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Robust option pricing: Hannan and Blackwell meet Black and Scholes ☆

Peter M. DeMarzo [a,*], Ilan Kremer [b,c], Yishay Mansour [d,e]

[a] Stanford University, United States
[b] Hebrew University, Israel
[c] University of Warwick, United Kingdom
[d] Tel Aviv University, Israel
[e] Microsoft Research, Israel

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Abstract

We apply methods developed in the literature initiated by Hannan and Blackwell on robust optimization, approachability and calibration, to price financial securities. Rather than focus on asymptotic performance, we show how gradient strategies developed to minimize asymptotic regret imply financial trading strategies that yield arbitrage-based bounds for option prices. These bounds are new and robust in that they do *not* depend on the continuity of the stock price process, complete markets, or an assumed pricing kernel. They depend only on the realized quadratic variation of the price process, which can be measured and, importantly, hedged in financial markets using existing securities. Our results also apply directly to a new class of options called timer options. Finally, we argue that the Hannan–Blackwell strategy is path dependent and therefore suboptimal with a finite horizon. We improve it by solving for the optimal path-independent strategy, and compare the resulting bounds with Black–Scholes.

JEL classification: C73; D81; G13

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* Corresponding author.
E-mail address: pdemarzo@stanford.edu (P.M. DeMarzo).

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1. Introduction

There is a growing literature in game theory on calibration, expert testing, learning in games, and the dynamic foundations of correlated equilibria; see Hart (2005), Foster et al. (1999), Fudenberg and Levine (1998), and Cesa-Bianchi and Lugosi (2006) for excellent surveys.[1] This literature is based on earlier work by Hannan (1957) and Blackwell (1956) who studied robust dynamic optimization and defined "approachability" or "regret minimization" for games under uncertainty. Regret is defined as the difference between the outcome of a strategy and that of the *ex-post* optimal strategy (within a given class).[2]

Many of the results in this literature are truly remarkable. For example, in many settings they imply that an agent is able to dynamically optimize, in an uncertain and even non-stationary environment, in a way that mimics the asymptotic performance of an agent with full knowledge of the underlying distribution of uncertainty. But despite the elegance of such results, questions remain regarding the economic relevance of this asymptotic performance in a meaningful economic context with a finite horizon and discounted payoffs.

The goal of this paper is to bridge this gap by applying the regret minimization methodology to financial economics, as monetary payoffs provide a tangible way to measure the performance of regret-minimizing strategies that we believe is more meaningful than the standard approach of evaluating asymptotic average performance. In particular, we focus on standard European-style options, which we can think of as contracts that, in exchange for an upfront premium, allow investors to minimize their regret when choosing an investment portfolio. We then show how results based on approachability translate to upper bounds for option prices. These bounds are both new and empirically relevant; moreover, they depend only on the quadratic variation of the stock price process, a generalization of the usual measure of volatility which allows for jumps. Importantly, the quadratic variation is easily measurable and often contracted upon in practice. Thus, these bounds are relevant in a trading context as financial instruments exist to hedge against fluctuations in quadratic variation, and so we can obtain an arbitrage-based link between the price of options and the value of these instruments. The bounds we compute can also be directly applied to provide assumption-free bounds for a new class of options called 'timer options' that have become increasingly popular (see Section 4).

How does our approachability-based approach compare to traditional option pricing theory? Classic option pricing methods, such as the Black–Scholes–Merton or Binomial option pricing models, rely on highly restrictive assumptions regarding stock price paths to guarantee market completeness. The binomial model assumes stock prices move discretely with jumps that have a known magnitude, while the Black–Scholes framework assumes a continuous price process with a constant volatility. Indeed, these restrictions on the set of allowable price paths are violated by essentially *all* observed price paths in practice.

To get around this problem and deal with reality of market incompleteness, an alternative approach used in the theoretical option pricing literature is to assume a specific form for the

[1] Recent contributions include Dekel and Feinberg (2006), Al-Najjar and Weinstein (2008) and Olszewski and Sandroni (2008).

[2] We use the terms "regret minimization" and "approachability" interchangeably. Formally, approachability is a more general concept whereas regret minimization is a classic application of it.

pricing kernel (equivalently, a risk-neutral measure). While doing so makes it possible to price any derivative without restricting attention to unrealistic price paths, such methods are equally non-robust in the sense that the assumed pricing kernel is almost-surely different than the (unobserved) true pricing kernel. And more importantly, these deviations are not observable, and therefore cannot be hedged using contracts.

In order to develop a "robust" option pricing theory, there are two possibilities: (i) we can put robust bounds on the allowable pricing kernel, or (ii) we can put robust bounds on the allowable price paths. Several papers in the literature have explored the first approach; see Section 1.2 for further discussion. However, there has been little research exploring the second approach. Our contribution is to show how the methods of approachability and calibration can be used to place robust bounds on option prices based on observed characteristics of stock price paths. Notably, while there are many papers that generalize Black–Scholes, our bounds are based only on assumptions regarding the realized quadratic variation of the stock price process, which can be measured and, more importantly, hedged in financial markets using existing securities. Thus, our results provide a new, model-free assessment of the quantitative impact of jumps, trading halts, or other trading irregularities on option prices.

Thus far, we have described how results from the robust optimization literature can be used to derive new results in option theory. Importantly, however, the contributions from developing the link between these literatures flow in both directions. In particular, in our analysis we also show how results from option pricing theory can be used to modify the standard approach to constructing regret minimizing strategies so as to improve their finite horizon performance. Ultimately, we show how to compute the tightest possible bounds for option prices based solely on a bound on quadratic variation, and demonstrate their empirical relevance.

To summarize, our main results are:

1) We consider the basic gradient strategies developed by Hart and Mas-Colell (2000) that follow Hannan (1957) and Blackwell (1956) to minimize asymptotic regret. We extend the strategies to a context with multiplicative, rather than additive, payoffs, and portfolio-based trading strategies in place of mixed strategies.
2) Based on the concept of "no arbitrage" we show that in a finite horizon economy any regret-minimizing strategy can be expressed as an upper bound for the value of a European call option.
3) Applying our results for gradient strategies, we provide a new upper bound for the value of a call option in terms of the quadratic variation of the stock's return. This bound holds for arbitrary stock price processes (which may be non-stationary and include jumps), and arbitrary trading environments (continuous or discrete).
4) While asymptotically efficient, we show that the gradient strategy is suboptimal in a finite horizon. The reason is that unlike Black–Scholes, the strategy is not memory-less. To sharpen this distinction we characterize and solve numerically for the optimal regret-minimizing strategy, and use it to provide the tightest possible option price bound absent assumptions of stationarity or continuous trading and prices.
5) We compare our bounds to existing option-pricing models such as Black and Scholes (1973) and Merton (1973) and show they are empirically meaningful. As we do not assume continuity, stationarity, or continuous trading opportunities, our bounds are necessarily weaker than Black–Scholes, but are much more robust.

1.1. Relationship to approachability and robust optimization

Approachability and regret minimization differ from the traditional approach to optimization under uncertainty by specifying a relative rather than an absolute objective function.[3] A regret-minimizing decision maker is not concerned about the absolute performance of her strategy, but rather how well it performs compared to a defined set of alternative strategies. Our purpose is not to evaluate whether regret minimization is an appropriate objective, but rather to consider the properties of the strategies that achieve it in a financial context.[4]

The roots of approachability and regret minimization in game theory can be traced to Hannan's (1957) and Blackwell's (1956) work on dynamic optimization when the decision maker has very little information about the environment. They considered an infinitely repeated decision problem in which in each period the agent chooses an action from some fixed finite set. Although the set of actions is fixed, the payoffs to these actions vary in a potentially non-stationary manner, so that learning is not possible. They showed that, given a uniform bound on the payoffs, there exist dynamic strategies that limit the agent's regret to grow with the square root of the number of periods. Thus, asymptotically, the average per period regret vanishes, and the agent earns an average payoff that is at least as high as that from the ex-post optimal static strategy in which the same action is taken repeatedly.[5] Thus, in terms of the long-run average payoff, the agent suffers no regret with respect to any static strategy.

We adapt these results to an investment setting with multiplicative, rather than additive payoffs, with portfolios replacing mixed strategies. We define the regret of a dynamic trading strategy as the ratio between the investor's wealth and the wealth obtained from a buy-and-hold investment in either the stock or the bond. The existence of a dynamic trading strategy that guarantees a high minimum ratio for *both* alternatives implies a no-arbitrage upper bound for option prices. To see why, suppose we have a strategy that guarantees a payoff no worse than 80% of the payoff from holding the stock or the bond (i.e., has a "maximum regret" of 20%). Let the interest rate on the bond be zero for simplicity. Then borrowing \$100 and investing \$125 in this strategy, we attain a payoff that is no worse than that of an at-the-money call option on \$100 worth of stock.[6] Thus, the value of the call option cannot exceed the initial investment of \$25, and so a "regret guarantee" of no more than 20% is equivalent to an upper bound of \$25 for an at-the-money call option.

In Section 2 of the paper, we use this link between dynamic performance guarantees and super-replicating portfolios to develop arbitrage-free price bounds for call options based on the

[3] Several papers consider regret minimization as an objective: Bergemann and Schlag (2005) examine a monopolist who minimizes regret. Bose et al. (2004) examine optimal auctions in this context. Milnor (1954) and later Hayashi (2005) provide axiomatic foundations for such preferences. In computer science, competitive analysis compares the performance of online algorithms to the optimal one (see Borodin and El-Yaniv (1998)).

[4] Maenhout (2004), Hansen and Sargent (1995) and Gilboa and Schmeidler (1989) are examples of a large related literature in economics and finance studying robust optimization when the agent considers a set of plausible stochastic processes and choose a policy that is robust in a 'max-min' sense. We consider a relative rather than an absolute objective: Rather than find the strategy with the best absolute payoff (in the worst case scenario), we look for a strategy that minimizes the loss relative to a fixed set of alternative policies. Our work shows why this criterion is useful for pricing options.

[5] This notion of regret corresponds to the concept of "external regret," where we compare the payoff of a strategy with that of the best *static* alternative.

[6] I.e., if S_t is the stock price on date t, the payoff of the strategy is at least $\$125 \times 80\% \times \max\{S_1/S_0, 1\} - 100 = (100/S_0) \times \max\{S_1 - S_0, 0\}$, matching the payoff of an at-the-money call option on \$100 worth of stock.

quadratic variation of the stock price path. These bounds are new, do not depend on specific assumptions about the price process or investor preferences, and are robust to jumps and trading halts.

The bounds we develop in Section 2 are based on the simple gradient strategy methods used in robust optimization. In Section 3 we consider how these bounds might be improved using dynamic programming techniques. We show that optimal dynamic strategies, unlike the gradient strategies, should be memory-less. We exploit this observation to compute the optimal option price bounds and show that these bounds represent a significant improvement to that derived using gradient strategies, and are tight enough to be empirically relevant.

1.2. Relationship to option pricing literature

Our paper provides important new results in the theory of option pricing. The new bounds we develop are independent of the specific price process, the timing of trade, and the underlying pricing kernel. Our bounds depend *only* on realized quadratic variation of the price process.

Of course, we are not the first to find bounds for option prices. In his classic paper Bob Merton (1973) obtained bounds for the value of an option absent *any* assumptions, and shows they are quite wide: if quadratic variation (equivalently, the realized variance) is unbounded then the best upper bound for the value of a call option is the stock price itself. At the other extreme, the Black–Scholes model assumes a given distribution of prices (geometric Brownian motion with a fixed volatility) and effectively complete markets (via continuous trade) to obtain a unique price. Clearly, there is a tradeoff between the strength of the assumptions made and the resulting option pricing bounds. Weaker assumptions necessarily lead to bounds that are less tight. There is no magic solution, and the approach one chooses depends largely on the intended application.

Table 1 describes this tradeoff for some of the key papers in the literature. The assumptions can be divided into i) assumptions that are made regarding the possible price path or the stochastic process, including its volatility, and ii) assumptions on the trading process and the pricing kernel (or equivalently, the preferences of a representative agent).

The first group of papers (Black–Scholes and the Binomial Model) provide exact option pricing results by assuming a specific price process and complete markets. Unfortunately, in practice the price processes assumed in these models do not hold, nor does the assumption of complete markets. The papers in group II relax the assumption of complete markets, by allowing for both stochastic volatility and jumps.[7] These papers provide exact option prices by assuming a specific price process together with a known pricing kernel. Of course, the pricing kernel is not known in practice. Papers in group III relax this assumption by assuming bounds on the pricing kernel, and as a result they provide bounds on the option price for a specific price process.[8] Papers in group IV do not assume a specific price process, but assume it is continuous with bounded volatility. The assumption of continuous trade then allows them to bound the option price.[9] Finally, the

[7] See Eraker (2004) for a useful summary of model specifications with empirical evidence on the importance of jumps for understanding both returns and option prices.

[8] In a related vein, Bertsimas et al. (2001) examine strategies that "almost" replicate the payoff of an option given a stochastic process for the underline stock.

[9] E.g. Mykland (2000) assumes the stock price is a continuous diffusion process, but allows the volatility to be stochastic. Nonetheless, he shows that the Black–Scholes price is an upper bound if we take the volatility parameter to be the upper bound over all realizations of the average stochastic volatility. (See also Lo (1987), Shreve et al. (1998), and Grundy and Wiener (1999) for related bounds based on the volatility process.)

Table 1
A comparison of option pricing methods.

	Method	Volatility	Price Process	Trading	Pricing Kernel	Result
I	Black-Scholes (1973)	known, constant	specific (GBM)	continuous	--	exact
	Binary Model (Cox et al 1979)	known	specific (binary lattice)	discrete	--	exact
II	Rubinstein (1976)	known, constant	specific (lognormal)	--	specific	exact
	Heston (1993)	stochastic (unbounded)	specific, continuous	--	specific	exact
	Pan (2002), Eraker et al (2003)	stochastic (unbounded)	specific, allows jumps	--	specific	exact
III	Bernardo-Ledoit (2000)	known, constant	specific (GBM)	--	bounded	bounds
	Cochrane-Saa Requejo (2000)	stochastic (unbounded)	specific, continuous	--	bounded	bounds
IV	Lyons (1995), Avellaneda et al (1995)	bounded (pointwise)	continuous	continuous	--	bounds
	Mykland (2000), Schafer-Vovk (2001)	bounded (total)	continuous	continuous	--	bounds
V	Frey-Sin (1999), Cvitanic et al (1999)	stochastic (unbounded)	continuous	continuous	--	trivial
	This paper	**bounded (total)**	**--**	**--**	**--**	**bounds**

papers in group V show that if volatility is unbounded and stochastic, then even with continuous price processes and continuous trading, option prices cannot be bounded beyond Merton's trivial bound (the stock price).

Other related work includes Cvitanic and Karatzas (1995), who examine optimal portfolios under drawdown constraints that also achieve long term optimal growth. Cvitanic (2000) analyzes optimal hedging policies for general contingent claims when markets are incomplete. Cvitanic et al. (1995) study bounds for option pricing with proportional transaction costs and show the only no-arbitrage bound is the trivial one of the stock price itself. Leland (1985) studies the pricing errors from discrete-time hedging with transaction costs, and their convergence as the trading frequency increases. While our model is not meant to handle transaction costs explicitly, our results may be useful in settings where fixed costs prevent continuous trade.

In Table 1, we have highlighted the assumptions that are strictly stronger than those made in this paper. Note that in almost all cases the existing models make stronger assumptions than we require. However, in contrast to our bound on quadratic variation, there are several papers – e.g. Pan (2002), and Eraker et al. (2003) – that allow for stochastic volatility and jumps for which the quadratic variation of the price process is potentially unbounded. But as the table shows, these results come at the heavy price of making strong assumptions regarding the pricing kernel or preferences as well as an assumption about a specific stochastic process. These assumptions are problematic in practice as they cannot be conclusively verified, nor hedged. Perhaps even more challenging for a trader using these models, *it is unclear how large the losses from a violation of these assumptions might be*.[10] On the contrary, our methods provide a precise estimate of the maximal potential loss if the quadratic variation is higher than anticipated, and moreover securities based on the realized quadratic variation (variance swaps) can be used to hedge this risk. We return to these points, and discuss further applications of our results, in our concluding remarks (Section 4).

[10] Unlike the aforementioned finance papers, there is related research in Computer Science/Statistics initiated by the seminal work of Cover (1991) which also takes a robust "model-free" approach to analyzing financial securities. Closest to our work is Cover and Ordentlich (1998), who consider portfolios that are constantly rebalanced to maintain fixed portfolio weights. They interpret the result in terms of the price of the exotic derivative that pays ex-post the best constant rebalanced portfolio. Although their results are mathematically very elegant, they cannot be applied to derive bounds for standard options.

2. Gradient strategies and option bounds

Von Neumann's (1928) famous minimax theorem establishes that there is a unique value that each player can guarantee himself in a two-person zero-sum game. In a seminal paper, Blackwell (1956) extends this idea to consider a two-player adversarial game in which the payoff is a vector, rather than a scalar. Rather than ask what minimum payoff a player can achieve, Blackwell asked whether a player's asymptotic average payoff can be guaranteed to lie in given convex set. Blackwell's remarkable "Approachability Theorem" provides necessary and sufficient conditions for such a result; more importantly, he constructs an adaptive strategy that achieves the guarantee. The intuition for the result is that, at each stage, we can define a scalar game in which the payoff corresponds to the change in the distance of the current average payoff to the desired set. By playing the minimax strategy for this stage game, the player can move closer to the desired set.

In closely related work, Hannan (1957) defined a notion of robust optimization in which a player compares the average payoff of his chosen dynamic strategy to that which would have been achieved by repeatedly playing a fixed strategy each round. The dynamic strategy exhibits "no external regret," or is "Hannan consistent," if its average payoff is, asymptotically, no worse than that of any fixed strategy. The connection between Hannan and Blackwell's work can be seen by letting the vector payoff in Blackwell's game be the loss of the agent's average payoff relative to that of each fixed strategy. Hannan consistency then amounts to the Blackwell approachability of the negative orthant.

These results are also closely related to the recent game theory literature on calibration, in which it is shown that even an *uninformed* forecaster can make frequency predictions that are asymptotically empirically correct: e.g., it is possible for a weatherman with no true forecasting ability to choose a dynamic reporting strategy so that, for the set of days the weatherman reports an X% chance of rain, it indeed rains approximately X% of the time. Foster and Vohra (1998), Foster (1999) and others have demonstrated that such calibration results can be deduced based on approachability. Hart and Mas-Colell (2000) show these results can be implemented via a simple and elegant gradient strategy.

In this section, we extend the approach of Hart and Mas-Colell (2000) in several ways. First, we apply it to a setting in which mixed strategies are replaced by portfolio choices, and so correspond to dynamic trading strategies. Second, rather than consider payoffs which are additive over time, we evaluate their (multiplicative) portfolio returns. Finally, we develop a characterization of the strategy's finite horizon, rather than asymptotic, performance.

2.1. A simple financial trading model

Consider a discrete-time N-period model where periods are denoted by $n \in \{0, 1, \dots, N\}$. There is a risky asset (e.g., a stock) whose value (price) in period n is given by S_n. Unless stated otherwise, we (without loss of generality) normalize the initial value to one, $S_0 = 1$, and assume that the asset does not pay any dividends. Denote by r_n the return between $n-1$ and n so that $S_n = S_{n-1}(1 + r_n)$. We call $r = r_1, \dots, r_N$ the *price path*. In addition to the risky asset we have a risk-free asset (e.g., a bond). We also treat the bond as numeraire and thereby normalize the risk-free rate to zero.

A dynamic trading strategy specifies a portfolio to hold each period. This portfolio has initial value $G_0 = 1$. Each period, the initial value of the portfolio G_{n-1} is invested in the assets, with some fraction $w_n \in [0, 1]$ in the stock and $1 - w_n$ in the bond. The portfolio weight w_n is specified as a function of the price path of the stock prior to period n. Since we assume a zero interest rate, at the end of period n its value is $G_n = (w_n G_{n-1})(1 + r_n) + (1 - w_n)G_{n-1} = G_{n-1}(1 + w_n r_n)$; its final value is G_N. Thus, a trading strategy determines a mapping $G_n(r)$ from price paths to payoffs each period. Our goal is to develop a dynamic trading strategy that performs well compared to a static investment in either the stock or the bond.[11]

2.2. The gradient trading strategy

To adapt the classical gradient strategy to our setting of financial trading, we begin by defining an appropriate loss measure. Recall that we want to compare the payoff of the trading strategy, G_N, with the payoff of the stock, S_N, and that of the bond, which has a risk-free value of 1. Because the model of returns is multiplicative, it is natural to compare log returns. Specifically, we define the cumulative loss or regret vector as

$$L_n \equiv \begin{bmatrix} \ln(S_n) \\ \ln(1) \end{bmatrix} - \begin{bmatrix} \ln(G_n) \\ \ln(G_n) \end{bmatrix} = \begin{bmatrix} \ln(S_n/G_n) \\ \ln(1/G_n) \end{bmatrix} \tag{1}$$

If we define the per-period payoffs to holding the stock and the bond as $\pi_{n,1} = \ln(1 + r_n)$ and $\pi_{n,2} = 0$, then (1) is equivalent to

$$L_{n,i} = \sum_{m=1}^{n} \pi_{m,i} - \sum_{m=1}^{n} \ln(1 + w_m r_m) \tag{2}$$

In the classical gradient strategy, we minimize asymptotic regret by choosing a mixed strategy each period in which the probabilities are proportional to the current regret. By doing so, we reduce the likelihood of falling further behind those strategies against which we are underperforming the most.

We extend this idea to a financial context by choosing portfolio weights proportional to the current loss vector. Specifically, if $G_{n-1} < \max\{S_{n-1}, 1\}$, then we choose the portfolio[12]

$$w_n = \frac{L^+_{n-1,1}}{L^+_{n-1,1} + L^+_{n-1,2}} = \frac{[\ln(S_{n-1}/G_{n-1})]^+}{[\ln(S_{n-1}/G_{n-1})]^+ + [-\ln(G_{n-1})]^+} \in [0, 1] \tag{3}$$

That is, the weight put on the stock versus the bond is proportional to the strategy's current loss relative to each. If there is no current loss – i.e. $G_{n-1} \geq \max\{S_{n-1}, 1\}$ – then we arbitrarily set the weights to be equal ($w_n = 1/2$).

To see the connection between the payoffs in our portfolio setting with multiplicative returns and the standard additively separable payoff structure for dynamic games, note that by Jensen's inequality,

$$\ln(1 + w_n r_n) \geq w_n \ln(1 + r_n) = w_n \pi_{n,1} + (1 - w_n)\pi_{n,2} \quad \text{for all } w_n \in [0, 1]. \tag{4}$$

Hence the log return of our portfolio each period is at least as great as the expected payoff from a mixed strategy in an additive setting with payoffs given by π. As a result, we can adapt the logic of the proof of Hart and Mas-Colell (2000) to this setting and obtain the following bound on the performance of the simple gradient strategy:

[11] The reason why consider just these two alternatives (stock, bond) is not only for simplicity but because it is the right set when we consider a European call option. That said, one should note that these strategies are the extreme elements in the set of all buy and hold strategies. As a result our bound remains the same even if we expand the set of alternatives to include all buy and hold strategies.

[12] We adopt the notation $x^+ = \max(x, 0)$ throughout the paper.

Proposition 1. *The gradient trading strategy implies*

$$G_N(r) \geq \exp\bigl(-q(r)\bigr)\max\{1, S_N\} \tag{5}$$

where $q^2(r)$ is the realized quadratic variation of the log returns:

$$q(r) \equiv \sqrt{\sum_{n=1}^{N}\bigl(\ln(1+r_n)\bigr)^2}. \tag{6}$$

Proof. Let $\Delta L_{n,i} \equiv L_{n,i} - L_{n-1,i} = \pi_{n,i} - \ln(1 + w_n r_n)$ and define $\Delta\hat{L}_{n,i} \equiv \pi_{n,i} - w_n \ln(1 + r_n)$. Note that (3) implies $\Delta\hat{L}_n \cdot L^+_{n-1} = 0$, and from (4), we know $(\Delta L_n - \Delta\hat{L}_n) \leq 0$. Therefore,

$$\Delta L_n \cdot L^+_{n-1} = (\Delta L_n - \Delta\hat{L}_n) \cdot L^+_{n-1} \leq 0. \tag{7}$$

Now because

$$\left\|L^+_n\right\|^2 = \left\|(L_{n-1} + \Delta L_n)^+\right\|^2 \leq \left\|L^+_{n-1} + \Delta L_n\right\|^2 = \left\|L^+_{n-1}\right\|^2 + \|\Delta L_n\|^2 + 2\Delta L_n \cdot L^+_{n-1},$$

we can use (7) to conclude

$$\left\|L^+_n\right\|^2 \leq \left\|L^+_{n-1}\right\|^2 + \|\Delta L_n\|^2 \tag{8}$$

Iterating the above expression recursively then yields

$$\left\|L^+_n\right\|^2 \leq \sum_{m=1}^{n}\|\Delta L_m\|^2.$$

Then, because $\|\Delta L_n\|^2 \leq (\ln(1 + r_n))^2$, we have that

$$\ln\bigl(\max\{S_N, 1\}/G_n\bigr) = \max_i L_{n,i} \leq \left\|L^+_n\right\| \leq \sqrt{\sum_{n=1}^{N}\|\Delta L_n\|^2} \leq \sqrt{\sum_{n=1}^{N}\bigl(\ln(1+r_n)\bigr)^2} = q(r).$$

The result then follows by rearranging and taking exponentials. □

Proposition 1 bounds the performance of the gradient strategy depending on the realized quadratic variation of the price path. Note that if the per period stock returns are bounded, then the loss rate q grows at rate $\sqrt{N}$, which implies

$$\lim_{N\to\infty}\frac{1}{N}\ln(G_N) \geq \lim_{N\to\infty}\frac{1}{N}\ln(S_N)^+.$$

That is, the asymptotic annualized returns of the gradient strategy are as good or better than those of both the bond and the stock, a result first shown by Cover (1991). Of course, our interest is in the finite horizon performance of the strategy, and its implications for option prices. But first, we develop a generalization of the simple gradient strategy that may improve its performance.

2.3. Generalized gradient strategies

In the simple gradient strategy, we set $L_0 = 0$. As a generalization, we can consider strategies which start with an initial "fictitious loss" $L_0 = x$, where $x = (x_1, x_2)$ is a parameter. In that case, we can define $L^x_n = L_n + x$, and choose portfolio weight w^x_n as in (3) but with L^x in place of L.

Doing so allows us to define a family of generalized gradient trading strategies indexed by the initial loss vector x.

We can characterize the performance of these trading strategy using the same methodology as of Proposition 1, though we must credit back the initial fictitious regret of x with respect to each strategy:

Proposition 2. *The generalized gradient trading strategy with initial loss x yields*

$$G_N(r) \geq \max\left\{\exp\left(x_1 - \sqrt{q^2(r) + x_1^2 + x_2^2}\right)S_N, \exp\left(x_2 - \sqrt{q^2(r) + x_1^2 + x_2^2}\right)\right\}$$

Proof. Using the identical logic to the proof of Proposition 1, we have

$$L^x_{n,i} \leq \sqrt{\sum_{n=1}^{N}\left(\ln(1+r_n)\right)^2 + \|x\|^2}.$$

Therefore,

$$\ln(S_N/G_n) = L^x_{n,1} - x_1 \leq \sqrt{\sum_{n=1}^{N}\left(\ln(1+r_n)\right)^2 + \|x\|^2} - x_1,$$

and similarly,

$$\ln(1/G_n) = L^x_{n,2} - x_2 \leq \sqrt{\sum_{n=1}^{N}\left(\ln(1+r_n)\right)^2 + \|x\|^2} - x_2.$$

The result then follows upon rearrangement. □

Proposition 2 shows that by raising x_i, we improve the performance of the gradient strategy with respect to fixed strategy i (but degrade its performance relative to the alternative strategy). More interestingly, starting from $x = 0$, by raising both x_1 and x_2 together we can reduce the sensitivity of the gradient strategy to early losses and improve performance with respect to *both* strategies. As we will see, these generalized strategies will allow us to optimize the price bounds we derive for stock options with arbitrary strike prices.

2.4. Price bounds for at-the-money options

In the previous section we described a family of trading strategies and characterized their regret conditional on the realized price path. We now show how we can use this result to provide upper bounds for the value of standard call options.

Let Φ be a set of possible price paths for the stock. Conditional on this set we assume that there is *no arbitrage* in prices. Namely, for any trading strategy G such that $G_n(r) > 1$ for some $r \in \Phi$, there exists another price path $r' \in \Phi$ such that $G_n(r') < 1$. Otherwise, investing in G and shorting the bond would lead to a profit in period n given path r with no possibility of a loss.

A European call option with strike price K that matures in period N has a final payoff of $\max\{0, S_N - K\}$. Let Φ be the set of feasible stock price paths, and let $C(S_0, K|\Phi)$ be the *highest* value of the call option at time 0 that is consistent with no arbitrage. Equivalently, $C(S_0, K|\Phi)$

is the *lowest* initial cost of a trading strategy whose payoff equals or exceeds that of the option for any price path in Φ.

Before proceeding we should discuss the importance of imposing restrictions on the price path given by Φ. Merton (1973) shows that without making additional assumptions on price paths, the best upper bound for the no-arbitrage value of a call option is

$$C(S_0, K|\Phi) \le S_0.$$

That is, the value of the option can be as high as that of the underlying asset. This is a very weak bound but cannot be improved if arbitrary price paths are allowed.

Our goal is to find bounds for the option value $C(S_0, K|\Phi)$ using the gradient trading strategies; in this section we begin by considering the case of an at-the-money option ($S_0 = K = 1$). We begin by formalizing the link between regret-limiting strategies and option prices discussed (as an example) in the Introduction:

Proposition 3. *Suppose we have a dynamic trading strategy that satisfies $G_N(r) \ge \beta(r) \max\{1, S_N\}$ for all $r \in \Phi$. Let $\beta^* = \inf_{r\in\Phi} \beta(r)$ be the minimal performance guarantee. Then no arbitrage implies the following upper bound for the value of an at-the-money call option:*

$$C(1, 1|\Phi) \le \frac{1}{\beta^*} - 1 \tag{9}$$

Proof. Investing $1/\beta^*$ in the trading strategy and borrowing \$1 leads to a payoff

$$\frac{G_N(r)}{\beta^*} - 1 \ge \frac{\beta(r)\max\{1, S_N\}}{\beta^*} - 1 \ge \max\{0, S_N - 1\}.$$

Because this payoff always exceeds that of the call option, the value of the option cannot exceed the cost $1/\beta^* - 1$ of this strategy. □

To gain some intuition, consider a very simple trading strategy. Suppose we decide to use a buy and hold strategy in which we invest a fraction $1/2$ in each asset. The future payoff of this static portfolio is

$$G_n = 0.5 + 0.5S_N \ge \max\{0.5, 0.5S_N\} = 0.5\max\{1, S_N\}$$

So in this case, $\beta^* = 1/2$ for any set of price paths. Using the above proposition we conclude that:

$$C(1, 1|\Phi) \le \frac{1}{0.5} - 1 = 1 = S_0.$$

As mentioned before, S_0 is the simple upper bound from Merton (1973). To improve upon this bound, we must consider a restricted set of price paths and strategies with better performance guarantees with higher β^*.

Consider first the simple gradient trading strategy and define $q(\Phi) = \sup_{r\in\Phi} q(r)$, the highest possible realized quadratic variation of the log returns for the price paths in Φ. Then from Proposition 1, $\beta(r) = \exp(-q(r))$ and thus $\beta^* = \exp(-q(\Phi))$. Hence Proposition 3 implies

$$C(1, 1|\Phi) \le \exp\big(q(\Phi)\big) - 1. \tag{10}$$

Now consider a generalized gradient trading strategy with $x_1 = x_2 = \bar{x}$. Applying Proposition 2, the performance guarantee becomes

$$\beta(r) = \exp\left(\bar{x} - \sqrt{q^2(r) + 2\bar{x}^2}\right). \tag{11}$$

We can see from (11) that the generalized strategy will improve the price bound as long as $0 \leq \bar{x} \leq 2q(r)$. Taking $q(\Phi)$ as given, and maximizing over $\bar{x}$, the tightest possible bound is achieved by choosing $\bar{x}^*(q) = q(\Phi)/\sqrt{2}$. Hence, we can improve the bound in (10) as follows:

Proposition 4. *Based on the generalized gradient trading strategy, the no arbitrage price of an at-the-money call option satisfies*

$$C(1, 1|\Phi) \leq \exp\left(\frac{1}{\sqrt{2}} q(\Phi)\right) - 1. \tag{12}$$

It is worth reviewing the meaning of the bound in the above proposition. Consider the set $\Phi(\bar{q})$ of *all* price paths r with quadratic variation $q(r)$ of at most $\bar{q}$. Proposition 4 then establishes a no-arbitrage price bound for an at-the-money call option given *any* probability distribution over the set of price paths $\Phi(\bar{q})$. In particular, we do not need to restrict attention to memory-less distributions (or moreover any properties of the distribution over these paths) or make any assumptions regarding the pricing kernel. Also, because our price paths are discrete, we automatically admit jumps. Finally, note that the bound does not depend directly on the number of trading periods, but only on the quadratic variation of the price path. Thus we can allow the number of periods to grow, and the time between trading opportunities to shrink, and extend these same bounds to settings in which trade is continuous, but price paths need not be (see Section 2.6).

2.5. Asymmetric trading strategies and alternative strike prices

Thus far we have developed price bounds for at-the-money options. In this section we show how the generalized trading strategies that we defined in Section 2.3 allow us to obtain bounds for options with different strike prices K and initial stock prices S_0. The starting point is the following generalization of Proposition 3:

Proposition 5. *Suppose we have a dynamic trading strategy that satisfies $G_N(r) \geq \beta(r)\max\{\alpha(r), S_N/S_0\}$ for all $r \in \Phi$, and let*

$$\beta^* = \inf_{r\in\Phi} \beta(r)\min\left\{1, \alpha(r)S_0/K\right\}. \tag{13}$$

Then no arbitrage implies the following upper bound for the value of a call option with initial stock price S_0 and strike price K:

$$C(S_0, K|\Phi) \leq \frac{S_0}{\beta^*} - K. \tag{14}$$

Proof. Investing S_0/β^* in the trading strategy and borrowing K leads to a payoff

$$\begin{aligned}\frac{S_0}{\beta^*}G_N(r) - K &\geq \frac{S_0}{\beta^*}\beta(r)\max\left\{\alpha(r), S_N/S_0\right\} - K = \frac{\beta(r)\max\{\alpha(r)S_0, S_N\}}{\beta^*} - K \\ &\geq \max\{K, S_N\} - K = \max\{0, S_N - K\}.\end{aligned}$$

The bound (14) is therefore implied by no arbitrage. □

In Proposition 2 we demonstrated that the generalized gradient trading strategy with initial regret of x provides a performance guarantee of

$$\beta(r) = \exp\left(x_1 - \sqrt{q^2(r) + x_1^2 + x_2^2}\right), \quad \text{and} \quad \alpha(r) = \exp(x_2 - x_1).$$

If we define $q(\Phi) = \sup_{r\in\Phi} q(r)$ and $k = \ln(K/S_0)$, then from (13),

$$\ln\beta^* = x_1 - \sqrt{q^2(\Phi) + x_1^2 + x_2^2} + \min\{0, x_2 - x_1 - k\}. \tag{15}$$

Equation (15) implies that to maximize β^*, we should choose $x_2 - x_1 = k$. For if $x_2 - x_1 > k$, we can increase β^* by decreasing x_2, and if $x_2 - x_1 < k$, we can increase β^* by decreasing x_1. Therefore, let $x_1 = \bar{x}$ and $x_2 = \bar{x} + k$. In that case, (15) reduces to

$$\ln\beta^* = \bar{x} - \sqrt{q^2(\Phi) + \bar{x}^2 + (\bar{x} + k)^2}. \tag{16}$$

Thus, the quality of the option price bound provided by the generalized gradient strategy depends upon $\bar{x}$. Maximizing (16), the optimal choice of $\bar{x}$ is given by

$$\bar{x}^*(q) = \frac{1}{2}\sqrt{2q^2 + k^2} - \frac{1}{2}k. \tag{17}$$

Substituting for $\bar{x}$ in (16) we find the best possible performance guarantee:

$$\ln\beta^* = -\frac{1}{2}\sqrt{2q^2(\Phi) + k^2} - \frac{1}{2}k. \tag{18}$$

Combining (18) and Proposition 5, the generalized gradient trading strategies imply the following no-arbitrage bound for the price of a call option:

Proposition 6. *The no-arbitrage price of a call option with initial stock price S_0 and strike price K satisfies*

$$C(S_0, K|\Phi) \le S_0 \exp\left(\frac{1}{2}\sqrt{2q^2(\Phi) + (\ln S_0/K)^2} - \frac{1}{2}\ln(S_0/K)\right) - K. \tag{19}$$

This bound is achieved by borrowing K and investing the total in the generalized gradient trading strategy with initial regret $\bar{x}^*(q)$ with respect to the stock and $\bar{x}^*(q) + k$ with respect to the bond. Finally, note that (17) and (19) generalize our earlier results in Section 2.4.

2.6. Comparison with Black–Scholes

What is remarkable about the arbitrage bounds in Proposition 4 and Proposition 6 is that they do not depend explicitly on the number of trading periods, N, or any properties of the stock price other than its volatility, as measured by its maximal quadratic variation.[13] The lack of dependence on the number of trading periods implies that we can subdivide a fixed time interval to allow for continuous trading in a manner analogous to the convergence of the binomial trading model to the Black–Scholes framework demonstrated by Cox et al. (1979). In the standard implementation of their model, given an option with expiration T, they subdivide the fixed time interval $[0, T]$ into N periods and assume that over each period the stock price changes by a factor of $\exp(\pm\sigma\sqrt{T/N})$. In that case, for each price path and for any N,

[13] Indeed, this feature of our model makes it particularly applicable to settings in which trading opportunities are limited (perhaps due to fixed costs), and reveals that the impact of limited trading on the cost of hedging comes through its effect on the realized quadratic variation.

$$q(r) \equiv \sqrt{\sum_{n=1}^{N} \left(\ln(1+r_n)\right)^2} = \sqrt{\sum_{n=1}^{N} \sigma^2 T/N} = \sigma\sqrt{T} \tag{20}$$

Thus, we can apply Proposition 6 in the setting of Black–Scholes by setting q equal to the anticipated volatility of the stock over the remaining life of the option.

More generally, we can extend our model to settings with continuous trading as follows: Let Φ be a set of price paths for the stock over the continuous time interval $[0, T]$, given by functions $S(t)$ (right continuous with left limits). Let P be any partition of the interval $[0, T]$, where $\#P$ denotes the number of periods and define $|P|$ as the maximal period length. Define r_n to be the stock's return over interval n of the partition; i.e., $1 + r_n = S(t_n)/S(t_{n-1}) - 1$. Then we measure the stock's volatility in terms of its quadratic variation as[14]

$$q^2(\Phi) \equiv \sup_{\Phi} \lim_{|P|\to 0} \sum_{n=1}^{\#P} \left(\ln(1+r_n)\right)^2 = \sup_{\Phi} \int_{t=0}^{T} \left(d \ln S(t)\right)^2 \tag{21}$$

With this extension of the definition of q, the arbitrage bounds in Proposition 4 and Proposition 6 continue to hold with continuous trading opportunities. Note that (21) holds even if the stock price path exhibits jumps. In the specialized setting of Black–Scholes, the stock price moves continuously as geometric Brownian motion with constant volatility σ, and thus $q^2(\Phi) = \sigma^2 T$.

The above results allow us compare the price bounds in Proposition 4 and Proposition 6 to the Black–Scholes option price. Obviously, our price bounds must be weaker, as we have relaxed the two key assumptions of Black–Scholes. First, we allow jumps, and second, we do not require that the stock price exhibit constant volatility. To get a sense of the impact of relaxing these assumptions, consider first the case of an at-the-money call option that is close to expiration, so that $q = \sigma\sqrt{T}$ is small. In that case, the Black–Scholes option price can be approximated as[15]:

$$C(1, 1|\Phi_{BS}) \approx q(\Phi_{BS})/\sqrt{2\pi} \quad \text{for small } q. \tag{22}$$

Alternatively, using the optimal generalized gradient trading strategy of Proposition 4 we have:

$$C(1, 1|\Phi) \leq \exp\left(q(\Phi)/\sqrt{2}\right) - 1 \approx q(\Phi)/\sqrt{2} \quad \text{for small } q. \tag{23}$$

Comparing equation (23) to equation (22), we can see that the option price bound provided by the generalized gradient trading strategy exceeds the Black–Scholes price by an amount equivalent to an increase in the stock's volatility by a factor of $\sqrt{\pi}$, or approximately 77%. We can interpret this increase as bounding the maximal impact on the option price from relaxing the Black–Scholes assumption of continuous price paths.

Using Proposition 6, we can perform a similar comparison for alternative strike prices. Because the Black–Scholes formula is monotone in the volatility σ, we can also interpret our price bound in terms of a corresponding "implied volatility" under Black–Scholes. Fig. 1 shows the generalized gradient bound in comparison with Black–Scholes when $q = 10\%$. The figure

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[14] More formally, let the price process S_t be a semimartingale, and given a partition $P = \{0 = t(0) \leq t(1) \leq \ldots \leq t(\#P) = T\}$ with $|P| = \max_n t(n) - t(n-1)$, then the limit in (21) exists and is well-defined under convergence in probability.

[15] To derive this result, let $f(\sigma)$ be the Black–Scholes value of an at-the-money call option with expiration T when the stock's price is 1, its volatility is σ, and the interest rate is zero. Then $f'(\sigma)$ is the corresponding "vega" (volatility sensitivity) of the option, and for σ small, $f(\sigma) \approx f(0) + f'(0)\sigma = 0 + (\frac{1}{\sqrt{2\pi}}\sqrt{T})\sigma$.

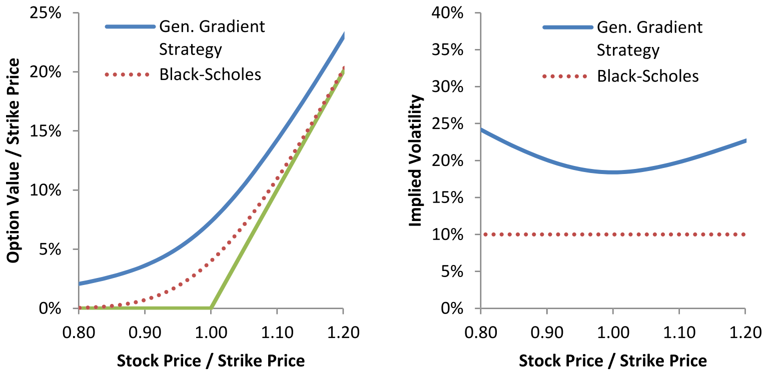

Fig. 1. Generalized gradient bounds versus Black–Scholes.

demonstrates the potential impact on the value of the option from dropping the Black–Scholes assumptions of a continuous price path with a constant volatility, expressed in terms of both the option price as well as its implied volatility.

Naturally, the price and implied volatility associated with the generalized gradient bound must be higher than with Black–Scholes, as value of the option increases when we allow for discontinuous jumps. This result is also consistent with the empirical observation that the Black–Scholes implied volatility tends to be significantly higher on average than the ex-post realized volatility of the stock price (see Bollen and Whaley, 2004).

Second, and not surprisingly, the figure shows that, the impact of jumps or time-varying volatility has a larger impact on the cost of hedging when the strike price is farther from the current stock price. This result is consistent with the observation that jumps and changes in volatility can help to explain the implied volatility "smile" that is observed empirically.

3. Optimal Q-based bounds

Thus far, we have demonstrated that we can obtain meaningful option price bounds by extending and generalizing Blackwell's gradient strategy to an investment trading environment. These bounds depend upon the volatility of the possible stock price paths, as measured by their maximal quadratic variation $q^2(\Phi)$. Importantly, the bounds do not depend on continuous trading, continuity of the stock price, or the constant volatility of its returns.

In this section, rather than start with a fixed set of price paths Φ, and a particular type of trading strategy, we start with the broadest set of price paths consistent with a given maximal quadratic variation and construct an *optimal* trading strategy. Specifically, we define the set of allowable stock price paths as all paths with quadratic variation less than some bound

$$\Phi\left(\bar{q}^2\right) = \left\{r | q^2(r) \le \bar{q}^2\right\}.$$

We then ask, what is the best bound available for the set $\Phi(\bar{q}^2)$? That is, what is $C(S_0, K|\Phi(\bar{q}^2))$, which we will denote more simply as $C(S_0, K|\bar{q}^2)$? We demonstrate in this section how to compute this bound recursively. This approach will allow us to further improve the bounds that we derived based on gradient strategies in Section 2.

3.1. Optimality and value independence

The Hannan and Blackwell gradient strategy that we examined in Section 2.2 is an example of a strategy that is performance dependent. That is, the portfolio chosen by the strategy depends on the strategy's past historical performance. In contrast, the trading strategy underlying the Black–Scholes formula is performance independent, in that it depends only on the current stock price and the volatility of the stock over the remainder of the option's life. In this section we argue that the optimal regret-minimizing strategy should also share this property of performance independence:

Definition. We say that a dynamic trading strategy is *performance independent* if the portfolio weights at time t depend only on the stock price at time t, S_t, and the possible future paths Φ_t.

To see why an optimal strategy should be independent of past performance, consider a scenario where in period n the stock price is given by S_n and the value of our portfolio is G_n. Let $\hat{G}_{n,N}$ denote the payoff in period N of a strategy that starts with a wealth of \$1 at n and follows a dynamic portfolio strategy $\hat{w}$. Then the regret-minimizing strategy must solve

$$\max_{\hat{w}} \min_{r\in\Phi_n} \underbrace{\left(\frac{G_n \hat{G}_{n,N}}{\max\{1, S_N\}}\right)}_{\substack{\text{regret from 0 to } N \\ \text{with strike } =1}} = \frac{G_n}{S_n} \times \max_{\hat{w}} \min_{r\in\Phi_n} \underbrace{\left(\frac{\hat{G}_{n,N}}{\max\{1/S_n, S_N/S_n\}}\right)}_{\substack{\text{regret from } n \text{ to } N \\ \text{with strike } =1/S_n}}. \tag{24}$$

Thus, the optimal strategy $\hat{G}_{n,N}$ is equal to the regret minimizing strategy initiated in period n with an initial stock price of S_n – or equivalently, given a stock price of 1 and a strike price $1/S_n$ – and is therefore independent of its past performance G_n.[16]

An immediate implication is that the Hannan–Blackwell gradient trading strategy is suboptimal.[17] One way of making the gradient strategy independent of past performance can be based on our generalized gradient strategy. Recall from Section 2.3 that we can interpret the generalized gradient strategy as though we started the game with an arbitrary initial relative performance vector L_0. Then in Section 2.5 we computed the optimal generalized gradient trading strategy, and showed that the optimal starting position depended only on the remaining quadratic variation of the price paths, and the ratio of the current stock price to the option strike price. In other words, it is optimal at each point in time to "forget" the actual past performance of the strategy, and choose the current portfolio in a path-independent manner. In the remainder of this section, we implement this idea by using backward induction to solve for the optimal regret-minimizing trading strategy.

[16] Note, of course, that the optimal strategy does depend on the future price paths Φ_t, and this set may depend on past realizations. Importantly, however, this is independent of the specific strategy used and its past performance.

[17] The claim extends to the standard framework with additive payoffs and mixed strategies; a similar argument shows that the optimal strategy at date n should depend only on the current cumulative payoff of each static strategy through date $n-1$ (i.e., $\sum_{m=1}^{n-1} \pi_m$), and the possible future payoff paths, but not on the past performance of the agent's actual strategy. Thus the Hannan and Blackwell gradient trading strategy does *not* minimize regret over a finite horizon. Of course, this criticism does not affect to their asymptotic results, as when the horizon is infinite the long-run average regret of their strategy *is* optimal.

3.2. The super-replicating portfolio

An optimal price bound for the call option is equal to the lowest cost of a super-replicating portfolio, which is a trading strategy for the stock and the bond whose value is *always* equal to or greater than the value of the call option after any move of the stock price on an allowed price path. Let $V(S, K, q^2)$ be the cost of a super-replicating portfolio, given a current stock price of S, an exercise price of K, and a future price path with a quadratic variation of at most q^2. Obviously, at the boundary $q^2 = 0$ we have:

$$V(S, K, 0) = \max\{0, S - K\}. \tag{25}$$

For $q^2 > 0$, the value of the lowest cost super-replicating portfolio should also satisfy[18]

$$\begin{aligned} V(S, K, q^2) \;=\; & \min_{\Delta, B} \Delta \times S + B \\ \text{s.t.} \quad & \Delta \times Se^r + B \geq V(Se^r, K, q^2 - r^2) \\ & \text{for all } r \text{ such that } r^2 \leq q^2 \end{aligned} \tag{26}$$

That is, V is the cost of a portfolio, which holds Δ shares of the stock and B units of the bond, such that given any feasible return for the stock, the new value of the portfolio will be sufficient to purchase the option given the new stock price and the remaining quadratic variation of the returns. Because (26) implicitly defines a strategy whose payoffs will equal or exceed that of the option, it is a sufficient condition for V to be an upper bound on the option's value.

Proposition 7. *If V satisfies (25) and (26), then $C(S, K|\bar{q}^2) \leq V(S, K, \bar{q}^2)$.*

The proof is straightforward, with the only minor complication that we must insure that if the option expires when there is quadratic variation remaining (relative to the initial q), then the strategy defined in (26) will still lead to an adequate payoff.

Proof. At each date t, let S_t be the current stock price and q_t^2 be the realized quadratic variation up to date t.[19] Given any (S, q), let $\Delta(S, q)$ and $B(S, q)$ solve (26). Then consider the dynamic trading strategy which invests, at each date t,

$$\Delta\big(S_t, \bar{q}^2 - q_t^2\big)S_t \text{ in the stock and, } B\big(S_t, \bar{q}^2 - q_t^2\big) \text{ in the bond.}$$

Note that implicit in (26) we assume the min is attained, and thus such Δ and B exist. Because the maximal quadratic variation is $\bar{q}$, at the expiration date T of the option, $\bar{q}^2 - q_T^2 \geq 0$, so the strategy remains well defined. The cost of this portfolio at each date is $V(S_t, K, \bar{q}^2 - q_t^2)$, and so by (26) this portfolio is self-financing (with perhaps some surplus) for any return process.

It remains to show that the payoff of the trading strategy at date T is no less than that of the option. This result is immediate from (25) if $q_T^2 = \bar{q}^2$, so suppose $q_T^2 < \bar{q}^2$. The constraint in (26) with $r = q = \sqrt{\bar{q}^2 - q_T^2}$, together with (25), implies that for any (S, q),

[18] Note that condition (26) implicitly requires that the minimum be attained; this condition will be satisfied in our construction and we therefore write it in this way at the outset to avoid introducing unnecessary technicalities.

[19] We use notation t (and T) here, rather than (n, N) because we are abstracting from explicit periods in our definition of a super-replicating portfolio – we would like it to hold for any trading frequency, as long as the bound on quadratic variation is met – and thus t corresponds to calendar time.

$$\Delta(S,q)Se^{q} + B(S,q) \geq \left(Se^{q} - K\right)^{+}, \quad \text{and} \quad \Delta(S,q)Se^{-q} + B(S,q) \geq \left(Se^{-q} - K\right)^{+}$$

Multiplying the first inequality by $(1 - e^{-q})/(e^{q} - e^{-q})$ and the second by $(e^{q} - 1)/(e^{q} - e^{-q})$, and using Jensen's inequality together with the convexity of the option payoff, implies

$$V\left(S, K, q^2\right) = \Delta(S,q)S + B(S,q) \geq (S - K)^{+}. \tag{27}$$

Therefore, the dynamic trading strategy given by (Δ, B) has initial cost $V(S, K, \bar{q})$, generates no intermediate losses, and has a final payoff no less than that of the option. Hence, by no arbitrage, $C(S, K|\bar{q}^2) \leq V(S, K, \bar{q}^2)$. □

While Proposition 7 characterizes a super-replicating portfolio, it does not provide a direct method to find such a strategy. We show next that we can calculate the optimal strategy recursively.

3.3. Computing the optimal bound

In general there may be multiple V which satisfy (25) and (26). But given distinct solutions V^1 and V^2, it is easy to see that $\min(V^1, V^2)$ is also a solution, and thus there is a unique minimal solution V^*. How can we compute this optimal bound?

We can compute V^* via backward induction on the number N of remaining price movements. The limiting case as N becomes large is equivalent to the solution when there is no limit to the number of price movements. Specifically, let $V^N(S, K, q^2)$ be the value with n periods remaining, and note that

$$V^0\left(S, K, q^2\right) = \max\{0, S - K\} \tag{28}$$

Then we can compute the bound for $N > 0$ inductively as follows:

$$\begin{aligned} V^N(S, K, q^2) \;\equiv\; & \min_{\Delta, B} \Delta \times S + B \\ & \text{s.t.} \quad \Delta \times Se^{r} + B \geq V^{N-1}(Se^{r}, K, q^2 - r^2) \\ & \qquad\quad \text{for all } r \text{ such that } r^2 \leq q^2 \end{aligned} \tag{29}$$

We have the following result:

Proposition 8. *The functions V^N are monotonically increasing in N so that $V^{N-1} \leq V^N \leq C = V^*$.*

Proof. The constraint in (29) with $r = 0$ immediately implies that V^N increases monotonically with N. We next show that for any N, $V^N(S, K, q^2) = C(S, K|\Phi_N(q^2))$, the maximum possible no-arbitrage value of the option given $\Phi_N(q^2)$, the set of N-period price paths with quadratic variation bounded by q^2. To do so, it is sufficient to construct a distribution over price paths in $\Phi_N(q^2)$ such that for all $n \leq N$, $E[S_n] = S_{n-1}$ (prices are a martingale), and $E[(S_N - K)^+] = V^N(S, K, q^2)$ (the option attains the value V^N).[20] Note that V^0 is trivially attainable, and by

[20] In other words, we are constructing a risk-neutral measure against which to evaluate the option. As long as the risk-neutral measure prices the stock correctly (i.e. the stock's return is equal to the risk-free rate of zero, and so its price is a martingale), then the expected value of any security with respect to this measure is a *possible* no-arbitrage price, and hence the upper bound for the set of no-arbitrage prices must exceed this value.

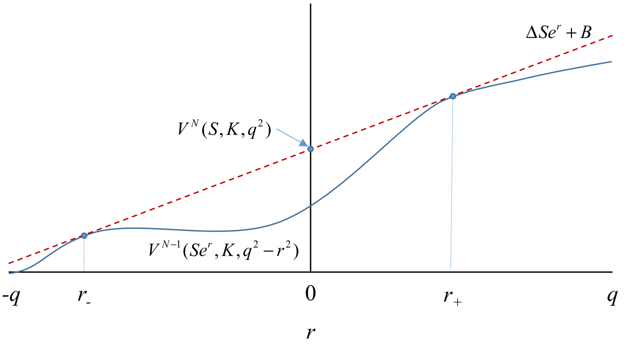

Fig. 2. Computing V^N.

induction, suppose we have such a construction for V^{N-1}. If $V^N(S,K,q^2) = V^{N-1}(S,K,q^2)$, then we can attain the value $V^N(S,K,q^2)$ by setting the initial return $r_1 = 0$ and then proceeding with the construction for V^{N-1}. Suppose instead $V^N(S,K,q^2) > V^{N-1}(S,K,q^2)$. Then the constraint in (29) must bind for some $r_+ > 0$ and some $r_- < 0$. See Fig. 2. Let r_1 have support $\{r_-, r_+\}$ with probabilities such that $E[e^{r_1}] = 1$. Then by induction,

$$\begin{aligned} V^N\big(S,K,q^2\big) &= E\big[V^{N-1}\big(Se^{r_1},K,q^2-r_1^2\big)\big] = E\big[E\big[(S_N-K)^+|S_1 = Se^{r_1}\big]\big] \\ &= E\big[(S_N-K)^+\big]. \end{aligned}$$

Hence, $V^N(S,K,q^2) = C(S,K|\Phi_N(q^2)) \le C(S,K|q^2) \le V^*(S,K,q^2)$, and since V^N converges to V^* from below, $V^* = C$. $\square$

It is easy to see by induction that V^N is homogeneous in (S,K) and convex in K. It is therefore also convex in S (by homogeneity). Thus V^* is also homogeneous and convex in S, and so almost everywhere differentiable. We have the following result, which allows us to identify the optimal trading strategy immediately from V^*:

Proposition 9. *Given V^*, the optimal trading strategy is given by $\Delta = V_S^*(S,K,q^2)$ and $B = V^*(S,K,q^2) - \Delta S$.*[21]

Proof. Recall that V^* must exceed the option value given any martingale return process. For any r with $r^2 \le q^2$, consider the process in which S jumps to Se^r with hazard rate $r/(e^r-1)$, while at the same time drifting at rate $-rS$. Then the expected rate of return of the stock is $-rS + S(e^r-1)r/(e^r-1) = 0$ as required. The expected return of the option under this strategy must be non-positive (or else the current option value would exceed V^*), and thus

$$-rSV_s^* + \big(V^*\big(Se^r,K,q^2-r^2\big) - V^*\big(S,K,q^2\big)\big)r/\big(e^r-1\big) \le 0.$$

Rearranging the above yields

$$V_s^*Se^r + \big(V^* - V_s^*S\big) \ge V^*\big(Se^r,K,q^2-r^2\big).$$

Because r was arbitrary, we have the required result. $\square$

[21] More generally, we can set Δ to be any subgradient of V_S^*.

An approximation

We will illustrate numerical results shortly. But first, to gain further insight into the form of the optimal bound, we derive a simple analytic lower bound below, which we calculate numerically to be within 1% of V^* for any $q < 50\%$.

Proposition 10. *Let $q_+ = e^q - 1$ and $q_- = 1 - e^{-q}$. Then*

$$V^*\big(S, K, q^2\big) \geq \begin{cases} V^*(1,1,q^2)K(S/K)^{1/q_-} & \text{for } S \leq K \\ V^*(1,1,q^2)K(S/K)^{-1/q_+} + S - K & \text{for } S \geq K \end{cases} \tag{30}$$

and $V^*(1,1,q^2) \geq V^1(1,1,q^2) = q_+q_-/(q_+ + q_-)$.

Proof. By homogeneity we can let $K = 1$ without loss of generality. To establish a lower bound for V^*, it is sufficient to construct a martingale price process such that the expected option payoff achieves the bound. First suppose $S < 1$. Consider a process in which the stock price drifts up slowly at rate $\hat{r}dt$ until the stock price equals 1, but with a risk-neutral hazard rate of $(\hat{r}/q_-)dt$ that the stock price will drop by q_- and stay there, making the option worthless. The instantaneous expected return is $\hat{r}dt - q_-(\hat{r}/q_-)dt = 0$, as required. If there is no drop, the stock price will reach 1 at time t such that $Se^{\hat{r}t} = 1$. Therefore, the risk-neutral probability that there is no drop before time t is $e^{-(\hat{r}/q_-)t} = S^{1/q_-}$. If the stock price reaches 1, because the smoothly drifting price path has zero quadratic variation (i.e., $\sum O(dt)^2 = O(dt)$), the maximum value for the option from that point onward is $V^*(1,1,q^2)$. Thus, following this price path the expected ex ante value of the option is $V^*(1,1,q^2)S^{1/q_-}$, providing the lower bound for V^* when $S < 1$. A symmetric calculation (assuming the stock price drifts down to 1 or jumps up by q_+) leads to the lower bound for V^* when $S > 1$, where we can use the put-call parity relation that the value of a call option is equal to a put option plus the stock and short bond.

Finally, the bound for $V^*(1,1,q^2)$ can be derived if we consider only the constraints for $r = \pm q$. Then, because the option value exceeds its intrinsic value (see (27)), the constraints become $\Delta \times Se^{\pm q} + B \geq \max\{0, Se^{\pm q} - K\}$. Letting $S = K = 1$ and taking a convex combination of these constraints, we find that for an at-the-money option,

$$V\big(1,1,q^2\big) \geq \frac{(e^q - 1)(1 - e^{-q})}{e^q - e^{-q}} = \frac{q_+q_-}{q_+ + q_-}. \qquad \square \tag{31}$$

In practice, Eq. (30) provides a useful analytic approximation to V^*. In addition, because it is a lower bound for V^*, we can use it in place of the initial condition (28) to speed the convergence of the numerical calculation. It also shows a clear limit to how much improvement is possible using the optimal strategy relative to the gradient strategy in Section 2. For an at the money option,

$$V\big(1,1,q^2\big) \geq \frac{q_+q_-}{q_+ + q_-} \approx \frac{q}{2} \quad \text{for small } q. \tag{32}$$

Comparing (32) with (22) and (23), we can see that while there is potential for significant improvement (a factor of $\sqrt{2}$) over the gradient strategy, the possibility of jumps implies that the best bound must still exceed the value from Black–Scholes by a factor of at least $\sqrt{\pi/2}$.

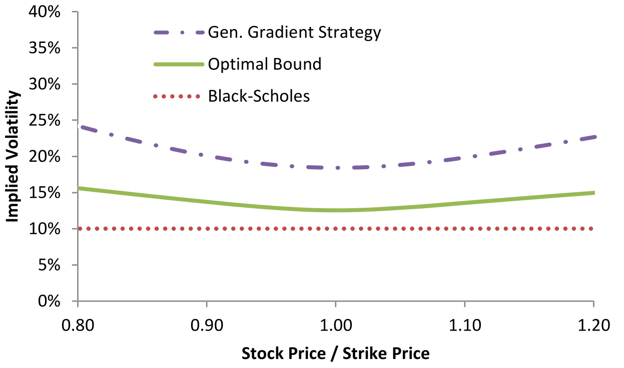

Fig. 3. Option price bounds versus Black–Scholes.

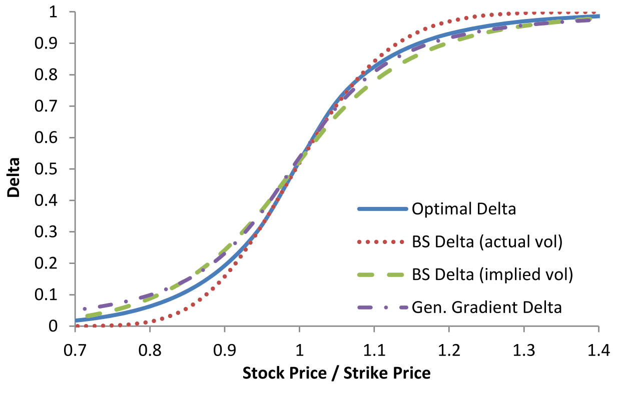

Fig. 4. Optimal versus Black–Scholes and generalized gradient delta hedging strategies.

3.4. Comparison with gradient strategies and Black–Scholes

We can compare the optimal q-based bound with the option price that would be obtained if the Black–Scholes assumptions were satisfied, as well as the generalized gradient strategy of Section 2. Fig. 3 shows the comparison in terms of the corresponding Black–Scholes implied volatility.

Finally, we compute the delta-hedging strategy of the optimal q-based bound using Proposition 9, and contrast it with the hedging strategy of Black–Scholes as well as the implied hedging strategy from the generalized gradient approach in Section 2. Note that we can compute the generalized gradient delta as follows: From Proposition 5, we invest S/β^* in the generalized gradient strategy, which has initial weight $w_1 = x_1/(x_1 + x_2)$ in the stock. Using the optimal choice of x and β^* from Proposition 6, we have

$$\Delta_{GG} = \frac{w_1}{\beta^*} = \frac{x_1}{x_1 + x_2} e^{x_2} = \left(\frac{1}{2} + \frac{\ln(S/K)}{2\sqrt{2q^2 + \ln(S/K)^2}} \right) e^{\frac{1}{2}\sqrt{2q^2+\ln(S/K)^2} - \frac{1}{2}\ln(S/K)} \quad (33)$$

The result is shown in Fig. 4 for $q = 10\%$. Here, we compute the Black–Scholes delta in two ways, first using volatility equal to the forecasted q of 10%, and second using the implied volatility of the option if it is priced according to the optimal q-based bound as shown in Fig. 3.[22] Note that the optimal delta is more sensitive to the stock price than would be indicated by Black–Scholes using the option's implied volatility, but is less sensitive for large price movements than Black–Scholes when using the forecasted volatility (q). The lower sensitivity of the optimal strategy makes sense given the potential for jumps – having delta to close to zero or one would be very costly in the event of a sudden increase or drop in the stock price. Both the generalized gradient method, and using Black–Scholes with the implied volatility, however, tend to over-correct for this possibility relative to the optimal strategy.

4. Concluding remarks

The goal of our paper has been to explore the link between the growing game theoretic literature on calibration and regret minimization and the financial theory of option pricing. Our analysis provides new insights for both literatures. In the context of the calibration literature, we provide new results regarding the finite horizon performance of the gradient strategy introduced by Hannan and Blackwell. We also extend their approach to a trading context in which the payoffs are multiplicative rather than additive. We further demonstrate that their strategy is generally suboptimal – first by adjusting the starting parameters of the algorithm, and then by showing that a truly optimal strategy must satisfy path independence. We use our approach to compute optimal strategies and option price bounds, which limit the cost of hedging as a function of the stock's realized quadratic variation in an environment with jumps or trading interruptions. Our results have both theoretical and practical relevance, as we discuss further below:

Relevance for trading The bounds we derive depend only on a single, measurable parameter, the quadratic variation. As a result, our approach offers an important practical advantage over existing methods. First, traders in these markets view themselves as taking positions on volatility (rather than on a specific price process or on underlying risk preferences). Furthermore, the bounds also indicate the maximal loss a trader might incur should the assumption be violated. Indeed, it is possible to trade derivative securities – variance swaps – with payoffs that are directly proportional to realized quadratic variation. Thus, traders can hedge the risk that their assumption regarding volatility is incorrect – unlike alternative pricing methods, as there are no equivalent markets to hedge against errors in the price process or the pricing kernel (indeed such securities would be difficult if not impossible to define).

In addition, market prices for variance swaps can be used to construct a completely *assumption-free* no arbitrage bound for the option price. In particular, suppose there are complete markets for variance swaps and let $P(q^2)$ be the price of a security that pays \$1 in the event that the realized quadratic variation over the life of the option is less than q^2. Then we can calculate the following assumption free no arbitrage bound for the option price:

$$C \leq \int_{q^2=0}^{\infty} V^*\big(S, q^2\big)\, dP\big(q^2\big). \tag{34}$$

Again, given market prices P, *this option price bound depends on <u>no</u> assumptions on the underlying stock price process or its volatility*.

[22] The latter approach corresponds to practice: delta is calculated from Black–Scholes based on the option's implied volatility.

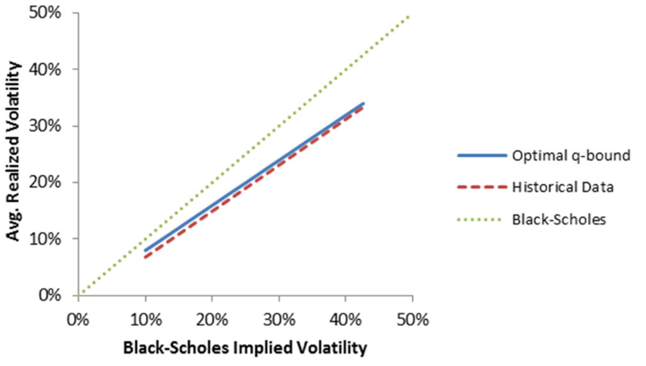

Fig. 5. Realized versus implied volatility for S&P 500.

Timer options In April 2007, Société Générale Corporate and Investment Banking (SG CIB) began marketing a new derivative instrument that has come to be known as a "timer option."[23] Timer call options are equivalent to standard European call options except that the expiration date is not fixed, but is determined based on when the realized quadratic variation of the underlying asset price hits a pre-specified target, referred to as the "variance budget." The securities have since become quite standard in the over-the-counter market.

The existing literature on these timer options have derived pricing formulas assuming *continuous* price processes and trading with various models of stochastic volatility (see Bernard and Cui (2011) and Carr and Lee (2009, 2010)). Because the realized quadratic variation at expiration is known for these securities, our results can be applied directly to bound the price of timer options *without any* assumptions on the underlying price process.

Reasonableness of bounds Of course, even though our price bounds may apply, if they are too wide then as with the original Merton bound, they will have little relevance in practice. But while our bounds allow for higher option prices than would be predicted by Black–Scholes, such deviations are also observed empirically. In particular, implied volatility derived from traded option prices is biased above average realized historical volatility, as shown in Fig. 5.[24] The dotted 45° line shows the relationship between implied volatility and realized volatility if Black–Scholes held perfectly, whereas the dashed line shows the fitted relationship between implied and realized volatility for S&P 500 index options.[25] Finally, the solid line shows the relationship that would hold given the optimal bounds we derive in Section 3 of the paper. As the figure illustrates, the deviations from Black–Scholes observed in practice are similar (and indeed slightly larger) than the maximal deviations we would obtain by dropping completely the continuity and continuous-trade assumptions implicit in Black–Scholes.

Future research We believe the methods we have developed in this paper have broader applicability for developing robust pricing results for a variety of derivative securities. The methods of Section 3 can be applied to alternative securities by changing the final payoff (25). More-

[23] N. Sawyer, "SG CIB Launches Timer Options," *Risk Magazine*, July 2007.

[24] See, e.g., Christensen and Hansen (2002), Jiang and Tian (2005), and Shu and Zhang (2003) for index options, and Neely (2009) for foreign exchange options.

[25] Based on Shu and Zhang (2003), using data from 1995–1999.

over, it would be easy to adapt our results to alternative restrictions on the stock price paths (e.g., a maximum jump size). Also, while we have measured quadratic variation in terms of log-returns, similar methods can be used to develop bounds based on alternative measures of stock price variation (e.g., using arithmetic returns).[26] Finally, our methodology can also be used to understand the characteristics of "worst case" price paths, as such price paths are derived implicitly in the numerical methods of Section 3.[27]

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[26] See e.g. DeMarzo et al. (2006).

[27] Indeed, we are pursuing a complete characterization of such price paths in related work.

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