EIGENVALUES AND EIGENVECTORS

$A \in \mathbb{R}^{n \times n}$

2 type of algorithms $\Rightarrow$ partial $\Rightarrow$ evaluate λ_1, λ_{max} $\Rightarrow$ relative eigenvectors

global $\Rightarrow$ complete spectrum + eigenvectors

Recap

$A \in \mathbb{R}^{n \times n}$ we want to find λ and a vector x s.t.

$$Ax = \lambda x$$

x is not unique $\forall\ \alpha x$

$$p_A(\lambda) = \det(A - \lambda I)$$

POWER METHOD (Metodo delle potenze)

Local method

$A \in \mathbb{R}^{n \times n}$

$$|\lambda_1| > |\lambda_2| > \dots |\lambda_n|$$

$\lambda_1 \neq \lambda_2$

x_1 eigenvector associated to λ_1

If the eigenvectors of A are linearly indipendent

$x^{(0)}$, set $y^0 = \frac{x^{(0)}}{\|x^{(0)}\|}$

$$x^{(k)} = A y^{(k-1)} \qquad y^k = \frac{x^{(k)}}{\|x^{(k)}\|}$$

$$\lambda^k = (y^{(k)})^T A y^{(k)}$$

$$y^{(k)} = \beta^{(k)} A^k y^{(0)}$$

$$\beta^{(k)} = \left(\prod_{i=1}^{k} \|x^i\| \right)^{-1}$$

$\{y^{(k)}\}$ for $k \to \infty$ alligns along x_1

$$|\lambda^k - \lambda^{k-1}| < \varepsilon |\lambda^k|$$

Convergence Analysis

$x_1, \dots, x_n$ of A

$\hookrightarrow$ a basis in $\mathbb{R}^n$

I can write $x^{(0)}$ and $y^{(0)}$

$$x^{(0)} = \sum_{i=1}^{n} \alpha_i x_i \qquad y^{(0)} = \beta^{(0)} \sum_{i=1}^{n} \alpha_i x_i$$

$$\beta^{(0)} = \frac{1}{\|x^{(0)}\|} \qquad \alpha_1 \in \mathbb{R}$$

$$x^{(1)} = A y^{(0)} = \beta^{(0)} A \sum_{i=1}^{n} \alpha_i x_i = \beta^{(0)} \sum_{i=1}^{n} \alpha_i \lambda_i x_i$$

$$y^{(1)} = \beta^{(1)} \sum_{i=1}^{n} \alpha_i \lambda_i x_i \qquad \beta^{(1)} = \frac{1}{\|x^{(0)}\| \|x^{1}\|}$$

$$y^{(k)} = \lambda_1^k \beta^k \left(a_1 x_1 + \sum_{i=2}^{n} a_i \left(\frac{\lambda_i}{\lambda_1} \right)^k x_i \right)$$

$$\left| \lambda_i / \lambda_1 \right| < 1 \quad \forall \quad i = 2, \dots, n$$

INVERSE POWER METHOD

eigenvalues of A^{-1} are the reciprocal of the eigenvalues of A

use the power method on A^{-1}

$$x^{(0)} \qquad y^{(0)} = \frac{x^{(0)}}{\|x^{(0)}\|}$$

$$x^{(k)} = A^{-1} y^{k-1}$$

$$y^k = \frac{x^k}{\|x^k\|} \qquad \mu^k = y^{(k)^T} A^{-1} y^{(k)}$$

$$\lim_{k \to \infty} \mu^{(k)} = \frac{1}{\lambda^n}$$

$$\left(\mu^{(k)} \right)^{-1} \to \lambda_n \text{ for } k \to \infty$$

$$\left[A x^{(k)} = y^{(k-1)} \right]$$

→ LU decompositions

POWER METHOD WITH SHIFT

Compute the eigenvalue which is closer to a given $\mu \in \mathbb{R}$

λ_μ $\qquad A_\mu = (A - \mu I)$

$$\lambda(A_\mu) = \lambda(A) - \mu$$

$\lambda_{min}(A_\mu) \Rightarrow$ eigenvalue of smallest value of A_μ

λ_μ of A closest to μ

$$\lambda_\mu = \lambda_{min}(A_\mu) + \mu$$

I apply inverse power method to A_μ

Gershgorin Theorems