# MSAI Probability Home Assignment 1

We have prepared 11 problems for you. You only need to solve **any 5** of them to obtain full score for this homework. Problem that you correctly solve extra to those 5 will be a bonus. Bonuses count towards exemption from exam, project and several last home assignments.

**Problem 1.** Prove that

$$\sum_{k=0}^{\infty} \binom{n}{k} = 2^n$$

**Problem 2.** (Vandermonde's identity) Prove that

$$\sum_{j=0}^{k} \binom{n}{j} \binom{m}{k-j} = \binom{m+n}{k}$$

**Problem 3.** A basket contains $m$ balls, out of which $m_1$ are white and $m_2$ are black ($m_1 + m_2 = m$). We extract $n$ balls from this basket **with replacement** and note their colors. Find the probability that out of these $n$ balls exactly $r$ were white.

**Problem 4.** A fair dice is rolled $n$ times. What is the probability that at least 1 of the 6 values never appears?

**Problem 5.** There are $(m+1)$ baskets and each basket has exactly $m$ balls in it. Additionally, for every $n = 0, 1, \dots, m$, we know that basket $n$ contains exactly $n$ white and $(m-n)$ black balls. We pick a basket at random and pick $k$ balls from it **with replacement**. Find the probability that $(k+1)$-th ball will be white if all $k$ balls were white.

**Problem 6.** A good student solves a problem correctly with probability 0.95, while a bad student — with probability 0.15. How many problems should the test include so that the probability that good student does not pass the test does not exceed 0.01, and the probability that the bad student passes the test does not exceed 0.1?

**Problem 7.** If you get a positive result on a COVID test that only gives a false positive with probability 0.001, what's the chance that you've actually got COVID, if

1. The probability that a person has COVID is 0.01
2. The probability that a person has COVID is 0.0001

**Problem 8.** Show that if events $A$ and $B$ are independent, then

1. Events $A$ and $\overline{B}$ are independent
2. Events $\overline{A}$ and $B$ are independent
3. Events $\overline{A}$ and $\overline{B}$ are independent

**Problem 9.** Show that if $\mathbb{P}(A|B) = \mathbb{P}\left(A|\overline{B}\right)$ for two non-zero events $A$ and $B$, then $\mathbb{P}(AB) = \mathbb{P}(A)\,\mathbb{P}(B)$ (i.e. $A$ and $B$ are independent).

**Problem** 10$^*$ **.** We will say that set $A \subset \mathbb{N}$ has asymptotic density $\theta$ if there exists the following limit:

$$\lim_{n\to\infty} \frac{|A \cap \{1, \dots, n\}|}{m}$$

Denote the family of such sets (which have asymptotic density) as $\mathcal{A}$. Is $\mathcal{A}$ a $\sigma$-algebra?

**Problem** 11$^*$ **.** (Bonferroni inequality) Prove that

$$\mathbb{P}\left(\bigcup_{k=1}^{n} A_n\right) \geqslant \sum_{k=1}^{n} \mathbb{P}(A_k) - \sum_{1 \leqslant i < j \leqslant n} \mathbb{P}(A_i \cap A_j)$$