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Time consistent multi-period robust risk measures and portfolio selection models with regime-switching

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ABSTRACT

To better describe the time-varying property of the dynamic investment risk and the ambiguity of the random return process, we propose two multi-period robust risk measures under the regime switching framework. Using regime-dependent dynamic uncertainty sets, we show that the multi-period robust portfolio selection problems under the two multi-period robust risk measures with regime switching can be transformed into second order cone programs, which can thus be efficiently solved in polynomial time. To show the generality of the dynamic uncertainty sets under the regime switching framework, we further consider multi-period robust risk measures under time-varying uncertainty sets with moments uncertainty and discuss the tractability of the corresponding multi-period robust portfolio selection problems. A series of empirical results demonstrate that the robust portfolio selection models with regime switching can flexibly help the investor make superior and robust investment strategies according to the switching of the market environment.

1. Introduction

The standard assumption in traditional risk measure and portfolio selection is that the probability distribution of the random return is known beforehand and only the realizations are unknown at the time of decision making. However, the underlying distribution of the random investment return or loss cannot be exactly specified in practice. The distributionally robust optimization technique was proposed to measure the ambiguity of the distribution, and has been successfully applied to many kinds of management problems under uncertainty. This technique gives a robust estimation with respect to the worst-case distribution from an uncertainty set.

Early research about distributionally robust optimization can be found in Scarf (1958) and Žáčková (1966). More recently, the distributionally robust optimization technique is widely used in the finance area, especially, it is useful for constructing worst-case risk measures and formulating robust portfolio selection models (Chen, He, & Zhang, 2011; El Ghaoui, Oks, & Oustry, 2003; Lobo & Boyd, 2000; Zhu & Fukushima, 2009). To make the robust risk measure reasonable and the resulting robust portfolio selection problem tractable, the proper choice of the uncertainty set is very important. The moment information is used frequently to construct the uncertainty set. However, for a medium-term or long-term investment problem, the investor can hardly estimate the moments which are suitable for any market environment in the future. Delage and Ye (2010) investigate the distributionally robust problem with uncertainty in terms of the distribution and of its moments, they show that the resulting solutions perform best over a set of distributions with high probability. Pflug and Wozabal (2007) and Wozabal (2014) consider an uncertainty set whose size is controlled by the probabilistic distance and use it to make the worst-case estimation of quite a lot risk measures. Liu and Chen (2014) adopt the regime switching method to describe the time-varying property of the uncertainty set specified by the first and second order moments, from which they propose two forms of static robust risk measures with regime switching. Haskell, Fu, and Dessouky (2016) study robust stochastic optimization problems for risk-averse decision makers, they consider both ambiguities in decision makers' risk preferences and the underlying probability distributions.

All the above robust risk measures and robust portfolio selection models are static. In practice, most investment problems are medium-term or long-term decision making problems, the investment risk should be measured and the investment decision should be adjusted dynamically according to the time-varying information (De Lara & Leclère, 2016; Homem-de Mello & Pagnoncelli, 2016). This requirement inspires us to discuss the distributionally robust risk measure when we face the ambiguity of the distributions in the multi-period case.

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The multi-period robust problem concerns the ambiguity of the multi-variate joint distribution stretching over multiple periods, which is much more difficult to quantify than that in the static case. Hence, tractability becomes one of the most important factors in constructing multi-period robust models. Ben-Tal, Margelit, and Nemirovski (2000) and Bertsimas and Pachamanova (2008) consider the multi-period robust portfolio optimization problem that can be viewed as an extension of the widely-adopted certainty equivalent controller procedure from dynamic programming (Bertsekas, 1995). Discussed in Ben-Tal, El Ghaoui, and Nemirovski (2009, Chapter 13) is the robust Markov decision process, where the system's dynamics is described by the state transition, and the uncertain transition probability is assumed to be within a given uncertainty set. Such a process is further extended to a more general case, called the adjustable robust approach in Ben-Tal et al. (2009, Chapter 14). The adjustable robust approach allows one to solve the multi-period robust problem in a computationally efficient way by adapting the dynamic programming technique (Shapiro, 2011). This method is applied to solve a multi-period robust portfolio selection problem with lower partial moment risk measure in Chen et al. (2011). Pınar (2016) derives a closed-form optimal portfolio for the multi-period robust mean–variance portfolio optimization under the framework of adjustable robust optimization with an ellipsoidal uncertainty set about the mean return vector. Liu, Chen, and Hui (2018) construct a time consistent multi-period worst-case risk measure using the framework of the separate expectation conditional mapping, which forms a good complement to existing multi-period robust portfolio selection models using the adjustable robust approach. They further derive the explicit optimal investment strategy for the corresponding multi-period robust portfolio selection problem.

In the current literature about multi-period robust portfolio selection problems, the uncertainty sets in different periods are usually assumed to be independent (Liu et al., 2018a). However, in practice, the uncertainty set often dynamically changes with time according to the market macro state. The regime switching technique has been used in many financial management studies to reflect the change of market states, and to describe the nonlinear dynamic relationship of market environments among different time periods. It can reflect the macro economic trend and the dynamic correlation of random returns or losses in different economic cycles. Through elaborately distinguishing economic states, regime switching can even describe the dynamic variation in the skewness, kurtosis and heavy-tail properties of the return (loss) distribution. One can refer to Hamilton (1989) about the introduction of regime switching and papers like Ma, MacLean, Xu, and Zhao (2011) and Elliott and Siu (2009) for its financial applications. Recently, Chen, Liu, and Hui (2017) use the regime switching technique to construct regime-dependent recursive risk measures and propose an efficient solution method for the corresponding multi-period portfolio selection problem.

Considering the above issues, we in this paper assume that the uncertainty set is time-varying under the regime switching framework, so that we can better describe the dynamic property of the uncertainty set with respect to different market environments. We define two kinds of regime-dependent robust multi-period risk measures, which are proved to be time consistent. When applying the new risk measures to the multi-period portfolio selection problem incorporating transaction costs, we consider several kinds of uncertainty sets based on given or unprecise moments information and show that these portfolio selection problems can be equivalently transformed into second order cone programs.

The main contributions of this paper can be summarized as follows. First of all, we propose new multi-period robust risk measures for markets with observable regime switching property, and apply them to multi-period portfolio selection problems; compared with the adjustable robust approach and its application in portfolio selection (Ben-Tal et al., 2009; Chen et al., 2011; Shapiro, 2011), our new robust model considers the time-varying robustness period-wise; compared with the static robust risk measure models in Liu and Chen (2014), the multi-period robust risk measure and associated multi-period portfolio selection model can better reflect the dynamic property of the investment risk and resulting investment policy; especially, the time consistency of the introduced multi-period robust risk measure guarantees the implementability of the optimal investment policy; furthermore, the introduction of regime switching makes the multi-period portfolio selection model more robust and flexible in terms of reflecting the constantly varying market environment; different from the usual period-wise independence assumption (Liu et al., 2018a), the dependence of dynamic uncertainty sets between adjacent periods is considered in our model, and different types of uncertainty sets are considered under different market regimes; at last, our multi-period portfolio selection models can be solved in polynomial time when the uncertainty sets are properly chosen. Therefore, the proposed new models can help the investor adjust his/her investment policy rapidly according to the latest information.

The remainder of the paper is organized as follows. After briefly reviewing the worst-case risk measure and the multi-period worst-case risk measure, Section 2 introduces two kinds of multi-period robust risk measures with regime switching. By taking CVaR as an example, we apply in Section 3 the proposed multi-period robust risk measures to multi-period portfolio selection problems and transform them into second order cone programs. In Section 4, we further consider two kinds of uncertainty sets with moments uncertainty and study the tractability of the resulting risk measures when applied to multi-period portfolio selection problems. Section 5 presents a series of empirical results to show the practicality and efficiency of our new models. Section 6 concludes the paper.

2. Multi-period robust risk measures

As a preparation for introducing multi-period robust risk measures with regime switching in the latter section, we briefly review the multi-period risk measure and the multi-period worst-case risk measure.

2.1. Multi-period (worst-case) risk measure

We consider the investment risk of a random loss process over an investment horizon $[0, T]$, which has $T+1$ time points: $0, 1, 2, \ldots, T$, and thus T consecutive investment periods. The loss process $\{x_t,\ t = 0, 1, \ldots, T\}$ is determined by a random return rate process $\{r_t,\ t = 1, 2, \ldots, T\}$ and a decision process $\{u_t,\ t = 0, 1, \ldots, T-1\}$. We define $\{x_t,\ t = 0, 1, \ldots, T\}$ on a probability space $(\Omega, \mathcal{F}, P)$, adapted to the filtration process $\{\mathcal{F}_t,\ t = 0, 1, \ldots, T\}$. We require $\mathcal{F}_0 = \{0, \Omega\}$, $\mathcal{F}_t \subseteq \mathcal{F}_{t+1}$, for $t = 0, 1, \ldots, T-1$, and $\mathcal{F}_T = \mathcal{F}$. The initial loss x_0 is a constant. We denote the probability space of the random loss x_t at period t as $\mathcal{L}_t = \mathcal{L}_p(\Omega, \mathcal{F}_t, P)$, $p \in [2, +\infty)$, $t = 1, \ldots, T$. For the random loss process $(x_1, \ldots, x_T)$, we denote $\mathcal{L}_{t,T} = \mathcal{L}_t \times \cdots \times \mathcal{L}_T$, and $x_{t,T} = (x_t, \ldots, x_T) \in \mathcal{L}_{t,T}$ for notational simplicity.

Under the multi-period setting, the investment risk of the random loss process between period $t+1$ and period T, observed at time point t, can be measured by a conditional risk mapping $\rho_{t,T}(\cdot) : \mathcal{L}_{t+1,T} \to \mathcal{L}_t$ (Ruszczyński, 2010). As a special case, $\rho_{t|\mathcal{F}_{t-1}}(\cdot) : \mathcal{L}_t \to \mathcal{L}_{t-1}$ corresponds to the single-period risk measure at period $t-1$. A sequence of conditional risk mappings $\rho_{t,T}$ from time point 0 to time point $T-1$ is called a multi-period risk measure.

A typical multi-period risk measure is the separable expected conditional (SEC for short) mapping introduced in Pflug and Römisch (2007):

$$\rho_{t,T}(x_{t+1,T}) = \sum_{i=t+1}^{T} \mathbb{E}_P\Big[\rho_{i|\mathcal{F}_{i-1}}(x_i)\Big|\mathcal{F}_t\Big],\ t = 0, 1, \dots, T-1.$$

Kovacevic and Pflug (2009) prove that the SEC mapping is time consistent under the definition in Wang (1999). This kind of time consistency requires an order relationship between risks at later periods and risks at earlier periods. It says, for any $0 \le \tau < \theta \le T-1$ and $x_{\tau+1,T}, y_{\tau+1,T} \in \mathcal{L}_{\tau+1,T}$, if $x_{\tau+1,\theta} \le y_{\tau+1,\theta}$ and $\rho_{\theta,T}(x_{\theta+1,T}) \le \rho_{\theta,T}(y_{\theta+1,T})$ imply $\rho_{\tau,T}(x_{\tau+1,T}) \le \rho_{\tau,T}(y_{\tau+1,T})$, then $\{\rho_{t,T}\}_{t=0}^{T-1}$ is called time consistent.

Similarly to the single-period case, the computation of a multi-period risk measure depends on the probability distributions at T periods. However, the precise prediction of the distribution process is harder than the prediction of the distribution in the static case, due to the dynamic relationship among periods. Hence, it is necessary to make a robust estimation for the multi-period risk measure to account for the estimation risk.

Liu et al. (2018a) propose the following multi-period worst-case risk measure, which reflects the distributionally robust counterpart of the conditional risk mapping period-wise.

Definition 1 (Multi-period worst-case risk measure)**.** For $t = 0, 1, \dots, T-1$ and $x_{t+1,T} \in \mathcal{L}_{t+1,T}$,

$$w\rho_{t,T}(x_{t+1,T}) = \sum_{i=t+1}^{T} \mathbb{E}_P\left[\sup_{P_i \in \mathscr{P}_i} \rho_{i|\mathcal{F}_{i-1}}(x_i)\middle|\mathcal{F}_t\right] \tag{1}$$

is called the conditional worst-case risk mapping. The sequence of the risk mappings $\{w\rho_{t,T}\}_{t=0}^{T-1}$ is called the multi-period worst-case risk measure.

Here, P_1 is the unconditional probability measure on $(\Omega, \mathcal{F}_1)$, and P_i is the conditional probability measure for period i, $i = 2, \dots, T$. P_i, $i = 1, 2, \dots, T$, forms a chain of transition probability measures, dissected from the measure P with respect to the filtration process $\{\mathcal{F}_i\}$, i.e., $P = P_1 \circ P_2 \circ \cdots \circ P_T$ (Pflug, 2009).

The multi-period worst-case risk measure is time consistent, and if $\rho_{i|\mathcal{F}_{i-1}}$ associated with any probability distribution $P_i \in \mathscr{P}_i$ is coherent, the corresponding multi-period worst-case risk measure is dynamic coherent (Artzner, Delbaen, Eber, Heath, & Ku, 2007).

2.2. Regime-dependent multi-period robust risk measures

In this section, we utilize the regime switching technique to make a robust measure of the multi-period investment risk, and propose two kinds of regime dependent multi-period robust risk measures.

We suppose the regime at the initial time is s_0, and the regimes during the following T investment periods are $s_1, \dots, s_T$. Like that in Ma et al. (2011) and Elliott and Siu (2009), we assume that the regime switching is Markovian and the set of possible regimes S_t at period t, $t = 1, \dots, T$, is constituted by J regimes: $s^1, \dots, s^J$. We denote the transition probability from s_{t-1} to s_t by $Q(s_t; s_{t-1})$. $Q\{s_t = s^j | s_{t-1} = s^i\}$ represents the transition probability from regime s^i at period $t-1$ to regime s^j at period t. Therefore, the probability measure Q corresponding to the regime switching process can be dissected into a chain of transition probabilities, $Q = Q_1 \circ Q_2 \circ \cdots \circ Q_T$, with

$$Q_t = \begin{pmatrix} Q\{s_t = s^1|s_{t-1} = s^1\} & Q\{s_t = s^2|s_{t-1} = s^1\} & \cdots & Q\{s_t = s^J|s_{t-1} = s^1\} \\ Q\{s_t = s^1|s_{t-1} = s^2\} & Q\{s_t = s^2|s_{t-1} = s^2\} & \cdots & Q\{s_t = s^J|s_{t-1} = s^2\} \\ \cdots & \cdots & \cdots & \cdots \\ Q\{s_t = s^1|s_{t-1} = s^J\} & Q\{s_t = s^2|s_{t-1} = s^J\} & \cdots & Q\{s_t = s^J|s_{t-1} = s^J\} \end{pmatrix}.$$

We denote the probability space the regime process belongs to as $(S, \mathcal{S}, Q)$, here $S = \{s^1, \dots, s^J\}$, and the corresponding filtration it generates is $\mathcal{S}_0 \subseteq \mathcal{S}_1 \subseteq \cdots \subseteq \mathcal{S}_T$.

By introducing the regime switching into the σ-filtration we defined in Section 2.1, we allow the random loss x_t to vary with the market regime s_t and treat here $\{x_t,\ t = 0, 1, \dots, T\}$ as defined on the product space $(\Omega \times S, \mathcal{F} \times \mathcal{S}, P)$. At each period t, $t = 0, 1, \dots, T$, x_t is adapted to the filtration $\mathcal{F}_t \times \mathcal{S}_t$. And hence we suppose $x_t \in \mathcal{L}_p(\Omega \times S, \mathcal{F}_t \times \mathcal{S}_t, P)$, $p \ge 2$. It is worth noting that P now is a probability measure on the product space (Elliott, 1967), with Q being its submeasure assigned to the observed regime switching process. Similarly to the case without regime switching, P can be dissected as $P = P_1 \circ P_2 \circ \cdots \circ P_T$.

To distinguish the influence of $\{\mathcal{F}_t\}$ and that of $\{\mathcal{S}_t\}$, we separate $\rho_{t|\mathcal{F}_{t-1}}(\cdot) : \mathcal{L}_t \to \mathcal{L}_{t-1}$ into two parts $\rho_{t|\mathcal{F}_{t-1}}(\cdot) = g_t(\rho_{s_t}(\cdot))$. The first basic element of the conditional risk mapping is associated with regime s_t,

$$\rho_{s_t}(\cdot) : \mathcal{L}_p(\Omega \times S, \mathcal{F}_t \times \mathcal{S}_t, P) \to \mathcal{L}_p(\Omega \times S, \mathcal{F}_{t-1} \times \mathcal{S}_t, P),$$

which corresponds to the usual one-period conditional risk measure with respect to the conditional probability measure P_t under regime s_t. Then the regime-dependent one-period risk measures are combined together through the function

$$g_t(\cdot) : \mathcal{L}_p(\Omega \times S, \mathcal{F}_{t-1} \times \mathcal{S}_t, P) \to \mathcal{L}_p(\Omega \times S, \mathcal{F}_{t-1} \times \mathcal{S}_{t-1}, P).$$

To provide a distributionally robust estimation of the multi-period investment risk, we first make a worst-case estimation of the one-period conditional risk measure, $w\rho_{s_t}(x_t) = \sup_{P_t \in \mathscr{P}_t(s_t)} \rho_{s_t}(x_t)$, which is also regime-dependent. $\mathscr{P}_t(s_t)$ is the uncertainty set including all possible distributions of x_t under regime s_t. After deriving the robust counterpart of the regime-dependent one-period risk mapping, we combine them together through the function $g_t(\cdot)$. In what follows, we propose two kinds of combination methods. The first strategy is to choose the largest risk value among all possible regimes as the robust measure of the risk at period t. It means $g_t(\cdot) = \sup_{s_t \in S_t}(\cdot)$. Finally, these robust risk measures at different periods are added together through the SEC mapping, which leads to the following multi-period worst-regime risk measure.

Definition 2 (Multi-period worst-regime risk measure)**.** For $t = 0, 1, \dots, T-1$ and $x_{t+1,T} \in \mathcal{L}_{t+1,T}$,

$$wr\rho_{t,T}(x_{t+1,T}; s_t) = \sum_{i=t+1}^{T} \mathbb{E}_P\left[\sup_{s_i \in S_i} \sup_{P_i \in \mathscr{P}_i(s_i)} \rho_{s_i}(x_i)\middle|\mathcal{F}_t \times \mathcal{S}_t\right] \tag{2}$$

is called the conditional worst-regime risk mapping. And the sequence of the conditional worst-regime risk mappings $\{wr\rho_{t,T}\}_{t=0}^{T-1}$ is called the multi-period worst-regime risk measure.

The multi-period worst-regime risk measure can be rewritten in the following dynamic equation form:

$$\begin{aligned} wr\rho_{t-1,T}(x_{t,T}; s_{t-1}) = {} & \left(\sup_{s_t \in S_t}\left(\sup_{P_t \in \mathscr{P}_t(s_t)} \rho_{s_t}(x_t)\right)\right) \\ & + \mathbb{E}_P[wr\rho_{t,T}(x_{t+1,T}; s_t)|\mathcal{F}_{t-1} \times \mathcal{S}_{t-1}],\ t = 1, 2, \dots, T. \end{aligned} \tag{3}$$

We can see from (2) and (3) that the multi-period worst-regime risk measure only cares about the worst regime in a period and ignores other regimes. Meanwhile, it does not take into account the probability that a regime could appear. Therefore, the multi-period worst-regime risk measure probably generates a very conservative risk evaluation. This inspires us to integrate all the risks under different regimes by incorporating their appearing probabilities. Concretely, we replace the supremum operator in the multi-period worst-regime risk measure with the weighted summation with respect to the regime transition probability, i.e., $g_t(\cdot) = \mathbb{E}_{Q_t}(\cdot|\mathcal{S}_{t-1})$. Under the finite state Markovian chain assumption for $\{s_t\}$, we have $\mathbb{E}_{Q_t}(x|\mathcal{S}_{t-1}) = \sum_{s_t \in S_t} Q(s_t; s_{t-1}) x(s_t)$. This gives us the following multi-period mixed worst-case risk measure:

Definition 3 (Multi-period mixed worst-case risk measure)**.** For $t = 0, 1, \dots, T-1$ and $x_{t+1,T} \in \mathcal{L}_{t+1,T}$,

$$mw\rho_{t,T}(x_{t+1,T}; s_t) = \sum_{i=t+1}^{T} \mathbb{E}_P\left[\mathbb{E}_{Q_i}\left[\sup_{P_i \in \mathscr{P}_i(s_i)} \rho_{s_i}(x_i)\middle|\mathcal{S}_{i-1}\right]\middle|\mathcal{F}_t \times \mathcal{S}_t\right] \tag{4}$$

is called the conditional mixed worst-case risk mapping. And the sequence of the conditional mixed worst-case risk mappings $\{mw\rho_{t,T}\}_{t=0}^{T-1}$ is called the multi-period mixed worst-case risk measure.

Like the previous risk measure, the multi-period mixed worst-case risk measure can also be rewritten in the following dynamic equation form:

$$mw\rho_{t-1,T}(x_{t,T}; s_{t-1}) = \left(\mathbb{E}_{Q_t}\left[\sup_{P_t \in \mathscr{P}_t(s_t)} \rho_{s_t}(x_t)\middle|\mathcal{S}_{t-1}\right]\right) + \mathbb{E}_P[mw\rho_{t,T}(x_{t+1,T}; s_t)|\mathcal{F}_{t-1} \times \mathcal{S}_{t-1}], t = 1, 2, \dots, T. \tag{5}$$

Remark 1. To the best of our knowledge, the inter-stage independence is assumed in almost all tractable multi-period robust models (Ben-Tal et al., 2009; Chen et al., 2011; Shapiro, 2011). Due to the introduction of regime switching, we can indirectly avoid the independence requirement for the dynamic uncertainty sets. Here, we suppose that the uncertainty set $\mathscr{P}_t(s_t)$ at period t is associated with the regime $s_t \in S_t$. The regime dependent uncertainty sets contain more information than the uncertainty sets used in Section 2.1, since the regime based uncertainty sets are of Bayesian nature: the random set $\mathscr{P}_t(s_t)$ appears with probability $Q(s_t; s_{t-1})$.

Based on the dynamic equations (3) and (5), we can establish the time consistency of the two multi-period robust risk measures.

Proposition 1 (Time consistency)**.** *If $\rho_{t|\mathcal{F}_{t-1}}$ associated with any probability distribution $P_t \in \mathscr{P}_t(s_t)$ is monotone for any s_t, $t = 1, 2, \dots, T$, then the corresponding multi-period robust risk measures $\{wr\rho_{t,T}\}_{t=0}^{T-1}$ and $\{mw\rho_{t,T}\}_{t=0}^{T-1}$ are time consistent.*

Proof. At any period t, suppose $x_t \le y_t$, $wr\rho_{t,T}(x_{t+1,T}; s_t) \le wr\rho_{t,T}(y_{t+1,T}; s_t)$ and $mw\rho_{t,T}(x_{t+1,T}; s_t) \le mw\rho_{t,T}(y_{t+1,T}; s_t)$, respectively. From the monotonicity of the expectation operator, we have

$$\mathbb{E}[wr\rho_{t,T}(x_{t+1,T}; s_t)|\mathcal{F}_{t-1} \times \mathcal{S}_{t-1}] \le \mathbb{E}[wr\rho_{t,T}(y_{t+1,T}; s_t)|\mathcal{F}_{t-1} \times \mathcal{S}_{t-1}]$$

and

$$\mathbb{E}[mw\rho_{t,T}(x_{t+1,T}; s_t)|\mathcal{F}_{t-1} \times \mathcal{S}_{t-1}] \le \mathbb{E}[mw\rho_{t,T}(y_{t+1,T}; s_t)|\mathcal{F}_{t-1} \times \mathcal{S}_{t-1}].$$

Moreover, from the monotonicity of ρ_{s_t} for any conditional probability distribution $P_t \in \mathscr{P}_t(s_t)$ and regime s_t, we have

$$\sup_{P_t \in \mathscr{P}_t(s_t)} \rho_{s_t}(x_t) \le \sup_{P_t \in \mathscr{P}_t(s_t)} \rho_{s_t}(y_t)$$

for any s_t. By taking the supremum or expectation operation to both sides of this inequality with respect to s_t, we have

$$\sup_{s_t \in S_t}\left(\sup_{P_t \in \mathscr{P}_t(s_t)} \rho_{s_t}(x_t)\right) \le \sup_{s_t \in S_t}\left(\sup_{P_t \in \mathscr{P}_t(s_t)} \rho_{s_t}(y_t)\right)$$

and

$$\mathbb{E}\left[\sup_{P_t \in \mathscr{P}_t(s_t)} \rho_{s_t}(x_t)\middle|\mathcal{S}_{t-1}\right] \le \mathbb{E}\left[\sup_{P_t \in \mathscr{P}_t(s_t)} \rho_{s_t}(y_t)\middle|\mathcal{S}_{t-1}\right].$$

Then it is easy to deduce from the above inequalities that $wr\rho_{t-1,T}(x_{t,T}; s_{t-1}) \le wr\rho_{t-1,T}(y_{t,T}; s_{t-1})$ and $mw\rho_{t-1,T}(x_{t,T}; s_{t-1}) \le mw\rho_{t-1,T}(y_{t,T}; s_{t-1})$, which gives the time-consistency at period t.

Applying these results recursively between any two adjacent periods gives the time consistency of the two multi-period robust risk measures. □

If ρ_{s_t} associated with any probability distribution $P_t \in \mathscr{P}_t(s_t)$ is coherent for any s_t, $t = 1, 2, \dots, T$, both $\{wr\rho_{t,T}\}_{t=0}^{T-1}$ and $\{mw\rho_{t,T}\}_{t=0}^{T-1}$ are dynamic coherent risk measures.

By specifying the single-period risk mapping ρ_{s_t}, we can obtain concrete multi-period robust risk measures. Especially, taking CVaR as an example, we can construct the multi-period worst-case CVaR, the multi-period worst regime CVaR and the multi-period mixed worst-case CVaR, we denote them as wCVaR, wrCVaR, and mwCVaR, respectively. From Definitions 2 and 3, we see that wrCVaR and mwCVaR degenerate to wCVaR when there is only one regime in the market.

3. Multi-period robust portfolio selection models under wrCVaR and mwCVaR

In this section, the introduced multi-period robust risk measures, wrCVaR and mwCVaR, are adopted to establish multi-period robust portfolio selection models. In order to obtain greater realism in our portfolio selection models, we take transaction costs and market constraints into account.

We suppose that there are n risky assets in the security market, and the investment horizon is divided into T consecutive periods. The investor joins the market at time 0 with an initial cash w_0, he/she can trade these assets at the beginning of each period. $r_t = [r_t^1, \dots, r_t^n]^\top$, $t = 1, 2, \dots, T$, is the random return rate vector of the n assets at period t, which is adapted to the information process $\{\mathcal{F}_t \times \mathcal{S}_t\}$. At the beginning of each period, the current wealth can be reallocated among the n risky assets, and $u_t = [u_t^1, \dots, u_t^n]^\top$ is the corresponding portfolio vector. We assume that u_t depends on the market regime and is predictable to the information process $\{\mathcal{S}_t\}$. w_t, $t = 1, 2, \dots, T$, is the resulting wealth process.

The investor has to pay proportional transaction costs to purchase or sell risky assets, with the transaction cost ratio vector being α and β, respectively. We assume that the investment process is self-financing, hence we have

$$w_0 = u_0^\top e + \alpha^\top (u_0)^+ + \beta^\top (u_0)^-, \tag{6}$$

$$w_t = u_t^\top e + \alpha^\top (u_t - u_{t-1})^+ + \beta^\top (u_t - u_{t-1})^-, \quad t = 1, \dots, T-1, \tag{7}$$

$$w_{t+1} = u_t^\top r_{t+1}, \quad t = 0, \dots, T-1, \tag{8}$$

where, for $\xi \in \mathbb{R}^n$, $(\xi)^+ = [\max\{0, \xi_1\}, \dots, \max\{0, \xi_n\}]^\top$ and $(\xi)^- = [-\min\{0, \xi_1\}, \dots, -\min\{0, \xi_n\}]^\top$ are the positive and negative parts of ξ, respectively. Moreover, in order to reflect the real market environment, such as the no-short selling restriction in some security markets or the quota for investments, we restrict u_t as follows:

$$\underline{u} \le u_t \le \overline{u}, \quad t = 0, \dots, T-1, \tag{9}$$

here $\underline{u} \in \mathbb{R}^n$ and $\overline{u} \in \mathbb{R}^n$ are the lower bound vector and the upper bound vector, respectively.

Similarly to the classic mean–variance framework, we consider a two-criteria approach with respect to the expected final wealth and the wrCVaR (mwCVaR) measure, and construct the following two multi-period portfolio optimization problems with regime switching:

$$\max_u \left\{\mathbb{E}[w_T; s_0] - \lambda \cdot wrCVaR_{0,T}(-w_{1,T}; s_0)\right\}, \tag{10}$$

$$\text{s.t. } (6)\text{–}(9). \tag{11}$$

and

$$\max_u \left\{\mathbb{E}[w_T; s_0] - \lambda \cdot mwCVaR_{0,T}(-w_{1,T}; s_0)\right\}, \tag{12}$$

$$\text{s.t. } (6)\text{–}(9). \tag{13}$$

here λ is the risk aversion coefficient. The confidence level at period t is $\epsilon_t(s_t)$, which changes with the regime s_t. We denote the above two robust mean-risk models as the mean-wrCVaR model and the mean-mwCVaR model, respectively, in what follows.

Due to the nonlinear relationship induced by regime switching between periods, it is impossible for us to find the closed-form optimal policies for the mean-wrCVaR and mean-mwCVaR models. To ensure its practicality and flexibility, we adopt the scenario tree technology to derive a tractable formulation for the above two optimization problems. Before that, we need to introduce some notation.

Firstly, we assume the regime switching during the T periods is a Markovian process and adopt a scenario tree to represent the structure of the regime process. The initial (root) node in the scenario tree represents the current regime s_0, each of its son nodes represents a possible regime s_1 appearing at period 1, here the branching probability is equal to the transition probability from s_0 to s_1, and so on. In the scenario tree, K^+ denotes the set of all nodes at periods $1, 2, \dots, T$, $N(K^+)$ is the number of nodes in K^+; K^- denotes the set of all nodes at periods $0, 1, \dots, T-1$, $N(K^-)$ is the number of nodes in K^-. For a node k in the scenario tree, we use $t(k)$ to denote the number of the period it belongs to, $s(k)$ to denote the corresponding regime, and $Q(k; s_0)$ to denote its appearing probability in the tree. For a node $k \in K^+$, it has exactly one direct predecessor, denoted as k^-.

For a node $k \in K^+$, let $\mu(k)$ denote the estimated mean vector of the return rate vector r_t of n risky assets at period $t(k)$, under the available information at its predecessor node k^-, and let $\Gamma(k)$ denote the estimation of the conditional covariance matrix, here we assume that $\Gamma(k)$ is positive definite. We assume that, at a node k, what we know about the return rates at the next period are only the information about their conditional first and second order moments, which depends on the current regime $s(k)$. Considering the ambiguity of the distribution of the return rate vector, we assume that the distribution of r_t measured at node k of period t belongs to an uncertainty set with the given conditional first two order moments:

$$\mathscr{P}(k) = \left\{P_t \,\middle|\, \mathbb{E}_{P_t}[r_t|\mathcal{F}_{t-1}, s_t = s(k)] = \mu(k),\right.$$
$$\left. \mathrm{Cov}_{P_t}[r_t|\mathcal{F}_{t-1}, s_t = s(k)] = \Gamma(k)\right\}. \tag{14}$$

Here, the uncertainty set is constructed by fixing the conditional expected returns and the conditional covariance matrix, which can be observed at period $t-1$. Therefore, this definition satisfies the basic requirement of the dynamic uncertainty set.

Given the regime switching scenario tree, we assume that the dynamic decision process $\{u_t\}$ adapts to the regime switching process. That is, at each node k in the scenario tree, a decision $u(k)$ is assigned. To coincide with the scenario tree, the dynamic equation for decision variables between two adjacent periods is rewritten as

$$u(k^-)^\top\mu(k) = u(k)^\top e + \alpha^\top(u(k) - u(k^-))^+ + \beta^\top(u(k) - u(k^-))^-, \quad k \in K^-\backslash\{0\} \tag{15}$$

which is a combination of (7) and (8). Here, $\mu(k)$ is the forecasted value of the return rate vector at node k of stage t. This formulation of the dynamic equation is known as the deterministic equivalence method (Bertsekas, 1995).

With the above notations, we can reformulate the mean-mwCVaR model (12) and (13) as a second order cone programming problem. Concretely,

Theorem 1. *With the uncertainty set (14) and the dynamic equation (15), the mean-mwCVaR model (12) and (13) is equivalent to the following cone programming problem:*

$$\begin{aligned}\max_{u,y,z,g,u^+,u^-} &\Big\{(1+T\lambda)w_0 \\ &+ \sum_{k\in K^+}(1 + (T - t(k^-) - 1)\lambda)Q(k; s_0)(\mu(k) - e)^\top u(k^-) \\ &- \lambda\sum_{k\in K^+}Q(k; s_0)y(k) - (1+T\lambda)\big(\alpha^\top u^+(0) + \beta^\top u^-(0)\big) \\ &- \sum_{k\in K^-\backslash\{0\}}(1 + (T - t(k))\lambda)Q(k; s_0)\big[\alpha^\top u^+(k) + \beta^\top u^-(k)\big]\Big\}\end{aligned} \tag{16}$$

$$\text{s.t. } \Gamma^{1/2}(k)u(k^-) = z(k), \quad k \in K^+, \tag{17}$$

$$(\mu(k) - e)^\top u(k^-) + y(k) = \kappa(k)g(k), \quad k \in K^+, \tag{18}$$

$$||z(k)||_2 \le g(k), \quad k \in K^+, \tag{19}$$

$$u(0) = u^+(0) - u^-(0), \tag{20}$$

$$w_0 = u(0)^\top e + \alpha^\top u^+(0) + \beta^\top u^-(0), \tag{21}$$

$$u(k) - u(k^-) = u^+(k) - u^-(k), \quad k \in K^-\backslash\{0\}, \tag{22}$$

$$u(k^-)^\top\mu(k) = u(k)^\top e + \alpha^\top u^+(k) + \beta^\top u^-(k), \quad k \in K^-\backslash\{0\}, \tag{23}$$

$$u^+(k), u^-(k) \ge 0, \quad k \in K^-, \tag{24}$$

$$\underline{u} \le u(k) \le \overline{u}, \quad k \in K^-, \tag{25}$$

with $(n+2)N(K^+) + 3nN(K^-)$ *variables,* $(n+1)N(K^+) + (n+1)N(K^-)$ *linear constraints and* $N(K^+)$ *standard second order cone constraints.*

Proof. We can deduce from (6)–(8) that

$$w_t = w_{t-1} + u_{t-1}^\top(r_t - e) - \alpha^\top(u_{t-1} - u_{t-2})^+ - \beta^\top(u_{t-1} - u_{t-2})^-. \tag{26}$$

Applying (26) recursively gives

$$\begin{aligned}w_t &= w_0 + \sum_{s=0}^{t-1}u_s^\top(r_{s+1} - e) - \sum_{s=1}^{t-1}(\alpha^\top(u_s - u_{s-1})^+ \\ &\quad + \beta^\top(u_s - u_{s-1})^-) - \alpha^\top(u_0)^+ - \beta^\top(u_0)^-.\end{aligned} \tag{27}$$

By substituting (26) into (4), we have

$$\begin{aligned}&mwCVaR_{0,T}(-w_{1,T}; s_0) \\ &= \sum_{t=1}^{T}\mathbb{E}\Bigg[\sum_{s_t\in S_t}\Big(Q(s_t; s_{t-1})\sup_{P_t\in\mathscr{P}_t(s_t)}CVaR_{t|\mathcal{F}_{t-1}}\Big(-w_{t-1} - u_{t-1}^\top(r_t - e) \\ &\quad + \alpha^\top(u_{t-1} - u_{t-2})^+ + \beta^\top(u_{t-1} - u_{t-2})^-\Big)\Big)\Bigg]\end{aligned}$$

$$= \sum_{t=1}^{T} \mathbb{E}\left[\sum_{s_t \in S_t} \left(Q(s_t; s_{t-1}) \sup_{P_t \in \mathscr{P}_t(s_t)} CVaR_{t|\mathcal{F}_{t-1}}\left(-u_{t-1}^\top (r_t - e) \right) \right) \right] - \sum_{t=0}^{T-1} \mathbb{E}[w_t]$$
$$+ \left(\alpha^\top (u_0)^+ + \beta^\top (u_0)^- \right) + \sum_{t=1}^{T-1} \mathbb{E}\left[\alpha^\top (u_t - u_{t-1})^+ + \beta^\top (u_t - u_{t-1})^- \right],$$

here we have utilized the translation invariance property of CVaR. And then, we take it back into (12), and find that the objective function in (12) can be divided into three parts: the risk part, the return part, and the transaction cost part. Specifically, we have

$$\mathbb{E}[w_T; s_0] - \lambda \cdot mwCVaR_{0,T}(-w_{1,T}; s_0) = Return - \lambda Risk - TC, \tag{28}$$

where

$$Return = (1 + T\lambda) w_0 + \sum_{t=0}^{T-1} (1 + (T - t - 1)\lambda) \mathbb{E}[u_t^\top (r_{t+1} - e)],$$
$$Risk = \sum_{t=1}^{T} \mathbb{E}\left[\sum_{s_t \in S_t} \left(Q(s_t; s_{t-1}) \sup_{P_t \in \mathscr{P}_t(s_t)} CVaR_{t|\mathcal{F}_{t-1}}\left(-u_{t-1}^\top (r_t - e) \right) \right) \right],$$
$$TC = (1 + T\lambda)\left(\alpha^\top (u_0)^+ + \beta^\top (u_0)^- \right) + \sum_{t=1}^{T-1} (1 + (T - t)\lambda) \mathbb{E}\left[\alpha^\top (u_t - u_{t-1})^+ + \beta^\top (u_t - u_{t-1})^- \right].$$

Firstly, we can reformulate the return part as follows

$$Return = (1 + T\lambda) w_0 + \sum_{k \in K^+} (1 + (T - t(k^-) - 1)\lambda) Q(k; s_0) (\mu(k) - e)^\top u(k^-), \tag{29}$$

which is a linear function of the decision variables.

Secondly, the risk part can be rewritten as

$$Risk = \min_{y} \sum_{k \in K^+} Q(k; s_0) y(k) \tag{30}$$

$$\text{s.t.} \sup_{P_{t(k)} \in \mathscr{P}(k)} CVaR_{t|\mathcal{F}_{t-1}}(-(r(k) - e)^\top u(k^-)) \le y(k), k \in K^+, \tag{31}$$

here, we introduce auxiliary variables $y(k)$, $k \in K^+$, which are adapted to $\mathcal{F}_t \times \mathcal{S}_t$.

Under $\mathscr{P}(k)$, the constraint (31) at node k is actually a single-period conditional worst-case CVaR constraint with the information about first two order moments. Then, we know from Chen et al. (2011, Theorem 2.9) that it is equivalent to

$$\kappa(k) ||\Gamma^{1/2}(k) u(k^-)||_2 - (\mu(k) - e)^\top u(k^-) \le y(k), \ k \in K^+, \tag{32}$$

where $\kappa(k) = \sqrt{\frac{1-\epsilon(k)}{\epsilon(k)}}$.

Through introducing auxiliary vectors $z(k) = [z_1(k), \dots, z_n(k)]^\mathrm{T}$, $k \in K^+$, and auxiliary variables $g(k)$, $k \in K^+$, the constraint (32) can further be equivalently described as the following two groups of linear constraints and a group of standard second order cone constraints:

$$\Gamma^{1/2}(k) u(k^-) = z(k), \ k \in K^+, \tag{33}$$

$$(\mu(k) - e)^\top u(k^-) + y(k) = \kappa(k) g(k), \ k \in K^+, \tag{34}$$

$$||z(k)||_2 \le g(k), \ k \in K^+. \tag{35}$$

Thirdly, the transaction cost part can be rewritten as:

$$Transaction = \min_{u^+, u^-} \left\{ (1 + T\lambda)\left(\alpha^\top u^+(0) + \beta^\top u^-(0) \right) + \sum_{k \in K^- \setminus \{0\}} (1 + (T - t(k))\lambda) Q(k; s_0) \left[\alpha^\top u^+(k) + \beta^\top u^-(k) \right] \right\} \tag{36}$$

$$\text{s.t. } u(0) = u^+(0) - u^-(0), \tag{37}$$

$$u(k) - u(k^-) = u^+(k) - u^-(k), \ k \in K^- \setminus \{0\}, \tag{38}$$

$$u^+(k), u^-(k) \ge 0, \ k \in K^-. \tag{39}$$

Finally, we can obtain the second order cone programming problem (16)–(25) by taking all (6), (15), (28), (29), (30), (33)–(35) and (36)–(39) into (12) and (13), which completes the proof. □

By using a similar demonstration as the proof of Theorem 1, we can show that the mean-wrCVaR model can also be transformed into a second order cone programming problem. It is summarized in the following theorem.

Theorem 2. *With the uncertainty set (14) and the dynamic equation (15), the mean-wrCVaR model (10) and (11) is equivalent to the following cone programming problem:*

$$\min_{u,y,z,g,u^+,u^-} \Bigg\{ (1 + T\lambda) w_0 + \sum_{k \in K^+} (1 + (T - t(k) - 1)\lambda) Q(k; s_0) (\mu(k) - e)^\top u(k^-) - \lambda \sum_{k \in K^-} Q(k; s_0) y(k) - (1 + T\lambda)\left(\alpha^\top u^+(0) + \beta^\top u^-(0) \right) + \sum_{k \in K^- \setminus \{0\}} (1 + (T - t(k))\lambda) Q(k; s_0) \left[\alpha^\top u^+(k) + \beta^\top u^-(k) \right] \Bigg\} \tag{40}$$

$$\text{s.t. } (\mu(k) - e)^\top u(k^-) + y(k^-) = \kappa(k) g(k), \ k \in K^+, \tag{41}$$

$$(17), (19)\text{–}(25). \tag{42}$$

with $(n+1)N(K^+) + (3n+1)N(K^-)$ variables, $(n+1)N(K^+) + (n+1)N(K^-)$ linear constraints and $N(K^+)$ standard second order cone constraints.

Theorems 1 and 2 mean that both of our robust portfolio selection models (12), (13) and (10), (11) can be efficiently solved in polynomial time by using interior point methods (Alizadeh & Goldfarb, 2003).

Remark 2. The multi-period worst-case risk measure is a special case of the multi-period worst regime risk measure or the multi-period mixed worst-case risk measure with the number of regimes being one. Therefore, if transaction costs and market constraints are taken into account, the corresponding multi-period portfolio selection model under the multi-period worst-case risk measure can also be efficiently solved by utilizing the second order cone programming problem (16)–(25) or (40)–(42).

4. Multi-period robust risk measures with moments uncertainty

In the last section, the uncertainty sets (14) with given first and second order moments are used to describe the ambiguity of conditional distributions at individual nodes in a scenario tree. Although the uncertainty sets at different nodes have the same structure, the parameters and the given first two order moments, can change with different nodes to reflect the variation of the statistics under different market regimes. To illustrate the flexibility of our time-varying uncertainty sets under the regime-switching framework, we consider in this section uncertainty sets with moments uncertainty.

In some complex situations, the first and second order moments could also be ambiguous. Hence, we discuss here the uncertainty sets that consider the uncertainty in terms of the distribution and of the mean vector and covariance matrix. For any node k in the scenario tree, we define

$$\mathscr{P}_U(k) = \left\{ P_t \left| \begin{array}{l} \mathbb{E}_{P_t}[r_t|\mathcal{F}_{t-1}, s_t = s(k)] = \bar{\mu}(k), \\ Cov_{P_t}[r_t|\mathcal{F}_{t-1}, s_t = s(k)] = \bar{\Gamma}(k), \\ (\bar{\mu}(k), \bar{\Gamma}(k)) \in U(\mu(k), \Gamma(k)). \end{array} \right. \right\}, \tag{43}$$

Here, $\mu(k)$ and $\Gamma(k)$ are the estimation values of the conditional mean vector and the conditional covariance matrix. We assume that $\mu(k)$ and $\Gamma(k)$ are different from the true conditional mean vector $\bar{\mu}(k)$ and the true conditional covariance matrix $\bar{\Gamma}(k)$. $U(\mu(k), \Gamma(k))$ is the uncertainty set for the conditional mean vector and covariance matrix. Two kinds of $U(\mu(k), \Gamma(k))$ are considered in this section:

$$U_1(\mu(k), \Gamma(k)) = \left\{ (\bar{\mu}(k), \bar{\Gamma}(k)) \left| \begin{array}{l} (\bar{\mu}(k) - \mu(k))^\top \Gamma(k)^{-1} (\bar{\mu}(k) - \mu(k)) \le \pi_1(k), \\ \bar{\Gamma}(k) \preceq \pi_2(k)\Gamma(k). \end{array} \right. \right\}, \tag{44}$$

and

$$U_2(\mu(k), \Gamma(k)) = \left\{ (\bar{\mu}(k), \bar{\Gamma}(k)) \left| \begin{array}{l} S(k)(\bar{\mu}(k) - \mu(k))^\top \Gamma(k)^{-1} (\bar{\mu}(k) - \mu(k)) \\ + \frac{S(k)-1}{2} \parallel \Gamma(k)^{-\frac{1}{2}} (\bar{\Gamma}(k) - \Gamma(k)) \Gamma(k)^{-\frac{1}{2}} \parallel_{tr}^2 \le \delta^2(k). \end{array} \right. \right\}. \tag{45}$$

Here $\parallel A \parallel_{tr}^2 = tr(AA^\top)$, $A \preceq B$ means that $B - A$ is positive semi-definite. $\pi_1(k)$, $\pi_2(k)$, $S(k)$ and $\delta(k)$ are parameters controlling the size of the uncertainty sets. These parameters can vary with the switching of the market regime. $U_1(\mu(k), \Gamma(k))$ is taken from Liu, Chen, Lisser, and Xu (2018b). $U_2(\mu(k), \Gamma(k))$ is taken from Lotfi and Zenios (2016) and includes a number of uncertainty sets based on moments information.

Theorem 3. *Suppose that the uncertainty set at a node k is $\mathscr{P}_U(k)$ with $U(\mu(k), \Gamma(k)) = U_1(\mu(k), \Gamma(k))$ or $U(\mu(k), \Gamma(k)) = U_2(\mu(k), \Gamma(k))$, then the worst-case counterpart of CVaR can be reformulated as follows:*

$$\sup_{P_{t(k)} \in \mathscr{P}_U(k)} CVaR_{t|\mathcal{F}_{t-1}}(-(r(k) - e)^\top u(k^-)) = \kappa(k) ||\Gamma^{1/2}(k) u(k^-)||_2 - (\mu(k) - e)^\top u(k^-),$$

where

$$\kappa(k) = \sqrt{\frac{1 - \epsilon(k)}{\epsilon(k)}} \sqrt{\pi_2(k)} + \sqrt{\pi_1(k)}$$

for $U_1(\mu(k), \Gamma(k))$ and

$$\kappa(k) = \max_{\tau \in [0,1]} \left(\sqrt{\frac{1 - \epsilon(k)}{\epsilon(k)}} \sqrt{1 + \delta(k) \sqrt{\frac{2(1 - \tau)}{S(k) - 1}}} + \delta(k) \sqrt{\tau / S(k)} \right)$$

for $U_2(\mu(k), \Gamma(k))$.

Proof. The conclusion for the case $U(\mu(k), \Gamma(k)) = U_1(\mu(k), \Gamma(k))$ can be established by applying Theorem 1 in Liu et al. (2018b) to this conditional distribution case. Similarly, the conclusion for the case $U(\mu(k), \Gamma(k)) = U_2(\mu(k), \Gamma(k))$ can be established by applying Theorem 2 in Lotfi and Zenios (2016). □

With Theorem 3, we can prove by using the similar demonstration as the proof of Theorem 1 that: if the uncertainty set (43) and (44), or (43) and (45), is adopted in the distributionally robust portfolio selection problems (10), (11) and (12), (13), both optimization problems can be reformulated as SOCP problems. The only difference among these reformulations is the values of $\kappa(k)$ for different uncertainty sets. Moreover, if we choose different uncertainty sets in different market regimes from the three types of discussed uncertainty sets, these optimization problems can also be reformulated as SOCP problems. Therefore, the resulting distributionally robust portfolio selection problems with $\mathscr{P}(k)$ or $\mathscr{P}_U(k)$ can efficiently solve complex multi-period investment problems in markets where the estimated moments could be regime-dependently imprecise.

5. Numerical results

In this section, we empirically show how to specify market regimes and to find optimal portfolios under the introduced mean-wCVaR, mean-wrCVaR and mean-mwCVaR models by using the real trading data in American stock markets. We examine the differences among the optimal investment policies. Furthermore, to demonstrate the practicability and efficiency of our new models, we will perform both in-sample and out-of-sample analyses, and compare proposed models with typical benchmarks models.

5.1. Data set and parameter estimation

We randomly choose 10 stocks from different industries in both Dow Jones Industrial Average and S&P 500 Indexes, they are DIS, DOW, ED, GE, IBM, MRK, MRO, MSI, PEP and JNJ. We use the adjusted daily close-prices of these stocks on every Monday to compute their weekly logarithmic return rates from February 14, 1977 to January 30, 2012. The original data are downloaded from Yahoo finance.[1] The historical weekly data are separated into two parts: the in-sample period is from February 22, 1977 to March 1, 2010, and the out-of-sample period is from March 1, 2010 to January 30, 2012.

We first compute basic statistics of the return rates of these assets with in-sample data, which are shown in Table 1. It can be seen from the values of their skewness and kurtosis that the return rates are left-skewed and fat-tailed with large leptokurtosis. Then we perform the Kolmogorov–Smirnov goodness-of-fit hypothesis test by using the ‘kstest’ function in Matlab. Given the null hypothesis that the return rate could have come from a normal distribution, the statistics and P-values are shown in the last two rows of Table 1. The null hypothesis is uniformly rejected at the 5% significance level for all stocks. Hence, we can conclude that the in-sample return rates of the stocks are significantly non-Gaussian distributed.

Quite a few empirical evidences in the literature (Liu, Xu, & Zhao, 2011; Ma et al., 2011) show that three regimes are sufficient for representing the states of American stock markets. Hence, we divide the American market into three market regimes: the bull regime, denoted as s^1, indicating that the market is going up; the consolidation regime, denoted as s^2, indicating that the market is in the transitional period between recovery and recession; and the bear regime, denoted as s^3, indicating that the market is going down. We assume that the regime switching process is stationary. That is, $Q_t \equiv Q$, $t = 1, \dots, T$.

In order to estimate the transition matrix Q and the relevant parameters $\mu(k)$ and $\Gamma(k)$ at the node k of the regime switching tree, we first divide the in-sample period into three parts, corresponding to the three market regimes by using a heuristic method introduced in what follows. Then the regime transition probabilities are estimated by counting the relevant historical transition

[1] http://finance.yahoo.com

Table 1
Statistics of return rates.

	DIS	DOW	ED	GE	IBM	MRK	MRO	MSI	PEP	JNJ
Mean (%)	0.1004	0.0681	0.1098	0.0970	0.0718	0.1046	0.1090	0.0676	0.1196	0.1147
Variance ($\times$ 10e−3)	0.3259	0.3355	0.1194	0.2489	0.2437	0.2645	0.3975	0.6187	0.2152	0.1938
Skewness	−0.5976	−0.1310	−0.1151	0.0518	−0.1548	−0.4145	−0.0928	−0.5688	−0.1642	−0.0909
Kurtosis	7.5037	6.5984	5.4945	9.5885	5.9615	6.4817	6.2720	7.4084	4.9321	5.5545
K–S test statistic[a]	0.0401	0.0642	0.0574	0.0579	0.0541	0.0434	0.0583	0.0491	0.0441	0.0421
P-value	0.0076	0.0000	0.0000	0.0000	0.0001	0.0029	0.0000	0.0005	0.0024	0.0043

[a] The critical value of the test is 0.0326.

times. Finally, the sample mean vector and covariance matrix under each regime are estimated by using the in-sample data belonging to the regime. And $\mu(k)$ and $\Gamma(k)$ are assigned to the sample mean vector and covariance matrix according to the market regime that the node k belongs to.

Motivated by Chen et al. (2017), we use the data of MKT-RF (Fama & French, 1993), a market index which is the value-weighted excess returns on all NYSE, AMEX and NASDAQ stocks minus the 30 day US T-Bill yield, to determine the market regime during each week. The corresponding weekly data of MKT-RF are downloaded from the Kenneth R. French Data Library.[2] In detail, to determine the regime of a specific week, we prescribe an effective time window with 28 weeks, centered on the examining week, and add all the data of MKT-RF in the effective time window together. If the summation is larger than 1.0, we say that the week is under the bull regime; if the summation is smaller than −1.0, we say that the week is under the bear regime; if the summation is between −1.0 and 1.0, we say that the week is under the consolidation regime. The time window approach can eliminate the influence of large short-term fluctuations and can improve the estimation accuracy in practice. By applying the above method to each week in the sample period, we can obtain a historical regime switching series. Then the regime transition probabilities can be estimated by counting the relevant historical transition times, which gives us the following transition matrix:

$$
\begin{array}{cc}
 & \begin{array}{ccc} \text{bull} & \text{consolidation} & \text{bear} \end{array} \\
Q = & \begin{bmatrix} 0.9475 & 0.0336 & 0.0189 \\ 0.3333 & 0.3148 & 0.3519 \\ 0.0471 & 0.0634 & 0.8895 \end{bmatrix} \begin{array}{c} \text{bull} \\ \text{consolidation} \\ \text{bear} \end{array}
\end{array}
$$

From the diagonal elements in Q, we see that it is stable to stay in the bull or bear regime, but there is a relatively high possibility to switch from the consolidation regime into the bull or bear regime. This phenomenon is consistent with the real market situation.

We consider a three-period investment problem, $T = 3$, with the corresponding regime switching process (tree) shown in Fig. 1.

After determining the regime for each week in the in-sample period, we can divide the in-sample data into three parts corresponding to the three regimes. For each market regime, s^j, we compute the sample mean value, $\mu(s^j)$, and sample covariance matrix, $\Gamma(s^j)$, of the historical return data in weeks belonging to that regime. Then we assign them to the expected return rates, $\mu(k)$, and the covariance matrix, $\Gamma(k)$, according to the market regime that the node k belongs to. The expected return rates of the 10 assets under the three regimes are shown in Table 2. Due to the space limitation, we only show in Table 3 the estimated variances of the return rates of the 10 assets under the three regimes.

We can see from Tables 2 and 3 that both the expected return rates and the variances change significantly among three different regimes. Under the bull regime, the expected return rates are the

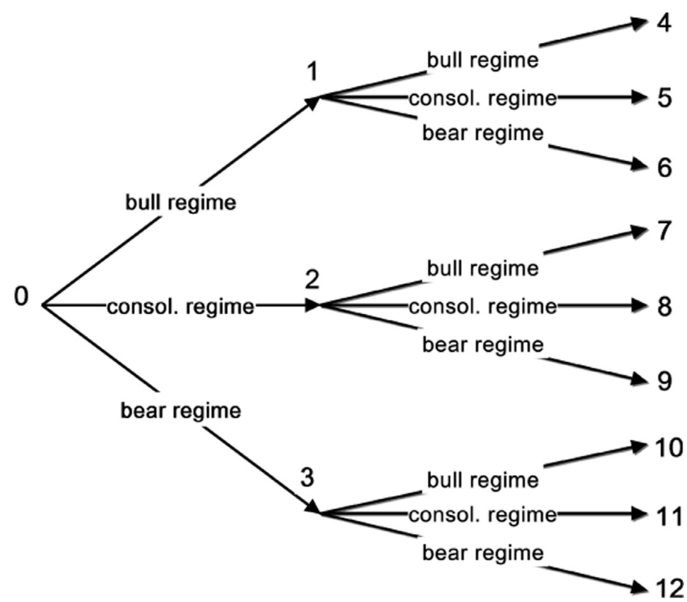

Fig. 1. The three-stage regime switching tree.

highest and always positive; under the bear regime, they are the lowest and mostly negative; and under the consolidation regime, they are in the middle. Correspondingly, the variances are always the largest under the bear regime, the smallest under the consolidation regime, and in the middle under the bull regime. These phenomena match the real market well: the investment under the bull (bear) regime is usually active, which leads to a high (low) return rate with large volatilities; while the investment under the consolidation regime is not that active, which leads to an intermediate return rate with a small volatility.

5.2. In-sample analysis

Having obtained the information about first and second order moments of the return rates under the three regimes, we can then determine the optimal investment policies under the mean-wrCVaR model and mean-mwCVaR model by solving problem (16)–(25) and problem (40)–(42), respectively. Correspondingly, we can also find the optimal investment policy under the mean-wCVaR model by solving problem (16)–(25) or (40)–(42) with $J = 1$, $\mu(s^1) = \mu$ and $\sigma^2(s^1) = \sigma^2$.

We assume that the initial wealth is 1.0; at each period, the confidence level for CVaR is chosen as $\epsilon_t(s_t) = 0.05$ under all three regimes; the risk aversion coefficient is set to $\lambda = 20$; the lower bounds for portfolio weights are 0.0, the short-selling is thus forbidden; the upper bounds for portfolio weights are 0.3. We use the Mosek package[3] in Matlab 7.6.0 (2008a) to solve problem (16)–(25) and problem (40)–(42), all the numerical experiments are

[2] http://mba.tuck.dartmouth.edu/pages/faculty/ken.french/data_library.html

[3] http://www.mosek.com/

Table 2
Expected return rates under the three regimes.

	DIS	DOW	ED	GE	IBM	MRK	MRO	MSI	PEP	JNJ
$\mu(s^1)$	0.2486	0.1845	0.1165	0.2260	0.1290	0.1884	0.1639	0.2291	0.1825	0.1511
$\mu(s^2)$	0.0206	−0.0116	0.1413	0.0110	−0.1879	0.1027	0.2251	0.0817	0.1653	0.1273
$\mu(s^3)$	−0.1921	−0.1583	0.0897	−0.1545	0.0035	−0.0691	−0.0274	−0.2706	−0.0199	0.0366

Table 3
Variances ($\times$ 10e−3) under the three regimes.

	DIS	DOW	ED	GE	IBM	MRK	MRO	MSI	PEP	JNJ
$\sigma^2(s^1)$	0.2677	0.3009	0.1012	0.2186	0.2182	0.2529	0.3545	0.4989	0.1954	0.1841
$\sigma^2(s^2)$	0.2531	0.1491	0.1065	0.1478	0.1667	0.2003	0.2842	0.4683	0.1982	0.1478
$\sigma^2(s^3)$	0.4489	0.4363	0.1599	0.3224	0.3099	0.2974	0.5075	0.8822	0.2577	0.2225

Table 4
Optimal root portfolios under mean-wCVaR, wrcVaR and mwCVaR models.

	DIS	DOW	ED	GE	IBM	MRK	MRO	MSI	PEP	JNJ
$u^*_{wCVaR}(s_0)$	0.0000	0.0000	0.3000	0.0000	0.3000	0.0000	0.0000	0.0000	0.2995	0.1005
$u^*_{wrCVaR}(s_0)$	0.0000	0.0000	0.3000	0.0000	0.3000	0.0000	0.0000	0.0000	0.1367	0.2633
$u^*_{mwCVaR}(s_0 = s^1)$	0.0000	0.0000	0.3000	0.0000	0.3000	0.0000	0.1385	0.0000	0.2615	0.0000
$u^*_{mwCVaR}(s_0 = s^2)$	0.0000	0.0000	0.3000	0.0000	0.3000	0.0000	0.0550	0.0000	0.3000	0.0450
$u^*_{mwCVaR}(s_0 = s^3)$	0.0000	0.0000	0.3000	0.0000	0.3000	0.0000	0.0000	0.0000	0.1492	0.2508

Table 5
Optimal portfolios at recourse periods under the mean-mwCVaR model when $s_0 = s^1$.

Node	Regime	DIS	DOW	ED	GE	IBM	MRK	MRO	MSI	PEP	JNJ
1	1	0.0000	0.0000	0.3000	0.0000	0.3000	0.0000	0.1461	0.0000	0.2553	0.0000
2	2	0.0000	0.0000	0.3000	0.0000	0.3000	0.0000	0.1243	0.0000	0.2771	0.0000
3	3	0.0000	0.0000	0.3000	0.0000	0.3000	0.0000	0.0000	0.0000	0.2615	0.1396
4	1	0.0000	0.0000	0.3000	0.0000	0.3000	0.0000	0.1553	0.0000	0.2476	0.0000
5	2	0.0000	0.0000	0.3000	0.0000	0.3000	0.0000	0.1461	0.0000	0.2568	0.0000
6	3	0.0000	0.0000	0.3000	0.0000	0.3000	0.0000	0.0000	0.0000	0.2553	0.1473
7	1	0.0000	0.0000	0.3000	0.0000	0.3000	0.0000	0.1258	0.0000	0.2771	0.0000
8	2	0.0000	0.0000	0.3000	0.0000	0.3000	0.0000	0.1243	0.0000	0.2785	0.0000
9	3	0.0000	0.0000	0.3000	0.0000	0.3000	0.0000	0.1243	0.0000	0.2771	0.0014
10	1	0.0000	0.0000	0.3000	0.0000	0.3000	0.0000	0.0014	0.0000	0.2615	0.1396
11	2	0.0000	0.0000	0.3000	0.0000	0.3000	0.0000	0.0000	0.0000	0.2629	0.1396
12	3	0.0000	0.0000	0.3000	0.0000	0.3000	0.0000	0.0000	0.0000	0.2615	0.1410

performed on a Lenovo PC with 2.98 gigabytes RAM and 2.93 gigahertz Dual Core CPU.

By setting the current regime (at node 0 in Fig. 1) to the bull, bear and consolidation regime in succession, we can find three groups of optimal portfolios under the mean-wCVaR, mean-wrCVaR and mean-mwCVaR models, respectively. The solution times for the three resulting SOCP problems are 0.44 seconds, 0.55 seconds and 0.50 seconds, respectively.

For the multi-period portfolio selection problem, the optimal portfolio at the current period is most important, because it will be applied immediately. Due to the definitions of wCVaR and wrCVaR, the optimal root portfolios under the mean-wCVaR or mean-wrCVaR model are the same whatever the current market regime is, bull, consolidation or bear. They are shown in the first and second rows of Table 4. Because of the definition of mwCVaR, the optimal root portfolios under the mean-mwCVaR model could vary with the current market regime. The resulting portfolios are shown in the third to fifth rows of Table 4. Furthermore, we show in Table 5 the optimal portfolios at recourse periods for the mean-mwCVaR model when the current regime is set to the bull regime, as an illustration to show the variation of optimal portfolios among periods.

We can derive at least the following three observations from the optimal portfolios in Tables 4 and 5.

Firstly, all the optimal portfolios determined under the three models allocate the available wealth among ED, IBM, MRO, PEP and JNJ. We know from Tables 2 and 3 that these five assets provide high return rates with small variances. As risk-averse investment models, it is rather natural that the optimal portfolios always choose assets which generate stable returns with small uncertainty.

Secondly, besides ED and IBM in which both the mean-wCVaR and mean-wrCVaR models invest with the maximum quota, the mean-wCVaR model invests most in PEP and the mean-wrCVaR model invests most in JNJ at the current period. From Tables 2 and 3, we know that PEP performs better (bad) in bull and consolidation regimes (in bear regime) than JNJ. This demonstrates that the mean-wrCVaR model pays much attention to the risk under the worst regime, while the mean-wCVaR model controls the risks under all three regimes simultaneously. Hence, the mean-wrCVaR model is more robust than the mean-wCVaR model in controlling extreme losses.

Thirdly, due to the properties of proposed multi-period risk measures, the mean-wCVaR (mean-wrCVaR) model generates the same optimal root portfolio at the current period no matter what the current market regime is; the mean-mwCVaR model provides us with three optimal root portfolios at the current period with respect to different regimes. Therefore, the optimal root portfolio under the mean-mwCVaR model is adaptively determined according to the present market regime, which is more reasonable. When the current regime is the bear regime, the optimal root portfolio obtained under the mean-mwCVaR model is similar to that obtained under the mean-wrCVaR model; when the current regime

Table 6
Statistics of out-of-sample return series got under three models.

Model	mean-wCVaR	mean-wrCVaR	mean-mwCVaR
Maximum (%)	1.1020	1.0683	1.2713
Minimum (%)	−1.4588	−1.4586	−1.2030
Mean (%)	0.1229	0.1234	0.1627
Variance (×10e−4)	0.2639	0.2688	0.2957
Skewness	−0.4449	−0.4343	−0.1873

is the consolidation regime, the root optimal portfolio obtained under the mean-mwCVaR model is similar to that portfolio got under the mean-wCVaR model; and when the current regime is the bull regime, the optimal root portfolio obtained under the mean-mwCVaR model invests more in MRO and PEP than those obtained with other two models. The change of the optimal root portfolio obtained under the mean-mwCVaR model with respect to a specific market regime is because the estimation of mwCVaR relies on the regime appearing probability, while wrCVaR only focuses on the risk under the worst regime and wCVaR views the market as a whole. Besides the optimal root portfolio at the current period, the optimal portfolios at recourse periods obtained under the mean-mwCVaR model also vary with respect to different regimes in a similar way, which can be observed from Table 5.

The optimal portfolios at recourse periods obtained under the mean-wCVaR or wrCVaR model are very similar to that at the current period, they do not change with respect to different regimes. Hence, we do not show them in detail, which can be provided upon requirement.

5.3. Out-of-sample performance tests

We now investigate the robustness of the mean-wCVaR, mean-wrCVaR and mean-mwCVaR models by examining the out-of-sample performance of the optimal root portfolio at the current period. We carry out the out-of-sample test in a rolling forward way. That is, we use the data in the in-sample period to determine the three optimal portfolios corresponding to the three models, respectively, just like what we did in the above analysis. We invest with the optimal root portfolio at the current period for the next one week starting from March 1, 2010, and compute the return rates of the three optimal portfolios in the first week of the out-of-sample period with the actual return data in that week. Then, we move the in-sample period one week forward by adding the new week and deleting the first week, re-solve the resulting three portfolio selection problems, find the optimal portfolios, and compute their return rates in the second week of the out-of-sample period, and so on. We carry out the out-of-sample test by rolling forward weekly until January 30, 2012, this provides us with three return rate series with 100 out-of-sample weekly return rates, corresponding to the mean-wCVaR, mean-wrCVaR and mean-mwCVaR models, respectively. All these optimization problems are solved within 0.6 seconds.

Table 6 shows typical statistic characteristics, including the maximum, the minimum, the mean, the variance and the skewness, of these three return series.

We can see from Table 6 that the optimal root portfolios got under the mean-wCVaR and mean-wrCVaR models have similar performance, while the optimal root portfolio got under the mean-mwCVaR model provides much higher return rate than the other two in terms of the maximum and mean. Although the variance of the return series obtained with the mean-mwCVaR model is a bit larger than those obtained with the other two models, the distribution of the out-of-sample return rates got with the mean-mwCVaR model is less left-skewed, as the value of skewness shows, and it has a smaller extreme loss. In modern risk management, controlling extreme losses is as important as earning high returns. Hence, the above out-of-sample result illustrates that the mean-mwCVaR model is more suitable for the investor to earn robust and superior profits, while the mean-wCVaR and mean-wrCVaR models are more suitable for the investor to reduce the investment volatility.

Table 7
Statistics of out-of-sample return rates under different regimes and different models.

Model	Regime	Bull	Consolidation	Bear
	Weight (weeks)	69	6	25
mean-wCVaR	Mean (%)	0.1421	0.2729	0.0339
	Variance (×10e−4)	0.2455	0.3133	0.3129
mean-wrCVaR	Mean (%)	0.1370	0.2401	0.0579
	Variance (×10e−4)	0.2542	0.3230	0.3129
mean-mwCVaR	Mean (%)	0.1938	0.2588	0.0535
	Variance (×10e−4)	0.2902	0.3421	0.3087

To illustrate the above results more intuitively, we compute the out-of-sample accumulative wealth processes under the obtained three return rate series and draw them in Fig. 2, which again show the significantly superior out-of-sample performance of the mean-mwCVaR model.

In order to examine the out-of-sample performances of the three models under different regimes, we divide the out-of-sample period into three parts with respect to the three regimes. And we estimate the mean and variance of the return rates in each of three parts. The results are listed in Table 7. We can see from this table that, under the consolidation regime, the performances of the optimal root portfolios got under the three models are similar. Under the bear regime, the optimal root portfolio obtained under the mean-wrCVaR model or mean-mwCVaR model performs better than that under the mean-wCVaR model. That is because the two models take the regime switching information into consideration and control the risk under the worst regime. Under the bull regime, the optimal root portfolio obtained under the mean-mwCVaR model performs much better than those got under the other two models. That is because the mean-mwCVaR model takes all the information under the three regimes into consideration, while the mean-wrCVaR model only controls the risk under the worst regime. From the above results, we can conclude that, the mean-wrCVaR model can provide more robust optimal portfolio than the mean-wCVaR model, while the mean-mwCVaR model can find regime dependent optimal portfolios which are suitable for different market environments. Both mean-wrCVaR and mean-mwCVaR models are good supplements to the mean-wCVaR model.

5.4. Comparison with typical benchmark models

In order to confirm the conclusions derived from the above last two sections and to show the superiority of our models, we further compare the out-of-sample performances of our models with the following three typical investment models:

(1) The distributionally robust portfolio selection model with a Wasserstein distance based uncertainty set under high ambiguity. Pflug, Pichler, and Wozabal (2012) have shown that, for this problem, the optimal policy is the $1/N$ investment strategy, i.e., investing in all risky assets with equal proportions.
(2) The dynamic MV model with risk aversion parameter being 2, proposed in Li and Ng (2000). The closed-form optimal solution can be derived by the conclusion in Section 3 of Li and Ng (2000).
(3) The multistage portfolio selection model with the robust second order lower partial moment as the risk measure

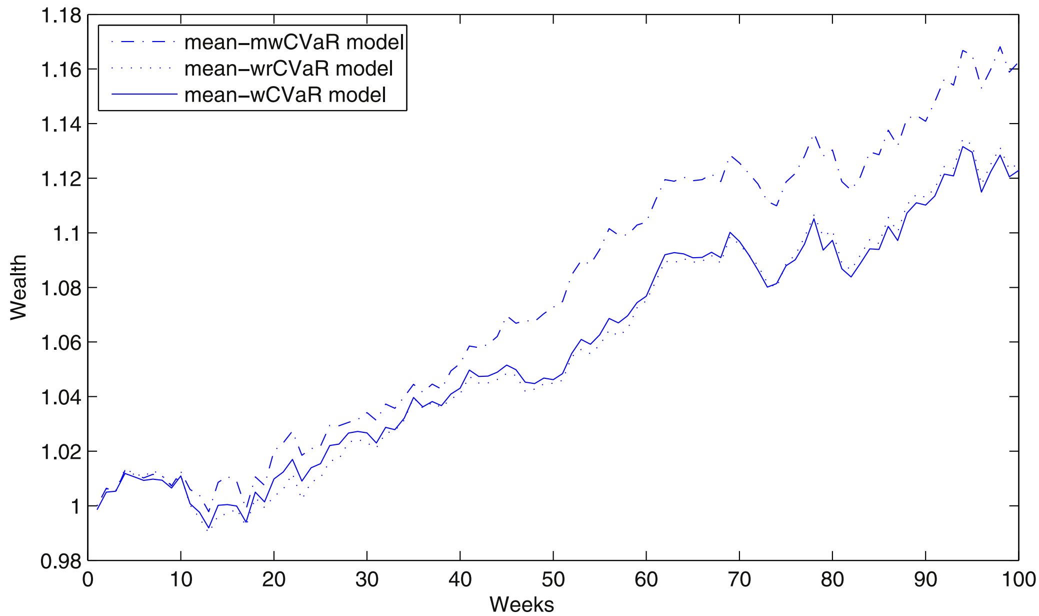

Fig. 2. The accumulative wealth series during 100 out-of-sample weeks.

Table 8
Statistics of out-of-sample return series in year 1994–1999.

Model	m-mwCVaR	m-wCVaR	m-wrCVaR	1/*N* strategy	LPM	MV
Maximum (%)	2.4191	2.8376	2.3679	3.2730	2.0500	5.9e+07
Minimum (%)	−2.4618	−2.3626	−2.3771	−2.3787	−2.2000	−5.5e+07
Mean (%)	0.1857	0.1711	0.1552	0.1679	0.1595	6.1e+05
Variance (×10e−4)	0.7460	0.7815	0.7978	0.6308	0.5922	2.4e+14
Skewness	−0.0666	0.0545	−0.0340	0.0712	−0.0900	0.3369

Table 9
Statistics of out-of-sample return series in year 2000–2005.

Model	m-mwCVaR	m-wCVaR	m-wrCVaR	1/*N* strategy	LPM	MV
Maximum (%)	3.2562	2.5002	3.8225	2.8235	2.8100	2.2e+07
Minimum (%)	−4.1175	−3.5762	−4.4372	−3.7912	−3.2000	−2.7e+07
Mean (%)	0.0868	0.0832	0.0893	0.0239	0.0501	2.0e+05
Variance (×10e−4)	0.8369	0.8012	0.8079	0.8167	0.6483	3.0e+13
Skewness	−0.5616	−0.4542	−0.2492	−0.5836	−0.4455	−0.0519

(LPM2 model for short), proposed by Chen et al. (2011). Here, we set the benchmark return to 0.5. The closed-form optimal solution can be derived by Theorem 3.1 in Chen et al. (2011).

For the above three models and the mean-wCVaR, mean-wrCVaR and mean-mwCVaR models, we set $T = 6$. We adopt the same stock pool and use the same test method as that in the previous section.

We carry out three groups of out-of-sample tests: in test 1, the in-sample period is from February 22, 1977 to November 27, 1993, and the out-of-sample period is from January 3, 1994 to November 27, 1999; in test 2, the in-sample period is from February 22, 1977 to November 27, 1999, and the out-of-sample period is from January 3, 2000 to November 26, 2005; in test 3, the in-sample period is from February 22, 1977 to November 26, 2005, and the out-of-sample period is from January 2, 2006 to January 30, 2012. Each of these out-of-sample periods is long enough to cover a short business cycle, thus different kinds of market regimes (Kitchin, 1923). For each test, we can get six out-of-sample return rate series corresponding to the six portfolio selection models. We compute the out-of-sample return rate series by a rolling forward method. Corresponding to three tests, Tables 8–10 show basic statistics, including the maximum, the minimum, the mean, the variance and the skewness, of the six return series.

Moreover, we merge the above three out-of-sample periods into a whole one, from January 3, 1994 to January 30, 2012, and then divide it into three parts according to the three regimes. We compute basic statistics of the return rates in each of the three parts. The results are listed in Tables 11–13.

We can observe from these six tables the following conclusions:

Most of the observations we obtain from Tables 6 and 7 still hold for the results of the mean-wCVaR, mean-wrCVaR and mean-mwCVaR models in Tables 8–13.

The dynamic MV model performs poorly in terms of risk control. One reason is that it has no robust control counterpart or the no-shorting constraints in the model. The other reason is that MV based models are extremely sensitive to the input parameters, the mean vector and the covariance matrix (Chopra & Ziemba, 1993; Kuhn, Parpas, Rustem, & Fonseca, 2009). When the assets' returns are highly correlated but their average returns are significantly different, the optimal portfolio becomes very aggressive, which may lead to extremely large loss.

The return variances of the optimal portfolios got under all the five robust portfolio selection models are less than one third of the return variance of the stocks. Hence, the robust mod-

Table 10
Statistics of out-of-sample return series in year 2006–2012.

Model	m-mwCVaR	m-wCVaR	m-wrCVaR	1/N strategy	LPM	MV
Maximum (%)	2.1493	2.7412	2.0925	4.6838	3.0500	3.9e+06
Minimum (%)	−5.1102	−5.8789	−5.1102	−8.0318	−6.1000	−4.3e+06
Mean (%)	0.0902	0.0624	0.0776	0.0285	0.0448	−3.1e+04
Variance (×10e−4)	0.6056	0.6483	0.5522	1.4847	0.6831	1.1e+12
Skewness	−1.1854	−1.5591	−1.4629	−0.9404	−1.6197	0.1867

Table 11
Statistics of out-of-sample return series under bull market in year 1994–2012.

Model	m-mwCVaR	m-wCVaR	m-wrCVaR	1/N strategy	LPM	MV
Maximum (%)	2.5547	2.8376	3.0697	4.6838	2.8100	5.9e+07
Minimum (%)	−2.9884	−2.7795	−2.7802	−3.2271	−2.3500	−5.5e+07
Mean (%)	0.1517	0.1418	0.1280	0.1560	0.1308	−3.2e+04
Variance (×10e−4)	0.6397	0.6154	0.6328	0.7284	0.5038	1.1e+14
Skewness	−0.2499	−0.1684	−0.1471	0.0648	−0.2200	0.0906

Table 12
Statistics of out-of-sample return series under consolidation market in year 1994–2012.

Model	m-mwCVaR	m-wCVaR	m-wrCVaR	1/N strategy	LPM	MV
Maximum (%)	2.1810	1.8163	1.8674	3.2730	1.8300	5.6e+07
Minimum (%)	−1.2063	−1.2063	−1.2063	−3.1567	−1.3200	−2.4e+07
Mean (%)	0.2186	0.2499	0.1912	0.0387	0.1367	7.8e+05
Variance (×10e−4)	0.6023	0.5206	0.4975	0.8877	0.4398	1.1e+14
Skewness	0.5256	0.3104	0.4616	−0.0715	0.3422	3.0879

Table 13
Statistics of out-of-sample return series under bear market in year 1994–2012.

Model	m-mwCVaR	m-wCVaR	m-wrCVaR	1/N strategy	LPM	MV
Maximum (%)	3.2562	2.7412	3.8225	4.0513	3.0500	5.1e+07
Minimum (%)	−5.1102	−5.8789	−5.1102	−8.0318	−6.1000	−3.9e+07
Mean (%)	0.0012	−0.0321	0.0140	−0.1191	−0.0498	5.7e+05
Variance (×10e−4)	0.9734	1.0685	0.9679	1.4247	0.9865	6.8e+13
Skewness	−0.8513	−0.8044	−0.7064	−1.2410	−1.0227	0.7279

els are efficient for controlling the volatility of the investment results.

The return means of the optimal portfolios got under the two benchmark robust models, LPM2 and 1/N, are larger than those of the mean-wrCVaR, mean-wCVaR and mean-mwCVaR models during the years 1994–1999, but less than those of the three models after year 2000. From the historical prices of S&P 500 index, we can find that the U.S. stock market was in bull during the years 1994–1999, it increased from 470 points to almost 1500 points. However, during the years 2000–2014, the stock index seesawed between 680 points and 1500 points and suffered two crashes in years 2000–2002 and years 2007–2009, respectively. From Tables 11–13, we can further find that the five robust models (except for MV) perform similarly in the bull market regime. But during the consolidation market regime or the bear market regime, the proposed mean-wrCVaR and mean-mwCVaR models perform better in terms of the portfolio return means. Especially, the mean-wrCVaR and mean-mwCVaR models are the only two models which have positive return means in all the three market regimes. These results illustrate the superiority of the mean-wrCVaR and mean-mwCVaR models in resisting stock market crashes.

The 1/N investment strategy and the LPM2 model suffer the largest and the second largest loss in all the out-of-sample tests. Meanwhile, the 1/N investment strategy and the mean-wrCVaR model gain the largest and the second largest return rate.

Lastly, the performance, in terms of the variance and the skewness, of the 1/N investment strategy varies significantly in different time periods. While, the performances of the mean-wrCVaR, mean-wCVaR, mean-mwCVaR and LPM2 models are more or less stable in all the time periods. Especially, the mean-wrCVaR and mean-mwCVaR models perform most stably in all the time periods and in all the market regimes.

In a word, comparing with the three benchmark investment models, the mean-wrCVaR and mean-mwCVaR models perform best overall. They can earn higher excess returns and can more efficiently resist market crashes in the out-of-sample tests.

6. Conclusions

In this paper, we apply the regime switching technique to robust optimization problems and propose two forms of multi-period robust risk measures with regime switching. The corresponding multi-period portfolio selection models under proposed risk measures are established, and their efficient solution methods are investigated. We demonstrate the flexibility of our dynamic uncertainty sets under the regime-switching framework by further considering multi-period robust risk measures with moments uncertainty and their application to portfolio selection. A series of empirical results demonstrate the practicality, efficiency and robustness of our new robust multi-period investment models with regime switching.

In this paper, we have specifically examined the wCVaR, wrCVaR and mwCVaR risk measures and their application to multi-period portfolio selection problems, mainly due to the good property of CVaR and the tractability of resulting portfolio selection problems. An interesting topic for future research is to apply the introduced multi-period robust risk measures with other forms of single-period risk mappings, such as LPM, two-sided coherent risk measure, and consider their application to multi-period portfolio selection problems.

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