Incorporating estimation errors into portfolio selection: Robust portfolio construction

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Incorporating Estimation Errors Into Portfolio Selection: Robust Portfolio Construction

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Portfolio managers who rely on mean-variance efficiency often find that their portfolios are unintuitive or do not behave as expected.

In this paper we discuss how estimation error can affect the quality of your portfolio and what you can do about it.

Incorporating Estimation Errors into Portfolio Selection: Robust Portfolio Construction

Sebastian Ceria [*] Robert A. Stubbs [†]

May 17, 2006

1 Introduction

More than 50 years have elapsed since Markowitz (1952) first introduced his Nobel Prize winning work on mean-variance portfolio optimization. His work led to the creation of the field now known as Modern Portfolio Theory (MPT). Throughout this time, MPT has had many followers, but has also been challenged from skeptics at academic and financial institutions alike. Today, even though MPT is still widely accepted as the primary theoretical framework for portfolio construction, its employment by investment professionals is not as ubiquitous as we might expect. There are several reasons for the lack of acceptance of MPT among practitioners, but perhaps the most significant is the argument that "optimal" portfolios obtained through the mean-variance approach are often "counterintuitive", "inexplicable", and "overly sensitive to the input parameters".

The fact that mean variance "optimal" portfolios are sensitive to small changes in input data is well documented in the literature. Chopra (1993) shows that even slight changes to the estimates of expected returns or risk can produce vastly different mean-variance optimized portfolios. Best and Grauer (1991) analyze the sensitivity of optimal portfolios to changes in expected return estimates. Instead of focusing on the weights of the assets in optimal portfolios, others have focused on the financial impact of mean-variance efficient portfolios computed from estimates. Jobson and Korkie (1981) show that even an equal-weighted portfolio can have a greater Sharpe ratio than an optimal mean-variance portfolio computed using estimated inputs. Broadie (1993) shows how the estimated efficient frontier overestimates the expected returns of portfolios for varying levels of estimation errors. Because of the ill-effects of estimation errors on optimal portfolios, portfolio optimization has been called "error maximization" (See Michaud, 1989). Michaud argues that mean-variance optimization overweights those assets with a large estimated return to estimated variance ratio (underweights those with a low ratio) and that these are precisely the assets likely to have large estimation errors.

It is widely believed that most of the estimation risk in optimal portfolios is due to errors in estimates of expected returns, and not in the estimates of risk. Chopra and Ziemba (1993) argue that cash-equivalent losses due to errors in estimates of expected returns are an order of magnitude greater than those for errors in estimates of variances or covariances. Many portfolio managers con-

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cur, saying that their confidence in risk estimates is much greater than their confidence in expected return estimates.

In order to cope with the effect of estimation errors in the estimates of expected returns, attempts have been made to create better and more stable mean-variance optimal portfolios by utilizing expected return estimators that have a better behavior when used in the context of the mean-variance framework. One of the more common techniques is the utilization of James-Stein estimators (see Jobson and Korkie, 1981). These estimators shrink the expected returns towards the average expected return based on the volatility of the asset and the distance of its expected return from the average. Jorion (1985) developed a similar technique that shrinks the expected return estimate towards the minimum variance portfolio. More recently, Black and Litterman (1990) have developed a new Bayesian approach for producing stable expected return estimates that combines equilibrium expected returns and investor's views on specific assets or weighted groups of assets. The area of robust statistics (see Cavadini et al., 2002) has recently been employed to create stable expected return estimates as well.

While strides have been made to improve estimates of expected returns, there will always be errors in these estimates because of the inherent stochastic nature of the asset return process. Even a portfolio manager employing Bayesian estimators such as James-Stein or Black-Litterman process will admit that estimation error remains a factor in the "corrected" estimates of expected returns, even if it is significantly less than that obtained without the use of these methods. In fact, the main premise of Bayesian statistics is that estimates do have distributions. This has led some authors to consider ways in which to account for estimation errors directly in the portfolio construction process.

One possible strategy for considering estimation error is to increase the risk-aversion parameter or modify the risk estimates by increasing the overall volatility. Since the estimated efficient frontier is an overestimate of the true efficient frontier because of the error-maximization property, it can be argued that by increasing the risk-aversion parameter, the resulting portfolio on the actual frontier will be closer to the true frontier. Horst et al. (2001) show how to create an optimal pseudo risk-aversion parameter to use in a mean-variance optimization problem rather than using the actual risk-aversion parameter. One problem with this approach is that it assumes that the covariance matrix of the estimation error is a constant multiple of the covariance matrix of returns, which is rarely the case in practice. Since expected return estimates are typically generated independent of the factor risk model, the distribution of the estimation error is likely to be quite different from that of the risk model.

Another development that has recently received much attention is the portfolio resampling methodology of Michaud (1999). Michaud introduces a statistical resampling technique that indirectly considers estimation error by averaging the individual optimal portfolios that result from optimizing with respect to many randomly generated expected-return and risk estimates. However, portfolio resampling is a somewhat ad hoc methodology that has many pitfalls (see Scherer, 2002). Because portfolio resampling is a simulation procedure in which each iteration involves a resampling of a time-series, creating mean-variance input estimators, and determining the optimal portfolio, it is overly time-consuming to compute. Like the modified risk-aversion parameter approach, portfolio resampling does not actually consider the portfolio manager's estimation error. It only considers the error of estimating a mean and covariance matrix from a simulated time-series from a stationary return process using the expected returns and covariance matrix to generate the time-series. Additionally, the resulting optimal portfolio does not necessarily satisfy all constraints. If a nonconvex constraint, e.g., a limit on the number of assets in the portfolio, is present, then the average of portfolios that satisfy the constraints individually will not necessarily satisfy the constraints.

Others have proposed adding constraints to control the ill-effects of estimation error on optimization generated portfolios. While some constraints can reduce the sensitivity of optimal portfolios to changes in inputs, we show that constraints can actually exacerbate the problem. Furthermore, we show that that the overestimate of expected return of an optimal portfolio can also be exacerbated by the presence of constraints.

In this paper, we discuss an optimization methodology known as robust optimization, which considers uncertainty in unknown parameters directly and explicitly in the optimization problem. It is generally concerned with ensuring that decisions are "adequate" even if estimates of the input parameters are incorrect. Robust optimization was introduced by Ben-Tal and Nemirovski (1997) for robust truss topology design. In a paper that describes several applications of robust optimization, Lobo et al. (1998) introduced the concept of considering the distribution of estimation errors of expected returns explicitly in a portfolio optimization problem. Since then, Goldfarb and Iyengar (2003) consider uncertainties in the factor exposure matrix of a factor risk model directly in the portfolio optimization problem.

Robust portfolio optimization is a fundamentally different way of handling estimation error in the portfolio construction process. Unlike the previously mentioned approaches, robust optimization considers the estimation error directly in the optimization problem itself. Here, we give a financial motivation for using robust portfolio optimization as a means of considering errors in the expected return estimates directly in the portfolio construction process. This motivation allows us to see that the "standard" formulation is only applicable in certain cases. In Section 4, we introduce modified forms of robust mean-variance optimization that are applicable in other commonly used portfolio management strategies. Many of the results in this paper were first presented at a practitioners conference in April of 2003 by Ceria (see Ceria and Stubbs, 2003). Since then, a number of other authors have independently proposed an approach similar to ours. Of particular relevance is the paper of Garlappi et al. (2004).

In this paper, we introduce an optimization methodology that significantly reduces some of the ill-effects of portfolio optimization that are caused by estimation error in expected return estimates. We show that errors in expected return estimates can lead to optimal portfolios whose weights are significantly different from those in the true optimal portfolio and whose expected return is significantly overestimated. We show that this can be particularly true in the presence of commonly found types of constraints. We discuss a "standard" robust optimization methodology to alleviate some these ill-effects and introduce new variants that more effectively handle the difficulties caused by estimation error in commonly used portfolio management strategies. Finally, we discuss some computational results that demonstrate the potentially significant economic benefits of investing in portfolios computed using standard robust optimization and the variants introduced here.

2 Estimation Errors and Classical Mean-Variance Optimization

It is a well documented fact in the investment management literature that mean-variance optimizers are very sensitive to small variations in expected returns. Slightly different expected return vectors can lead to drastically different portfolios. The seemingly unexplainable changes in asset weights due to small perturbations in expected returns is not the only pitfall of classical mean-variance optimization. Because of the error-maximization effect, it is typically the case that the expected return is significantly overestimated.

In order to better understand the effect of estimation error in expected returns on optimal portfolios, consider the following example. Suppose we have two assets where the objective is to maximize expected return subject to a budget constraint that forces full investment between the two assets, and a constraint that limits the total active risk to be no more than 10% with respect to the benchmark

Asset	Benchmark Weights	Alpha 1 (%)	Alpha 2 (%)	True Alpha (%)	Return Std. Dev. (%)	Alpha Std. Dev. (%)
Asset 1	0.5	2.4	2.5	2.48	0.42	0.5
Asset 2	0.5	2.5	2.4	2.42	0.33	0.5

Table 1: Expected Returns and Standard Deviations for Example 1

Attribute	Folio A	Folio B	Folio C	Folio D
Alpha	1	2	1	2
Budget	✓	✓		
Asset 1 Weight	0.169	0.831	0.5253	0.5546
Asset 2 Weight	0.831	0.169	0.7796	0.7503

Table 2: Optimal Portfolios for Example 1

portfolio (shown as point 'M' in Figure 1). The estimates of expected returns and standard deviations of the two assets are given in Table 1. We assume that the correlation between the two assets is 0.7. The feasible region of this example is illustrated in Figure 1 as the intersection of the shaded ellipsoidal region and the budget constraint, i.e., the feasible region of this example is simply the line segment between points A and B.

Using column "Alpha 1" from Table 1 as the estimates of expected returns, the optimal portfolio is at point A in Figure 1. Using the slightly different expected returns given in column "Alpha 2," the optimal portfolio is at point B. (The values of the portfolio weights are given in Table 2.) This example shows that with only a very small change in the estimates of expected returns of the assets, the weights of the assets in the optimal portfolios changed dramatically. The true optimal solution is at point B with an expected return of 2.46986%. The estimated expected return of points A and B using "Alpha 1" and "Alpha 2", respectively, are both 2.4831%. In this example, the expected returns of both optimal portfolios evaluated with respect to their expected return estimates overestimate the true expected return. From this example, it is clear why portfolio managers find "optimized" portfolios to be counterintuitive and impractical.

Some changes in optimal weights should be expected when using different estimates of expected returns. However, most of the variations in asset weights arise due to optimizers exacerbating the estimation error problem by significantly overweighting assets with an error to the upside and underweighting assets with an error to the downside. Though this behavior has been described before, we are not aware of any authors that have given a precise and intuitive explanation of the error exacerbating effect. We claim that the cause of the "error maximization" property of mean-variance optimizers is not only the presence of estimation error, but also the interaction of the estimation error in expected returns with the constraints present in the portfolio optimization problem.

If we reconsider our example, but drop the budget constraint, then the optimal portfolios with respect to "Alpha 1" and "Alpha 2" are points C and D, respectively. From Figure 1 and Table 2, we can see that the change in optimal portfolio weights with respect to the same small change in expected return estimates is much smaller when the budget constraint is dropped. The true optimal solution is at a point on the boundary of the ellipsoid between points C and D with an expected return of 3.191258%. The estimated expected return of points C and D using "Alpha 1" and "Alpha 2", respectively, are 3.20972% and 3.18722%. In this example, the expected return of portfolio C evaluated with respect to its expected return estimate overestimates the true expected return, but

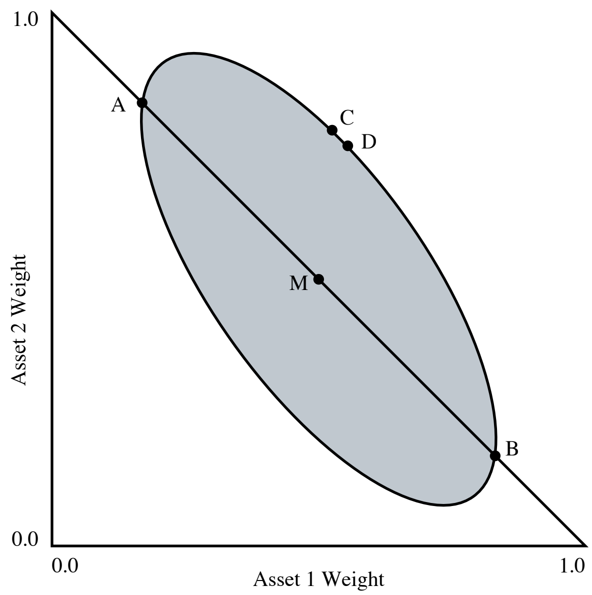

Figure 1: Feasible Region for Example 1

portfolio D evaluated with respect to its expected return estimate actually underestimates the true expected return. This situation is extremely rare when constraints are present, particularly in higher dimensions.

This simple example was created to geometrically illustrate how slightly different expected return estimates can lead to very different portfolios and how this phenomenon can be exacerbated by the introduction of constraints. It also shows how the error in expected returns is optimized so that the estimated expected return of a portfolio typically overestimates the true expected return. In this small example, the change in expected returns of the portfolios was small, but this was only a two-asset example. To better illustrate the error-maximization effect, we consider efficient frontiers in a more realistic investment scenario.

As defined by Broadie (1993), we use the terms *true frontier*, *estimated frontier*, and *actual frontier* to refer to the efficient frontiers computed using the true expected returns (unobservable), estimated expected returns, and true expected returns of the portfolios on the estimated frontier, respectively. Specifically, we refer to the frontier computed using the true, but unknown, expected returns as the true frontier. Similarly, we refer to the frontier computed using estimates of the expected returns and the true covariance matrix as the estimated frontier. Finally, we define the actual frontier as follows. We take the portfolios on the estimated frontier and then calculate their expected returns using the true expected returns. Since we are using the true covariance matrix, the variance of a portfolio on the estimated frontier is the same as the variance on the actual frontier. By definition, the actual frontier will always lie below the true frontier. The estimated frontier can lie anywhere with respect to the other frontiers. However, if the errors in the expected return estimates have a mean of zero, then the estimated frontier will lie above the true frontier with extremely high

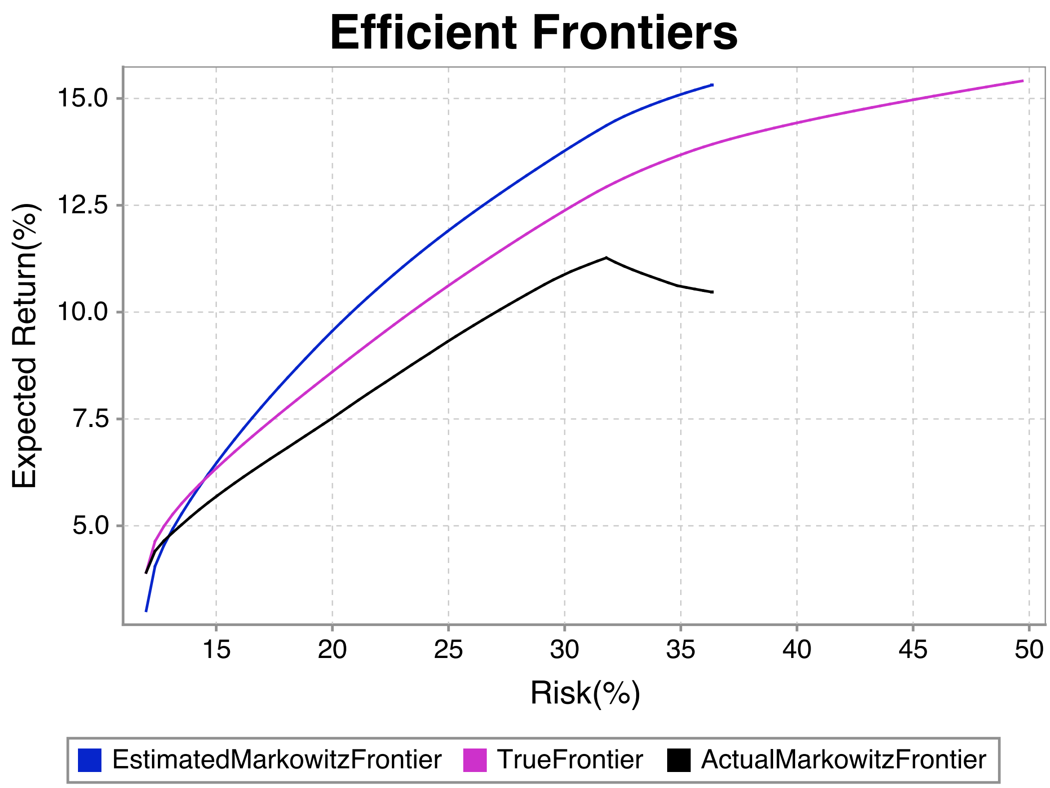

Figure 2: Markowitz Efficient Frontiers

probability, particularly when the investment universe is large.

Using the covariance matrix and expected return vector from Idzorek (2002), we randomly generated a time-series of normally distributed returns and computed the average to use as estimates of expected returns. Using this computed expected-return estimate and the true covariance matrix, we generated an estimated efficient frontier of active risk versus active return where the portfolios were subject to no-shorting constraints and a budget constraint that forces the sum of the weights to be one. Similarly, we generated the true efficient frontier using the original covariance matrix and expected return vector. Finally, we generated the actual "frontier" by computing the expected return and risk of the portfolios on the estimated frontier with the true covariance and expected return values. The actual "frontier" is not necessarily concave, since it is not computed as the result of any optimization, but rather by applying the true expected returns and true covariance to the efficient portfolios in the estimated efficiency frontier. These three frontiers are illustrated in Figure 2. Using the same estimate of expected returns, we also generate active risk versus active return where we also constraint the active holdings of the assets to be +/- 3% of the benchmark holding of each asset. These frontiers are illustrated in Figure 3. Note how the estimated frontiers significantly overestimate the expected return for most risk levels in both types of frontiers. More importantly, note that the actual frontier lies far below the true frontier in both cases. This shows that the "optimal" mean-variance portfolio is not necessarily a good portfolio, i.e., it is not "mean-variance efficient".

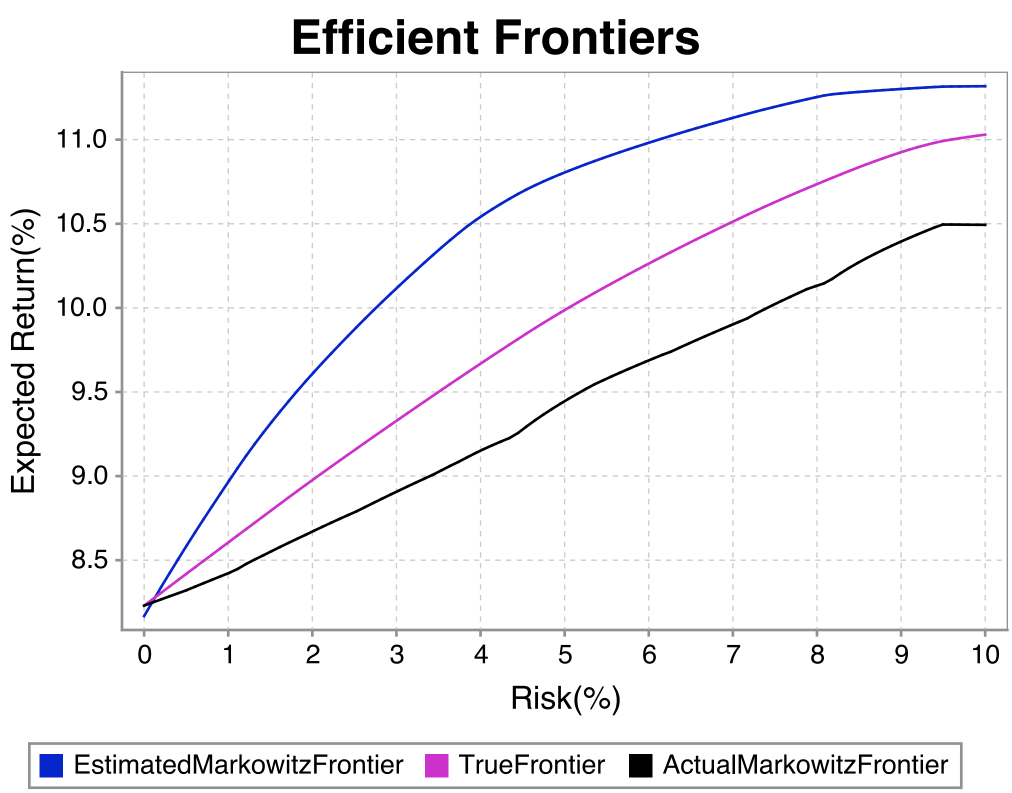

Figure 3: Markowitz Benchmark-Relative Efficient Frontiers

Portfolio	Expected Return (2.5, 2.4)	Expected Return (2.2, 2.7)
A	2.4169%	2.6155%
B	2.4831%	2.2845%

Table 3: Extreme Expected Returns for Optimal Portfolios

In general, we do not know how far the actual expected return may be from the expected return of the mean-variance optimal portfolio. Returning to our example, suppose that the true expected return estimate is some convex combination[1] of the expected return estimates, $(\alpha_1, \alpha_2) = (2.5, 2.4)$ and $(\alpha_1, \alpha_2) = (2.2, 2.7)$ and that one value is no more likely to occur than another. Depending on the point estimate of expected returns used in the mean-variance optimization problem, the optimal portfolio will be either portfolio A or portfolio B. The actual expected returns of these portfolios for the two extreme expected return estimates are given in Table 3. Suppose that our estimate of the expected returns leads to optimal portfolio B. In the best scenario, portfolio B will have an expected return that is 0.0662 percentage points greater than that of portfolio A. However, in the worst-case, portfolio B will have an expected return that is 0.331 percentage points less than that of portfolio A. So, by investing in a portfolio that may have an expected return that is 0.0662 percentages points greater than an alternative, there is the risk that the expected return may be as much as 0.331 percentage points less. Since we assumed a uniform distribution of expected returns between the two extreme values, we would argue that portfolio A is a better, more robust portfolio. That is, the portfolio performs better under more situations within the range of uncertainty of expected returns.

3 Robust Portfolio Optimization

As we have already discussed, the actual frontier resulting from classical mean-variance optimization can be far away from both the true and estimated frontiers because of estimation error. The estimated frontier generally lies well above the actual frontier. We will now analyze just how far apart the estimated and actual frontiers can be over a specified confidence region of the true expected return. We assume that the n-dimensional vector of true expected returns, α, is Normally distributed. Given an estimate of expected returns, $\bar{\alpha}$, and a covariance matrix Σ[2], of the estimates of expected returns, we assume the true expected returns lies inside the confidence region

$$(\alpha - \bar{\alpha})^T \, \Sigma^{-1} \, (\alpha - \bar{\alpha}) \leq \kappa^2, \tag{1}$$

with probability $100\eta\%$ where $\kappa^2 = \chi_n^2(1-\eta)$ and χ_n^2 is the inverse cumulative distribution function of the chi-squared distribution with n degrees of freedom.[3].

If the covariance matrix of returns, Q, is full rank, then we can compute points on the efficient frontier by solving the maximum expected return problem,

$$\begin{array}{ll} \text{maximize} & \bar{\alpha}^T w \\ \text{subject to} & w^T Q w \leq v, \end{array} \tag{2}$$

[1]A convex combination of two vectors a and b is defined to be $\lambda a + (1-\lambda) b$ where $0 \leq \lambda \leq 1$.

[2]Σ is a symmetric positive definite matrix.

[3]We do not need to assume normality. We only require that the distribution is elliptical. An elliptic distribution is a symmetric distribution such that any minimum volume confidence region of the distribution is defined by an $n-$dimensional ellipsoid of the form of equation (1)

for varying values of v, where α is the expected return estimate, Q is the covariance matrix of returns, and v is the target portfolio variance. It is easy to show that the optimal holdings to this maximum expected return problem are given by

$$w = \sqrt{\frac{v}{\alpha^T Q^{-1} \alpha}} Q^{-1} \alpha.$$

Let α^* be the true, but unknown, expected return vector and $\bar{\alpha}$ be an expected return estimate. Recall that the actual frontier is constructed using the true expected return, α^*. That is, the true expected return of a portfolio on the estimated frontier is computed as

$$\sqrt{\frac{v}{\bar{\alpha}^T Q^{-1} \bar{\alpha}}} {\alpha^*}^T Q^{-1} \bar{\alpha}.$$

Let $\tilde{w}$ be the optimal portfolio on the estimated frontier for a given target risk level. Generally, the estimated expected return is greater than the actual expected return because of the "error maximization" effect of the optimizer. The question we address is "How great can the difference be?" To answer this, we consider the maximum difference between the estimated expected return and the actual expected return of $\tilde{w}$. This difference can be written as

$$\bar{\alpha}^T \tilde{w} - {\alpha^*}^T \tilde{w}.$$

For a 100η %-confidence region of α, the maximum difference between the expected returns on the estimated efficient frontier and the actual efficient frontier is computed by solving

$$\begin{array}{ll} \text{maximize} & \bar{\alpha}^T \tilde{w} - \alpha^T \tilde{w} \\ \text{subject to} & (\alpha - \bar{\alpha})^T \Sigma^{-1} (\alpha - \bar{\alpha}) \leq \kappa^2. \end{array} \tag{3}$$

Note that $\tilde{w}$ is fixed in problem (3). We are optimizing over the variable α. The optimal solution to (3) can be shown to be

$$\alpha = \bar{\alpha} - \sqrt{\frac{\kappa^2}{\tilde{w}^T \Sigma \tilde{w}}} \Sigma \tilde{w}. \tag{4}$$

Therefore, the lowest possible value of the actual expected return of the portfolio over the given confidence region of true expected returns is computed as

$$\alpha^T \tilde{w} = \bar{\alpha}^T \tilde{w} - \kappa \|\Sigma^{1/2} \tilde{w}\|, \tag{5}$$

and the maximum difference between the estimated frontier and the actual frontier is

$$\bar{\alpha}^T \tilde{w} - \left(\bar{\alpha}^T \tilde{w} - \kappa \|\Sigma^{1/2} \tilde{w}\| \right) = \kappa \|\Sigma^{1/2} \tilde{w}\|. \tag{6}$$

(Throughout this paper, $\| \cdot \|$ refers to the 2-norm.)

Naturally, we would like for this difference to be as small as possible. This would reduce the error-maximization effect, bring the estimated and actual frontiers closer together, and thus create portfolios that are closer to the true efficient frontier. However, simply minimizing the distance between the two frontiers will drive the optimal portfolio towards a portfolio that minimizes the estimation risk. Clearly, this is not what we want to do. There is no point in considering estimation error if we do not consider the estimates. Instead, we simultaneously want to continue to maximize the expected return of the portfolio so that we are minimizing the estimation risk for a given level of estimated expected return. In order to do this, we solve an optimization problem where we maximize an objective of the form of (5). With this optimization problem, we will bring the actual and estimated frontiers closer together. However, we will not be able to guarantee that these frontiers are actually closer to the true frontier.

In this problem, w, the vector of optimal holdings, is not fixed. We optimize over w to find the optimal asset weights. Additionally, any set of portfolio constraints can be added. For instance, a long-only robust portfolio satisfying a budget constraint and a variance constraint can be written as

$$\begin{array}{ll} \text{maximize} & \bar{\alpha}^T w - \kappa \|\Sigma^{1/2} w\| \\ \text{subject to} & e^T w = 1 \\ & w^T Q w \leq v \\ & w \geq 0, \end{array} \tag{7}$$

where v is a variance target. Note that this problem is exactly the same as a classical mean-variance optimization problem except for the $\kappa\|\Sigma^{1/2}w\|$ term in the objective. This term is related to the estimation error and its inclusion in the objective function reduces the effect of estimation error on the optimal portfolio.

There is an important distinction between Q and Σ. Q is the covariance matrix of returns, while Σ is the covariance matrix of estimated expected returns, which is related to the estimation error arising from the process of estimating α, the vector of expected returns. This distinction is even more relevant in practice, where typically Q is obtained from a risk model provider, and is completely independent from Σ which is the result of a proprietary estimation process for α of which the risk model provider is not even aware.

Let us consider just how this additional objective term affects an optimal solution. If we consider equation (4), we see that the expected returns of those assets with positive weights will be effectively[4] downwards.[5] Similarly, the expected returns of those assets with negative weights, i.e., short holdings, will be adjusted upwards. The size of the adjustment is controlled by the size of κ, i.e., the size of the confidence region. Note that the alpha correction term in equation (4) is exactly the constant κ multiplied by the marginal contribution to estimation risk of the assets. Therefore, for a portfolio with a lot of estimation risk in a single asset, the expected return for that asset is effectively adjusted so as to reduce the marginal contribution to estimation risk of that asset. The purpose of adjusting the expected returns estimates in this way is to counter the error-maximization effects of portfolio optimization.

We have just described what we refer to as a robust objective problem. The other forms of classical mean-variance optimization can also be modeled using robust optimization. For instance, the maximum utility form of the problem can be written as

$$\text{maximize} \quad \bar{\alpha}^T w - \kappa\|\Sigma^{1/2} w\| - \rho/2 w^T Q w. \tag{8}$$

Similarly, the minimum volatility form of the problem can be written as

$$\begin{array}{ll} \text{minimize} & w^T Q w \\ \text{subject to} & \bar{\alpha}^T w - \kappa\|\Sigma^{1/2} w\| \geq r. \end{array} \tag{9}$$

[4] We defined the effective alpha, or effectively adjusted expected return, to be the value of α determined by equation (4) or the related equation for a variant of (6) for the optimal solution to the robust optimization problem.

[5] When considering adjustments of expected returns, we assume that Σ is a diagonal matrix so that we can easily conceptualize the effective expected-return adjustments without worrying about any interactions between the adjustments. That is, assuming a diagonal matrix, Σ, means that the adjustment to the expected return for asset i is dependent upon the weight of asset i, but no others. This is not to say that the adjustments are truly independent though. Constraints may force one weight to go up if another goes down which implicitly creates an interrelationship between the alpha adjustments. Because Σ is positive definite, a diagonal Σ will have all positive elements.

Problem (7) and its variants cannot be solved by a standard mean-variance optimizer or even a general-purpose quadratic optimizer because the estimation-error term is a 2-norm which contains a square root and cannot be reformulated as a pure quadratic problem. This robust optimization problem must be solved by either an optimizer capable of handling general convex expressions or a symmetric second-order cone optimizer. Second-order cone optimization is a relatively new branch of optimization and special purpose optimizers have been created to solve problems of this type. These specialized solvers can optimize robust optimization problems in roughly the same amount of time that a mean-variance optimizer can solve the classical problem.

4 Alternative Forms of Robust Portfolio Optimization

The robust optimization problem introduced in Section 3 will only adjusts the estimates of expected returns downwards if long-only constraints are present. Assuming that each expected return estimate overestimates the true expected return and adjusting all estimates downwards is too pessimistic. Even though there are errors in an expected return estimate, it is not likely that the expected return estimate of each asset is an overestimate of the actual expected return. Similarly, when managing an active fund, the expected returns will be adjusted downwards for any asset with a positive weight. This really doesn't make sense because we are interested in active returns. We would not expect our alpha to be adjusted downwards for an asset that already has a negative active weight.

In this section, we introduce new variants of robust optimization that deal with these issues. It should be noted that the two variants introduced here do not necessarily cover all real-world situations. These variants, along with the standard formulation, do provide ways of handling most commonly found portfolio management strategies. At the end of this section, we describe a more general framework under which to view these alternative forms of robust optimization. This framework can be used to develop other extensions for applicable circumstances.

4.1 Zero Net Alpha-Adjustment Frontiers

In the standard robust optimization discussed in the previous section, we considered the maximum possible difference between the estimated frontier and the actual frontier. We then minimized this maximum difference. Depending on the goals of the portfolio manager, this approach can potentially be too conservative since the net adjustment to the estimated expected return of a portfolio will always be downwards. However, if the manager's expected returns are symmetrically distributed around the point estimate, then we would expect that there are approximately as many expected returns above their estimated values as there are below the true values. It may be more natural and less conservative to build this expectation into our model.

In order to incorporate a zero net alpha-adjustment into the robust problem, we modify (6) by adding the linear constraint

$$e^T D(\alpha - \bar{\alpha}) = 0 \tag{10}$$

for some symmetric invertible matrix D to obtain the following:

$$\begin{array}{ll} \text{maximize} & \bar{\alpha}^T \tilde{w} - \alpha^T \tilde{w} \\ \text{subject to} & (\alpha - \bar{\alpha})^T \Sigma^{-1} (\alpha - \bar{\alpha}) \leq \kappa^2 \\ & e^T D(\alpha - \bar{\alpha}) = 0. \end{array} \tag{11}$$

For now, assume that $D = I$ in which case (10) forces the total net adjustment to the expected returns to be zero. That is, for every basis point decrease in an expected return of an asset, there must be a corresponding gross basis point increase in the expected return of other assets.

We can show that the optimal solution to problem (11) is

$$\alpha = \bar{\alpha} - \sqrt{\frac{\kappa^2}{\left(\Sigma\tilde{w} - \frac{e^T D\Sigma\tilde{w}}{e^T D\Sigma D^T e}\Sigma D^T e\right)^T \Sigma^{-1}\left(\Sigma\tilde{w} - \frac{e^T D\Sigma\tilde{w}}{e^T D\Sigma D^T e}\Sigma D^T e\right)}}\left(\Sigma\tilde{w} - \frac{e^T D\Sigma\tilde{w}}{e^T D\Sigma D^T e}\Sigma D^T e\right) \tag{12}$$

Therefore,

$$\alpha^T\tilde{w} = \bar{\alpha}^T\tilde{w} - \kappa\left\|\left(\Sigma - \frac{1}{e^T D\Sigma D^T e}\Sigma D^T e e^T D\Sigma\right)^{1/2}\tilde{w}\right\|. \tag{13}$$

Instead of having a zero net adjustment of the alphas, we could restrict the alpha region to have a zero net adjustment in standard deviations of the alphas. To do this, we set $D = L^{-1}$, where $\Sigma = LL^T$ is the Cholesky decomposition of Σ. This forces every standard deviation of upward adjustment in the alphas to be offset by an equal downward adjustment of one standard deviation. Similarly, we could restrict the alpha region to have a zero net adjustment in the variance of alphas in which case we set $D = \Sigma^{-1}$.

Now, let us consider how this objective is effectively adjusting alphas when $D = \Sigma^{-1}$. In this case the adjustment term becomes

$$\alpha = \bar{\alpha} - \sqrt{\frac{\kappa^2}{\left(\tilde{w} - \frac{e^T\tilde{w}}{e^T\Sigma^{-1}e}\Sigma^{-1}e\right)^T \Sigma\left(\tilde{w} - \frac{e^T\tilde{w}}{e^T\Sigma^{-1}e}\Sigma^{-1}e\right)}}\,\Sigma\left(\tilde{w} - \frac{e^T\tilde{w}}{e^T\Sigma^{-1}e}\Sigma^{-1}e\right). \tag{14}$$

For a problem with a dollar-neutral constraint, i.e., $e^T\tilde{w} = 0$, the zero-net alpha adjustment form of robust optimization is equivalent to the standard form. However, in a fully invested problem, there will be a budget constraint of the form $e^T\tilde{w} = 1$. In this case, the term

$$\frac{e^T\tilde{w}}{e^T\Sigma^{-1}e}\Sigma^{-1}e$$

is exactly the portfolio that minimizes estimation error subject to being fully invested. In this case, if a portfolio weight is above that which minimizes estimation error, then the effective alpha is adjusted downwards. Similarly, if the weight of an asset is below that which minimizes estimation error, then the effective alpha is adjusted upwards.

4.2 Robust Active Return / Active Risk Frontiers

Thus far, we have discussed the classical efficient frontier that demonstrates the tradeoff between the expected values of total return and total risk. Active managers are more interested in an efficient frontier comparing the expected values of active return and active risk. For a $100\eta\%$-confidence region of α, the most that the difference between the expected active returns on the estimated efficient frontier and the actual frontier can be is computed by

$$\begin{array}{ll} \text{maximize} & \bar{\alpha}^T(\tilde{w} - b) - \alpha^T(\tilde{w} - b) \\ \text{subject to} & (\alpha - \bar{\alpha})^T\Sigma^{-1}(\alpha - \bar{\alpha}) \leq \kappa^2, \end{array} \tag{15}$$

where b is the benchmark holdings. The optimal solution to this problem is

$$\alpha = \bar{\alpha} - \sqrt{\frac{\kappa^2}{(\tilde{w} - b)^T\Sigma(\tilde{w} - b)}}\,\Sigma(\tilde{w} - b) \tag{16}$$

which implies that

$$\alpha^T (\tilde{w} - b) = \bar{\alpha}^T (\tilde{w} - b) - \kappa \|\Sigma^{1/2} (\tilde{w} - b) \|. \tag{17}$$

This gives the following robust optimization problem for long-only active funds:

$$\begin{array}{ll} \text{maximize} & \bar{\alpha}^T w - \kappa \|\Sigma^{1/2} (w - b) \| \\ \text{subject to} & e^T w = 1 \\ & (w - b)^T Q (w - b) \leq v \\ & w \geq 0, \end{array} \tag{18}$$

Now, let us see how this variant of robust objective function effectively adjusts expected return estimates. If the holding in an asset is below the benchmark weight, then the α is adjusted upwards. Similarly, if the holding in an asset is above the benchmark weight, then the α for that particular asset is adjusted downwards. This behavior is much more intuitive and performs much better in practice for active strategies.

4.3 General Robust Optimization Framework

All three forms of robust portfolio optimization discussed thus far can all be cast in a single generalized form. Note that the only difference between equations (4), (12), and (16) is the model portfolio that is compared to the vector of portfolio holdings, $\tilde{w}$, in constructing the expected return adjustments. Let z be the generic "model" portfolio. Then the generic expected return adjustment can be written as

$$\alpha = \bar{\alpha} - \sqrt{\frac{\kappa^2}{(\tilde{w} - z)^T \Sigma (\tilde{w} - z)}} \Sigma (\tilde{w} - z). \tag{19}$$

In equations (4), (12), and (16), z is

$$0, \; \frac{e^T D \Sigma \tilde{w}}{e^T D \Sigma D^T e} D^T e, \text{ and } b,$$

respectively. Note that z can be dependent on $\tilde{w}$ as it is for the zero-net alpha case.

This generic framework allows for the construction of other alternative forms of robust portfolio optimization. Both of the alternatives introduced were created to prevent robust optimization from adjusting alphas based on anything other than estimation error. For example, in the case of the active manager that measures performance relative to a benchmark, we compare the portfolio weights to the benchmark to expected returns from being adjusted because of the active manager's constraints. Similarly, in the case of a fully-invested fund, we introduced the zero-net alpha adjustment that compares the portfolio weights to the fully-invested minimum estimation error portfolio. The adjustment prevents the expected returns from always being adjusted downwards because of the fully-invested constraint.

For different investment strategies, other constraints may force the expected returns to be adjusted in a particular way even if it isn't suggested by estimation error. In these cases, the general form can be used to create an effective robust portfolio construction strategy.

5 Numerical Experiments

In order to measure the effect of the proposed methodology on the efficient frontiers we reran the experiments used to produce Figure 2 using robust optimization. Using $D = I$, we generated efficient frontiers using both the standard mean-variance problem and the equivalent robust optimization

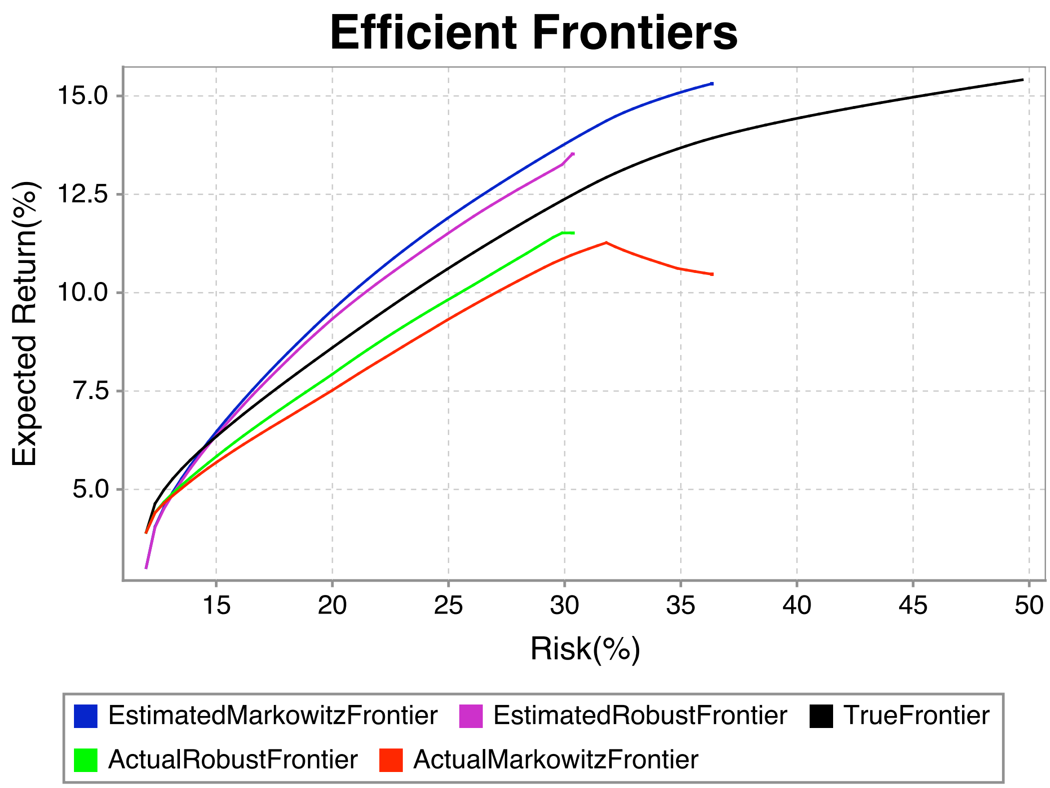

Figure 4: Robust Efficient Frontiers

problem and compared them to the true frontier as we did in Figure 2. These frontiers are illustrated in Figure 4. Similarly, we generated efficient frontier of active risk versus active return using both classical mean-variance optimization and the equivalent robust counterpart. These frontiers are illustrated in Figure 5. By incorporating the estimation error into the portfolio construction process, we significantly reduced its effect on the optimal portfolio. In both cases, the predicted return for any given risk level was not exaggerated nearly as much. More importantly, the actual robust frontiers are much closer to the true frontiers than are the actual mean-variance frontiers.

As expected, the computational experiments show that when using robust optimization, the actual and estimated frontiers lie closer to each other. This is due to the objective function in the robust optimization problem being based on reducing the distance between the predicted and actual frontiers. However, the real goal is to get these frontiers not only closer together, but also closer to the true efficient frontier. We believe that this result will be very difficult to establish theoretically, and for this reason we demonstrated it empirically by running a very large number of computational experiments that are outlined below.

While frontiers help illustrate the effect of robust optimization, they only represent one rebalancing period. We cannot say that portfolios constructed using robust optimization will outperform those constructed using classical mean-variance optimization each month with certainty. However, we argue that portfolios constructed using robust optimization do outperform those constructed us-

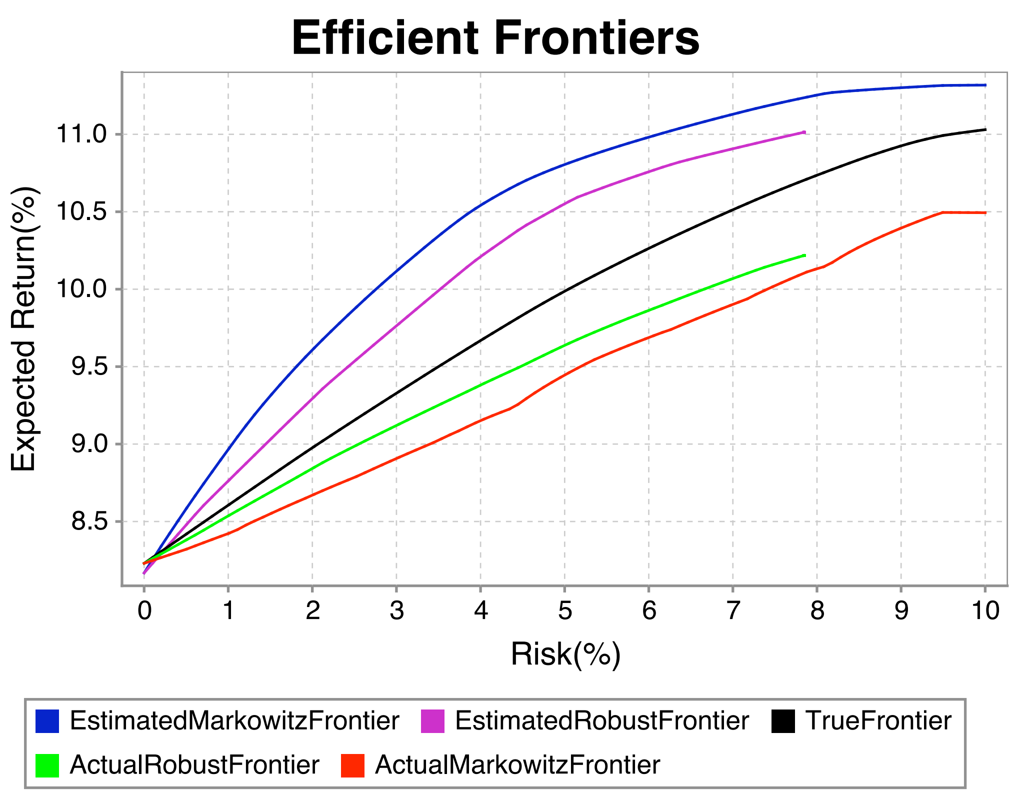

Figure 5: Robust Active Return Efficient Frontiers

Lambda	Kappa	Markowitz Ann. Ret.	Robust Ann. Ret.	Robust Win Percentage
0.025	1	2.06%	2.90%	82%
0.025	2	2.06%	3.62%	80%
0.025	3	2.06%	4.22%	83%
0.025	4	2.06%	4.71%	83%
0.050	1	10.53%	11.30%	79%
0.050	2	10.53%	11.92%	78%
0.050	3	10.53%	12.38%	76%
0.050	4	10.53%	12.66%	76%

Table 4: Backtest Results for Long-Short Dollar-Neutral Strategy

ing classical mean-variance optimization the majority of the time. To demonstrate this, we ran simulated backtests using the various forms of robust optimization described in this paper.

For each simulated backtest, we generated a time-series of monthly returns using the excess expected returns and covariance matrix from Idzorek (2002) for thirty US equities. For each month, a mean vector of returns, μ, is computed using the previous number of historical periods, T, specified in the backtest. The sample covariance matrix of returns, S, is computed over the same time horizon. For each month, we computed an expected return estimate $\alpha = (1-\lambda)\mu + \lambda r$, where r is that month's realized returns and $0 \leq \lambda \leq 1$ is a parameter specified in the backtest. The value of α used in the backtests intentionally contains some look-ahead bias that is designed to simulate portfolio managers' information. The estimation error matrix, $\Sigma = \frac{(1-\lambda)}{T} S$, is used in the robust objective term for each backtest. The value of κ used in each backtest is specified in the results tables. All results are based on 100 different runs of the backtests using different seeds for the random number generation. All backtests cover 120 periods, or 10 years of monthly rebalancings.

The first set of backtests simulate a long-short dollar-neutral strategy with a limit on the total risk of 10%. Asset weights were constrained to be within +/- 25% of the amount invested. We also restrict the portfolio so that the maximum total value of th long positions is equal to the amount invested in order to restrict leverage. The value of T in each of the backtests was 120. The results are shown in Table 4. The columns labeled "Ann. Ret." give the average annualized return over all 100 simulations. The column "Robust Win Percentage" gives the percentage of the simulations in which the total return using robust optimization was greater than the total return using classical mean-variance optimization. In these tests, the total excess return for the robust backtests were greater than the total excess return for the classical tests between 76 and 83 percent of the time. Also note that the average annualized return of the robust portfolios is between 84 and 265 basis points greater than the average annualized return of the classical portfolios.

The second set of backtests simulate a long-only maximum return strategy. Here, we maximize expected returns in a fully-invested long-only portfolio with a limit of 20% expected total risk. In these backtests, we also limited the monthly roundtrip turnover to be at most 15% by imposing a linear constraint in the portfolio construction problem. The zero-net alpha adjustment version of robust optimization introduced in Section 4.1 was used to construct the robust portfolios. The value of T in each of the backtests was 120. The results are shown in Table 4. In these tests, the total excess return for the robust backtests were greater than the total excess return for the classical tests between 68 and 87 percent of the time. Also note that the average annualized excess return of the robust portfolios is between 37 and 93 basis points greater than the average annualized excess return

Lambda	Kappa	Markowitz Ann. Ret.	Robust Ann. Ret.	Robust Win Percentage
0.075	1	11.93%	12.30%	87%
0.075	3	11.93%	12.59%	81%
0.075	5	11.93%	12.78%	75%
0.075	7	11.93%	12.77%	68%
0.100	1	14.04%	14.46%	84%
0.100	3	14.04%	14.81%	81%
0.100	5	14.04%	14.97%	74%
0.100	7	14.04%	14.97%	68%

Table 5: Backtest Results for Long-Only Maximum Total Return Strategy

Lambda	Kappa	Markowitz Ann. Act. Ret.	Robust Ann. Act. Ret.	Robust Win Percentage
0.025	1	1.59%	1.69%	69%
0.025	3	1.59%	1.85%	68%
0.025	5	1.59%	1.98%	70%
0.025	7	1.59%	2.02%	65%
0.050	1	3.35%	3.48%	76%
0.050	3	3.35%	3.68%	78%
0.050	5	3.35%	3.80%	75%
0.050	7	3.35%	3.80%	65%

Table 6: Backtest Results for Long-Only Active Strategy

of the classical portfolios.

The last set of backtests simulate a long-only active strategy. The portfolios are constrained to be fully invested, have at most a 3% active risk, and the asset weights must be within +/- 10% of the investment size of the benchmark weights. In these backtests, we also limited the monthly roundtrip turnover to be at most 15%. The active return/active risk version of robust optimization introduced in Section 4.2 was used to construct the robust portfolios. Again, the value of T in each of the backtests was 120. The results of these backtests are given in Table 6. The results are based on active returns rather than excess returns, but otherwise show the same type of information as the previous table showed. Again, the total return for the robust backtests were greater than the total return for the classical tests. This time, the robust portfolios were superior between 65 and 78 percent of the time. The average annualized active return of the robust portfolios is between 10 and 45 basis points greater than the average annualized active return of the classical portfolios.

6 Conclusions

We believe that one of the main reasons why modern portfolio theory is not being fully utilized in practical portfolio management is the fact that the result from a classical mean-variance framework are unstable and too sensitive to expected return estimates. We argue that these ill-effects of classical portfolio optimization are caused by the error-maximization property. The robust optimization technology described in this paper directly addresses these issues.

Our frontier illustrations show that portfolios generated using robust optimization may be closer to the true efficient frontier. Our backtesting results indicate that portfolios constructed using robust optimization outperformed those created using traditional mean-variance optimization in the majority of cases. The realized returns were greater when using robust optimization. We believe that the reason for this is that more information is transfered to the portfolios when constructing them using robust optimization. Classical optimization will tend to overweight assets with positive estimation error in the expected returns. Because of this, portfolios constructed using mean-variance optimization typically represent less information from the true expected returns. That is, the portfolios constructed using robust optimization usually have a higher correlation between the true expected returns and the alphas implied from the portfolio than do those portfolios constructed using robust optimization.

Robust optimization is a fairly new optimization methodology that has not yet found widespread use in the financial community. Robust portfolio optimization problem is indeed a more complex optimization problem, but one that can be efficiently handled by a class of interior-point optimizers that are capable of handling second-order cone constraints. Therefore, based on the computational results in this paper, we believe that robust portfolio optimization is a practical and effective portfolio construction methodology.

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