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Worst-case robust decisions for multi-period mean–variance portfolio optimization

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Abstract

In this paper, we extend the multi-period mean–variance optimization framework to worst-case design with multiple rival return and risk scenarios. Our approach involves a min–max algorithm and a multi-period mean–variance optimization framework for the stochastic aspects of the scenario tree. Multi-period portfolio optimization entails the construction of a scenario tree representing a discretised estimate of uncertainties and associated probabilities in future stages. The expected value of the portfolio return is maximized simultaneously with the minimization of its variance. There are two sources of further uncertainty that might require a strengthening of the robustness of the decision. The first is that some rival uncertainty scenarios may be too critical to consider in terms of probabilities. The second is that the return variance estimate is usually inaccurate and there are different rival estimates, or scenarios. In either case, the best decision has the additional property that, in terms of risk and return, performance is guaranteed in view of all the rival scenarios. The ex-ante performance of min–max models is tested using historical data and backtesting results are presented.

Keywords: Stochastic programming; Nonlinear programming; Risk management; Finance; Worst-case design; Uncertainty modelling; Scenario tree

1. Introduction

In financial portfolio management, the maximization of return for a level of risk is the accepted approach to decision making. A classical example is the single-period mean–variance optimization model in which expected portfolio return is maximized and risk measured by the variance of portfolio return is minimized (Markowitz, 1952). Consider n risky assets with random rates of return $r_1, r_2, \ldots, r_n$. Their expected values are denoted by $E(r_i)$ or $\bar{r}_i$, for $i = 1, \ldots, n$. The single period model of Markowitz considers a portfolio of n assets defined in terms of a set of weights w^i for $i = 1, \ldots, n$, which sum to unity. Given an expected rate of portfolio return $\bar{r}_p$, the optimal portfolio is defined in terms of the solution of the following quadratic programming problem:

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$$\min_{\mathbf{w}}\{\langle \mathbf{w}, \Lambda\mathbf{w}\rangle | \mathbf{w}'\bar{\mathbf{r}} \geqslant \bar{r}_p, \mathbf{1}'\mathbf{w} = 1, \mathbf{w} \geqslant 0\},$$

where Λ is the covariance matrix of asset returns. The quadratic program yields the minimum variance portfolio. Note that the classical stochastic linear programming formulation maximizes the expected return but takes no account of risk.

A multi-period framework to reformulate the single stage asset allocation problem as an adaptive multi-period decision process has been developed by using multi-period stochastic programming, see for example Birge and Louveaux (1997), Kall (1976), Kall and Wallace (1994) and Prekopa (1995). In the multi-stage case, the investor decides based on expectations and/or scenarios up to some intermediate times prior to the horizon. These intermediate times correspond to rebalancing or restructuring periods. The mean or the variance of total wealth at the end of the investment horizon is modelled by either linear stochastic programming or quadratic stochastic programming in Gülpınar et al. (2002, 2003).

Multi-period portfolio optimization entails the construction of a scenario tree representing a discretised estimate of uncertainties and associated probabilities in future stages. The multi-period stochastic mean–variance approach takes account of the approximate nature of the discrete set of scenarios by considering a variance term around the return scenarios. Hence, uncertainty on return values of instruments is represented by a discrete approximation of a multivariate continuous distribution as well as the variability due to the discrete approximation. The mean–variance framework is based on a single forecast of return and risk. In reality, however, it is often difficult or impossible to rely on a single forecast; there are different rival risk and return estimates, or scenarios. Two sources of further uncertainty might require a strengthening of the robustness of the mean–variance decision. The first is that some rival uncertainty scenarios may be too critical to consider in terms of probabilities. A worst-case optimal strategy would yield the best decision determined simultaneously with the worst-case scenario. The second is that the return variance estimate is usually inaccurate and there are different rival estimates, or scenarios. A worst-case optimal strategy protects against risk of adopting the investment strategy based on the wrong scenario. In either case, the best decision has the additional property that, in terms of risk and return, *performance is guaranteed* in view of all the rival risk and return scenarios. The min–max optimal performance will improve for any scenario other than the worst-case. This guaranteed performance and noninferiority property of min–max are discussed further in Section 3.

In this paper, multi-period mean–variance optimization framework is extended to the robust worst-case design problem with multiple return and risk scenarios. Our approach involves a min–max algorithm and a multi-period mean–variance optimization framework for the stochastic aspects of the scenario tree, (Gülpınar and Rustem, 2004). Since optimal investment decision is based on the min–max strategy, the robustness is ensured by the non-inferiority of min–max. The optimal portfolio is constructed (relative to benchmark) simultaneously with the worst-case to take account of all rival scenarios. The portfolio is balanced at each time period incorporating scalable (not fixed) transaction cost and its relative performance is measured in terms of returns and the volatility of returns.

The rest of the paper is organized as follows. In Section 2, the multi-period mean variance optimization problem is described. In Section 3, we introduce multi-period discrete min–max formulations of multi-period mean–variance optimization problem for robust, optimal investment strategies in view of rival return and risk scenarios (which are input scenarios in the min–max formulation). Section 4 focuses on the generation of scenario tree and forecasting rival risk and return scenarios. In Section 5, we present our computational results which are based on worst-case risk-return frontiers and backtesting (out-of-sample).

2. Problem statement

The central problem considered in this paper is to determine multi-period discrete-time optimal portfolio strategies over a given finite investment horizon. Therefore, we start with the definition of returns and uncertainties. Subsequently, we present the model constraints, the expected return and risk formulations based on the scenario tree. For the more details of multi-period mean–variance optimization framework, the reader is referred to Gülpınar et al. (2003).

We consider n risky assets and construct a portfolio over an investment horizon T. After the initial investment ($t = 0$), the portfolio may be restructured at discrete times $t = 1, \ldots, T-1$ in terms of both return and

Table 1
Notation

Input parameters	
n	Number of investment assets
T	Planning horizon
$\mathbf{h}$	Current portfolio position (i.e. at $t = 0$, before optimization)
$\bar{\mathbf{w}}_*$	Market benchmark
$\mathbf{c}_b$	Vector of unit transaction costs for buying
$\mathbf{c}_s$	Vector of unit transaction costs for selling
α_t	Weight for each time period
γ	Scaling constant determining the level of risk aversion
λ	Penalty parameter
$\Lambda_i \in \mathscr{R}^{n\times n}$	Covariance matrices for $i = 1,\ldots,I_\mathbf{e}$ (or I_t and I_k)
$\mathscr{W}_t$	Expected total wealth at time $t = 1,\ldots,T$
Scenario tree	
$\boldsymbol{\rho}_t$	Vector of stochastic data observed at time t, $t = 0,\ldots,T$
$\boldsymbol{\rho}^t$	$\equiv\{\boldsymbol{\rho}_0,\ldots,\boldsymbol{\rho}_t\}$—history of stochastic data up to t
$\mathscr{N}$	Set of all nodes in the scenario tree
$\mathscr{N}_t$	Set of nodes of the scenario tree representing possible events at time t
$\mathscr{N}_I$	$\equiv \mathscr{N} - (\mathscr{N}_0 \cup \mathscr{N}_T)$, i.e. set of all *interior* nodes of the scenario tree
s	Index denoting a scenario, i.e. path from root to leaf in the scenario tree
$\mathbf{e} \equiv (s,t)$	Index denoting an event (node of the scenario tree), which can be identified by an ordered pair of scenario and time period
$a(\mathbf{e})$	Ancestor of event $\mathbf{e} \in \mathscr{N}$ (parent in the scenario tree)
$p_\mathbf{e}$	Branching probability of event $\mathbf{e} : p_\mathbf{e} = \text{Prob}[\mathbf{e}\vert a(\mathbf{e})]$
$P_\mathbf{e}$	Probability of event $\mathbf{e}$: if $\mathbf{e} = (s,t)$, then $P_\mathbf{e} = \prod_{i=1\ldots t} p_{(s,i)}$
$\mathbf{r}_t(\boldsymbol{\rho}^t)$	Stochastic vector of return values for the n assets, $t = 1,\ldots,T$
$\mathbf{r}_\mathbf{e}$	Stochastic realization of $\mathbf{r}_t$ in event $\mathbf{e} : \mathbf{r}_\mathbf{e} \sim N(\mathbf{r}_t(\boldsymbol{\rho}^t),\Lambda)$
$\hat{\mathbf{r}}_\mathbf{e}$	$\equiv E(\mathbf{r}_\mathbf{e}(\boldsymbol{\rho}_t\vert\boldsymbol{\rho}^{t-1}))$—expectation of $\mathbf{r}_t(\boldsymbol{\rho}^t)$ for event $\mathbf{e}$, conditional on $\boldsymbol{\rho}^{t-1}$
Decision variables	
$\mathbf{w}_*$	Decision vector indicating asset balances
$\mathbf{b}_*$	Decision vector of "buy" transaction volumes
$\mathbf{s}_*$	Decision vector of "sell" transaction volumes
v_*	Worst-case risk
μ	Worst-case return
$*$	Vectors $\mathbf{w}$, $\bar{\mathbf{w}}$, $\mathbf{b}$, and $\mathbf{s}$ can be indexed either with $t = 1,\ldots,T$ (in which case they represent stochastic quantities with an implied dependence on $\boldsymbol{\rho}^t$), or $\mathbf{e} \in \mathscr{N}$ (in which case they represent specific realizations of those quantities). Variable v is indexed with $\mathbf{e}$, t, and k (rival return scenarios)
$E[\cdot]$	Expectation with respect to ρ
$\mathbf{1}$	$\equiv(1,1,1,\ldots,1)'$
$\mathbf{u} \circ \mathbf{v}$	$\equiv(u_1v_1, u_2v_2,\ldots,u_nv_n)'$

risk, and redeemed at the end of the period ($t = T$). A description of our notation is given in Table 1. All quantities in boldface represent vectors in $\mathscr{R}^n$. The transpose of a vector or matrix will be denoted with the symbol $'$.

2.1. Scenario tree

Let $\boldsymbol{\rho}^t \equiv \{\boldsymbol{\rho}_1,\ldots,\boldsymbol{\rho}_t\}$ be stochastic events at $t = 1,\ldots,T$. The decision process is non-anticipative (i.e. decision at a given stage does not depend on the future realization of the random events). The decision at t is dependent on $\boldsymbol{\rho}_{t-1}$. Thus, constraints on a decision at each stage involve past observations and decisions. A scenario is defined as a possible realization of the stochastic variables $\{\boldsymbol{\rho}_1,\ldots,\boldsymbol{\rho}_T\}$. Hence, the set of scenarios corresponds to the set of leaves of the scenario tree, $\mathscr{N}_T$, and nodes of the tree at level $t \geqslant 1$ (the set $\mathscr{N}_t$) correspond to possible realizations of $\boldsymbol{\rho}^t$. We denote a node of the tree (or event) by $\mathbf{e} = (s,t)$, where s is a scenario (path from root to leaf), and time period t specifies a particular node on that path. The root of the tree is $\mathbf{0} = (s,0)$ (where s can be any scenario, since the root node is common to all scenarios). The ancestor (parent) of event $\mathbf{e} = (s,t)$ is denoted $a(\mathbf{e}) = (s,t-1)$, and the branching probability $p_\mathbf{e}$ is the conditional probability of event $\mathbf{e}$, given its parent event $a(\mathbf{e})$. The path to event $\mathbf{e}$ is a partial scenario with probability $P_\mathbf{e} = \prod p_\mathbf{e}$ along

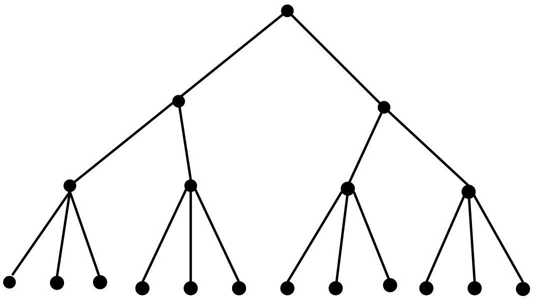

Fig. 1. A scenario tree.

that path; since probabilities $p_{\mathbf{e}}$ must sum to one at each individual branching, probabilities $P_{\mathbf{e}}$ will sum up to one across each layer of tree-nodes $\mathcal{N}_t$; $t = 0, 1, \ldots, T$.

Each node at a level t corresponds to a decision $\{\mathbf{w}_t, \mathbf{b}_t, \mathbf{s}_t\}$ which must be determined at time t, and depends in general on $\boldsymbol{\rho}^t$, the initial wealth $\mathbf{w}_0$ and past decisions $\{\mathbf{w}_j, \mathbf{b}_j, \mathbf{s}_j\}, j = 1, \ldots, t-1$. This process is adapted to $\boldsymbol{\rho}^t$ as $\mathbf{w}_t$, $\mathbf{b}_t$, $\mathbf{s}_t$ cannot depend on future events $\boldsymbol{\rho}_{t+1}, \ldots, \boldsymbol{\rho}_T$ which are not yet realized. Hence $\mathbf{w}_t = \mathbf{w}_t(\boldsymbol{\rho}^t)$, $\mathbf{b}_t = \mathbf{b}_t(\boldsymbol{\rho}^t)$, and $\mathbf{s}_t = \mathbf{s}_t(\boldsymbol{\rho}^t)$. However, for simplicity, we shall use the terms $\mathbf{w}_t$, $\mathbf{b}_t$ and $\mathbf{s}_t$, and assume their implicit dependence on $\boldsymbol{\rho}^t$. Notice that $\boldsymbol{\rho}_t$ can take only finitely many values. Thus, the factors driving the risky events are approximated by a discrete set of scenarios or sequence of events. Given the event history up to a time t, $\boldsymbol{\rho}^t$, the uncertainty in the next period is characterized by finitely many possible outcomes for the next observation $\boldsymbol{\rho}_{t+1}$. This branching process is represented using a scenario tree. An example of scenario tree with three time periods and 2-2-3 branching structure is presented in Fig. 1. We also model a continuous perturbation in addition to the discretised uncertainty; see also Frauendorfer (1995), so that the vector of return values at time t has a multivariate normal distribution, with mean $\mathbf{r}_t(\boldsymbol{\rho}^t)$, and specified covariance matrix Λ.

2.2. Mean–variance optimization

The single period mean–variance optimization problem can be extended to multi-stage programming. In this case, referring to Fig. 2, after the initial investment we can rebalance our portfolio (subject to any desired bounds) to maximize profit at the investment horizon and minimize the risk at discrete time periods and redeem at the end of the period.

An initial benchmark portfolio is specified as $\bar{\mathbf{w}}_0$ (possibly $= \mathbf{h}$), and benchmarks at later time periods derive from $\bar{\mathbf{w}}_0$ by accruing returns, but not allowing any reallocation:

$$\bar{\mathbf{w}}_t(\boldsymbol{\rho}^t) = \mathbf{r}_t(\boldsymbol{\rho}^t) \circ \bar{\mathbf{w}}_{t-1}(\boldsymbol{\rho}^{t-1}), \quad t = 1, \ldots, T.$$

Henceforth, $\bar{\mathbf{w}}_t$ will be referred to without its implicit dependence on $\boldsymbol{\rho}^t$.

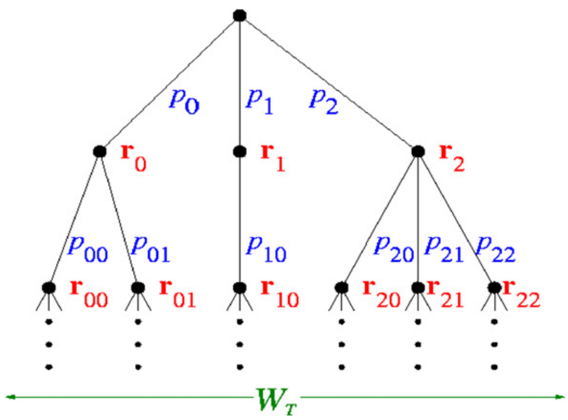

Fig. 2. Mean–variance optimization on a scenario tree.

At $t = 0$, the initial budget is normalized to 1. If the investor currently has holdings of assets $1, \ldots, n$, then vector $\mathbf{h}$ (scaled so that $\mathbf{1}'\mathbf{h} = 1$) represents his current position. If the investor currently has no holdings (wishing to buy in at time $t = 0$), then $\mathbf{h} = \mathbf{0}$. The allocation of the initial budget of 1 can be represented with the following constraints:

$$\mathbf{h} + (1 - \mathbf{c}_b)\mathbf{b}_0 - (1 + \mathbf{c}_s)\mathbf{s}_0 = \mathbf{w}_0, \tag{1}$$

$$\mathbf{1}'\mathbf{b}_0 - \mathbf{1}'\mathbf{s}_0 = 1 - \mathbf{1}'\mathbf{h}. \tag{2}$$

Constraint (2) enforces the initial budget of unity (whether it is new investment or re-allocation). The net total amount of buying makes up the shortfall of the original portfolio beneath the budget. It is important to note that our model allows money to be added to the portfolio in this way only in the initial time period.

The decision at time $t > 0$ is clearly dependent on $\boldsymbol{\rho}^t$ and $\mathbf{w}_t(\boldsymbol{\rho}^t)$ depends on the past and not the future. Therefore, the policy $\{\mathbf{w}_0, \ldots, \mathbf{w}_t\}$ is called non-anticipative. Furthermore, $\boldsymbol{\rho}_0$ is observed before the initial decision and is thus treated as deterministic information. The dynamic structure of the model is captured by investment strategy which are defined by asset weights at each interior node of the scenario tree as follows:

$$r_{\mathbf{e}} \circ \mathbf{w}_{a(\mathbf{e})} + (1 - \mathbf{c}_b) \circ \mathbf{b}_{\mathbf{e}} - (1 + \mathbf{c}_s) \circ \mathbf{s}_{\mathbf{e}} = \mathbf{w}_{\mathbf{e}}, \quad \mathbf{e} \in \mathcal{N}_I. \tag{3}$$

We require subsequent transactions (buy $= \mathbf{b}_t$, sell $= \mathbf{s}_t$) not to alter the wealth within the period t. Buy and sell decisions are made in view of (3) and subject to Eqs. (6)–(8). Hence, we have the condition

$$\mathbf{1}'\mathbf{b}_{\mathbf{e}} - \mathbf{1}'\mathbf{s}_{\mathbf{e}} = 0, \quad \mathbf{e} \in \mathcal{N}_I. \tag{4}$$

Since transactions of buying and selling at the last time period are not allowed, asset weights at $t = T$ are computed as

$$r_{\mathbf{e}} \circ \mathbf{w}_{a(\mathbf{e})} = \mathbf{w}_{\mathbf{e}}, \quad \mathbf{e} \in \mathcal{N}_T. \tag{5}$$

In the mean–variance optimization framework, bounds on decision variables prevent the short sale and enforce further restrictions imposed by the investor. Box constraints are defined on $\mathbf{w}_{\mathbf{e}}$, $\mathbf{b}_{\mathbf{e}}$, $\mathbf{s}_{\mathbf{e}}$ for each event $\mathbf{e} \in \mathcal{N}_I$ as

$$\mathbf{w}_{\mathbf{e}}^{\mathrm{L}} \leqslant \mathbf{w}_{\mathbf{e}} \leqslant \mathbf{w}_{\mathbf{e}}^{\mathrm{U}}, \tag{6}$$

$$\mathbf{0} \leqslant \mathbf{b}_{\mathbf{e}} \leqslant \mathbf{b}_{\mathbf{e}}^{\mathrm{U}}, \tag{7}$$

$$\mathbf{0} \leqslant \mathbf{s}_{\mathbf{e}} \leqslant \mathbf{s}_{\mathbf{e}}^{\mathrm{U}}. \tag{8}$$

The objective of an investor is to minimize portfolio risk while maximizing expected portfolio return on investment, or achieving a prescribed expected return. Given the possible events $\mathbf{e} \in \mathcal{N}_t$ (the discretisation of $\boldsymbol{\rho}_t$), the expected wealth at time t, relative to benchmark $\bar{\mathbf{w}}_t$, is given by

$$\begin{aligned}
\mathscr{W}_t &= \mathrm{E}[\mathbf{1}'(\mathbf{w}_t - \bar{\mathbf{w}}_t)] = \mathrm{E}[\mathbf{r}_t(\boldsymbol{\rho}_t|\boldsymbol{\rho}^{t-1})'(\mathbf{w}_{t-1} - \bar{\mathbf{w}}_{t-1})] \\
&= \mathrm{E}\left[\sum_{\mathbf{e} \in \mathcal{N}_t} P_{\mathbf{e}}(\mathbf{r}_{\mathbf{e}}'(\mathbf{w}_{a(\mathbf{e})} - \bar{\mathbf{w}}_{a(\mathbf{e})}))\right] \\
&= \sum_{\mathbf{e} \in \mathcal{N}_t} P_{\mathbf{e}}(\hat{\mathbf{r}}_{\mathbf{e}}'(\mathbf{w}_{a(\mathbf{e})} - \bar{\mathbf{w}}_{a(\mathbf{e})})),
\end{aligned} \tag{9}$$

where $\hat{\mathbf{r}}_{\mathbf{e}}$ and $\mathbf{w}_{\mathbf{e}}$ ($\mathbf{e} \in \mathcal{N}_t$) are the various realizations of stochastic quantities $\mathbf{r}_t(\boldsymbol{\rho}_t|\boldsymbol{\rho}^{t-1})$ and $\mathbf{w}_{t-1}(\boldsymbol{\rho}^{t-1})$.

Risk, for any realization of $\boldsymbol{\rho}^t$, is measured as the variance of the portfolio return relative to the benchmark $\bar{\mathbf{w}}$. Once $\mathscr{W}_t$ is known, the variance of the wealth at time t (relative to the benchmark) can be similarly calculated:

$$\begin{aligned}
&\mathrm{Var}[\mathbf{r}_t(\boldsymbol{\rho}_t|\boldsymbol{\rho}^{t-1})'(\mathbf{w}_{t-1} - \bar{\mathbf{w}}_{t-1})] \\
&\quad = \mathrm{E}[(\mathbf{r}_t(\boldsymbol{\rho}_t|\boldsymbol{\rho}^{t-1})'(\mathbf{w}_{t-1} - \bar{\mathbf{w}}_{t-1}))^2] - (\mathscr{W}_t)^2
\end{aligned}$$

$$
\begin{aligned}
&= \mathrm{E}\left[\sum_{\mathbf{e}\in\mathcal{N}_t} P_{\mathbf{e}}(\mathbf{r}'_{\mathbf{e}}(\mathbf{w}_{a(\mathbf{e})} - \bar{\mathbf{w}}_{a(\mathbf{e})}))^2\right] - (\mathscr{W}_t)^2 \\
&= \sum_{\mathbf{e}\in\mathcal{N}_t} P_{\mathbf{e}}((\mathbf{w}_{a(\mathbf{e})} - \bar{\mathbf{w}}_{a(\mathbf{e})})'(\Lambda + \hat{\mathbf{r}}_{\mathbf{e}}\hat{\mathbf{r}}'_{\mathbf{e}})(\mathbf{w}_{a(\mathbf{e})} - \bar{\mathbf{w}}_{a(\mathbf{e})})) - (\mathscr{W}_t)^2. \qquad (10)
\end{aligned}
$$

The multi-period portfolio reallocation problem can be expressed as the minimization of risk subject to linear constraints (1)–(8) (which describe the growth of the portfolio along all the various scenarios, bounds on the variables) and the performance constraint.

$$
\begin{aligned}
\min_{\mathbf{w},\mathbf{b},\mathbf{s}} \quad & \sum_{t=1}^{T} \alpha_t \sum_{\mathbf{e}\in\mathcal{N}_t} P_{\mathbf{e}}[(\mathbf{w}_{a(\mathbf{e})} - \bar{\mathbf{w}}_{a(\mathbf{e})})'(\Lambda + \hat{\mathbf{r}}_{\mathbf{e}}\hat{\mathbf{r}}'_{\mathbf{e}})(\mathbf{w}_{a(\mathbf{e})} - \bar{\mathbf{w}}_{a(\mathbf{e})})] \\
\text{subject to} \quad & \text{Constraints (1)–(8)}, \\
& \sum_{\mathbf{e}\in\mathcal{N}_t} P_{\mathbf{e}}(\hat{\mathbf{r}}'_{\mathbf{e}}(\mathbf{w}_{a(\mathbf{e})} - \bar{\mathbf{w}}_{a(\mathbf{e})})) \geqslant \mathscr{W}.
\end{aligned}
$$

The performance constraint describes the final expected wealth to a particular constant parameter $\mathscr{W}$. The optimization model above finds the lowest-variance (least risky) investment strategy to achieve that specified expected wealth, $\mathscr{W}$. The $\mathscr{W}$ range varies from the solution of linear programming problem (provides the total risk-seeking investment strategy)

$$
\begin{aligned}
\max_{\mathbf{w},\mathbf{b},\mathbf{s}} \quad & \sum_{\mathbf{e}\in\mathcal{N}_t} P_{\mathbf{e}}(\hat{\mathbf{r}}'_{\mathbf{e}}(\mathbf{w}_{a(\mathbf{e})} - \bar{\mathbf{w}}_{a(\mathbf{e})})) \\
\text{subject to} \quad & \text{Constraints (1)–(8)}
\end{aligned}
$$

to the value of $\mathscr{W}$ corresponding to the solution of quadratic programming problem

$$
\begin{aligned}
\min_{\mathbf{w},\mathbf{b},\mathbf{s}} \quad & \sum_{t=1}^{T} \alpha_t \sum_{\mathbf{e}\in\mathcal{N}_t} P_{\mathbf{e}}[(\mathbf{w}_{a(\mathbf{e})} - \bar{\mathbf{w}}_{a(\mathbf{e})})'(\Lambda + \hat{\mathbf{r}}_{\mathbf{e}}\hat{\mathbf{r}}'_{\mathbf{e}})(\mathbf{w}_{a(\mathbf{e})} - \bar{\mathbf{w}}_{a(\mathbf{e})})] \\
\text{subject to} \quad & \text{Constraints (1)–(8)},
\end{aligned}
$$

which provides total risk-aversion investment strategy. Varying $\mathscr{W}$ between the risk-seeking and the risk-averse investment strategies and reoptimizing generate optimal portfolios on the risk-return efficient frontier.

3. Min–max optimization

In the presence of the uncertainty, a robust approach is to compute expectations using the worst-case probability distribution. Aoki (1967) provides a detailed discussion of this approach within the context of general dynamical systems. In this paper, the mean value of the portfolio is described in terms of rival return scenarios, or rival scenario trees, which represent rival views of the future. We assume that all rival scenarios are reasonably likely and that it is not possible to rule out any of these using statistical analysis. Wise decision making would therefore entail the incorporation of all scenarios to provide an integrated and consistent framework. We propose worst-case robust optimal strategies which yield guaranteed performance. In risk management terms, such robust strategy would ensure that performance is optimal in the worst-case and this will further improve if the worst-case scenarios do not materialize. We note that there may well be more than one scenario corresponding to the worst-case. This noninferiority is the guaranteed performance character of min–max. To illustrate this, consider the problem

$$
\min_{w} \max_{i\in I} \{f^i(w)\},
$$

where I is the (discrete) index of rival scenarios and f^i be a general objective function corresponding to the ith scenario. Let w^* be the solution of this problem. Then, by optimality, we have the inequality

$$\min_{w} \max_{i\in I}\{f^i(w)\} = \max_{i\in I}\{f^i(w^*)\} \geqslant f^i(w^*) \quad \forall i \in I.$$

Hence, if the worst-case does not materialize, then performance is guaranteed to improve.

The mean–variance portfolio allocation problem presented in Section 2 considers a scenario tree for the stochastic structure of uncertainty and a covariance matrix as an input. It is well known that asset return forecasts and risk estimates are inherently inaccurate. The inaccuracy in forecasting and estimation can be addressed through the specification of rival scenarios. These are used with forecast pooling by stochastic programming; e.g. see Kall (1976), Lawrence et al. (1986) and Makridakis and Winkler (1983). Robust pooling using min–max has been introduced in Rustem et al. (2000) and Rustem and Howe (2002). A min–max algorithm for stochastic programs based on a bundle method is discussed in Breton and El Hachem (1995). Min–max optimization is more robust to the realization of worst-case scenarios than considering a single scenario or an arbitrary pooling of scenarios. It is suitable for situations which need protection against risk of adopting the investment strategy based on the wrong scenario since The min–max mean–variance optimization model constructs an optimal portfolio simultaneously with the worst-case scenario.

In this section, we present multi-period min–max mean–variance optimization models with multiple rival risk and return scenarios. Three alternative min–max formulations of the multi-period mean–variance optimization are introduced. The difference between these formulations is characterized by the way the rival risk scenarios are integrated. An investor might wish to consider the same rival risk scenarios at different stages such as at each node of scenario tree, any/all intermediate time periods of the investment horizon, or with each rival return scenarios. Rival return scenarios are determined by sub-scenario trees rooted at the first time period of the scenario tree since the investor wishes to survive at the first time period. For example, the scenario tree in Fig. 3 consists of three rival return scenarios defined by number of events realized at the first time period. There are different forecasting methods for rival risk and return scenarios which are discussed in detail in Section 4.

3.1. Model 1: Risk scenarios at each node

Assume that the risk scenarios are considered at each event of the future realizations. Let I be the number of covariance matrices defining rival risk scenarios and K denote the number of rival return scenarios.

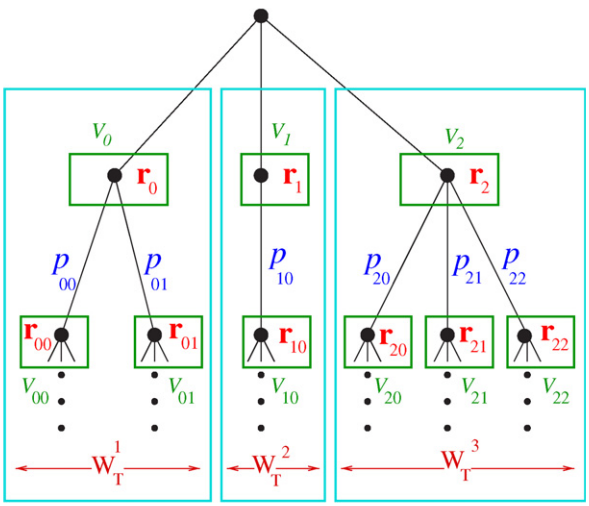

Fig. 3. Worst-case scenario tree for Model 1.

At each node of the scenario tree, $\mathbf{e} \in \mathcal{N}_t$ for $t = 1, \ldots, T$, we have the same number of covariance matrices, denoted by $I_\mathbf{e}$. The selection of covariance matrices is the user dependent and an input to the min–max model.

Given the rival risk and return scenarios (or scenario tree), the general min–max formulation for multi-period portfolio management problem is as follows;

$$\min_{\mathbf{w},\mathbf{b},\mathbf{s}} \left\{ \gamma \sum_{t=1}^{T} \alpha_t \sum_{\mathbf{e}\in\mathcal{N}_t} \max_i [P_\mathbf{e}(\mathbf{w}_{a(\mathbf{e})} - \bar{\mathbf{w}}_{a(\mathbf{e})})'(\Lambda_i + r'_\mathbf{e} r_\mathbf{e})(\mathbf{w}_{a(\mathbf{e})} - \bar{\mathbf{w}}_{a(\mathbf{e})})] \right\}$$
$$+ \min_{\mathbf{w},\mathbf{b},\mathbf{s}} \left\{ -\min_k \left[\sum_{\mathbf{e}\in\mathcal{N}_T^k} P_\mathbf{e}(\mathbf{w}_{a(\mathbf{e})} - \bar{\mathbf{w}}_{a(\mathbf{e})})'\mathbf{r}_\mathbf{e} \right] \right\} \equiv \min_{\mathbf{w},\mathbf{b},\mathbf{s}} \left\{ \gamma \sum_{t=1}^{T} \alpha_t \sum_{\mathbf{e}\in\mathcal{N}_t} \max_i [J^i_\mathbf{e}(\mathbf{w})] - \min_k [R^k(\mathbf{w})] \right\},$$

where

$$J^i_\mathbf{e}(\mathbf{w}) = P_\mathbf{e}(\mathbf{w}_{a(\mathbf{e})} - \bar{\mathbf{w}}_{a(\mathbf{e})})'(\Lambda_i + r'_\mathbf{e} r_\mathbf{e})(\mathbf{w}_{a(\mathbf{e})} - \bar{\mathbf{w}}_{a(\mathbf{e})}),$$
$$R^k(\mathbf{w}) = \sum_{\mathbf{e}\in\mathcal{N}_T^k} P_\mathbf{e}(\mathbf{w}_{a(\mathbf{e})} - \bar{\mathbf{w}}_{a(\mathbf{e})})'\mathbf{r}_\mathbf{e}$$

for $i \in I_\mathbf{e}$, $k = 1, \ldots, K$, $t = 1, \ldots, T$ and $\mathbf{e} \in \mathcal{N}_t$. Let $v_\mathbf{e}$ and μ be the worst-case risk at node $\mathbf{e} \in \mathcal{N}_t$ and the worst-case return, respectively. In order to solve the min–max problem above, we reformulate it as a quadratically constraint mathematical program as

$$\min_{\mathbf{w},\mathbf{b},\mathbf{s}} \quad \gamma \sum_{t=1}^{T} \alpha_t \sum_{\mathbf{e}\in\mathcal{N}_t} v_\mathbf{e} - \mu$$

subject to Constraints (1)–(8),

$$\sum_{\mathbf{e}\in\mathcal{N}_T^k} P_\mathbf{e}(\mathbf{w}_{a(\mathbf{e})} - \bar{\mathbf{w}}_{a(\mathbf{e})})'\mathbf{r}_\mathbf{e} \geqslant \mu, \quad k = 1, \ldots, K, \tag{11}$$

$$P_\mathbf{e}(\mathbf{w}_{a(\mathbf{e})} - \bar{\mathbf{w}}_{a(\mathbf{e})})'(\Lambda_i + r'_\mathbf{e} r_\mathbf{e})(\mathbf{w}_{a(\mathbf{e})} - \bar{\mathbf{w}}_{a(\mathbf{e})}) \leqslant v_\mathbf{e}, \; i \in I_\mathbf{e}, \mathbf{e} \in \mathcal{N}_t, \; t = 1, \ldots, T. \tag{12}$$

The level of risk aversion optimized for is determined by the scaling constant γ. When $\gamma = 0$, the pure risk-seeking investment strategy (at the highest end of the efficient frontier) is obtained by solving a linear programming problem (by ignoring the constraint (12)). When $\gamma = \infty$, completely risk-averse strategy (at the lowest end of the efficient frontier) is obtained by solving the quadratic programming problem (by ignoring the constraint (11)).

The worst-case risk $v_\mathbf{e}$ for each event $\mathbf{e} \in \mathcal{N}_t$ is calculated as the maximum risk value, which is computed by implementing the min–max strategy on specific rival risk scenarios and selecting maximum one among Λ_i, $i \in I_\mathbf{e}$. The worst-case return μ is obtained as the minimum expected return at the last time periods among $W^1_T, \ldots, W^K_T$ corresponding to each sub-scenario tree (see Fig. 3). Notice that the number of quadratic constraints, $\sum_{t=1}^{T} \sum_{\mathbf{e}\in\mathcal{N}_t} I_\mathbf{e}$, depends on the number of rival risk scenarios considered and the topology of the scenario tree. However, the number of risk variables only depends on the structure of the scenario tree since we have one variable $v_\mathbf{e}$ associated with node $\mathbf{e}$ on the scenario tree.

3.2. Model 2: Risk scenarios at each time period

One may consider the same number of risk scenarios at each time period of investment horizon. In this case, the general min–max formulation of the multi-period mean–variance optimization problem becomes

$$\min_{\mathbf{w},\mathbf{b},\mathbf{s}} \left\{ \gamma \sum_{t=1}^{T} \alpha_t \max_i \left[\sum_{\mathbf{e}\in\mathcal{N}_t} P_\mathbf{e}(\mathbf{w}_{a(\mathbf{e})} - \bar{\mathbf{w}}_{a(\mathbf{e})})'(\Lambda_i + r'_\mathbf{e} r_\mathbf{e})(\mathbf{w}_{a(\mathbf{e})} - \bar{\mathbf{w}}_{a(\mathbf{e})}) \right] \right\}$$
$$+ \min_{\mathbf{w},\mathbf{b},\mathbf{s}} \left\{ \max_k \left[-\sum_{\mathbf{e}\in\mathcal{N}_T^k} P_\mathbf{e}(\mathbf{w}_{a(\mathbf{e})} - \bar{\mathbf{w}}_{a(\mathbf{e})})'\mathbf{r}_\mathbf{e} \right] \right\} \equiv \min_{\mathbf{w},\mathbf{b},\mathbf{s}} \left\{ \gamma \sum_{t=1}^{T} \alpha_t \max_i [J^i_t(\mathbf{w})] - \min_k [R^k(\mathbf{w})] \right\},$$

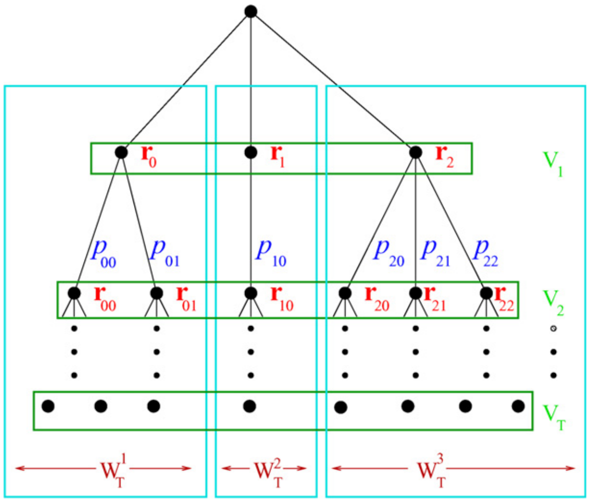

Fig. 4. Worst-case scenario tree for Model 2.

where

$$J_t^i(\mathbf{w}) = \sum_{\mathbf{e}\in\mathcal{N}_t} P_{\mathbf{e}}(\mathbf{w}_{a(\mathbf{e})} - \bar{\mathbf{w}}_{a(\mathbf{e})})'(\Lambda_i + r_{\mathbf{e}}' r_{\mathbf{e}})(\mathbf{w}_{a(\mathbf{e})} - \bar{\mathbf{w}}_{a(\mathbf{e})})$$

$$R^k(\mathbf{w}) = \sum_{\mathbf{e}\in\mathcal{N}_T^k} P_{\mathbf{e}}(\mathbf{w}_{a(\mathbf{e})} - \bar{\mathbf{w}}_{a(\mathbf{e})})'\mathbf{r}_{\mathbf{e}}$$

for $i \in I_t$, $k = 1,\ldots,K$, $t = 1,\ldots,T$. As can be seen from the scenario tree presented in Fig. 4, in Model 2, we have the worst-case risk v_t at each time period $t = 1,\ldots,T$. The overall worst case risk is computed as the sum of all worst-case risks determined at t. An equivalent quadratic programming formulation of min–max problem for Model 2 is as follows:

$$\begin{aligned}
\min_{\mathbf{w},\mathbf{b},\mathbf{s}} \quad & \gamma\sum_{t=1}^{T}\alpha_t v_t - \mu \\
\text{subject to} \quad & \text{Constraints (1)–(8)}, \\
& \sum_{\mathbf{e}\in\mathcal{N}_T^k} P_{\mathbf{e}}(\mathbf{w}_{a(\mathbf{e})} - \bar{\mathbf{w}}_{a(\mathbf{e})})'\mathbf{r}_{\mathbf{e}} \geqslant \mu, \quad k = 1,\ldots,K, \\
& \sum_{\mathbf{e}\in\mathcal{N}_t} P_{\mathbf{e}}(\mathbf{w}_{a(\mathbf{e})} - \bar{\mathbf{w}}_{a(\mathbf{e})})'(\Lambda_i + r_{\mathbf{e}}' r_{\mathbf{e}})(\mathbf{w}_{a(\mathbf{e})} - \bar{\mathbf{w}}_{a(\mathbf{e})}) \leqslant v_t, \quad i \in I_t,\ t = 1,\ldots,T.
\end{aligned}$$

Notice that the difference between Model 1 and Model 2 is characterized by the quadratic constraints. In Model 2, the number of quadratic constraints $\sum_{t=1}^{T} I_t$ depends on the number of rival risk scenarios specified at time period t and length of the investment horizon. The number of risk variables is determined as T since there is one risk variable v_t associated with time period $t = 1,\ldots,T$. The optimization problem in Model 2 becomes more dense than the one in Model 1, since in Model 2 the quadratic constraint at t consists of sum of all worst-case risk terms at node $\mathbf{e} \in \mathcal{N}_t$ at period t.

3.3. Model 3: Risk scenarios for each rival return scenario

We also consider the same number of rival risk scenarios associated with each rival return scenarios, which are defined by the sub-scenario trees rooted at nodes of the first time period. Let K be the number of rival return scenarios and I_j be the number of rival risk scenarios associated with each sub-tree $j = 1,\ldots,K$. Referring to Fig. 5, we have three (rival return scenarios) sub-scenario trees rooted at a node in the first time period, consequently, three variables represent the worst-case risk corresponding to each sub-tree.

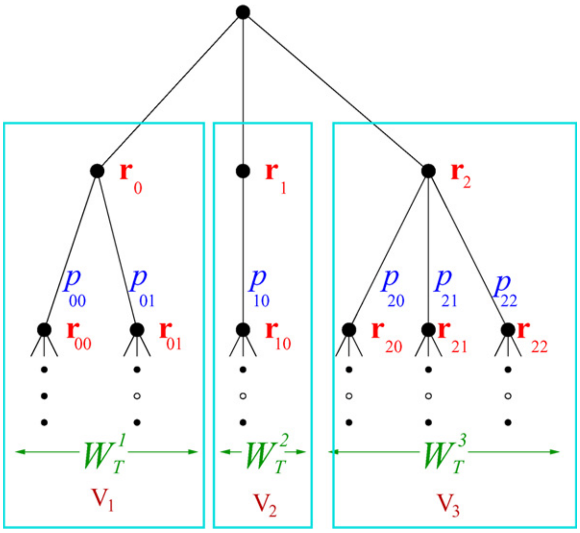

Fig. 5. Worst-case scenario tree for Model 3.

The general min–max problem for Model 3 is formulated as follows:

$$\min_{\mathbf{w},\mathbf{b},\mathbf{s}} \left\{ \gamma \sum_{j=1}^{K} \max_{i,j} \left[\sum_{t=2}^{T} \sum_{\mathbf{e}\in\mathcal{N}_t^j} \alpha_t P_{\mathbf{e}} (\mathbf{w}_{a(\mathbf{e})} - \bar{\mathbf{w}}_{a(\mathbf{e})})'(\Lambda_i + r'_{\mathbf{e}} r_{\mathbf{e}})(\mathbf{w}_{a(\mathbf{e})} - \bar{\mathbf{w}}_{a(\mathbf{e})}) \right] \right\}$$
$$+ \min_{\mathbf{w},\mathbf{b},\mathbf{s}} \left\{ - \min_{k} \left[\sum_{\mathbf{e}\in\mathcal{N}_T^k} P_{\mathbf{e}} (\mathbf{w}_{a(\mathbf{e})} - \bar{\mathbf{w}}_{a(\mathbf{e})})' \mathbf{r}_{\mathbf{e}} \right] \right\} \equiv \min_{\mathbf{w},\mathbf{b},\mathbf{s}} \left\{ \gamma \sum_{j=1}^{K} \max_{i,j} [J_j^i(\mathbf{w})] - \min_k [R^k(\mathbf{w})] \right\},$$

where for $i \in I_j$, $j, k = 1, \ldots, K$,

$$J_j^i(\mathbf{w}) = \sum_{t=2}^{T} \sum_{\mathbf{e}\in\mathcal{N}_t^j} \alpha_t P_{\mathbf{e}} (\mathbf{w}_{a(\mathbf{e})} - \bar{\mathbf{w}}_{a(\mathbf{e})})'(\Lambda_i + r'_{\mathbf{e}} r_{\mathbf{e}})(\mathbf{w}_{a(\mathbf{e})} - \bar{\mathbf{w}}_{a(\mathbf{e})}),$$
$$R^k(\mathbf{w}) = \sum_{\mathbf{e}\in\mathcal{N}_T^k} P_{\mathbf{e}} (\mathbf{w}_{a(\mathbf{e})} - \bar{\mathbf{w}}_{a(\mathbf{e})})' \mathbf{r}_{\mathbf{e}}.$$

An equivalent formulation of min–max Model 3 is presented by the following quadratic programming problem:

$$\begin{aligned}
\min_{\mathbf{w},\mathbf{b},\mathbf{s}} \quad & \gamma \sum_{k=1}^{K} v_k - \mu \\
\text{subject to} \quad & \text{Constraints (1)–(8)}, \\
& \sum_{\mathbf{e}\in\mathcal{N}_T^k} P_{\mathbf{e}} (\mathbf{w}_{a(\mathbf{e})} - \bar{\mathbf{w}}_{a(\mathbf{e})})' \mathbf{r}_{\mathbf{e}} \geqslant \mu, \quad k = 1, \ldots, K, \\
& \sum_{t=2}^{T} \sum_{\mathbf{e}\in\mathcal{N}_t^k} \alpha_t P_{\mathbf{e}} (\mathbf{w}_{a(\mathbf{e})} - \bar{\mathbf{w}}_{a(\mathbf{e})})'(\Lambda_i + r'_{\mathbf{e}} r_{\mathbf{e}})(\mathbf{w}_{a(\mathbf{e})} - \bar{\mathbf{w}}_{a(\mathbf{e})}) \leqslant v_k, \quad i \in I_k, \ k = 1, \ldots, K.
\end{aligned}$$

In Model 3, the number of quadratic constraints, $\sum_{k=1}^{K} I_k$, depends on not only the number of branching at the first time period but also the number of covariance matrices specified by the investor. The number of worst-case risk variables v_k, is K.

4. Scenario tree generation

Multi-period stochastic programming requires a coherent representation of uncertainty. This is expressed in terms of multivariate continuous distributions. Hence, a decision model is generated with internal sampling or a discrete approximation of the underlying continuous distribution. In the multi-period mean–variance framework, return uncertainty is expressed by a multivariate continuous distribution which is represented by finitely many discrete approximations. Therefore, it is assumed that $\boldsymbol{\rho}_{t+1}$ at time period $t+1$ can take only finitely many values, given the event history up to time t. A scenario tree represents this branching process. Finding a set of discrete outcomes of the random variable depends on the scenario tree generation method. The root node of a scenario tree corresponds to the decision made "today" and the nodes further down represent the conditional decisions at later stages. The arcs linking the nodes represent various realizations of the uncertain variables. The dynamic of decision making is captured as the decisions are adjusted according to realizations of uncertainty.

Scenario tree generation plays a very important role in the performance of multi-period stochastic programming, for instance see Dempster (1993). The discrete approximations of the underlying distribution and probability space must take into account all extreme events of the problem under consideration so that the future is adequately represented. For the multi-period worst-case portfolio problem, we generate scenario trees by an approach based on probabilistic simulation. A discussion of the relative merits of this approach is beyond the scope of this paper. Therefore, the method is briefly described here; for a detailed description of this and alternative approaches, the reader is referred to Gülpınar et al. (2004).

In the simulation based scenario tree generation approach, the return scenarios at each time period are generated as the centroids of simulations generated in parallel or sequentially. Given the topology of the tree to generate, mean growth rates, and a covariance matrix obtained by fitting the history to the expected mean growth, one time period of growth from "today" to "tomorrow" in each scenario is simulated. Large numbers of randomly generated simulations are then clustered into a number of nodes defined by the branching at that time period. If k branches are desired from the current scenario tree node, then k clusters need to be formed. Initially, the seed points around which the clusters are built might as well be chosen to be the first k scenarios, since the scenarios are independently generated, and are in arbitrary order. If the resulting clustering fails to meet the criteria applied in the test stage, new seed points are chosen, and the clustering process repeated, until the criteria is met. The distance measure to determine which seed each scenario is the closest to can be chosen with great flexibility as well as the way the simulations are generated. In this way, the centre point of each clustering is chosen as the best representative return scenario.

5. Computational results

5.1. Design of experiments

The multi-period min–max mean–variance optimization models explained in the previous section are implemented and integrated with a software package called *mrobust*. *Mrobust* is written in C++ and uses the interior point linear/quadratic solver BPMPD (Mészáros, 1997, 1998) to optimize the linear and quadratic programming problems. *Mrobust* has the ability to handle simple box constraints on the decision variables, as well as percentage constraints. All computational experiments are carried out on a 3 GHz Pentium 4, running Linux with 1.5 GB RAM.

A scenario tree must be an input to the mean–variance optimization models. For the simulation based scenario tree generation method, the topology of the scenario tree, the number of random simulations to generate, and statistical parameters for the simulation of scenarios need to be provided. In the computational experiments, the statistical parameters were measured from historical data consisted of monthly price data of randomly selected 5 FTSE 100 stocks from 1988 to mid-2000. The historical data was fitted to an exponential growth curve. The obtained monthly growth rates were annualized, and used to simulate future growth. A covariance matrix was measured from the residuals of the exponential fit, and used in the simulation-based scenario generation method.

The scenario trees input to *mrobust* can have arbitrary branching structure and depth (limited only by computer memory). In order to test the performance of the min–max models, we generate a scenario tree with

Table 2
The problem statistics in terms of variables and constraints

	Model 1	Model 2	Model 3
Variables	224	199	200
Linear constraints	192	192	192
Quadratic constraints	84	9	12

three time periods forecast and 4-2-2 branching structure. It has 28 nodes, three time periods and four subscenario trees defining rival return scenarios. Given the topology of the scenario tree, the problem statistics of the Models 1, 2 and 3 are presented in Table 2 in terms of the number of variables and constraints.

The specification of rival covariance matrices can be realized by observing the data during different periods in the past. Dividing a given observation period into a number of subperiods and measuring volatility in each one is an effective way of estimating risk scenarios. It is well known that the estimates corresponding to each subperiod can be substantially different, and employing the worst-case scenario arising from this consideration yields a robust strategy. Other covariance estimation methods such as ARCH-GARCH models and bootstrapping are also clearly useful.

We consider three rival risk scenarios which are obtained from the historical data. The first covariance matrix is obtained by an exponential fit of 150 monthly price historical data. For the second and the third risk scenarios, we divide the past price data into two parts. Then covariance matrices (corresponding to the second and third rival risk scenarios) are measured from the residuals of the exponential fit of the first part and the second part of the data.

5.2. Results

The performance of multi-period min–max optimization models are measured in terms of worst-case risk return efficient frontiers. Fig. 6 compares worst-case investment strategies with single scenario tree versus multiple rival return scenarios. The worst-case efficient frontiers at the bottom are obtained by the min–max optimization models over four rival return and three risk scenarios. Each portfolio on the worst-case efficient frontiers at the top shows the min–max strategies provided by optimization models with each of the four rival return scenarios and three rival risk scenarios.

Fig. 7 illustrates a cross evaluation of a single return scenario with Models 1, 2 and 3. For illustrative purposes, we only present the results of rival return scenario R2. The performance of the portfolio strategies if any other scenarios are realized is evaluated with the optimal investment strategy obtained by min–max Models 1, 2 and 3 with the single return scenario, R2. The min–max model constructs an optimal portfolio simultaneously with the worst-case scenario. Hence, the worst-case optimal investment strategy has the best lower bound performance which can only be improved when any scenario other than the worst-case is realized. Therefore, non-inferiority of min–max optimization ensures the robustness of the strategy. Non-inferiority of min–max strategy obtained by Models 1, 2 and 3 is shown in Fig. 8. The min–max efficient frontier over all rival risk and return scenarios is the one at the bottom and the other curves are obtained by implementing the min–max strategy on other single rival return scenarios. This confirms a consequent feature of min–max optimization: if the worst-case does not occur, then the performance of min–max portfolio can only be better.

The benefit of using min–max optimization with Models 1, 2 and 3 is displayed in Fig. 9. The top four curves in Fig. 9 represent the efficient frontiers obtained from optimizing with each of the four return scenarios alone and three rival risk scenarios. The middle curve is the min–max strategy obtained by optimization models over rival four return and three risk scenarios. The four curves at the bottom show what actually would happen if the worst-case of the four return scenarios with respect to optimized portfolio was actually realized.

The min–max implementations of Models 1, 2 and 3 are also backtested by generating scenario trees from historical price data, then using multi-period min–max portfolio optimization software *mrobust* to determine optimal investment strategies (for specified levels of risk), and measuring the success of those strategies by their performance with the historical data. See Gülpınar et al. (2004)), for more specific details of the backtesting procedures, but a summary follows.

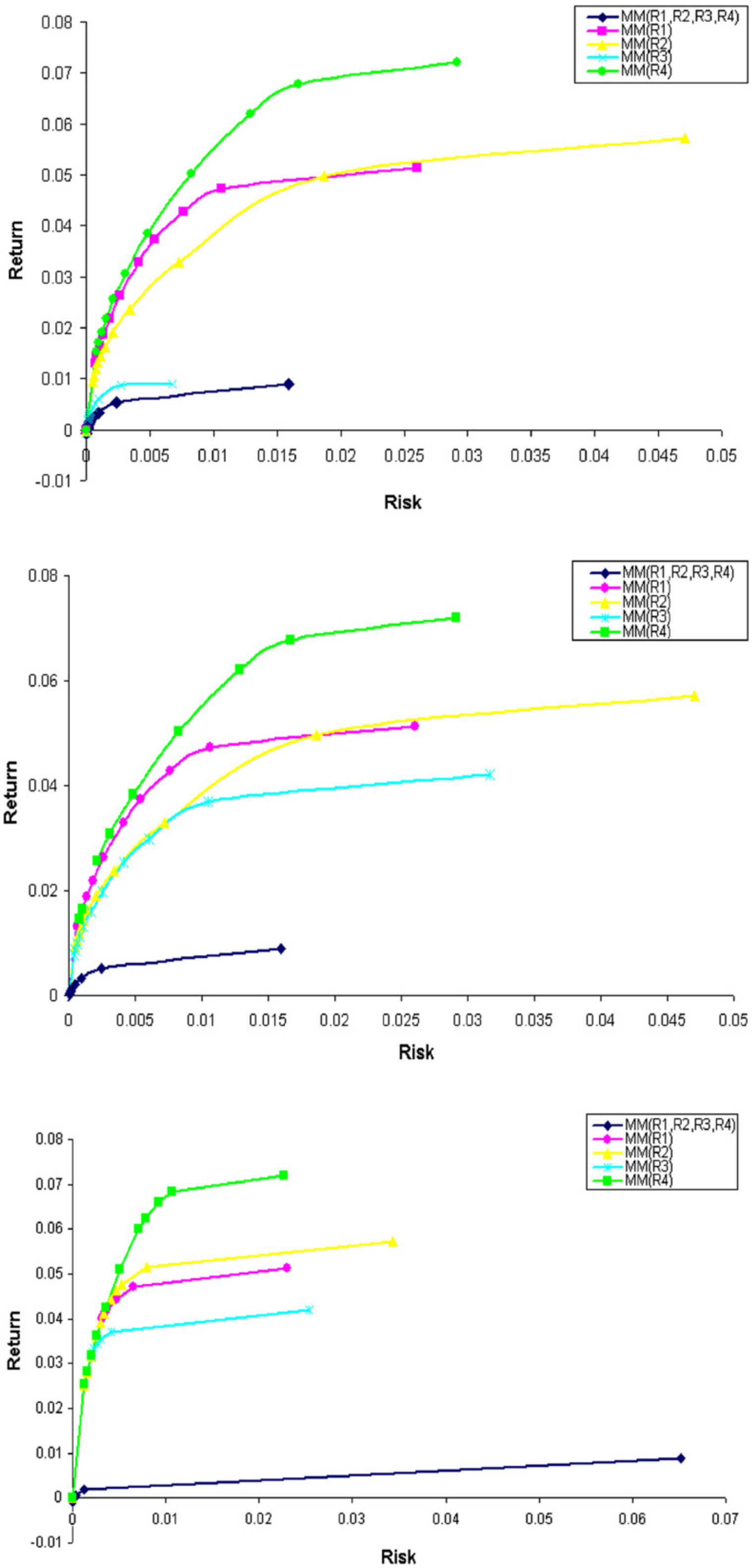

Fig. 6. Min–max strategies over single rival return scenario and multiple rival return scenarios obtained by Models 1, 2 and 3.

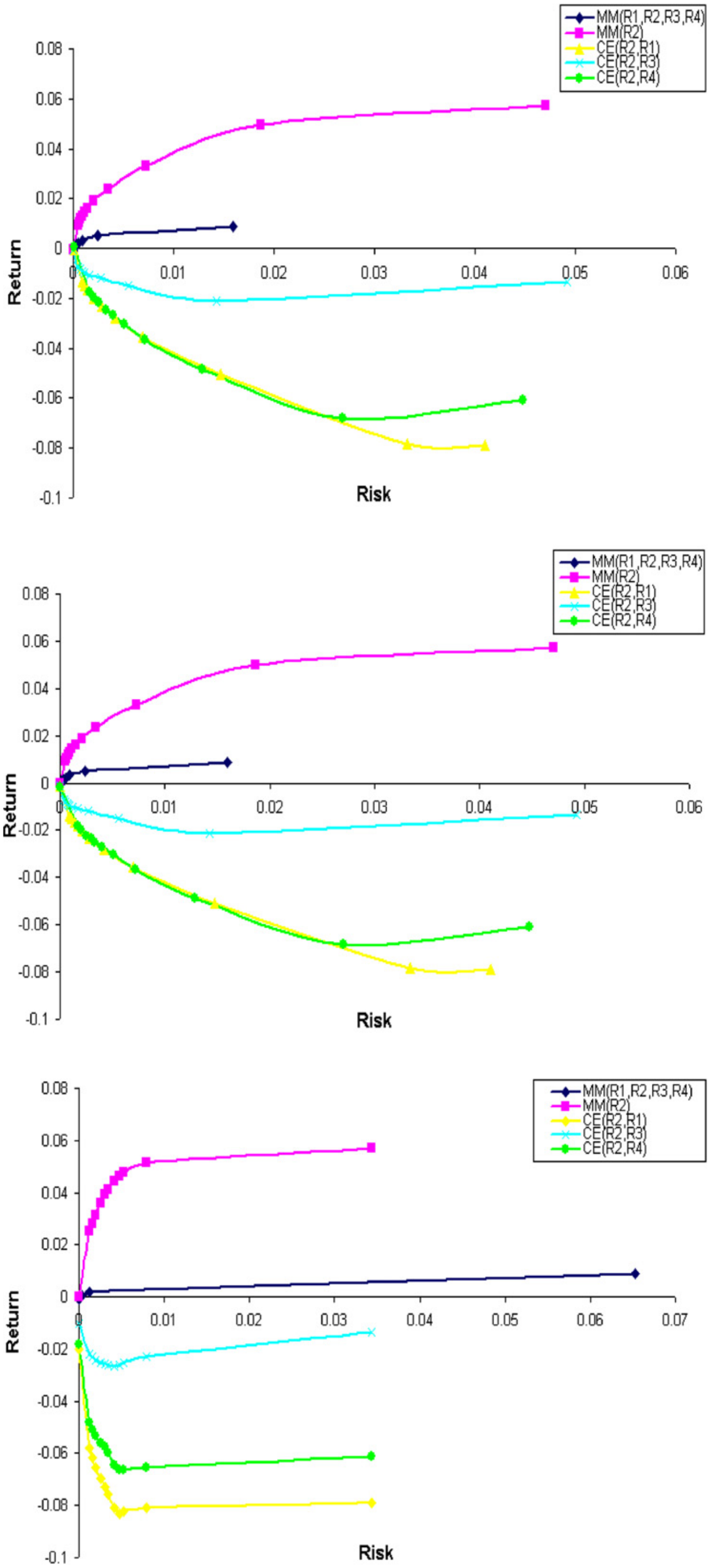

Fig. 7. Cross evaluation of single return scenario with Models 1, 2 and 3.

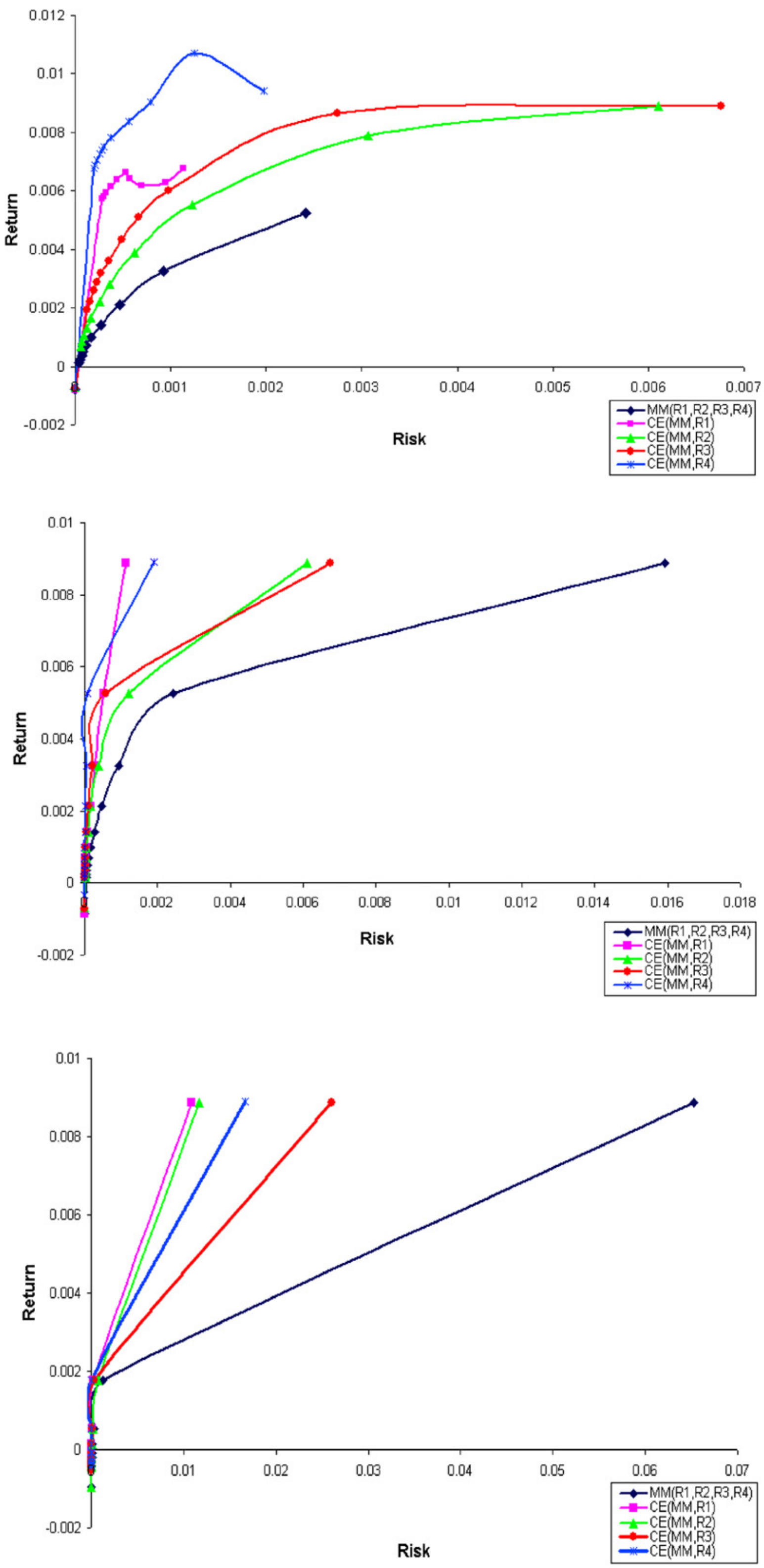

Fig. 8. Non-inferiority of min–max obtained by Models 1, 2 and 3.

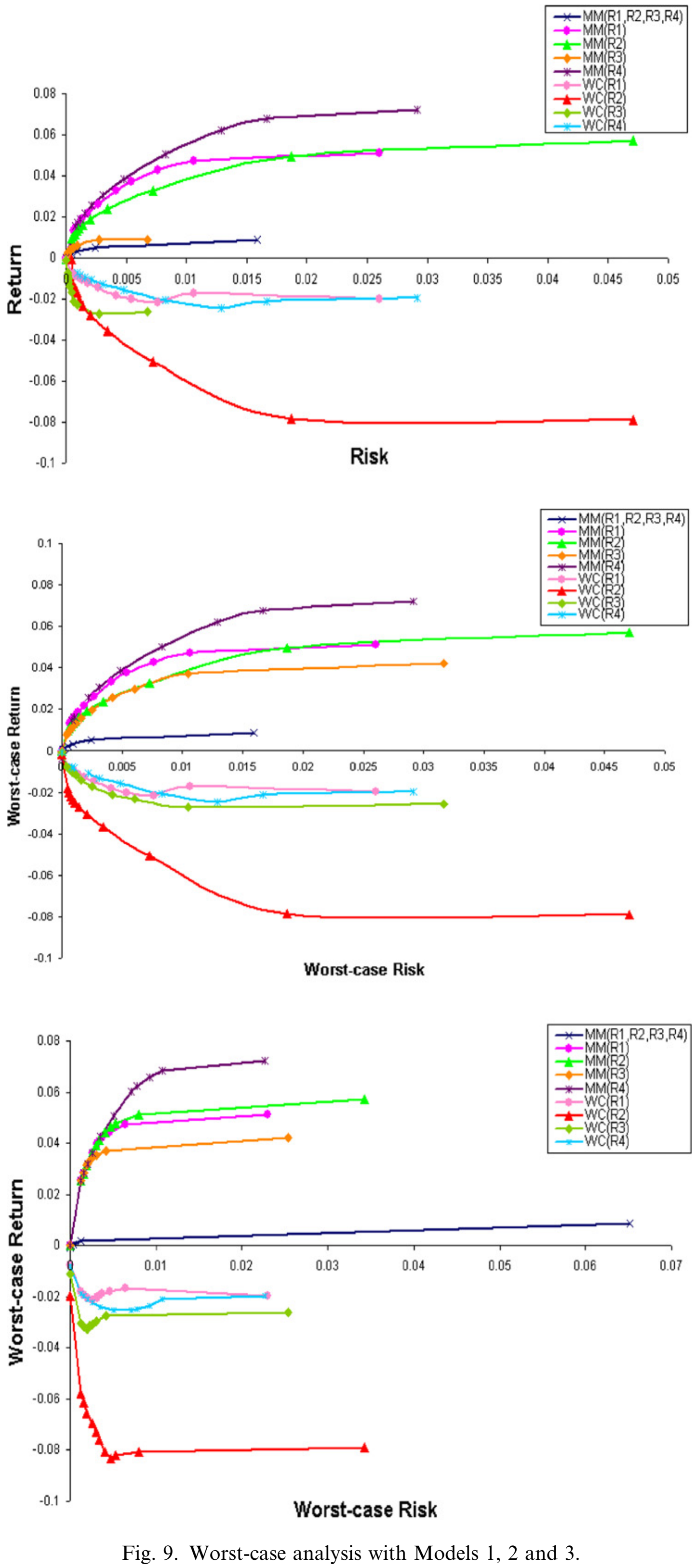

Fig. 9. Worst-case analysis with Models 1, 2 and 3.

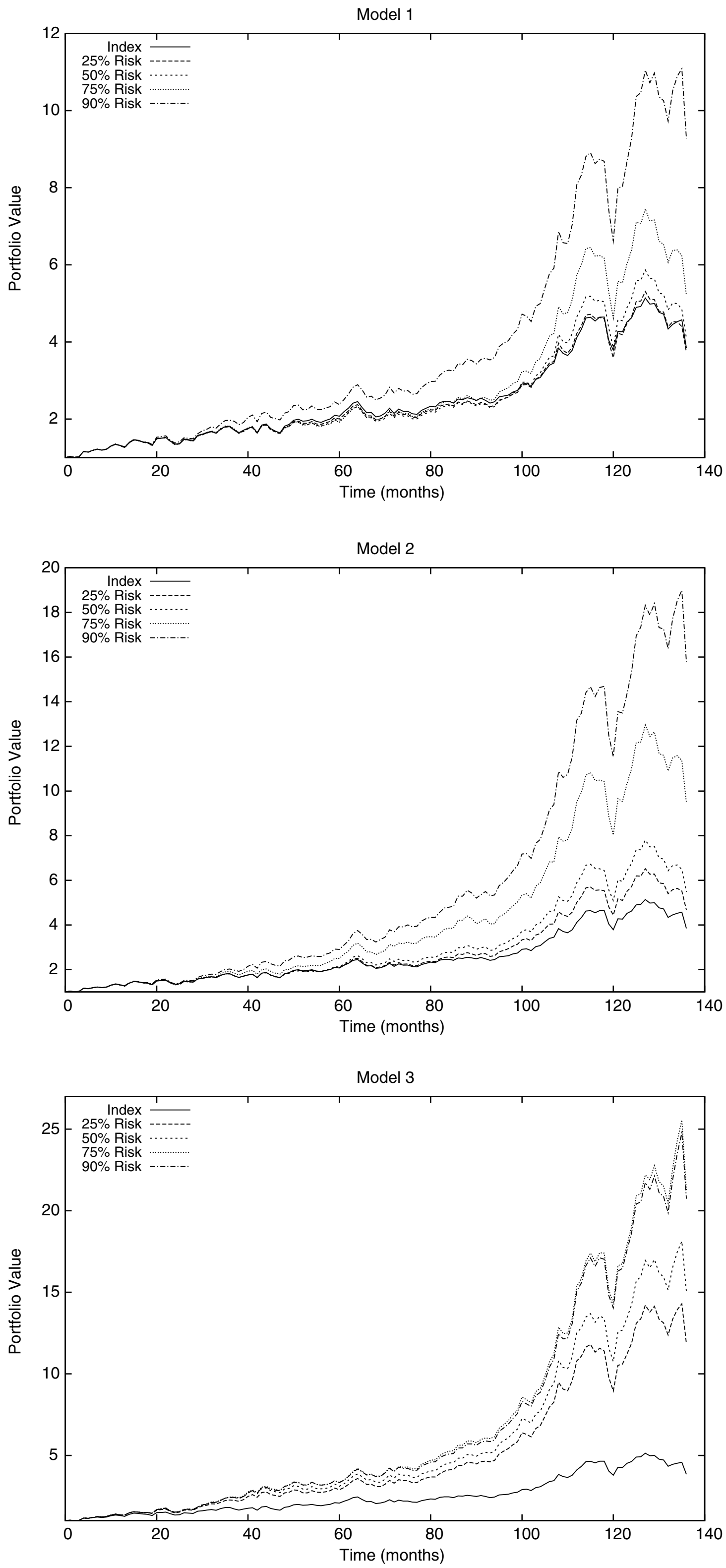

Fig. 10. Backtesting results obtained by Models 1, 2 and 3 at different risk levels.

The historical data consists of monthly price data of ten arbitrarily selected FTSE 100 stocks through the 1990s. At any particular ''present'' time, the previous 10 time periods of past history are fitted to exponential growth curves that model the expected current ''growth'' rate in the market. A scenario tree with three periods forecast, with branching of 3-2-2 is generated by simulation based scenario generation method (Gülpınar et al., 2004). The multi-period worst-case optimization software *mrobust* is used to find an optimal investment strategy over scenario trees. The resulting investment strategy is implemented at the ''present'' prices, and portfolio value is updated according to ''tomorrow'' prices. Then the present is moved forward one time period, and the process is repeated. Note that, for any given scenario tree, the optimizer can yield the entire range of efficient strategies, from risk-seeking to risk-averse (in other words from the highest expected return to the lowest risk). So, the desired risk level is another parameter that needs to be specified during a backtesting experiment. Risk levels are measured as follows: 100% risk corresponds to the solution of the LP results from total-risk seeking behaviour (i.e. ignoring the variance term) and 0% risk level corresponds to the strategy that exclusively minimizes risk with no consideration of performance. In our computational experiments, we consider 25%, 50%, 75% and 90% risk levels and the results are plotted against an equally-weighted index of the

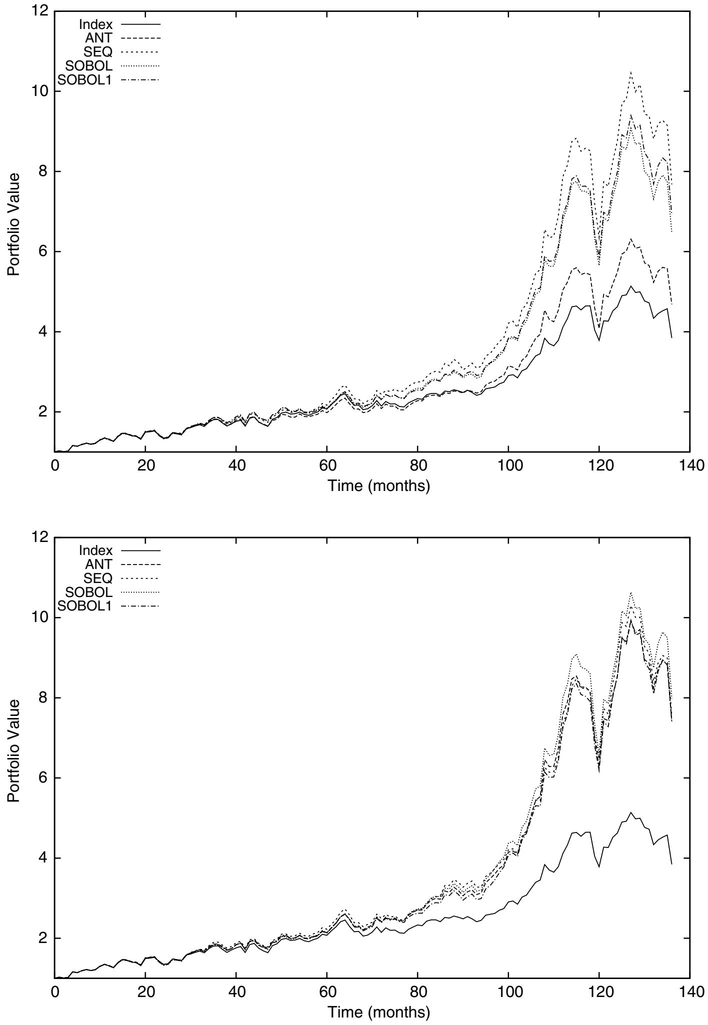

Fig. 11. Backtesting results obtained by Model 1 with different scenario trees at 75% risk levels with three periods 2-2-2 and 4-2-2 branching.

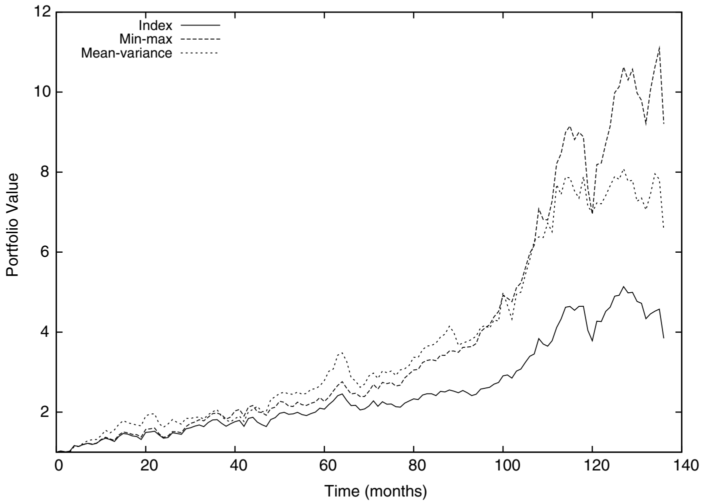

Fig. 12. Comparison of backtesting obtained by min–max and mean–variance optimization strategies at 90% risk level.

assets involved in Fig. 10. We can see a fairly clear ordering of the results of the varying risk strategies with Models 1, 2 and 3.

We have also backtested the performance of min–max models on different scenario trees, with three time periods 2-2-2 and 4-2-2 branching topologies, generated by antithetic variates (ANT), sequential method (SEQ) and Sobol low-discrepancy (SOBOL) random generator. In Fig. 11, we only present the results obtained by Model 1 at 75% risk level for illustrative purposes. These results also show that the forecasting technique as well as the topology of the scenario trees play an important role on the performance of the multi-period min–max optimization models.

Finally, we compare the relative performance of min–max with the mean–variance approach. For min–max optimization, we choose the scenario tree generated by antithetic variates (with three time periods and 2-2-2 branching topology) and three rival risk scenarios. In the mean–variance framework, the same scenario tree is used with a single risk scenario chosen among the rivals. We consider a high risk portfolio strategy (90% risk) which would normally yield high expected return but would also be vulnerable to undesirable effects of uncertainty. It is possible to show that risk taking is sometimes severely penalized by unexpected events. The period in question corresponds to month 120 which includes the financial markets crash in late 1998. Under these circumstances, we expect that a cautious approach like min–max will help to avoid excessive losses. Indeed, Fig. 12 shows that the min–max strategy performs better in such circumstances. Hindsight (of choosing a period that includes a juncture such as '1998') is usually not an acceptable analysis procedure. However, economists can identify periods of unsustainable growth in the markets and choose to follow a robust strategy that integrates several rival scenarios.

6. Conclusions

In this paper, we present min–max formulations of multi-period mean–variance optimization problem with multiple rival risk and return scenarios. The advantage of the worst-case analysis is to provide a guaranteed performance. The issue of inaccuracy in asset return forecasting and risk estimation is addressed in the min–max framework.

Our computational experiments illustrate that the worst-case strategy is robust. The specification of rival risk and return forecasts plays an important role on the performance of the worst-case optimal strategy. The approach also provides some flexibility in the modelling of rival risk scenarios in a multi-stage process. Out-of-sample backtesting results indicate that relying on a single scenario among the rivals might lead to worse performance. The min–max is an alternative approach which yields a cautious strategy to avoid excessive losses.

Acknowledgements

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