Review Problems

Math 324F
Advanced Multivariable Calculus

Part 1

1) Find the volume of the solid tetrahedron with vertices $(0,0,0)$, $(0,0,1)$, $(0,2,0)$ and $(2,2,0)$.

Answer: $\frac{2}{3}$

2) Find the volume of the solid bounded by the cylinder $x^2 + y^2 = 4$ and the planes $z = 0$ and $y + z = 3$.

Answer: 12π

3) Find the volume of one of the wedges cut from the cylinder $x^2 + 9y^2 = a^2$ by the planes $z = 0$ and $z = mx$.

Answer: $\frac{2}{9}ma^3$

4) Find the volume of the solid above the paraboloid $z = x^2 + y^2$ and below the half cone $z = \sqrt{x^2 + y^2}$.

Answer: $\frac{\pi}{6}$

5) Compute $\iiint\limits_E y^2 z^2 \, dV$ where E is bounded by the paraboloid $x = 1 - y^2 - z^2$ and the plane $x = 0$.

Answer: $\frac{\pi}{96}$

6) Compute $\iiint\limits_E yz \, dV$ where E lies above the plane $z = 0$, below the plane $z = y$, and inside the cylinder $x^2 + y^2 = 4$.

Answer: $\frac{64}{15}$

7) Evaluate $\iiint\limits_H z^3\sqrt{x^2+y^2+z^2}\,dV$ where H is the solid hemishphere that lies above the xy plane and has center the origin and radius 1.

Answer: $\frac{\pi}{14}$

8) The center of mass $(\bar{x},\bar{y},\bar{z})$ of a solid E is defined as follows :

$\bar{x} = \dfrac{\iiint\limits_E x\rho(x,y,z)\,dV}{\iiint\limits_E \rho(x,y,z)\,dV}$

$\bar{y} = \dfrac{\iiint\limits_E y\rho(x,y,z)\,dV}{\iiint\limits_E \rho(x,y,z)\,dV}$

$\bar{z} = \dfrac{\iiint\limits_E z\rho(x,y,z)\,dV}{\iiint\limits_E \rho(x,y,z)\,dV}$

where $\rho(x,y,z)$ is the density distribution of the solid E.

Find the center of mass of the solid tetrahedron with vertices $(0,0,0)$, $(1,0,0)$, $(0,2,0)$ and $(0,0,3)$ and density function $\rho(x,y,z) = x^2+y^2+z^2$.

Answer: $(\frac{4}{21},\frac{11}{21},\frac{8}{7})$

9) Rewrite the integral $\int\limits_{-1}^{1}\int\limits_{x^2}^{1}\int\limits_{0}^{1-y} f(x,y,z)\,dz\,dy\,dx$ as an iterated integral in the order $dx\,dy\,dz$.

Answer: $\int\limits_0^1\int\limits_0^{1-z}\int\limits_{-\sqrt{y}}^{\sqrt{y}} f(x,y,z)\,dx\,dy\,dz$

10) Give 5 other iterated integrals that are equal to $\int\limits_0^2\int\limits_0^{y^3}\int\limits_0^{y^2} f(x,y,z)\,dz\,dx\,dy$.

Answer: $\int\limits_0^8\int\limits_{x^{1/3}}^{2}\int\limits_0^{y^2} f(x,y,z)\,dz\,dy\,dx = \int\limits_0^4\int\limits_{\sqrt{z}}^{2}\int\limits_0^{y^3} f(x,y,z)\,dx\,dy\,dz = \int\limits_0^2\int\limits_0^{y^2}\int\limits_0^{y^3} f(x,y,z)\,dx\,dz\,dy$

$= \int\limits_0^8\int\limits_0^{x^{2/3}}\int\limits_{x^{1/3}}^{2} f(x,y,z)\,dy\,dz,\,dx + \int\limits_0^8\int\limits_{x^{2/3}}^{4}\int\limits_{\sqrt{z}}^{2} f(x,y,z)\,dy\,dz\,dx$

$= \int\limits_0^4\int\limits_0^{z^{3/2}}\int\limits_{\sqrt{z}}^{2} f(x,y,z)\,dy\,dx\,dz + \int\limits_0^4\int\limits_{z^{3/2}}^{8}\int\limits_{x^{1/3}}^{2} f(x,yz)\,dy\,dx\,dz.$

11) Use the transformation $u = x-y$ and $v = x+y$ to evaluate $\iint\limits_R \frac{x-y}{x+y}\,dA$ where R is the square with vertices $(0,2)$, $(1,1)$, $(2,2)$ and $(1,3)$.

Answer: $-\ln 2$

12) Use the transformation $x = u^2$, $y = v^2$, $z = w^2$ to find the volume of the region bounded by the surface $\sqrt{x} + \sqrt{y} + \sqrt{z} = 1$ and the coordinate planes.

Answer: $\frac{1}{90}$

13) Let f be continuous on $[0,1]$ and let R be the triangular region with vertices $(0,0)$, $(1,0)$ and $(0,1)$. Show that $\iint\limits_R f(x+y)\,dA = \int\limits_0^1 u f(u)\,du$

14) Assume all the given functions are differentiable. If $z = f(x,y)$, where $x = r\cos\theta$ and $y = r\sin\theta$, find $\frac{\partial z}{\partial r}$ and $\frac{\partial z}{\partial \theta}$ and show that

$(\frac{\partial z}{\partial x})^2 + (\frac{\partial z}{\partial y})^2 = (\frac{\partial z}{\partial r})^2 + \frac{1}{r^2}(\frac{\partial z}{\partial \theta})^2$.

15) One side of a triangle is increasing at a rate of 3 cm/s and a second side is decreasing at a rate of 2 cm/s. If the area of the triangle remains constant, at what rate does the angle between the sides change when the first side is 20 cm long, the second side is 30 cm, and the angle is $\frac{\pi}{6}$.

Answer: $\frac{-1}{12\sqrt{3}}$ radians/s

16) Use chain rule to find out the indicated partial derivative for $z = x^4 + x^2 y$,
$x = s + 2t - u$, $y = stu^2$;
$\frac{\partial z}{\partial s}$, $\frac{\partial z}{\partial t}$, $\frac{\partial z}{\partial u}$ when $s = 4$, $t = 2$, $u = 1$.

Answer: 1582, 3164, -700

17) Find all the points at which the direction of fastest change of $f(x,y) = x^2 + y^2 - 2x - 4y$ is $\mathbf{i} + \mathbf{j}$.

Answer: Any point on the line $y - x = 1$

18) Suppose that over a certain region of space the electric potential V is given by $V(x,y,z) = 5x^2 - 3xy + xyz$.
(a) Find the rate of change of the potential at $P(3,4,5)$ in the direction of the vector $\mathbf{i} + \mathbf{j} - \mathbf{k}$.
(b) In which direction does V change most rapidly at P?
(c) What is the maximum rate of change at P?

Answer: a) $\frac{32}{\sqrt{3}}$, b) 38$\mathbf{i}$ + 6$\mathbf{j}$ + 12$\mathbf{k}$, c) $2\sqrt{406}$

Part 2

19) Find the equation of the tangent plane and normal line to the surface $xyz^2 = 6$ at $(3,2,1)$.

Answer: a) $2x+3y+12z=24$, b) $(2t+3,3t+2,12t+1)$

20) Show that every plane that is tangent to the cone $z^2 = x^2 + y^2$ passes through the origin.

21) Find the work done by the force field $\mathbf{F}(x,y) = x\mathbf{i} + (y+2)\mathbf{j}$ in moving an object along an arch of the cycloid $\mathbf{r}(t) = (t-\sin(t))\mathbf{i} + (1-\cos(t))\mathbf{j}$, $0 \le t \le \pi/2$.

Answer: $2\pi^2$

22) An object with mass m moves with position function $\mathbf{r}(t) = a\sin(t)\mathbf{i} + b\cos(t)\mathbf{j} + ct\mathbf{k}$, $0 \le t \le \pi/2$. Find the work done on the object during this time.

(Hint: What will $\mathbf{F}$ be?)

Answer: $\frac{m}{2}(b^2 - a^2)$

23) Show that the line integral is independent of the path and evaluate the integral. $\int\limits_C 2xe^{-y}\,dx + (2y - x^2e^{-y}\,dy$ where C is any path from $(1,0)$ to $(2,1)$.

Answer: -2

24) Evaluate the line integral $\int\limits_C y^3\,dx + x^2\,dy$ where C is the arc of the parabola $x = 1-y^2$ from $(0,-1)$ to $(0,1)$.

Answer: 4/15

25) Use Green's theorem to evaluate the line integral along the given positively oriented curve. $\int\limits_C (y + e^{\sqrt{x}})\,dx + (2x + \cos(y^2))\,dy$ where C is the boundary of the region enclosed by the parabolas $y = x^2$ and $x = y^2$.

Answer: 1/3

26) Use Green's theorem to evaluate the line integral along the given positively oriented curve. $\int\limits_C \cos(y)\,dx + x^2\sin(y)\,dy$ where C is the rectangle with vertices $(0,0)$, $(5,0)$, $(5,2)$ and $(0,2)$.

Answer: $30(1-\cos 2)$

27) Use Greens theorem to evaluate $\mathbf{F} \cdot d\mathbf{r}$ where $\mathbf{F}(x,y) = (y - \cos y)\mathbf{i} + (x \sin y)\mathbf{j}$ and C is the circle $(x-3)^2 + (y+4)^2 = 4$ oriented clockwise.

Answer: 4π

28) Use Greens theorem to evaluate $\mathbf{F} \cdot d\mathbf{r}$ where $\mathbf{F}(x,y) = (e^{-x} + y^2)\mathbf{i} + (e^{-y} + x^2)\mathbf{j}$ and C consists of the arc of the curve $y = \cos x$ from $(-\pi/2, 0)$ to $(\pi/2, 0)$, and the line segment from $(\pi/2, 0)$ to $(-\pi/2, 0)$.

Answer: $\pi/2$

29) Find the curl and divergence of $\mathbf{F}(x,y,z) = e^{xy} \sin z\,\mathbf{j} + y \tan^{-1}(x/z)\,\mathbf{k}$.

Answer: $curl\ \mathbf{F} = [\tan^{-1}(x/z) - e^{xy}\cos z]\mathbf{i} - \frac{yz}{x^2+z^2}\mathbf{j} + ye^{xy}\sin z\mathbf{k}$; $div\ \mathbf{F} = xe^{xy}\sin z - \frac{xy}{x^2+z^2}$

30) Determine if the given vector field $\mathbf{F}$ is conservative or not. If it is conservative, find a scalar function f so that $\mathbf{F} = \nabla f$. $\mathbf{F}(x,y,z) = y^2z^3\mathbf{i} + 2xyz^3\mathbf{j} + 3xy^2z^2\mathbf{k}$.

Answer: Yes. $\nabla f = \mathbf{F}$ where $f(x,y,z) = xy^2z^3 + C$

31) Is there a vector field $\mathbf{G}$ on $\mathbb{R}^3$ such that $curl\,\mathbf{G} = < xyz, -y^2z, yz^2 >$? Explain.

Answer: No

32) Find the area of part of the sphere $x^2 + y^2 + z^2 = 4z$ that lies inside the paraboloid $z = x^2 + y^2$.

Answer: 4π

33) Find the area of part of the sphere $x^2 + y^2 + z^2 = a^2$ that lies inside the cylinder $x^2 + y^2 = ax$.

Answer: $2a^2(\pi - 2)$

34) Find the area of the surface $\frac{2}{3}(x^{3/2} + y^{3/2})$, $0 \le x \le 1$, $0 \le y \le 1$.

Answer: $\frac{4}{15}(3^{2.5} - 2^{3.5} + 1)$

35) Find a parametric representation for the part of the plane $z = x + 3$ that lies inside the cylinder $x^2 + y^2 = 1$.

Answer: $x = r\cos\theta, y = r\sin\theta, z = 3 + r\cos\theta,\ 0 \le r \le 1,\ 0 \le \theta \le 2\pi$

36) Evaluate the surface integral $\iint\limits_S x^2z^2\,dS$, where S is part of the cone $z^2 = x^2 + y^2$ that lies between the planes $z = 1$ and $z = 3$.

Answer: $\frac{364\sqrt{2}\pi}{3}$

37) Evaluate the surface integral $\iint\limits_S (x^2+y^2+z^2)\,dS$, where S is part of the cylinder $x^2+y^2=9$ between the planes $z=0$ and $z=2$, together with its top and bottom disks.

Answer: 241π

38) Evaluate the surface integral $\iint\limits_S \mathbf{F}\cdot\mathbf{dS}$ with $\mathbf{F}(x,y,z) = xy\mathbf{i} + 4x^2\mathbf{j} + yz\mathbf{k}$, S is the surface $z = xe^{xy}$, $0 \leq x, y \leq 1$ with positive (upward) orientation.

[Hint: What would up upward orientation for a surface given by the graph of a function $z = g(x,y)$ be?]

Answer: 1 - e

39) Evaluate $\iint\limits_S \mathbf{F}\cdot\mathbf{dS}$, where $\mathbf{F}(x,y,z) = x^2\mathbf{i} + xy\mathbf{j} + z\mathbf{k}$ and S is part of the paraboloid $z = x^2+y^2$ below the plane $z=1$ with upward (positive) orientation.

[Hint: You can use divergence theorem but you need to be a little careful]

Answer: $\pi/2$

40) Verify that the Stoke's theorem is valid for the vector field $F(x,y,z) = x^2\mathbf{i}+y^2\mathbf{j}+z^2\mathbf{k}$ where S is part of the paraboloid $z = 1-x^2-y^2$ that lies above the xy plane and has upward orientation.

Answer: 0

41) Use Stoke's theorem to evaluate $\iint\limits_S curl\,\mathbf{F}\cdot\mathbf{dS}$ where $\mathbf{F}(x,y,z) = x^2yz\mathbf{i}+yz^2\mathbf{j}+z^3e^{xy}\mathbf{k}$, S is part of the sphere $x^2+y^2+z^2=5$ that lies above the plane $z=1$, and is oriented upward.

Answer: -4π

42) Use the Divergence theorem to calculate the surface integral $\iint\limits_S \mathbf{F}\cdot\mathbf{dS}$, where $\mathbf{F}(x,y,z) = x^3\mathbf{i} + y^3\mathbf{j} + z^3\mathbf{k}$ and S is the surface of the solid bounded by the cylinder $x^2+y^2=1$ and the planes $z=0$ and $z=2$.

Answer: 11π

43) Verify that the Divergence theorem is true for the vector field $\mathbf{F}(x,y,z) = x\mathbf{i}+y\mathbf{j}+z\mathbf{k}$, where E is the unit ball $x^2+y^2+z^2 \leq 1$.

Answer: 4π

44) Use the Divergence theorem to calculate the surface integral $\iint\limits_S curl\,\mathbf{F}\cdot\mathbf{dS}$, where $\mathbf{F}(x,y,z) = x\sin y\mathbf{i} + x\cos y\mathbf{j} - xz\sin y\mathbf{k}$ where S is the surface $x^8 + y^8 + z^8 = 1$

Answer: 0

45) Use Divergence theorem to evaluate $\iint\limits_S (2x + 2y + z^2)\,dS$ where S is the sphere $x^2 + y^2 + z^2 = 1$.

Answer: $\frac{4\pi}{3}$