The Out-of-sample Performance of Robust Portfolio Optimization

André Alves Portela Santos*

Abstract

Robust optimization has been receiving increased attention in the recent few years due to the possibility of considering the problem of estimation error in the portfolio optimization problem. A question addressed so far by very few works is whether this approach is able to outperform traditional portfolio optimization techniques in terms of out-of-sample performance. Moreover, it is important to know whether this approach is able to deliver stable portfolio compositions over time, thus reducing management costs and facilitating practical implementation. We provide empirical evidence by assessing the out-of-sample performance and the stability of optimal portfolio compositions obtained with robust optimization and with traditional optimization techniques. The results indicated that, for simulated data, robust optimization performed better (both in terms of Sharpe ratios and portfolio turnover) than Markowitz's mean-variance portfolios and similarly to minimum-variance portfolios. The results for real market data indicated that the differences in risk-adjusted performance were not statistically different, but the portfolio compositions associated to robust optimization were more stable over time than traditional portfolio selection techniques.

***Keywords*: ** estimation error; Sharpe ratio; portfolio turnover; mean-variance; minimum variance.

JEL codes: G11.

Resumo

A otimização robusta de carteiras tem recebido grande interesse nos últimos anos devido a possibilidade de considerar o erro de estimação dentro do problema da seleção de carteiras. Uma questão analisada por poucos estudos é se esta nova abordagem é capaz de gerar uma performance fora-da-amostra superior aos métodos tradicionais. Além disso, é importante saber esta abordagem é capaz de gerar carteiras mais estáveis ao longo do tempo, contribuindo assim para reduzir os custos de administração e facilitando a implementação na prática. Este estudo analisa a performance fora-da-amostra e a estabilidade das composições ótimas das carteiras obtidas com otimização robusta e com métodos tradicionais. Os resultados indicam que, para dados simulados, a otimização robusta obtém uma performance superior aos métodos tradicionais tanto em termos de índice de Sharpe como em termos de portfolio turnover. Os resultados para dados reais de indicam que a performance ajustada ao risco foi estatisticamente equivalente, entretanto as composições ótimas foram mais estáveis do que as obtidas através de métodos tradicionais de otimização.

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*Department of Statistics, Universidad Carlos III de Madrid, Madrid, Espanha.
E-mail: andre.alves@uc3m.es

1. Introduction

The portfolio optimization approach proposed by Markowitz (1952) is undoubtedly one of the most important models in financial portfolio selection. The idea behind Markowitz's work (hereafter mean-variance optimization) is that individuals will decide their portfolio allocation based on the fundamental trade-off between expected return and risk. Under this framework, individuals will hold portfolios located in the efficient frontier, which defines the set of Pareto-efficient portfolios. This set of optimal portfolios is usually described using a two-dimensional graph that plots their expected return and standard deviations. Therefore, an efficient portfolio is the one that maximizes the expected return for a desired level of risk, which is usually understood as standard deviation.

In order to implement the mean-variance optimization in practice, one needs to estimate means and covariances of asset returns and then plug these estimators into an analytical or numerical solution to the investor's optimization problem. This leads to an important drawback in the mean-variance approach: the estimation error. Since means and covariances are sample estimates, one should always expect some degree of estimation error. Nevertheless, it interesting to note that under the hypothesis of normality the sample estimates of means and covariances are maximum likelihood estimates (MLE), which are the most efficient estimators for the assumed distribution. Therefore, as DeMiguel and Nogales (2009) argue, if those estimates are the most efficient ones, where does the estimation error come from? The answer is that the performance of MLE based on normality assumptions is highly sensitive to deviations of the empirical or sample distribution from the assumed normal distribution. Taking into account that stock returns usually violate the normality assumption, we should expect the estimation error to affect the performance of optimization techniques that rely on sample estimates. In fact, it is well known in the financial literature that the mean-variance optimization suffers from the problem of estimation error, since it uses estimated means and covariances as inputs. Michaud (1989), for instance, refers to the traditional mean-variance approach as an "error-maximization" approach.

An issue closely related to the problem of estimation error in the mean-variance framework is the sensitivity to small changes in the means of the individual assets. For instance, Best and Grauer (1991) found that for a 100-asset portfolio the elasticities of the portfolio weights were on the order of 14,000 times the magnitude of the average elasticity of any of the portfolio returns. As a consequence, mean-variance portfolios usually display radical changes in their compositions within a certain time period. This high portfolio turnover increases management costs and makes difficult the practical implementation of the strategy. In this sense, the stability of the portfolio composition is an important question that, together with performance, should to be taken into account when evaluating a portfolio selection strategy.

The literature on asset allocation models has evolved towards numerous extensions of the mean-variance paradigm, in both model formulation and econometric estimation mainly designed to reduce the effect of estimation error. Jagannathan and Ma (2003) propose a minimum-variance portfolio with a short selling restriction. They claim that, since the estimation errors in the means are much larger than the estimation errors in covariances, the minimum-variance portfolio weights should be more stable than the traditional mean-variance portfolio weights. Another common approach is the James-Stein shrinkage estimator (Jobson and Korkie, 1981, Jorion, 1986). This estimator "shrinks" the sample means toward a common value, which is often chosen to be the grand mean across all variables. Therefore, the estimation errors that may occur in the cross-section of individual means might be reduced, resulting in a lower overall variance of the estimators. Optimality results regarding the shrinkage estimator for the sample covariance matrix are obtained in Ledoit and Wolf (2003), Ledoit and Wolf (2004a), and Ledoit and Wolf (2004b). Thus, the shrinkage estimator can be used in the plug-in procedure in order to find optimal portfolio weights with improved properties. More recently, in Mendes and Leal (2005) and DeMiguel and Nogales (2009) have also proposed the use of alternative robust estimators of risk in order to reduce estimation error.

Another approach able to consider the estimation error that has been receiving increased attention is the robust portfolio optimization (see Tütüncü and Köenig (2004) and Goldfarb and Iyengar (2003) for seminal references). According to Tütüncü and Köenig (2004) robust optimization is an emerging branch in the field of optimization that consists in finding solutions to optimization problems with uncertain input parameters. In the approach proposed by Tütüncü and Köenig (2004), uncertainty is described using an uncertainty set which includes all, or most, possible realizations of the uncertain input parameters. This yields a *worst-case optimization*, in the sense that for each choice of the decision variable (in this case the portfolio weights) it is considered the worst case realization of the data and evaluated the corresponding objective value, finally picking the set of values for the variables with best worst-case objective. However, as Ceria and Stubbs (2006) point out, this approach can be rather too conservative. The authors argue that if expected returns are expected to be symmetrically distributed around the estimated mean, one would expect that there are as many expected returns above the estimated as there are below the true value. Therefore, in order to alleviate the problem of an excessive conservative (pessimistic) view of expected returns, Ceria and Stubbs (2006) propose an adjustment to the traditional formulation of the robust portfolio in order to accommodate a less conservative view of expected returns. In their simulations, the portfolios constructed using robust optimization outperformed those created using traditional mean-variance optimization in the majority of cases.

The robust optimization approach, however, has been also subjected to criticisms. Scherer (2007), for instance, shows that the robust optimization approach of Tütüncü and Köenig (2004) is equivalent to a Bayesian shrinkage estimator and, therefore, offers no additional marginal value. Besides, the author argues that the parameter that controls the dimension of the uncertainty set is difficult to control/calibrate. In contrast to the results reported by Ceria and Stubbs (2006), Scherer (2007) found that robust optimization underperformed even simple mean-variance portfolios.

This paper aims at shedding light on the recent debate concerning the importance of the estimation error and weights stability in the portfolio allocation problem, and the potential benefits coming from robust portfolio optimization in comparison to traditional techniques. We will empirically compare two versions of robust portfolio optimization, the standard approach and the zero net alpha-adjusted robust optimization proposed by Ceria and Stubbs (2006) (hereafter adjusted robust optimization), with two well-established traditional techniques: Markowitz's mean-variance portfolio and minimum-variance portfolio. We will evaluate the out-of-sample performance of those portfolio allocation approaches according to the methodology of rolling horizon proposed in DeMiguel and Nogales (2009). 4 different data sets composed of US equity portfolios will be used. However, Scherer (2007) points out that a particular (real) sample path might have characteristics that put an unfair advantage to a particular method. Therefore, in order to have a perfect control of the data generating process we will also use simulated data in the comparison among portfolio allocation techniques.

The remainder of the paper is organized as follows. In the section 2 we explain the portfolio optimization techniques used in the work. Section 3 describes the data sets used as well as the methodology used in the out-of-sample evaluation. Section 4 brings the results and discussion. Finally, section 5 concludes.

2. Portfolio Optimization Methods

2.1 Traditional Markowitz approach

When computing optimal mean-variance portfolios, it is important to note that the choice of the desired risk premium depends on the investor's tolerance to risk. Risk-loving investors might be willing to accept a higher volatility in their portfolios in order to achieve a higher risk premium while risk-averse investors will prefer less volatile portfolios, therefore penalizing performance. To incorporate the investor's optimal trade-off between expected return and risk, consider N risky assets with random return vector R_{t+1} and a risk-free asset with known return R_t^f. Define the excess return $r_{t+1} = R_{t+1} - R_t^f$ and denote their conditional means (or risk premia) and covariance matrix by μ_t and Σ_t, respectively. Therefore the mean-variance problem can be formulated as,

$$\begin{aligned}&\min_x x'\Sigma x - \tfrac{1}{\lambda}E[r_{p,t+1}]\\ &\text{subjected to}\\ &\iota' x = 1\end{aligned} \tag{1}$$

where λ measures the investor's level of relative risk aversion, $x \in \Re^N$ is the vector of portfolio weights and ι is a vector of ones. One can also consider adding a no-short selling in this formulation, i.e., $x_i \geq 0$. We refer to this as a constrained policy in opposite to an unconstrained policy. The minimum-variance portfolio is the solution to the following optimization problem:

$$\begin{aligned} &\min_{x} \ x'\Sigma x \\ &\text{subjected to} \\ &\iota' x = 1 \end{aligned} \tag{2}$$

where, in this case, one can also consider adding a no-short selling constraint, yielding a constrained portfolio policy. Note that the portfolio selection problem in (2) has a closed form solution given by:

$$x^*_{\text{min-var}} = \frac{\Sigma^{-1}\iota}{\iota'\Sigma^{-1}\iota}. \tag{3}$$

It is also worth noting that the minimum-variance portfolio is the mean-variance portfolio corresponding to an infinite risk aversion parameter. Jagannathan and Ma (2003) point out that this portfolio has interesting properties since the estimation error of the covariances is smaller than the estimation error of the means. Moreover, the authors show that adding a no-short selling constraint in this formulation improves the stability of the weights. Finally, there is empirical evidence that shows the minimum-variance portfolio usually performs better out-of-sample than any other mean-variance portfolio even when Sharpe ratio or other performance measures related to both the mean and variance are used for the comparison (DeMiguel and Nogales, 2009).

2.2 Robust mean-variance portfolio optimization

A very active area of research in optimization is called robust portfolio optimization. This approach explicitly recognizes that the result of the estimation process is not a single-point estimate, but rather an uncertainty set, where the true mean and covariance matrix of asset returns lie with certain confidence; see, for instance, (Goldfarb and Iyengar, 2003, Tütüncü and Köenig, 2004, Garlappi et al., 2007). A robust portfolio is then one that optimizes the worst-case performance with respect to all possible values the mean and covariance matrix may take within their corresponding uncertainty sets.

A robust portfolio is achieved by optimizing with respect to the worst-case performance, but the uncertainty sets are obtained by traditional estimation procedures. That is, in robust optimization the uncertainty is not stochastic (no random variables), it is deterministic and based on sets (bounded and convex). It is a *worst-case scheme* but the associated problems can be solved in an efficient way: they are *tractable*.

A robust portfolio is then one that is designed to optimize the worst-case performance within the set of values for the mean and for the covariance matrix in the corresponding uncertainty set. We consider the case where Σ is known. The robust approach is:

$$\begin{aligned} &\text{minimize} \quad f(x) = \sup_{\omega \in \Omega} F(x, \omega) \\ &\text{subjected to } x \in \chi. \end{aligned} \tag{4}$$

For the portfolio selection problem, suppose that $\mu \sim N(\bar{\mu}, \Sigma)$ and for a know Σ we have the following ellipsoidal uncertainty set for the unknown μ:

$$\Omega = \left\{ \mu : \ (\mu - \bar{\mu})' \, \Sigma^{-1} \, (\mu - \bar{\mu}) \leq \kappa^2 \right\} \tag{5}$$

where the parameter κ defines the confidence region, i.e., $\kappa^2 = \chi^2_N(1-\alpha)/T$ and χ^2_N is is the inverse cumulative distribution function of the chi-squared distribution with N degrees of freedom.

The previous equation can be reformulated as $\beta^T \beta \leq \kappa^2$ where $\beta = \Sigma^{-1} \, (\mu - \bar{\mu})$. Therefore, we can express the robust approach applied to the portfolio selection problem as:

$$\begin{aligned} &\text{maximize}_x \quad \min_{\mu \in \Omega} \mu' x \\ &\text{subjected to} \quad \iota' x = 1. \end{aligned} \tag{6}$$

It is possible to provide an equivalent formulation of this problem with a linear objective function:

$$\begin{aligned} &\text{maximize}_x \quad t \\ &\text{subjected to} \quad \min \ \mu' x \geq t \\ &\qquad\qquad\qquad \beta' \beta \leq \kappa^2 \\ &\qquad\qquad\qquad \iota' x = 1. \end{aligned} \tag{7}$$

Straightforward manipulation shows that the constraint $\min \ \mu' x \geq t$ is equivalent to $\bar{\mu}' x + \beta' \Sigma^{1/2} x \geq t$. Moreover, since the first term of this expression is constant with respect to β, the optimization problem can be now written as

$$\begin{aligned} &\text{minimize}_\beta \quad \beta' \Sigma^{1/2} x \\ &\text{subjected to} \quad \beta' \beta \leq \kappa^2 \end{aligned} \tag{8}$$

and it can be shown by using Karush-Kuhn-Tucker (KKT) conditions that the optimal objective value is $-\kappa \left\| \Sigma^{1/2} x \right\|$. Ceria and Stubbs (2006) observe that this term is related to the estimation error and its inclusion in the objective function reduces the effect of estimation error on the optimal portfolio. Finally, the robust counterpart of the traditional mean-variance optimization problem is

$$\begin{aligned} &\text{maximize}_x \quad \bar{\mu}' x - \kappa \left\| \Sigma^{1/2} x \right\| \\ &\text{subjected to} \quad \iota' x = 1 \end{aligned} \tag{9}$$

which is refereed in the optimization literature as a Second Order Cone Programming (SOCP) problem. One important drawback in this approach refers to the choice of the parameter κ. Scherer (2007) points out that, so far, there is no way to consistency determine the value of this parameter; usually, this parameter is determined heuristically.

Ceria-Stubbs adjusted robust optimization

Ceria and Stubbs (2006) introduced what is called the zero net alpha-adjustment to the standard robust optimization. This adjustment is proposed in order to consider a less pessimistic view of expected returns. Specifically, it is assumed that there are as many realization of returns above their expected value as there are below their expected value (thus the name zero net alpha-adjustment). The way Ceria and Stubbs (2006) proposed to consider this assumption is to include the following restriction in the optimization problem:

$$\iota' D\left(r - \bar{r}\right) = 0 \tag{10}$$

where D is some symmetric invertible matrix. Assuming that $D = I$ will force the net adjustment of expected returns to be zero. If we want the expected returns to have a zero adjustment in the variance of returns, we set $D = \Sigma^{-1}$. Finally, if we want this zero adjustment to be in the standard deviation of returns, we set $D = L^{-1}$ where L comes from the Cholesky decomposition of the covariance matrix, i.e. $\Sigma = LL^{-1}$. Following the same notation of Ceria and Stubbs (2006), the adjusted robust maximization problem can now be written as:

$$\begin{array}{ll} \text{maximize} & \bar{r}'w - r'w \\ \text{subjected to} & \left(r - \bar{r}\right)' \Sigma^{-1} \left(r - \bar{r}\right) \leq \kappa^2 \\ & \iota' D\left(r - \bar{r}\right) = 0. \end{array} \tag{11}$$

It can be shown that the optimal solution to the problem in (11) is:

$$r'w = \bar{r}'w - \kappa \left\| \left(\Sigma - \frac{1}{\iota' D \Sigma D' \iota} \Sigma D' \iota \iota' D \Sigma \right) w \right\|. \tag{12}$$

Therefore, we can rewrite the optimization problem in (9) as

$$\begin{array}{ll} \text{maximize} & \bar{r}'w - \kappa \left\| \left(\Sigma - \frac{1}{\iota' D \Sigma D' \iota} \Sigma D' \iota \iota' D \Sigma \right) w \right\| \\ \text{subjected to} & \iota' x = 1. \end{array} \tag{13}$$

Again, a no-short selling constraint can be imposed in this optimization problem in order to obtain a constrained portfolio policy. The numerical experiments reported by Ceria and Stubbs (2006) indicated that the portfolio constructed using robust portfolio optimization outperformed those created using traditional mean-variance optimization in most of the analyzed cases.

3. Methodology

3.1 Description of the out-of-sample evaluation

In order to compare the performance of robust optimization approaches detailed in the previous section with traditional mean-variance and minimum-variance portfolios, we use a rolling horizon procedure similar as in DeMiguel and Nogales (2009). First, the sample estimates of mean returns and covariances are made using an estimation window of T=150 observations, which for monthly data corresponds to 12.5 years.[1] Two, using these samples estimates we compute the optimal portfolio policies according to each strategy (mean-variance, minimum-variance and robust). Three, we repeat this procedure for the next period, by including the data for the new date and dropping the data for the earliest period. We continue doing this until the end of the data set is reached. At the end of this process, we have generated $L - T$ portfolio weight vectors for each strategy, where L is the total number of observations in the data set. This procedure is repeated to each data set.

In the case of the traditional mean-variance optimization, we considered an investor with risk aversion parameter $\lambda = 1$[2]. In the case of the robust portfolio optimization, we followed Ceria and Stubbs (2006) and performed our simulations with four different values for the parameter κ: 1, 3, 5 and 7. Three different approaches for the adjustment matrix D will be applied (see equation 10) : $D = \Sigma^{-1}$ (inverse covariance), $D = I$ (identity) and $D = L^{-1}$ (Choleski decomposition of the covariance matrix).

The out-of-sample performance of each strategy is evaluated according to the following statistics: mean excess returns, variance, Sharpe ratio and portfolio turnover. Holding the portfolio w_t gives the out-of-sample excess return in period $t + 1$: $\hat{r}_{t+1} = w_t^T r_t$ where $\hat{r}_{t+1}$ is the return in excess to the risk-free rate. After computing the $L - T$ excess returns, the out-of-sample mean, variance, Sharpe ratio (SR), and portfolio turnover are:

[1] To the best of our knowledge, there is no common prescription for choosing the length of the estimation window used to compute sample estimates. An important trade off, however, is that an estimation window with few observations will lead to more estimation error in the sample moments, while an estimation window with many observations will capture less time variation in sample moments. The assessment of this trade off is beyond the scope of this paper. Therefore, we used a similar estimation window of $T = 150$ observations as in previous studies, such as DeMiguel and Nogales (2009).

[2] We performed simulations using other values for the risk aversion parameter, obtaining similar results. Therefore, we decided to keep this parameter equal to one.

$$\begin{aligned}
&\hat{\mu} = \tfrac{1}{L-T} \sum_{t=T}^{L-1} w_t' r_{t+1} \\
&\hat{\sigma}^2 = \tfrac{1}{L-T-1} \sum_{t=T}^{L-1} (w_t' r_{t+1} - \hat{\mu})^2 \\
&\widehat{SR} = \tfrac{\hat{\mu}}{\hat{\sigma}} \\
&\text{Turnover} = \tfrac{1}{L-T-1} \sum_{t=T}^{L-1} \sum_{j=1}^{N} (|w_{j,t+1} - w_{j,t}|)
\end{aligned} \tag{14}$$

where $w_{j,t}$ is the portfolio weight in asset j at time $t+1$ but *before* rebalancing and $w_{j,t+1}$ is the desired portfolio weight in asset j at time $t+1$. Therefore, the portfolio turnover is a measure of the variability in the portfolio holdings and can indirectly indicate the magnitude if the transaction costs associated to each strategy. For instance, a portfolio turnover of 0.1 associated to some portfolio selection strategy indicates that, on average, the investor has to change 10% of his/her portfolio composition in each rebalancing date. Clearly, the smaller the turnover, the smaller the transaction costs associated to the implementation of the strategy. We also note that the portfolio turnover computed as in (14) takes into account not only changes in portfolio weights but also changes in asset prices.

In order to assess the statistical significance for the difference in Sharpe ratios among the methods employed in this study, we use a bootstrapping methodology proposed by Ledoit and Wolf (2008) which is designed for the case in which portfolio returns have fat tails and are of a time series nature. Thus, in order to test the hypothesis $H_0 : SR_{Portfolio} - SR_{Benchmark} = 0$ we compute a two-sided p-value using the studentized circular block bootstrap proposed in Ledoit and Wolf (2008) with B=1,000 bootstrap resamples and a block size equal to b=5. The p-values for the differences in Sharpe ratios will be computed for each strategy with respect to the one obtained under the mean-variance approach, which will be taken as a benchmark.

We provide further evidence of the weight stability by plotting the time-varying portfolio weights obtained under each strategy for the $L-T$ out-of-sample periods . Finally, we will focus our analysis only on constrained policies, i.e. with a no-short selling constraint in each optimization problem. All simulations were run on a Intel PC with 2.8GHz and 1Gb RAM. The Matlab system `CVX` for convex optimization (Grant and Boyd, 2008) was used in the implementation of some optimization problems. The total computational time was approximately 96 hours (4 days).

3.2 Data

Simulated data

In order to have a perfect control of the data generating process, we simulate several data sets with different size and statistical distributions. First, we simulate three data sets with $N = \{10, 50, 100\}$ assets from a multivariate normal distribution with annualized mean of 12% and standard deviation of 16%. Second, in order to capture stylized facts such as fat tails we simulate three data sets with $N = \{10, 50, 100\}$ assets from a multivariate Student's t distribution with 7 degrees of freedom. In this case, we assume that the correlation matrix among simulated assets is the identity matrix. Each simulated data set has a total of 203 observations. We also generate a (constant) risk free asset assuming an annualized mean of 6%. Finally, the evolution of the simulated asset returns is illustrated in Figure 1.

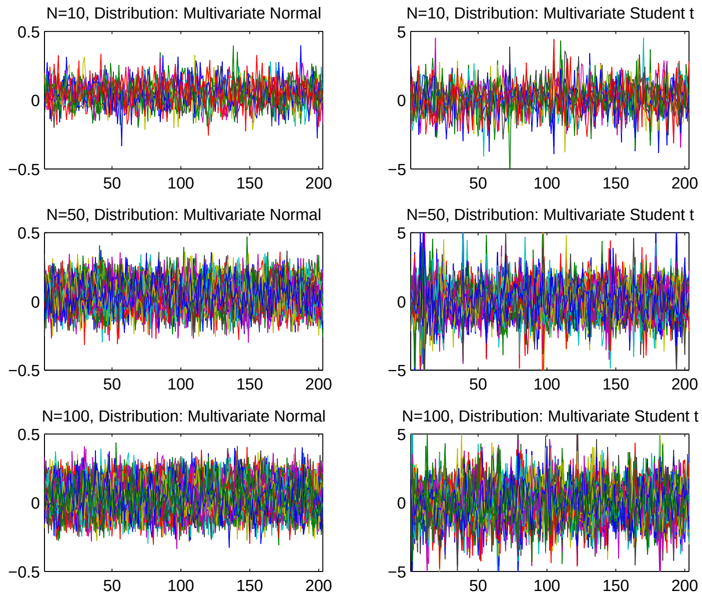

Figure 1
Data simulated from a multivariate normal distribution with annualized mean of 12% and standard deviation of 16% (left) and from a multivariate Student's t distribution with 7 degrees of freedom (right).

Real market data

Five portfolios-of-portfolios commonly used in the financial literature (for instance, Fama and French (1996) were employed in the empirical evaluation of the portfolio policies under consideration[3]. The data sets are:

- Fama-French 25 size and book-to-market sorted portfolios (FF25);
- Fama-French 100 size and book-to-market sorted portfolios (FF100);
- 38 industry portfolios (38IND);
- 5 industry portfolios (5IND);

Our sample goes from Jan. 1990 to Dec. 2006 (203 monthly observations). The period used to evaluate the portfolio optimization techniques according to methodology detailed in section 3 goes from Jul. 2002 to Dec. 2006 (53 monthly observations). The risk-free rate used to compute excess returns was the three month US T-Bills.

4. Results

4.1 Performance with simulated data

Tables 1 and 2 report the results of robust and traditional optimization techniques, respectively, when applied to the simulated data from a multivariate Normal distribution. We can check that all robust specifications delivered Sharpe ratios statistically higher than than the one obtained with the mean-variance portfolio policy, and similar to the one obtained minimum-variance portfolio policy regardless the size of the data sets. The performance of robust methods, however, significantly improved over traditional approaches as the size of the simulated portfolios increased. For instance, for the case of the simulated portfolio with $N = 100$ assets, the Sharpe ratio of the adjusted robust optimization with $D = L^{-1}$ and kappa=3 was 2.187 whereas the Sharpe ratio for the mean-variance and minimum-variance policies were 0.352 and 1.714, respectively. Moreover, the portfolio turnover of the robust methods was smaller than those of traditional approaches in the majority of the specifications. For instance, in the case of the simulated portfolio with $N = 100$ assets the portfolio turnover of the robust portfolio optimization with $D = L^{-1}$ and kappa=3 was 0.097 whereas the portfolio turnover of the mean-variance and minimum-variance policies was 0.244 and 0.105, respectively.

[3]All data sets were downloaded from the web site of Kenneth French (`http://mba.tuck.dartmouth.edu/pages/faculty/ken.french/data_library.html`)

Table 1
Out-of-sample performance (mean returns, variance of returns, portfolio turnover, Sharpe ratio (SR), and p-value for the difference in SR with respect to the mean-variance strategy) of robust portfolio optimization methods. Data simulated from a multivariate Normal distribution with annualized mean of 12%, standard deviation of 16%

	Robust optimization			Adjusted robust optimization								
				$D = \Sigma^{-1}$			$D = I$			$D = L^{-1}$		
	N=10	N=50	N=100	N=10	N=50	N=100	N=10	N=50	N=100	N=10	N=50	N=100
$kappa = 1$												
Mean	0.032	0.027	0.024	0.032	0.028	0.024	0.032	0.025	0.025	0.031	0.027	0.025
Variance	0.001	0.000	0.000	0.001	0.000	0.000	0.001	0.000	0.000	0.001	0.000	0.000
Turnover	0.027	0.080	0.123	0.018	0.064	0.111	0.001	0.001	0.001	0.013	0.040	0.122
SR	1.018	1.361	1.760	1.047	1.566	1.917	1.090	1.677	2.415	1.100	1.586	2.130
p-value	0.003	0.002	0.001	0.015	0.008	0.001	0.010	0.001	0.001	0.011	0.001	0.001
$kappa = 3$												
Mean	0.032	0.028	0.024	0.032	0.027	0.023	0.032	0.025	0.025	0.031	0.027	0.025
Variance	0.001	0.000	0.000	0.001	0.000	0.000	0.001	0.000	0.000	0.001	0.000	0.000
Turnover	0.018	0.062	0.109	0.018	0.064	0.108	0.001	0.001	0.001	0.013	0.039	0.097
SR	1.044	1.576	1.825	1.047	1.558	1.758	1.090	1.677	2.415	1.100	1.592	2.187
p-value	0.008	0.004	0.002	0.016	0.014	0.001	0.014	0.001	0.001	0.022	0.003	0.001
$kappa = 5$												
Mean	0.032	0.028	0.023	0.032	0.027	0.023	0.032	0.025	0.025	0.031	0.027	0.026
Variance	0.001	0.000	0.000	0.001	0.000	0.000	0.001	0.000	0.000	0.001	0.000	0.000
Turnover	0.018	0.062	0.109	0.018	0.064	0.106	0.001	0.001	0.001	0.013	0.039	0.094
SR	1.046	1.573	1.767	1.047	1.556	1.742	1.090	1.677	2.415	1.100	1.593	2.184
p-value	0.015	0.006	0.002	0.020	0.010	0.002	0.006	0.001	0.001	0.020	0.001	0.001
$kappa = 7$												
Mean	0.032	0.027	0.023	0.032	0.027	0.023	0.032	0.025	0.025	0.031	0.027	0.026
Variance	0.001	0.000	0.000	0.001	0.000	0.000	0.001	0.000	0.000	0.001	0.000	0.000
Turnover	0.018	0.063	0.108	0.018	0.064	0.106	0.001	0.001	0.001	0.013	0.039	0.093
SR	1.046	1.569	1.753	1.047	1.556	1.735	1.090	1.677	2.415	1.100	1.593	2.183
p-value	0.019	0.009	0.003	0.024	0.014	0.003	0.009	0.001	0.001	0.010	0.001	0.001

Table 2
Out-of-sample performance (mean returns, variance of returns, portfolio turnover, Sharpe ratio (SR), and p-value for the difference in SR with respect to the mean-variance strategy) of traditional portfolio optimization methods. Data simulated from a multivariate Normal distribution with annualized mean of 12%, standard deviation of 16%

	Mean-Var	Min-Var
$N = 10$		
Mean	0.034	0.032
Variance	0.004	0.001
Turnover	0.118	0.018
SR	0.523	1.047
p-value	1.000	0.012
$N = 50$		
Mean	0.021	0.027
Variance	0.004	0.000
Turnover	0.198	0.064
SR	0.335	1.554
p-value	1.000	0.013
$N = 100$		
Mean	0.020	0.023
Variance	0.003	0.000
Turnover	0.244	0.105
SR	0.352	1.714
p-value	1.000	0.006

Tables 3 and 4 report the performance of the robust and traditional portfolio optimization methods when applied to the data simulated from a Student's t distribution. It is worth highlighting four main results. First, the risk-adjusted performance of all methods got worse in comparison to the results previously obtained with the simulated data from a Normal distribution. This result is expected since all portfolio optimization methods considered in this paper depend on the use of the same critical input – the sample covariance matrix –, which is the MLE under the assumption of normality. Considering that the simulated data comes from a Student's t distribution, the sample estimate is no longer a MLE and will carry more estimation error. Second, the robust methods performed worse than traditional methods in some specifications, specially when the size of the simulated data set was small ($N = 10$). This result is mainly due to the fact that the theory behind the robust portfolio optimization is based on uncertainty ellipsoidal sets of a multivariate Normal distribution. The simulated data, however, does not come from a Normal distribution, which explains why the risk-adjusted performance of the robust methods was worse than the one of traditional methods in some of the cases. Third, the difference in risk-adjusted performance among robust and traditional methods vanished as the size of the simulated data sets increased. In fact, the performance of robust methods was better than that of traditional methods in some of the cases. For instance, when $N = 100$, the Sharpe ratio of the adjusted robust optimization with $D = I$ was 0.408 whereas the Sharpe ratio of the mean-variance and minimum-variance methods was 0.173 and 0.326, respectively. This result suggests that the robust methods are more prone to treat the estimation error in large systems, even if the data has departures from normality. Fourth, the portfolio turnover obtained with robust methods was smaller than that of traditional methods in the vast majority of the specifications, regardless the size of the data sets.

Table 3
Out-of-sample performance (mean returns, variance of returns, portfolio turnover, Sharpe ratio (SR), and p-value for the difference in SR with respect to the mean-variance strategy) of robust portfolio optimization methods. Data simulated from a multivariate Student's t distribution with 7 degrees of freedom

	Robust Optimization			Adjusted Robust Optimization								
				$D=\Sigma^{-1}$			$D=I$			$D=L^{-1}$		
	N=10	N=50	N=100	N=10	N=50	N=100	N=10	N=50	N=100	N=10	N=50	N=100
$kappa=1$												
Mean	0.075	0.004	0.039	0.046	0.011	0.045	0.069	0.033	0.045	0.058	0.016	0.043
Variance	0.187	0.043	0.034	0.183	0.042	0.033	0.179	0.027	0.012	0.178	0.035	0.033
Turnover	0.037	0.090	0.120	0.030	0.065	0.118	0.008	0.008	0.009	0.024	0.044	0.129
SR	0.174	0.017	0.209	0.108	0.052	0.247	0.162	0.201	0.408	0.138	0.085	0.235
p-value	0.151	0.991	0.744	0.070	0.781	0.561	0.435	0.215	0.102	0.209	0.582	0.648
$kappa=3$												
Mean	0.056	0.008	0.056	0.046	0.013	0.060	0.069	0.033	0.045	0.058	0.018	0.041
Variance	0.182	0.041	0.036	0.183	0.043	0.037	0.179	0.027	0.012	0.178	0.036	0.032
Turnover	0.030	0.069	0.108	0.030	0.063	0.109	0.008	0.008	0.009	0.024	0.042	0.120
SR	0.130	0.038	0.294	0.108	0.065	0.312	0.162	0.201	0.408	0.138	0.097	0.229
p-value	0.105	0.851	0.459	0.081	0.741	0.477	0.435	0.205	0.102	0.201	0.572	0.747
$kappa=5$												
Mean	0.052	0.010	0.059	0.046	0.014	0.062	0.069	0.033	0.045	0.058	0.019	0.041
Variance	0.182	0.042	0.037	0.183	0.043	0.038	0.179	0.027	0.012	0.178	0.036	0.032
Turnover	0.029	0.065	0.110	0.030	0.063	0.112	0.008	0.008	0.009	0.024	0.042	0.122
SR	0.121	0.050	0.309	0.108	0.066	0.318	0.162	0.201	0.408	0.138	0.099	0.226
p-value	0.103	0.825	0.462	0.097	0.734	0.472	0.435	0.205	0.102	0.248	0.552	0.758
$kappa=7$												
Mean	0.050	0.011	0.061	0.046	0.014	0.063	0.069	0.033	0.045	0.058	0.019	0.040
Variance	0.182	0.042	0.038	0.183	0.043	0.039	0.179	0.027	0.012	0.178	0.036	0.033
Turnover	0.029	0.064	0.111	0.030	0.063	0.112	0.008	0.008	0.009	0.024	0.042	0.122
SR	0.117	0.056	0.313	0.108	0.067	0.321	0.162	0.201	0.408	0.138	0.099	0.223
p-value	0.103	0.780	0.459	0.087	0.720	0.446	0.435	0.205	0.102	0.215	0.556	0.801

Table 4
Out-of-sample performance (mean returns, variance of returns, portfolio turnover, Sharpe ratio (SR), and p-value for the difference in SR with respect to the mean-variance strategy) of traditional portfolio optimization methods. Data simulated from a multivariate Student's t distribution with 7 degrees of freedom

	Mean-Var	Min-Var
$N=10$		
Mean	0.087	0.046
Variance	0.193	0.183
Turnover	0.042	0.030
SR	0.197	0.108
p-value	1.000	0.102
$N=50$		
Mean	0.004	0.014
Variance	0.057	0.043
Turnover	0.115	0.063
SR	0.018	0.069
p-value	1.000	0.714
$N=100$		
Mean	0.046	0.065
Variance	0.069	0.039
Turnover	0.150	0.114
SR	0.173	0.326
p-value	1.000	0.440

As an illustration, Figure 2 plots the time-varying portfolio weights for each optimization technique for the case of a simulated data with a multivariate Normal with $N = 10$ assets. We can check that this figure corroborates the main findings by showing the high instability associated to the time-varying compositions of mean-variance portfolios, in contrast to the relative stability in the composition of robust and minimum-variance optimized portfolios.

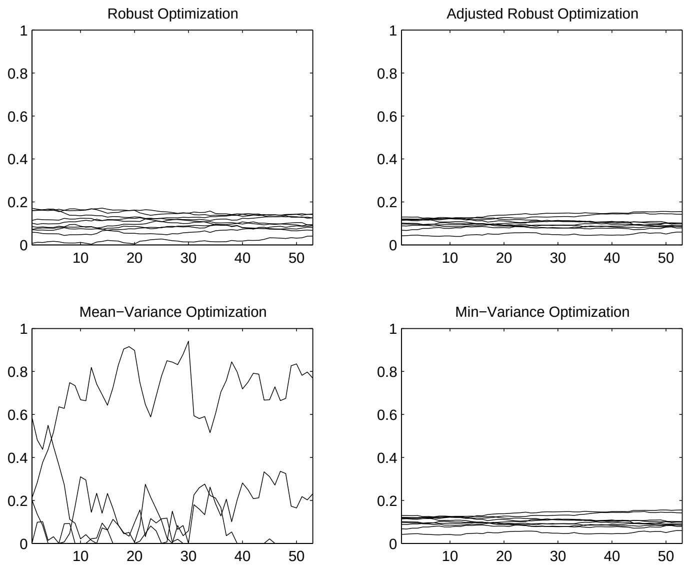

Figure 2
Time-varying portfolio weights for robust, adjusted robust, mean-variance and minimum-variance optimization techniques. Data simulated from a multivariate Normal distribution with annualized mean of 12%, standard deviation of 16% and 10 simulated assets

4.2 Performance with real market data

Tables 5 and 6 show the results for all portfolio optimization techniques when applied to the FF25 data set. In terms of Sharpe ratio, the best strategy was the traditional mean-variance (Sharpe ratio = 0.298). Among the robust portfolio optimization techniques, the standard approach of Tütüncü and Köenig (2004) performed slightly better than the adjusted approach of Ceria and Stubbs (2006) (Sharpe ratios of 0.277 and 0.271, respectively). Those values were obtained by setting $\kappa = 1$ and adjustment matrix $D = \Sigma^{-1}$. The p-values for the differences in Sharpe ratios indicated that none of the strategies yielded significantly higher Sharpe ratios. In terms of portfolio turnover, the strategy that yielded the smaller value was the adjusted robust optimization with adjustment matrix $D = I$.

Table 5
Out-of-sample performance (mean returns, variance of returns, portfolio turnover, Sharpe ratio (SR), and p-value for the difference in SR with respect to the mean-variance strategy) of robust optimization methods. Data set: Fama-French 25 size and book-to-market sorted portfolios

	Robust Optimization	Adjusted Robust Optimization		
		$D = \Sigma^{-1}$	$D = I$	$D = L^{-1}$
$kappa = 1$				
Mean	0.010	0.010	0.011	0.010
Variance	0.001	0.001	0.002	0.002
Turnover	0.036	0.030	0.000	0.030
SR	0.277	0.271	0.247	0.234
p-value	0.706	0.639	0.370	0.263
$kappa = 3$				
Mean	0.009	0.009	0.011	0.010
Variance	0.001	0.001	0.002	0.002
Turnover	0.028	0.029	0.000	0.028
SR	0.265	0.263	0.247	0.231
p-value	0.601	0.585	0.375	0.249
$kappa = 5$				
Mean	0.009	0.009	0.011	0.010
Variance	0.001	0.001	0.002	0.002
Turnover	0.029	0.029	0.000	0.028
SR	0.262	0.261	0.247	0.230
p-value	0.554	0.536	0.351	0.242
$kappa = 7$				
Mean	0.009	0.009	0.011	0.010
Variance	0.001	0.001	0.002	0.002
Turnover	0.029	0.029	0.000	0.028
SR	0.261	0.260	0.247	0.230
p-value	0.547	0.534	0.351	0.242

Table 6
Out-of-sample performance (mean returns, variance of returns, portfolio turnover, Sharpe ratio (SR), and p-value for the difference in SR with respect to the mean-variance strategy) of traditional optimization methods. Data set: Fama-French 25 size and book-to-market sorted portfolios

	Mean-Var	Min-Var
Mean	0.014	0.009
Variance	0.002	0.001
Turnover	0.113	0.030
SR	0.298	0.258
p-value	1.000	0.535

Finally, Figure 3 displays the time-varying portfolio weights for all optimization techniques. We can see that the mean-variance optimization concentrated the allocation in only two portfolios out of 25 available, and the allocation between these two portfolios radically changed in the period analyzed. This is reflected in the high portfolio turnover achieved associated to the mean-variance optimization (0.113). The robust optimization approach, on the other hand, yielded a more diversified and stable strategy over time, with lower portfolio turnover.

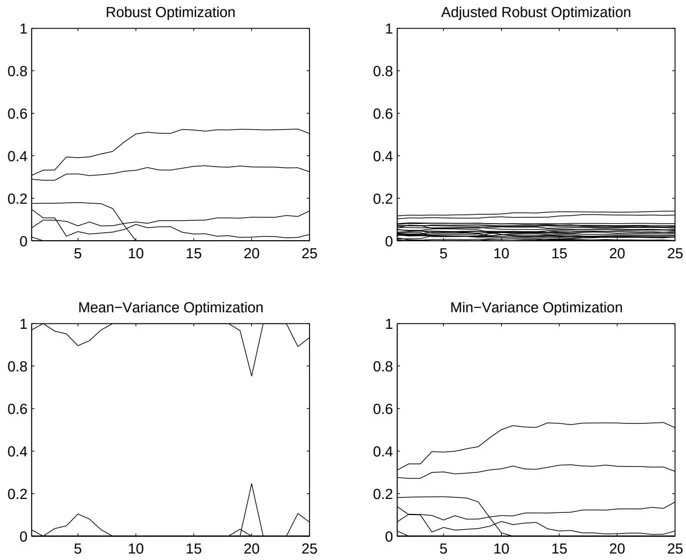

Figure 3
Time-varying portfolio weights for robust, adjusted robust, mean-variance and minimum-variance optimization techniques. Data set: Fama-French 25 size and book-to-market sorted portfolios

The results for the portfolio FF100 are show in Tables 7 and 8. The minimum-variance and robust optimization yielded similar Sharpe ratios (0.348), which is approx 41% higher than the Sharpe ratio obtained under the mean-variance approach (Sharpe ratio of 0.246) even tough the statistical significance of this difference is not so conclusive (p-value of 0.12). In terms of portfolio turnover, the robust and minimum-variance optimization achieved a similar portfolio turnover (0.06), both smaller than the one achieved by the mean-variance optimization (0.08). Those findings are corroborated by the visual inspection of Figure 4 which shows the time-varying portfolio weights of each optimization technique. We can check that both robust and minimum-variance portfolios provided an improved stability in the portfolio composition over mean-variance portfolios.

Table 7
Out-of-sample performance (mean returns, variance of returns, portfolio turnover, Sharpe ratio (SR), and p-value for the difference in SR with respect to the mean-variance strategy) of robust optimization methods. Data set: Fama-French 100 size and book-to-market sorted portfolios

	Robust Optimization	Adjusted Robust Optimization		
		$D = \Sigma^{-1}$	$D = I$	$D = L^{-1}$
$kappa = 1$				
Mean	0.012	0.012	0.011	0.012
Variance	0.001	0.001	0.002	0.002
Turnover	0.115	0.106	0.127	0.222
SR	0.326	0.328	0.262	0.284
p-value	0.191	0.191	0.734	0.446
$kappa = 3$				
Mean	0.012	0.012	0.011	0.011
Variance	0.001	0.001	0.002	0.002
Turnover	0.076	0.074	0.127	0.222
SR	0.340	0.341	0.262	0.264
p-value	0.161	0.154	0.734	0.748
$kappa = 5$				
Mean	0.012	0.012	0.011	0.011
Variance	0.001	0.001	0.002	0.002
Turnover	0.068	0.067	0.127	0.224
SR	0.345	0.346	0.262	0.260
p-value	0.149	0.139	0.734	0.796
$kappa = 7$				
Mean	0.012	0.012	0.011	0.011
Variance	0.001	0.001	0.002	0.002
Turnover	0.066	0.065	0.127	0.224
SR	0.347	0.348	0.262	0.258
p-value	0.116	0.138	0.734	0.800

Table 8
Out-of-sample performance (mean returns, variance of returns, portfolio turnover, Sharpe ratio (SR), and p-value for the difference in SR with respect to the mean-variance strategy) of traditional optimization methods. Data set: Fama-French 100 size and book-to-market sorted portfolios

	Mean-Var	Min-Var
Mean	0.013	0.012
Variance	0.003	0.001
Turnover	0.083	0.062
SR	0.246	0.348
p-value	1.000	0.129

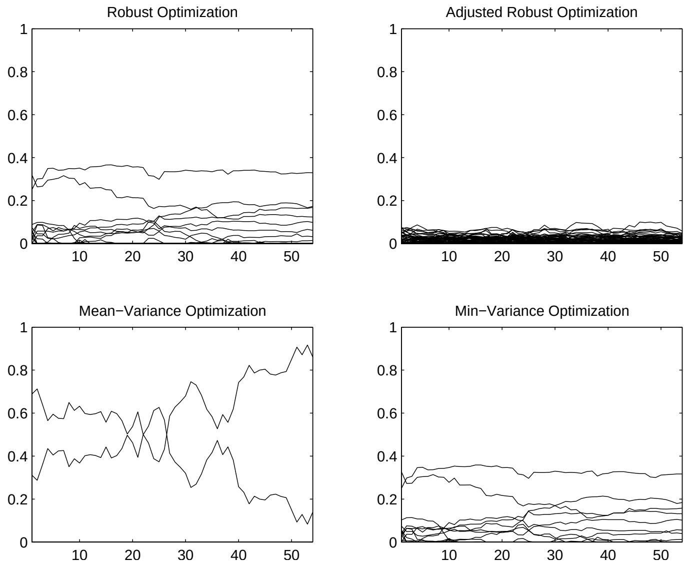

Figure 4
Time-varying weights for robust, adjusted robust, mean-variance and minimum-variance optimization techniques.
data set: Fama-French 100 size and book-to-market sorted portfolios

For the data set FF38, the highest Sharpe ratio were obtained by the adjusted robust optimization (with adjustment matrix $D = L^{-1}$), as shown in Table 9. However the differences in Sharpe ratios obtained were not significant in all cases. The adjusted robust optimization was also able to deliver the smallest portfolio turnover among all competing strategies. Figure 5 also indicates that the allocation among available portfolios was more diversified under the robust and minimum-variance strategy. As in the previous case, the changes in the portfolio composition associated to the mean-variance optimization were substantial.

Table 9
Out-of-sample performance (mean returns, variance of returns, portfolio turnover, Sharpe ratio (SR), and p-value for the difference in SR with respect to the mean-variance strategy) of robust optimization methods. Data set: Fama-French 38 industry portfolios

	Robust Optimization	Adjusted Robust Optimization		
		$D=\Sigma^{-1}$	$D=I$	$D=L^{-1}$
$kappa=1$				
Mean	0.006	0.006	0.010	0.010
Variance	0.001	0.001	0.001	0.001
Turnover	0.073	0.060	0.000	0.038
SR	0.197	0.199	0.256	0.278
p-value	0.841	0.836	0.298	0.234
$kappa=3$				
Mean	0.006	0.006	0.010	0.010
Variance	0.001	0.001	0.001	0.001
Turnover	0.057	0.055	0.000	0.038
SR	0.200	0.201	0.256	0.279
p-value	0.820	0.833	0.298	0.227
$kappa=5$				
Mean	0.006	0.007	0.010	0.010
Variance	0.001	0.001	0.001	0.001
Turnover	0.055	0.055	0.000	0.039
SR	0.201	0.201	0.256	0.278
p-value	0.795	0.807	0.298	0.236
$kappa=7$				
Mean	0.007	0.007	0.010	0.009
Variance	0.001	0.001	0.001	0.001
Turnover	0.055	0.055	0.000	0.039
SR	0.201	0.202	0.256	0.278
p-value	0.809	0.806	0.298	0.245

Table 10
Out-of-sample performance (mean returns, variance of returns, portfolio turnover, Sharpe ratio (SR), and p-value for the difference in SR with respect to the mean-variance strategy) of traditional optimization methods. Data set: Fama-French 38 industry sorted portfolios

	Mean-Var	Min-Var
Mean	0.009	0.007
Variance	0.002	0.001
Turnover	0.157	0.055
SR	0.176	0.204
p-value	1.000	0.772

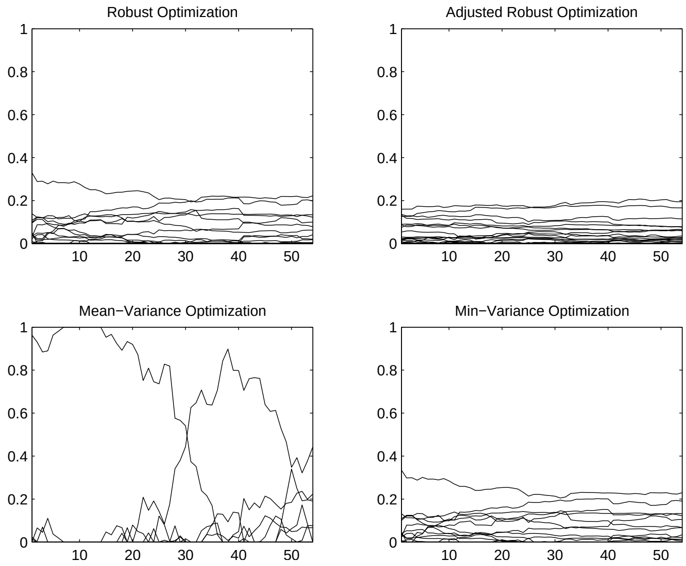

Figure 5
Time-varying portfolio weights for robust, adjusted robust, mean-variance and minimum-variance optimization techniques. Data set: Fama-French 38 industry portfolios

For the data set FF5, the results shown in Tables 11 and 12 indicate that the two highest Sharpe ratios were obtained by the mean-variance and standard robust optimization approaches (0.281 and 0.258), with no statistical significance for the difference (p-value of 0.6). As in the previous cases, the higher portfolio turnover was achieved by the mean-variance optimization (average change of 17% in portfolio composition in each rebalancing date). The smallest turnover was achieved by the adjusted portfolio optimization. Figure 6 also indicates that the allocation among the five available portfolios under the mean-variance policy is rather unstable over time, in contrast to the allocations delivered by the standard and adjusted robust optimization (and also the minimum-variance policy), which clearly appear to be much more stable over time.

Table 11
Out-of-sample performance (mean returns, variance of returns, portfolio turnover, Sharpe ratio (SR), and p-value for the difference in SR with respect to the mean-variance strategy) of robust optimization methods. Data set: Fama-French 5 size and book-to-market sorted portfolios

	Robust Optimization	Adjusted Robust Optimization		
		$D = \Sigma^{-1}$	$D = I$	$D = L^{-1}$
$kappa = 1$				
Mean	0.008	0.008	0.007	0.006
Variance	0.001	0.001	0.001	0.001
Turnover	0.028	0.023	0.000	0.014
SR	0.258	0.252	0.214	0.194
p-value	0.695	0.656	0.336	0.219
$kappa = 3$				
Mean	0.008	0.008	0.007	0.006
Variance	0.001	0.001	0.001	0.001
Turnover	0.024	0.024	0.000	0.012
SR	0.252	0.250	0.214	0.192
p-value	0.654	0.632	0.336	0.221
$kappa = 5$				
Mean	0.008	0.008	0.007	0.006
Variance	0.001	0.001	0.001	0.001
Turnover	0.024	0.024	0.000	0.012
SR	0.251	0.249	0.214	0.191
p-value	0.662	0.636	0.336	0.197
$kappa = 7$				
Mean	0.008	0.008	0.007	0.006
Variance	0.001	0.001	0.001	0.001
Turnover	0.024	0.024	0.000	0.012
SR	0.250	0.249	0.214	0.191
p-value	0.635	0.624	0.336	0.204

Table 12
Out-of-sample performance (mean returns, variance of returns, portfolio turnover, Sharpe ratio (SR), and p-value for the difference in SR with respect to the mean-variance strategy) of traditional optimization methods. Data set: Fama-French 5 size and book-to-market sorted portfolios

	Mean-Var	Min-Var
Mean	0.008	0.008
Variance	0.001	0.001
Turnover	0.170	0.024
SR	0.281	0.249
p-value	1.000	0.623

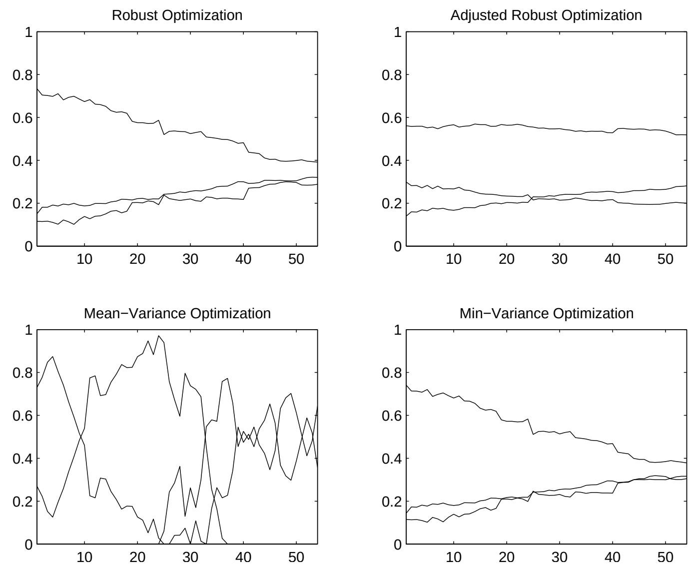

Figure 6
Time-varying weights for robust, adjusted robust, mean-variance and minimum-variance optimization techniques. Data set: Fama-French 5 industry portfolios

Interestingly, in all data sets analyzed, including the simulated ones, the adjusted robust optimization with adjustment matrix $D = I$ delivered the smallest portfolio turnover in all specifications. Further investigation revealed that in fact this allocation corresponded to an equally-weighted allocation. This particular result can be probably stated theoretically; further investigation will be conducted.

4.3 Discussion

From the results presented in this section, some important implications for investment decisions based on portfolio selection policies can be pointed out. First, the empirical evidence based on simulated data (in which the data generating process in under perfect control) shows that the robust methods significantly outperformed the mean-variance optimization in terms of Sharpe ratios and portfolio turnover in the majority of the specifications. This result is in contrast with the empirical evidence of Scherer (2007), who also used simulated data. We offer two explanations for this contrast. First, the differences in our results can be attributed to differences in type of the data used. In this work, the assets contained in each data set have similar investment risks, thus rewarding portfolio diversification.[4] In fact, our results for both simulated and real market data show that robust methods delivered more diversified (and stable) portfolios in comparison to

[4]We thank Bernd Scherer for clarifying this difference in our results.

the mean-variance approach. Therefore, portfolio diversification might be playing an important role in our results. Second, and in contrast to Scherer (2007), we analyzed alternative versions of robust methods, such as the adjusted robust optimization of Ceria and Stubbs (2006). This last approach delivered better results in comparison to the standard robust approach of Tütüncü and Köenig (2004) due to the inclusion of a less pessimistic view of expected returns.

We also note that the minimum-variance portfolio also performed better than the mean-variance portfolio. This result is in line with previous studies such as Jagannathan and Ma (2003). The authors show that the global minimum-variance portfolio has a shrinkage effect on the sample covariance matrix, therefore reducing estimation error. In fact, our results show that the minimum-variance portfolio delivered a similar performance in relation to robust methods in several specifications. In this sense, the minimum-variance approach appeared to be also an effective, simple technique to deal with the problem of estimation error.

The robust optimization approaches were able to deliver lower portfolio turnover, meaning that the management costs associated to the implementation if this strategy is lower in comparison to the competing alternatives. The empirical evidence using real market data indicated that, even tough the difference in performances between robust optimization techniques and traditional techniques did not seem to be statistically significant, we found that robust were able to deliver more stability in the portfolio weights in comparison to the mean-variance approach. The main implication of this finding is that, if we assume equal performance across techniques, investors will be better off by choosing a strategy that does not require radical changes in the portfolio composition over time. These substantial changes in portfolio composition are rather difficult to implement in practice due to (i) management costs and (ii) negative cognitive aspects perceived by investors and/or investment managers. Second, portfolios selected based on robust optimization seemed to be more diversified than portfolio selected by mean-variance portfolios. This last approach tended to concentrate the allocation on a small subset of all investment opportunities. The diversification presented of robust portfolios made the technique suitable for practical implementation, since in many cases investors and/or investment companies require some level of portfolio diversification.

5. Concluding Remarks

Robust optimization is one of the most recent fields in the area of portfolio selection and optimization under uncertainty. The importance devoted to this technique is due to the possibility of including into the optimization problem the estimation error, which is a well known problem that makes the portfolio selection problem harder to solve. The empirical evidence provided in this work, comparing the most two recent robust approaches with traditional, well established techniques such as Markowitz's mean-variance and minimum-variance approaches indicate that robust optimization is indeed an effective way to treat the problem of estimation error in the means. When simulated data is used, robust optimization performed better than mean-variance optimized portfolios both in terms of Sharpe ratios and portfolio turnover. When real data was used, the performance of robust optimization in terms of Sharpe ratios was statistically equal to the competing techniques. However, robust optimization was able to deliver portfolios with lower turnover, which facilitates the practical implementation of this strategy. The minimum-variance portfolio also performed similarly in relation to robust alternatives, indicating that this approach is a simple alternative able to alleviate the effects of estimation errors in means.

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