Lecture 2.
Sorting algorithms (continued), Binary search.

Algorithms and Data Structures
Ivan Solomatin
MIPT 2021

Outline

- Linear $O(N)$ sorting algorithms
  - Why do we need additional limitations to get better complexity than $O(N \log N)$
  - Counting sort
- Binary Search
  - Binary search in an array
  - L/R binary search (in an array)
  - Binary search on a function
  - Binary search by answer

Linear $O(N)$ sorting algorithms

- **Additional limitations (why?)**
- Counting sort

Linear $O(N)$ sorting algorithms

Additional limitations

Actually, sorting problem **cannot** be solved faster than $O(N \log N)$ if using problem statement given on previous lecture.

Let's prove it using problem statement in terms of permutations:

Given input:

$$x_0, x_1, \dots, x_{N-1}\colon \;\; x_i \in X$$

Output:

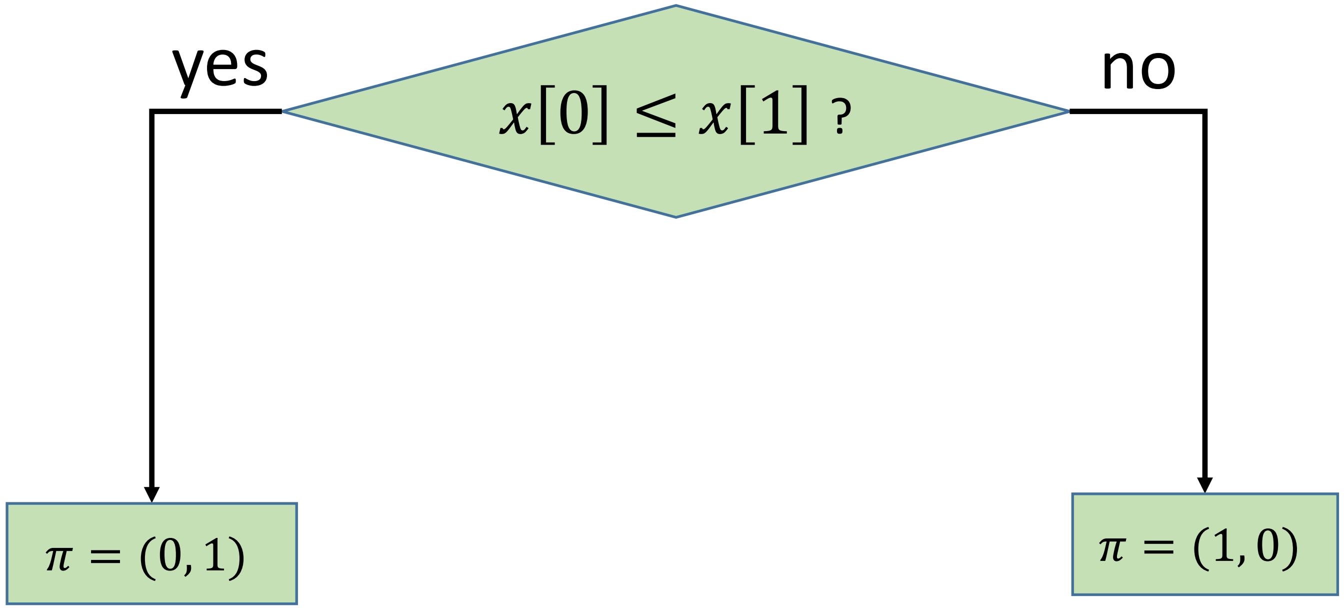
$$\pi = (i_0, i_1, \dots i_{N-1})\colon x_{i_0} \le x_{i_1} \le \cdots \le x_{i_{N-1}}$$

Linear $O(N)$ sorting algorithms

Additional limitations

Any algorithm which uses pairwise comparison of elements compares pair of elements and makes decisions according to the result:

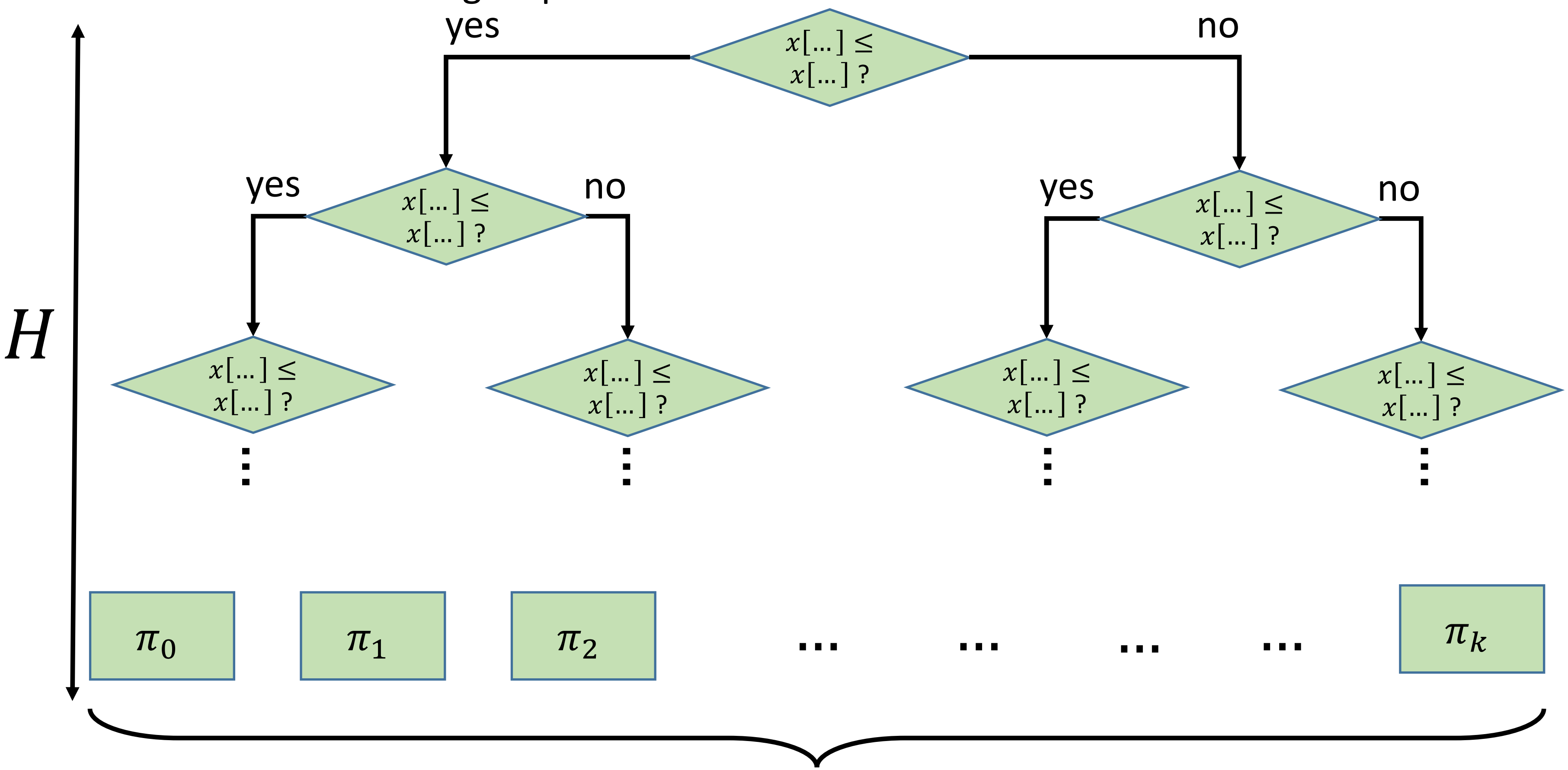
$N = 2$



Linear $O(N)$ sorting algorithms

Additional limitations

Decision tree for sorting sequence of N elements:



$H \geq \log_2 k$ $\qquad$ $k \geq N!$

Linear $O(N)$ sorting algorithms

Additional limitations

$$H \geq \log_2 k \geq \log_2 N! \geq \log_2 \left(\frac{N}{2}\right)^{\frac{N}{2}} = \frac{N}{2}\log_2 \frac{N}{2} =$$

$$= \frac{1}{2}(N \log_2 N - N) \; = \boldsymbol{\Omega(N \log N)}$$

$$N! = \underbrace{N(N-1)(N-2)\ldots\left\lceil\frac{N}{2}\right\rceil}_{\geq \frac{N}{2} \text{ elements, each } \geq \frac{N}{2}} \ldots 2 * 1 \;\; \geq \left(\frac{N}{2}\right)^{\frac{N}{2}}$$

So, this product is $\geq \left(\frac{N}{2}\right)^{\frac{N}{2}}$

Stirling's formula:

$$N! \sim \sqrt{2\pi N}\left(\frac{N}{e}\right)^{n}$$

Linear $O(N)$ sorting algorithms

Additional limitations

So, it's impossible to create an algorithm which operates only by comparing elements pairwise with complexity better than $O(N \log N)$.

But if we have additional limitations on X, it's possible. Let's suppose that we have $<$ and $=$ relations defined on X and that X is finite and rather small set: $|X| = M \sim N$.

Linear $O(N)$ sorting algorithms

- Additional limitations (why?)
- **Counting sort**

Counting sort

Idea

Let's suppose that $|X| = M \sim N$.

Let's enumerate elements of X with integer numbers within small range: $[0, M)$ according to their order.

Now we need to sort these integer numbers.

Let's just calculate number of times each value occurs in source array.

Counting sort

Idea

Let’s just calculate number of times each value occurs in source array.

$$M = 3$$

	0	1	2	3	4	5	6	7
x	0	2	1	1	0	1	0	0

Let’s create an accumulator array of M elements:

	0	1	2
a	4	3	1

$a[i]$ – number of elements equal to i

Now we can restore desired array:

sorted(x)	0	0	0	0	1	1	1	2

Counting sort

Implementation

```
def counting_sort(x):
    N = len(x)
    M = max(x) + 1
    a = [0] * M
    for v in x:
        a[v] += 1
    res = []
    for i in range(M):
        for j in range(a[i]):
            res.append(i)
    return res
```

$N + M + N =$
$O(N + M) = O(N)$

$X-$ integer numbers within $[0, M)$.

Counting sort

Modification

This implementation is quite easy but has one disadvantage: we restore elements directly from accumulator array **a**.

Actually, X is a set of values being sorted. But we can have our x array consisting of tuples and use first element of this tuple to performing counting sort.

So if we have object/tuples in x array, we'll lose them.

Counting sort

Modification

Let's modify the algorithm. Having accumulator array **a** we can easily calculate cumulative array:

$$c[i] = \sum_{j=0}^{i} a[j]$$

$a[i]$ – number of elements $= i$

$c[i]$ – number of elements $\leq i$

Counting sort

Modification

$c[i]$ – number of elements $\leq i$

Now, let's remember sorting problem statement:

	0	1	2	3	4	5	6	7
sorted(x)	1	2	3	4	5	6	7	8

$\leq \quad \leq \quad \leq \quad \leq \quad \leq \quad \leq \quad \leq$

You can notice that number of elements $\leq x[i]$ actually denotes it's position in sorted array (we just need to subtract 1, because $x[i] \leq x[i]$ as well).

To deal with similar values, let's just decrease $c[i]$ value after adding to sorted array.

Counting sort

Modification

	0	1	2	3	4	5	6	7
x	0	2	1	1	0	1	0	0

	0	1	2
a	4	3	1

	0	1	2
c	4	7	8

	0	1	2	3	4	5	6	7
sorted(x)	0	0	0	0	1	1	1	2

$$\underbrace{0\quad 0\quad 0\quad 0}_{0}\;\underbrace{1\quad 1\quad 1}_{1}\;\underbrace{2}_{2}$$

Counting sort

Implementation 2

```
def counting_sort2(x):
    N = len(x)
    M = max(x) + 1
    c = [0] * M
    for v in x:
        c[v] += 1
    for i in range(1, M):
        c[i] += c[i - 1]
    res = [None] * N
    for i in range(N):
        position = c[x[i]] - 1
        res[position] = x[i]
        c[x[i]] -= 1
    return res
```

$$N + M + N = O(N + M)$$
$$= O(N)$$

$X-$ integer numbers within $[0, M)$.

Binary search

- **Binary search in an array**
- L/R binary search (in an array)
- Binary search on a function
- Binary search by answer

Binary search in an array

Idea

We want to check if array x contains an element with given key value.

$$key = 7$$

	0	1	2	3	4	5	6	7
x	2	4	5	1	7	3	4	8

	0	1	2	3	4	5	6	7
$sorted(x)$	1	2	3	4	5	6	7	8

$$\leq \quad \leq \quad \leq \quad \leq \quad \leq \quad \leq \quad \leq$$

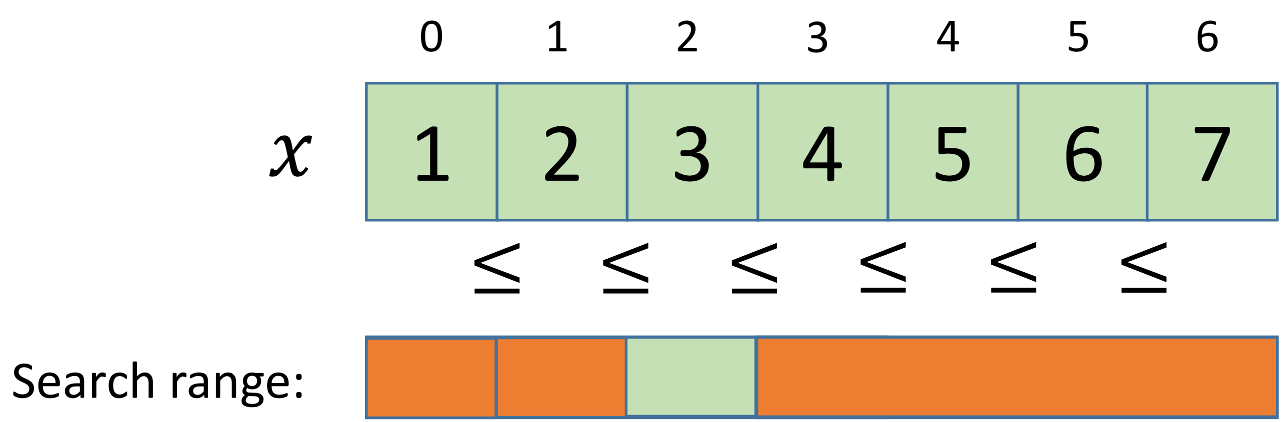
Binary search in an array

Idea

Let's iteratively compare middle element (right to the middle if in current search range with key value and reduce search range using inequality of sorted array.

When length of search is reduced to 1, we can just compare this element with key value.

$$key = 3$$



$$3 = 3$$

Binary search in an array

Implementation

Let's denote search range as $[l, r)$:

```
def binsearch_exists(x, key):
    l = 0
    r = len(x)
    while r - l > 1:
        m = (l + r) // 2
        if x[m] <= key:
            l = m
        else:
            r = m
    return x[l] == key
```

$$O(\log N)$$

Binary search

- Binary search in an array
- **L/R binary search (in an array)**
- Binary search on a function
- Binary search by answer

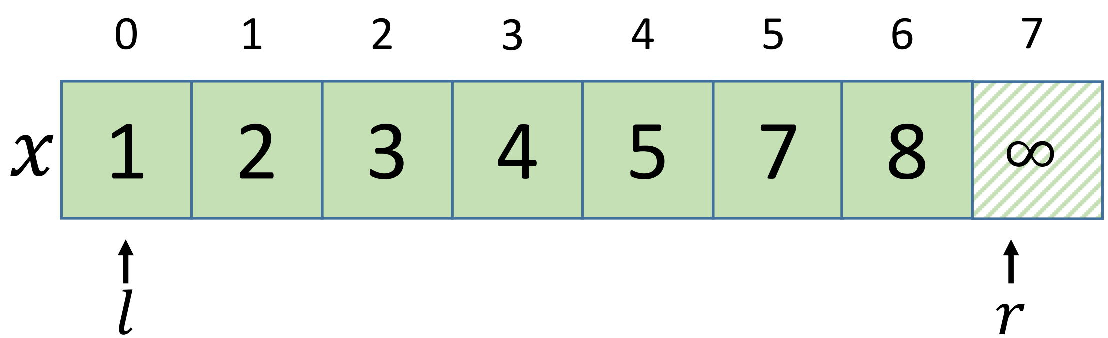
Left/right binary search

Idea

Now, let’s see what will happen if key is not in array.

```
def binsearch_exists(x, key):
  l = 0
  r = len(x)
  while r - l > 1:
    m = (l + r) // 2
    if x[m] <= key:
      l = m
    else:
      r = m
  return x[l] == key
```

$$key = 6$$



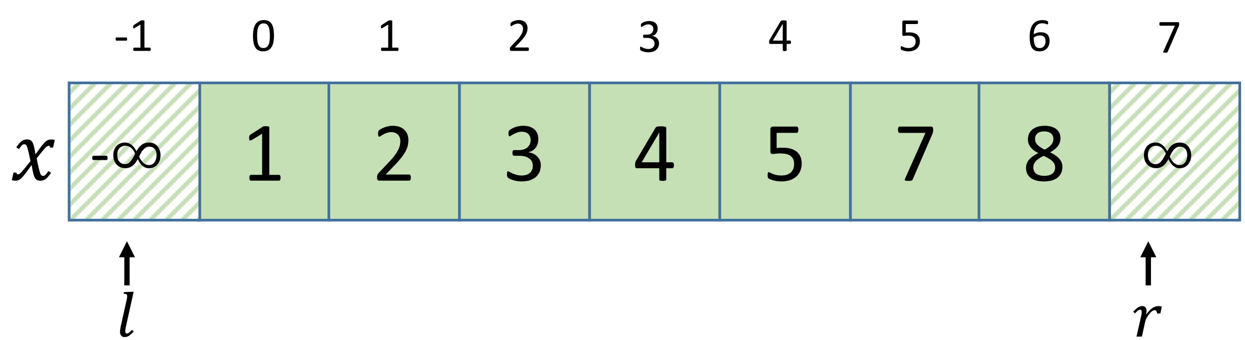
$$\left[\begin{array}{l} x[r] > key \\ r = len(x) \end{array}\right. \qquad \left[\begin{array}{l} x[l] \le key \\ x[0] > key \end{array}\right.$$

Left/right binary search

Idea

```python
def binsearch(x, key):
  l = -1
  r = len(x)
  while r - l > 1:
    m = (l + r) // 2
    if x[m] <= key:
      l = m
    else:
      r = m
  ...
```

$$key = 6$$

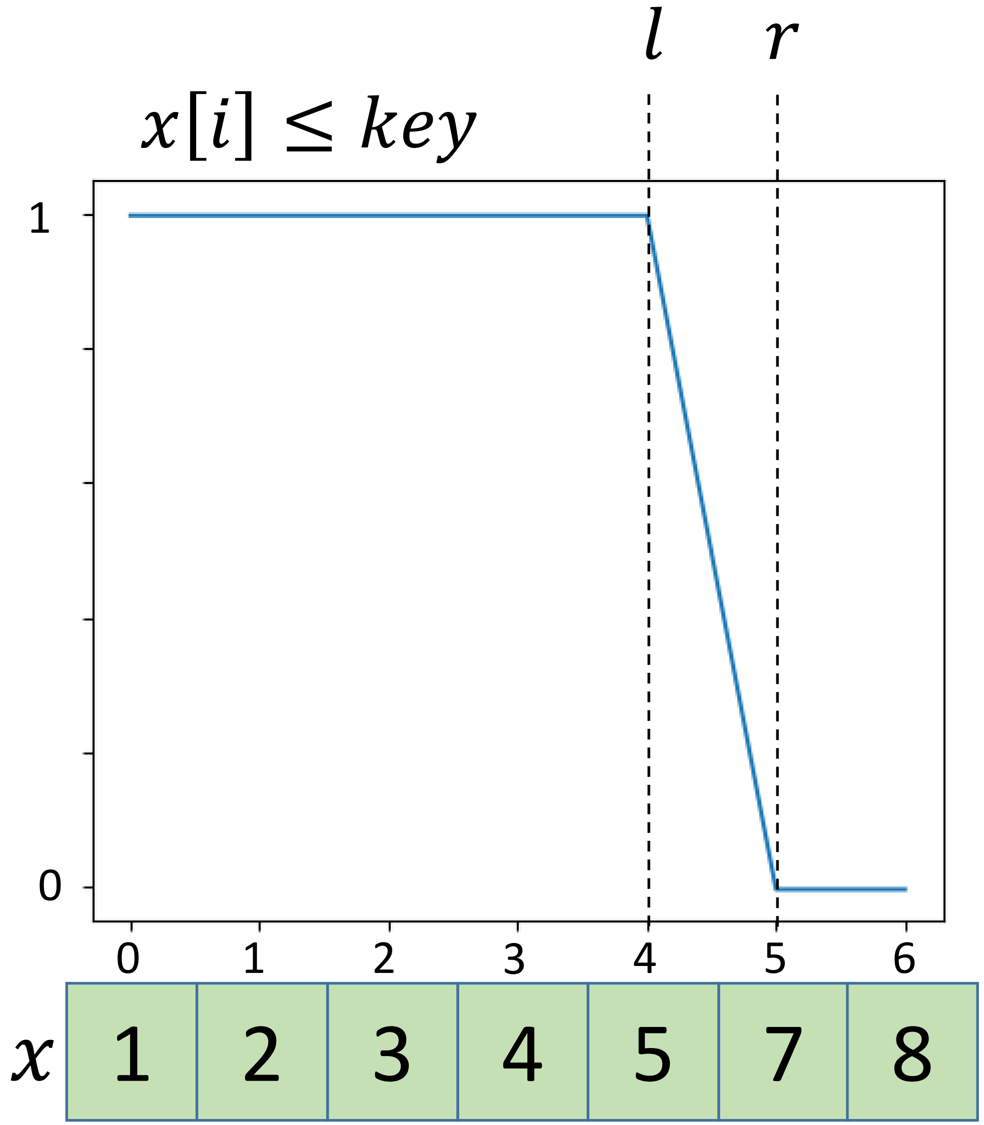


$$\left[\begin{array}{l} x[r] > key \\ r = len(x) \end{array}\right. \qquad \left[\begin{array}{l} x[l] \leq key \\ l = -1 \end{array}\right.$$

Left/right binary search

Idea

$key = 6$



Left/right binary search

Idea

What if we have several identical values: $key = 3$

	0	1	2	3	4	5	6
x	1	2	3	3	3	4	8
$x[i] \le key$	1	1	1	1	1	0	0
$x[i] < key$	1	1	0	0	0	0	0

$l_< \mid r_<$ $\qquad$ $l_\le \mid r_\le$

$\forall i \in [r_<, r_\le): x[i] = key$ $\qquad$ $|i: x[i] = key| = r_\le - r_<$

Left/right binary search

Implementation

```
def bsearch_l(x, key):
  l = -1
  r = len(x)
  while r - l > 1:
    m = (l + r) // 2
    if x[m] < key:
      l = m
    else:
      r = m
  return r
```

```
def bsearch_r(x, key):
  l = -1
  r = len(x)
  while r - l > 1:
    m = (l + r) // 2
    if x[m] <= key:
      l = m
    else:
      r = m
  return r
```

`lower_bound`

Returns index of first element $\geq\ key$

`upper_bound`

Returns index of first element $> key$

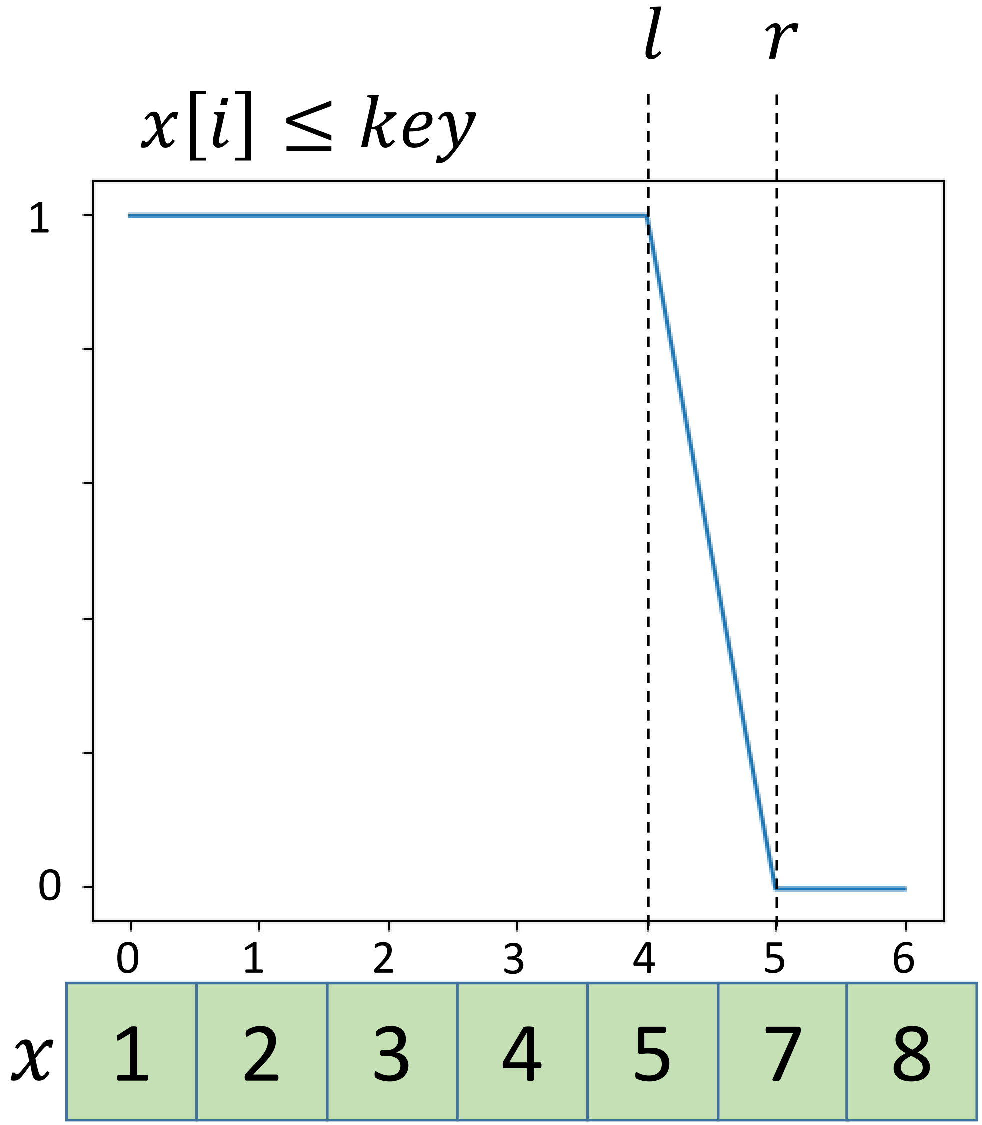
Binary search

- Binary search in an array
- L/R binary search (in an array)
- **Binary search on a function**
- Binary search by answer

Binary search on a function

Idea

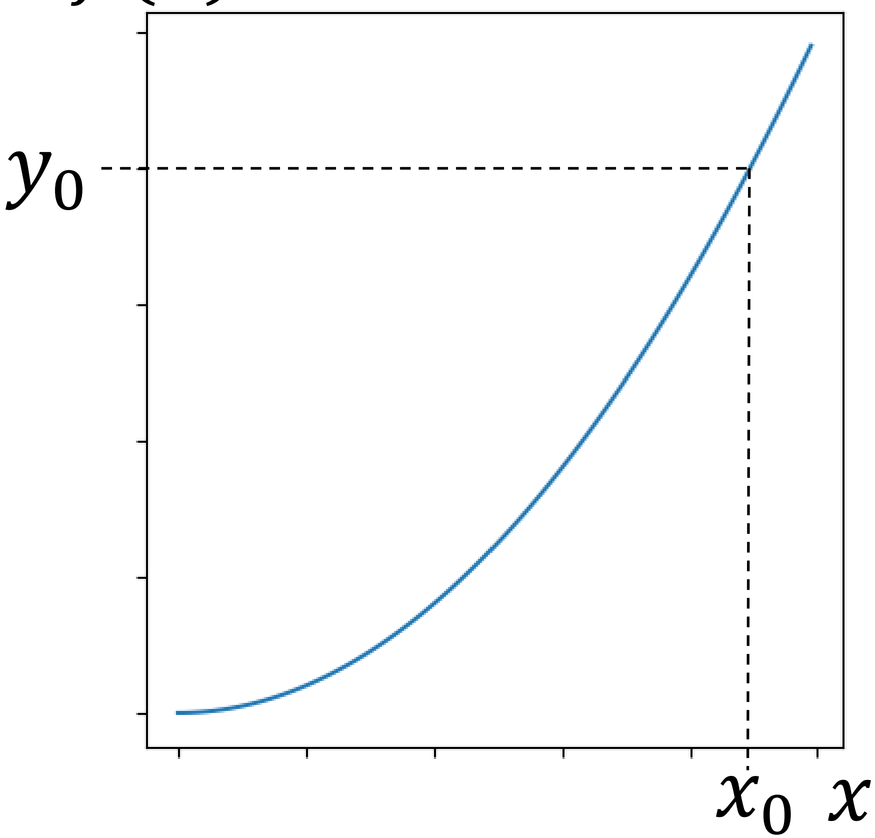
$key = 6$



Binary search on a function

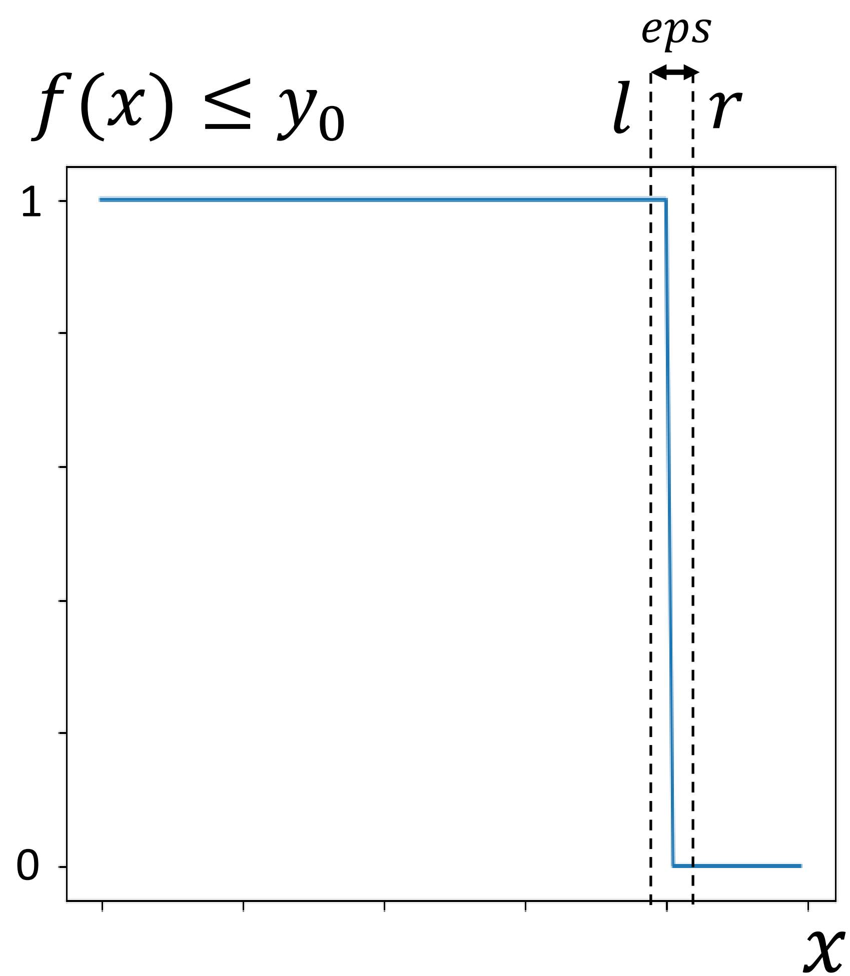
Idea

$f(x)$



$$f(x_0) = y_0$$

$f(x) \le y_0$

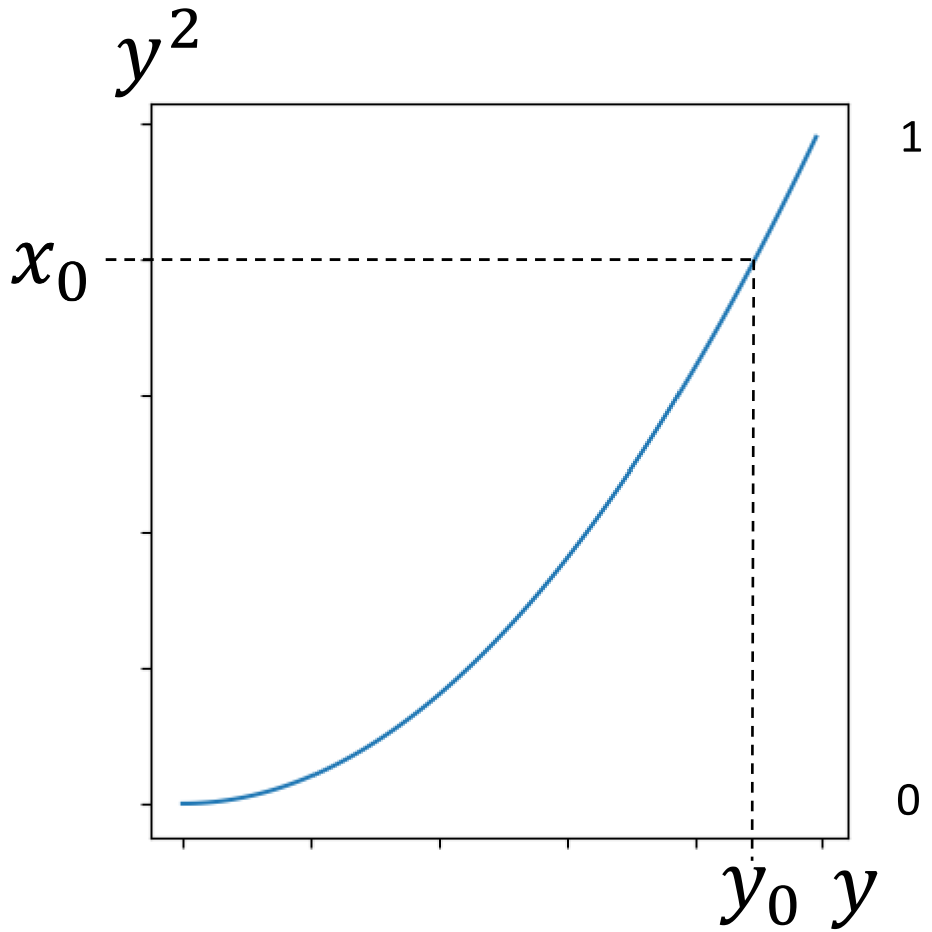


Binary search on a function

Example: sqrt

$$y_0 = \ sqrt(x_0)$$

$$y_0^2 = x_0$$



Binary search on a function

Sqrt implementation

```
def sqrt(x, eps):
    l = 0.
    r = max(1, float(x))
    while r - l > eps:
        m = (l + r) / 2
        if m ** 2 < x:
            l = m
        else:
            r = m
    return (l + r) / 2
```

$\log_2 \frac{x}{eps}$

iterations

Binary search

- Binary search in an array
- L/R binary search (in an array)
- Binary search on a function
- **Binary search by answer**

Binary search by answer

Idea

Let's suppose that we need to find minimum (maximum) value which satisfy given boolean requirements: $f(x) = 1$

Also, let's suppose that these requirements are monotonous:

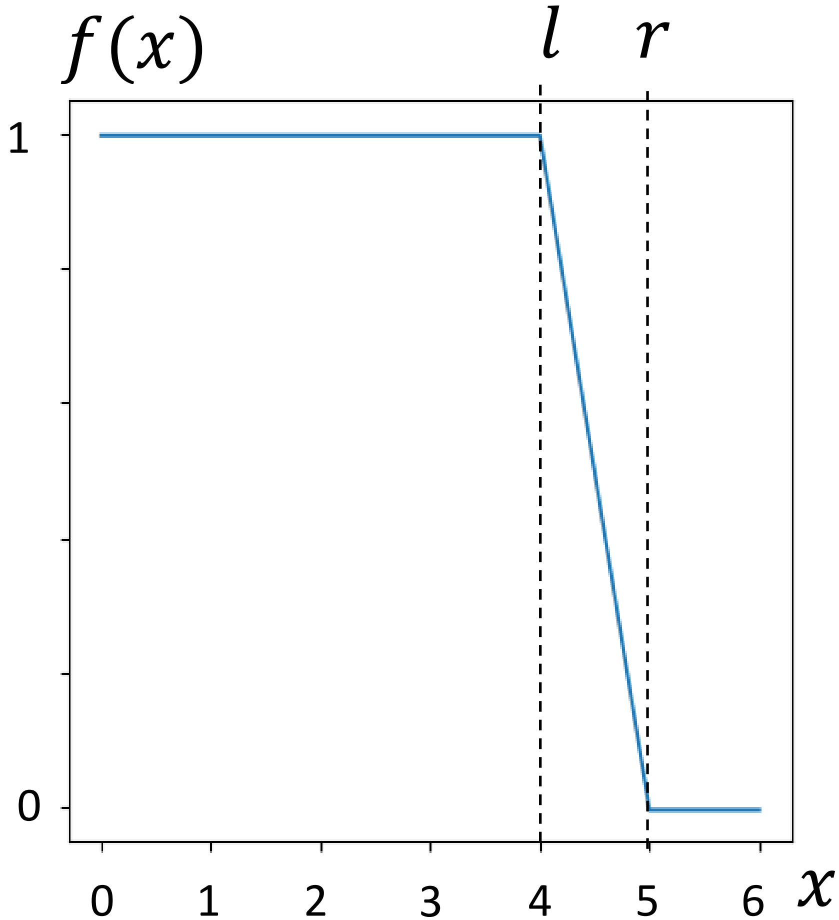
$$f(x) = 1 \rightarrow f(x+1) = 1$$
$$or$$
$$f(x) = 0 \rightarrow f(x+1) = 0$$

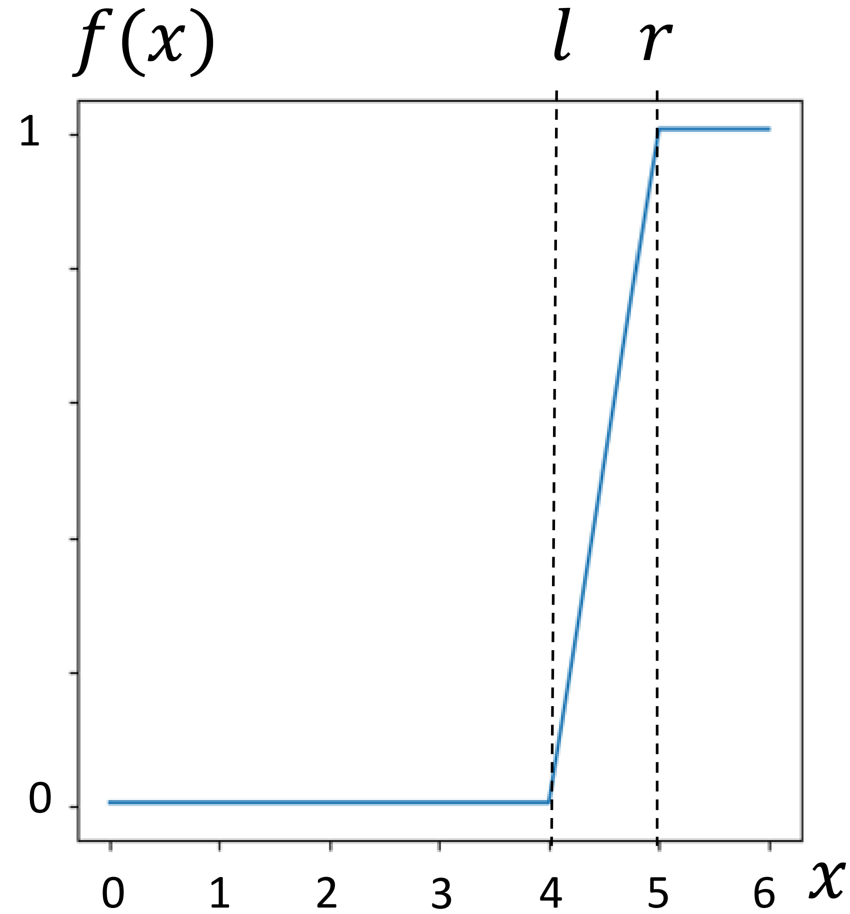
We can use binary search approach to find this value.

Binary search by answer

Idea

1. Function is monotonous
2. Search range





Binary search by answer

Example: Xerox

Problem: We have two copies of the document. And we have two copiers. One copies a document in x seconds, another – in y seconds. How long will it take to obtain N copies?

Changed statement: Find minimum number of seconds T_0 such that number of copies we can make in this time is not less than N:

$$K(T_0) \geq N$$

Solution: How much copies can we make in T seconds?

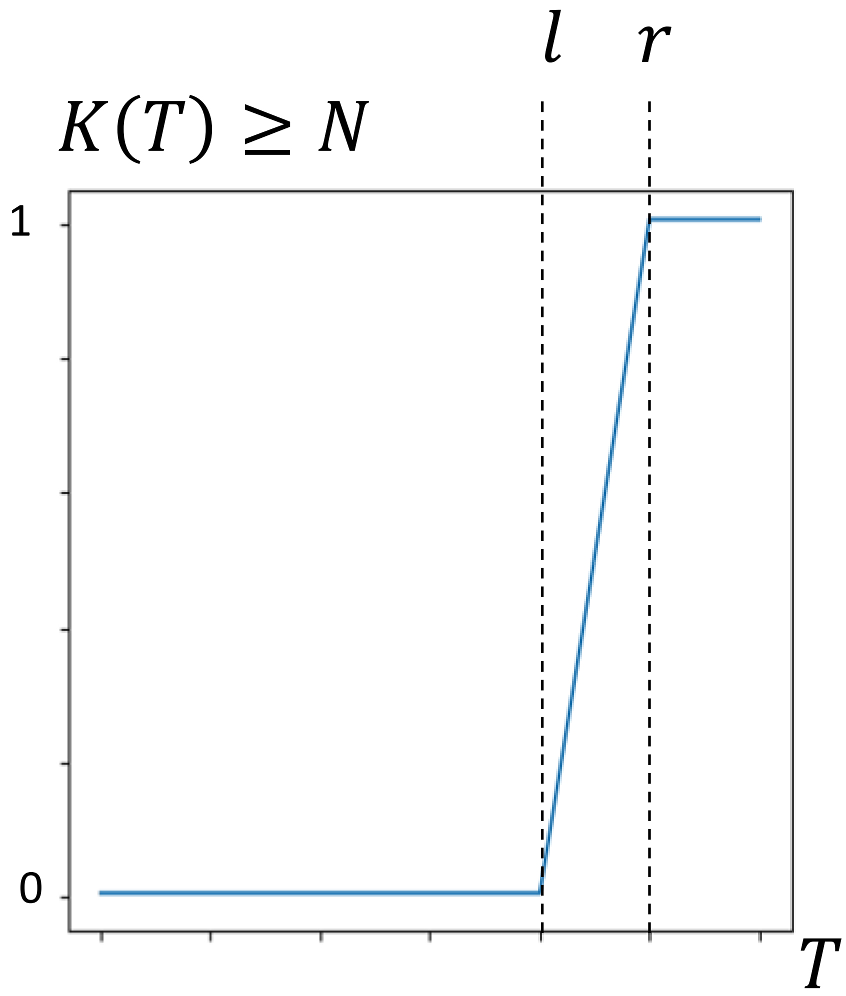
$$K(T) = \left\lfloor \frac{T}{x} \right\rfloor + \left\lfloor \frac{T}{y} \right\rfloor$$

$K(T)$ function is obviously monotonous, so, our requirements are also monotonous. So, we can use binary search to find minimum T_0: $\mathrm{K}(\mathrm{T}_0) \geq N$.

Initial range: $[0, N \min(x, y)]$

Binary search by answer

Example: Xerox



$$\boldsymbol{T_0 = r}$$

Conclusion

Python built-ins

left bin.search (lower_bound):

`bisect.`**`bisect_left`**

right bin.search (upper_bound):

`bisect.`**`bisect_right`**

Thank you for watching!