Seminar 7

Recap of random variables

A **random variable** is a function from sample space to the real numbers $X : S \to \mathbb{R}$.

Recap of distributions

For a random variable $X : S \to \mathbb{R}$, its distribution acts on numbers in $\mathbb{R}$ in the same way as probability function P acts on outcomes.

Recap of functions describing distributions

- For any distribution we have **cumulative distribution function** (CDF) $F_X(x) = \mathbb{P}(X \leqslant x)$
- For discrete distributions we have **probability mass function** (PMF) $\mathbb{P}_X(x) = \mathbb{P}(X = x)$

Probability density function

- If X has a discrete distribution, then F_X has a countable number of jumps $p_i = \mathbb{P}(X = x_i)$ and at $x = x_i$ it is continuous

- If X has absolutely continuous distribution, then F_X is differentiable a.e. and can be recovered from its derivative:

$$F_X(x) = \int\limits_{-\infty}^{x} f_X(t)dt$$

  where $f_X(t)$ is the probability density function and $f_X(t) = F'_X(x)$ a.e.

Example 1

We say that random variable X is distributed uniformly on $[a, b]$ and write $X \sim U([a, b])$ if

$$f_X(x) = \begin{cases} \frac{1}{b-a}, a \leqslant x \leqslant b, \\ 0, \text{else} \end{cases}$$

What is $F_X(x)$?

$$F_X(x) = \int\limits_{-\infty}^{x} \frac{1}{b-a} \mathrm{d}x = \frac{1}{b-a} \int\limits_{a}^{x} \mathrm{d}x = \frac{x-a}{b-a}$$

e.g. if $X \sim U([0,1])$, then $F_X(x) = x$.

Continuous convolution formula

Consider X and Y independent random variables with PDFs f_X and f_Y respectively. Then, their sum $Z = X + Y$ has absolutely continuous distribution with density

$$f_Z(z) = \int f_X(x) f_Y(z-x) dx$$

Example 2

Let $X, Y \sim U([0,1])$ and $Z = X + Y$. Find $f_Z(z)$.

Solution 2

$$f_Z(z) = \int\limits_{0}^{1} f_X(x) f_Y(z-x) dx = \int\limits_{0}^{1} f_Y(z-x) dx = \begin{cases} z, & 0 \leqslant z \leqslant 1, \\ 2-z, & 1 \leqslant z \leqslant 2, \\ 0, & \text{else} \end{cases}$$

Functions of continuous random variables

Random variables transform like functions, i.e. if $Y = \varphi(X)$, then $Y(\omega) = \varphi(X(\omega))$.

For a smooth φ, the density will be:

$$f_Y(y) = \sum_{\varphi(x) = y} \frac{f_X(x)}{|\varphi'(x)|}$$

LOTUS for continuous random variables

If X is a continuous r.v. with PDF $f_X(x)$ and $g(\cdot)$ is a function, then,

$$\mathbb{E}[g(X)] = \int g(x) f_X(x) \mathrm{d}x$$

Example 3

Let X be a **normally distributed** random variable with parameters m and σ^2:

$$f_X(x) = \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left(-\frac{(x-m)^2}{2\sigma^2}\right)$$

Find PDF of $Y = X^2$.

Solution 3

According to the formula,

$$f_Y(y) = \sum_{x^2=y} \frac{f_X(x)}{|\varphi'(x)|} = \sum_{x=\pm\sqrt{y}} \frac{f_X(x)}{|2x|} = \frac{f_X(-\sqrt{y}) + f_X(\sqrt{y})}{2\sqrt{y}}$$

Recap of expectation

- If X is discrete, then

$$\mathbb{E}\left[X\right] = \sum_k x_k \mathbb{P}(X = x_k)$$

- If X is continuous, then

$$\mathbb{E}\left[X\right] = \int_{-\infty}^{+\infty} x f_X(x) dx$$

Recap of variance

We call **variance** the following quantity of a r.v. X with finite expectation:

$$\mathbb{V}\mathrm{ar}(X) = \mathbb{E}\left[(X - \mathbb{E}[X])^2\right]$$

Example 4

Find the expectation and variance of uniform distribution. Draw its CDF and PDF.

Solution 4

Let $X \sim U[a,b]$, then $f_X(x) = \frac{1}{b-a}$

$$\mathbb{E}\left[X\right] = \int\limits_a^b x\frac{1}{b-a}dx = \frac{1}{b-a}\left(\frac{b^2}{2} - \frac{a^2}{2}\right) = \frac{a+b}{2}$$

$$\mathbb{E}\left[X^2\right] = \int\limits_a^b x^2\frac{1}{b-a}dx = \frac{1}{b-a}\left(\frac{b^3}{3} - \frac{a^3}{3}\right) = \frac{1}{3}\frac{b^3-a^3}{b-a}$$

$$\mathbb{V}\mathrm{ar}\left(X\right) = \frac{1}{3}\frac{b^3-a^3}{b-a} - \frac{1}{4}(a+b)^2 = \frac{(b-a)^2}{12}$$

Universality of Uniform distribution

Let X be an r.v. with CDF F. Find the distribution of r.v. $Y = F(X)$.

$$F_Y(y) = \mathbb{P}\left(Y \leqslant y\right) = \mathbb{P}\left(F(X) \leqslant y\right) = \mathbb{P}\left(X \leqslant F^{-1}(y)\right) = F(F^{-1}(y)) = y$$

What does it mean about distribution of Y?,

It means $Y \sim U([0,1])$

Example 5

Find the expectation and variance of normal distribution. Draw its CDF and PDF.

Solution 5

$Z \sim \mathcal{N}(0,1)$ means it has PDF $\varphi(z) = \frac{1}{\sqrt{2\pi}}e^{-z^2/2}$. To prove that it is correct PDF:

$$\left(\int\limits_{-\infty}^{\infty}\varphi(x)dx\right)\left(\int\limits_{-\infty}^{\infty}\varphi(y)dy\right) = \frac{1}{2\pi}\int\limits_{-\infty}^{\infty}\int\limits_{-\infty}^{\infty}e^{-(x^2+y^2)}dxdy = \frac{1}{2\pi}\int\limits_0^{\infty}\int\limits_0^{2\pi}e^{-r^2}rdrd$$

$$\Phi(z) = \frac{1}{\sqrt{2\pi}}\int\limits_{-\infty}^{z}e^{-z^2/2}dz$$

$$\mathbb{E}\left[Z^2\right] = \frac{1}{\sqrt{2\pi}} \int\limits_{-\infty}^{\infty} z^2 e^{-z^2/2} dz = \frac{2}{\sqrt{2\pi}} \int\limits_{0}^{\infty} z^2 e^{-z^2/2} dz = -\frac{2}{\sqrt{2\pi}} \left(z e^{-z^2/2} \Big|_0^{\infty} - \int\limits_{0}^{\infty} \epsilon \right.$$