in H^{m-j-1/2}(\partial\Omega)$, $v_h \in H^m(\Omega)$ is any extension of h so that $\gamma v_h = h$.

- $\mathcal{N}_{A,q}$ is well-defined and bounded as an operator

$$\prod_{j=0}^{m-1} H^{m-j-1/2}(\partial\Omega) \to \left(\prod_{j=0}^{m-1} H^{m-j-1/2}(\partial\Omega) \right)'$$

Main result

Theorem (Assylbekov-I '17)

Let $\Omega \subset \mathbb{R}^n$, $n \geq 3$ be a bounded open set with C^∞ boundary, and let $m \geq 2$ be an integer. Let $0 < \delta < 1/2$. Suppose that A_1, A_2 satisfy (A) *and q_1, q_2 satisfy* (q) *and* 0 *is not in the spectrum of $\mathcal{L}_{A_1,q_1}$ and $\mathcal{L}_{A_2,q_2}$. If $\mathcal{N}_{A_1,q_1} = \mathcal{N}_{A_2,q_2}$, then $A_1 = A_2$ and $q_1 = q_2$.*

Prior work

Krupchyk, Lassas and Uhlmann in 2014 proved a similar result for $A \in W^{1,\infty}(\mathbb{R}^n) \cap \mathcal{E}'(\bar{\Omega})$ and $q \in L^\infty(\Omega)$.

Assylbekov in 2016 relaxed regularity assumptions of A and q for $m < n$.

Our main contribution is to remove the restriction $m < n$ and enlarge the class to which A and q belong even further.

Key steps in the proof

- Let $\Omega^{'}$ be a smooth bounded open set such that $\Omega \subset\subset \Omega'$. If $\mathcal{N}_{A_1,q_1} = \mathcal{N}_{A_2,q_2}$, then $\mathcal{N}'_{A_1,q_1} = \mathcal{N}'_{A_2,q_2}$.

(IId) If $\mathcal{N}'_{A_1,q_1} = \mathcal{N}'_{A_2,q_2}$, then

$$\langle D_{A_2-A_1}(u_2), \bar{v}\rangle_{\Omega'} + \langle m_{q_2-q_1}(u_2), \bar{v}\rangle_{\Omega'} = 0$$

for any $u_2, v \in H^m(\Omega^{'})$ satisfying $\mathcal{L}_{A_2,q_2} u_2 = 0$ in $\Omega^{'}$ and $\mathcal{L}^*_{A_1,q_1} v = 0$ in $\Omega^{'}$, respectively.

- $\mathcal{L}^*_{A,q} = \mathcal{L}_{\bar{A},\bar{q}+D\cdot\bar{A}}$ is the formal adjoint of $\mathcal{L}_{A,q}$.

Carleman estimates

Before we go on to exploit the integral identity to get the desired uniqueness, we need some ~~coffee~~ estimates.

- Carleman estimates : For $0 < h \ll 1$,we have

$$||u||_{H^{m/2}_{scl}(\mathbb{R}^n)} \lesssim \frac{1}{h^m}||e^{\phi/h}(h^{2m}\mathcal{L}_{A,q})e^{-\phi/h}u||_{H^{-3m/2}_{scl}(\mathbb{R}^n)}$$

  for all $u \in C_0^\infty(\Omega^{'})$.

- Here, ϕ is the so-called Limiting Carleman Weight for $-h^2\Delta$ in $\tilde{\Omega} \supset\supset \Omega^{'}$ and $H^s_{scl}(\mathbb{R}^n)$ are certain weighted Sobolev spaces.

CGO solutions

Proposition

Let $\zeta \in \mathbb{C}^n$ be such that $\zeta \cdot \zeta = 0$, $\zeta = \zeta_0 + \zeta_1$ with ζ_0 independent of h and $\zeta_1 = \mathcal{O}(h)$ as $h \to 0$. For all $h > 0$ small enough, there exists $u(x, \zeta; h) \in H^{m/2}(\Omega^{'})$ solving $\mathcal{L}_{A,q} u = 0$ of the form

$$u(x, \zeta; h) = e^{\frac{ix\cdot\zeta}{h}}(a(x, \zeta_0) + h^{m/2} r(x, \zeta; h)),$$

where $a(\cdot, \zeta_0) \in C^\infty(\overline{\Omega^{'}})$ satisfies

$$(\zeta_0 \cdot \nabla)^2 a = 0 \quad \text{in} \quad \Omega^{'}.$$

and the correction term r is such that $||r||_{H^{m/2}_{scl}(\Omega^{'})} = \mathcal{O}(1)$ as $h \to 0$.

- Let $\xi, \mu_1, \mu_2 \in \mathbb{R}^n$ such that $|\mu_1| = |\mu_2| = 1$ and $\mu_1 \cdot \mu_2 = \xi \cdot \mu_1 = \xi \cdot \mu_2 = 0$. For $h > 0$, set

$$\zeta_2 = \frac{h\xi}{2} + \sqrt{1 - h^2\frac{|\xi|^2}{4}}\mu_1 + i\mu_2,$$
$$\zeta_1 = -\frac{h\xi}{2} + \sqrt{1 - h^2\frac{|\xi|^2}{4}}\mu_1 - i\mu_2.$$

- We have $\zeta_2 = \mu_1 + i\mu_2 + \mathcal{O}(h)$, $\zeta_1 = \mu_1 - i\mu_2 + \mathcal{O}(h)$, $\zeta_j \cdot \zeta_j = 0$, $j = 1, 2$ and $\zeta_2 - \bar{\zeta}_1 = h\xi$.

- We use $n \geq 3$ at this step in the proof.

Proof of Approximation Result

- Construct $u_2(\cdot,\zeta_2;h)$ and$v(\cdot,\zeta_1;h)$ in $H^m(B)$ solving $\mathcal{L}_{A_2,q_2}u_2 = 0$ and $\mathcal{L}^*_{A_1,q_1}v = 0$ in $\Omega^{'} = B$ and plug them in to the integral identity (IId).
- Let $h \to 0$ to get $A_1 = A_2$.
- To show $q_1 = q_2$, plug in $A_1 = A_2$ and $a_1 = a_2 = 1$ in (IId) and let $h \to 0$.

Summary

- CGO solutions differ from the one in Krupchyk, Lassas and Uhlmann. In particular, we have a decay of $h^{\frac{m}{2}}$ which is stronger.
- Using continuity of multiplication in certain Sobolev spaces and decay of $h^{\frac{m}{2}}$ for the remainder term we get the regularity and integrability indices for A and q by working backwards.
- For $m = 1$, there is a natural obstruction to uniqueness. We can expect uniqueness only modulo a gauge invariance. However if $A = 0$, one can recover q uniquely.