Transformations

Examples of transformations

- Linear transformations of random variables and vectors $Y = aX + b$
- Non-linear invertible transformations of random variables $Y = g(X)$
- Sums $Y = X_1 + X_2$

Transformations previously

We have a technique for computing the expectation of transformed random variable. What is its name?

LOTUS:

$$\mathbb{E}[g(X)] = \sum_x g(x)\mathbb{P}(X = x)$$

It works with continuous r.v.s:

$$\mathbb{E}[g(X)] = \int g(x) f_X(x) dx$$

And it works in multiple dimensions:

$$\mathbb{E}[g(X, Y)] = \sum_x \sum_y g(x, y)\mathbb{P}(X = x, Y = y)$$

Transformations now

Now we want not just the expected value, but the whole distribution (CDF, PMF, PDF). The approach depends on whether the distribution is discrete or continuous.

Discrete case

The formula:

$$\mathbb{P}(g(X) = y) = \sum_{x \text{ such that } g(x)=y} \mathbb{P}(X = x)$$

If $g(\cdot)$ is one-to-one, it simplifies to:

$$\mathbb{P}(g(X) = y) = \mathbb{P}(X = g^{-1}(y))$$

Example 1

Let $X \sim Bin(n, p)$. Find the PDF of $Y = \exp(X)$.

Solution 1

So $g(x) = \exp(x)$, it's one-to-one and the inverse is $g^{-1}(x) = \log x$.

$$\mathbb{P}(Y = y) = \mathbb{P}(X = g^{-1}(y)) = \mathbb{P}(X = \log y)$$

Continuous case

In the continuous case, when additionally $g(\cdot)$ is one-to-one, continuous and strictly increasing, we have the following relation for CDF:

$$F_{g(X)}(y) = \mathbb{P}(g(X) \leqslant y) = \mathbb{P}(X \leqslant g^{-1}(y)) = F_X(g^{-1}(y))$$

To get the PDF, we need to differentiate this relation at every point with the chain rule:

$$f_{g(X)}(g(x))d\left(g(x)\right) = f_X(x)dx$$

To account for the case when $g(\cdot)$ is strictly decreasing, we add the modulus

$$f_{g(X)}(g(x)) = f_X(x)\left|\frac{dg(x)}{dx}\right|^{-1}$$

This is called 1D change of variables formula. Similarly to sicrete case, we can extend it to non one-to-one $g(\cdot)$ using the sum over x such that $g(x) = y$.

Example 2

Let $X \sim Exp(1)$. Find the PDF of $Y = \exp(-X)$.

Solution 2

So $g(x) = \exp(-x)$, it is one-to-one, and $g^{-1}(y) = -\log y$. Let's find

$$\frac{dg(x)}{dx} = -\exp(-x)$$

So,

$$f_Y(y) = f_X(x)\left|\frac{dg(x)}{dx}\right|^{-1} = f_X(-\log y)\exp(x) =$$
$$= f_X(-\log y)\exp(-\log y) = \frac{1}{y}f_X(-\log y)$$

Example 3

Let $X \sim \mathcal{N}(0,1)$. Find the PDF of $Y = X^2$. This distribution is called chi-square distribution.

Solution 3

So, $g(x) = x^2$. It is not one-to-one, so we need the sum:

$$f_Y(y) = \sum_{x=\{-\sqrt{y},\sqrt{y}\}} f_X(x)\left|\frac{dg(x)}{dx}\right|^{-1} = \sum_{x=\{-\sqrt{y},\sqrt{y}\}} f_X(x)2x =$$
$$= f_X(\sqrt{y})\frac{1}{2\sqrt{y}} + f_X(-\sqrt{y})\frac{1}{2\sqrt{y}} =$$
$$= 2f_X(\sqrt{y})\frac{1}{2\sqrt{y}}$$

Example 4

Let X be a random variable with PDF f_X. Find PDF of $Y = aX + b$.

Solution 4

So, $g(x) = ax + b$, it is one-to-one, and the inverse is $g^{-1}(y) = \frac{1}{a}(y - b)$. We need to calculate

$$\frac{dg(x)}{dx} = \frac{d(ax+b)}{dx} = a$$

$$f_Y(y) = f_X(x)\left|\frac{dg(x)}{dx}\right|^{-1} = f_X\left(\frac{y-b}{a}\right)\frac{1}{|a|}$$

Multivariate transformations

Consider n-dimensional random vector $X \in \mathbb{R}^n$ with continuous distribution with PDF f_X. Let $g : A_0 \to B_0$ be invertive one-to-one function from open subset A_0 containing support of X to an open subset B_0 containing the range of $g(\cdot)$. Denote $Y = g(X)$ and $y = g(x)$. Suppose that all partial derivatives $\frac{\partial y_i}{\partial x_j}$ exist and are continuous. Then, we can form the Jacobian matrix:

$$\frac{\partial y}{\partial x} = \begin{pmatrix} \frac{\partial y_1}{\partial x_1} & \frac{\partial y_1}{\partial x_2} & \cdots & \frac{\partial y_1}{\partial x_n} \\ \vdots & & \ddots & \vdots \\ \frac{\partial y_n}{\partial x_1} & \frac{\partial y_n}{\partial x_2} & \cdots & \frac{\partial y_n}{\partial x_n} \end{pmatrix}$$

Assume that this matrix is non-degenerate. Then, the PDF of Y is:

$$f_Y(y) = f_X(g^{-1}(y))\left|\det\frac{\partial y}{\partial x}\right|^{-1}$$

If the function is not one-to-one, we add sum.

Example 5

Consider n-dimensional random vector $X \in \mathbb{R}^n$ with continuous distribution, a non-degenerate matrix $A \in \mathbb{R}^{m\times n}$ and a vector $b \in \mathbb{R}^m$, then define random vector $Y = AX + b$. What is the expected value, the covariance matrix and the PDF of Y?

Solution 5

Its expected value will be $\mathbb{E}[Y] = A\mathbb{E}[X] + b$ and its covariance matrix will be:

$\Sigma_Y = \mathbb{E}\left[(AX - A\mathbb{E}[X])(AX - A\mathbb{E}[X])^\top\right] = A\mathbb{E}\left[(X - \mathbb{E}[X])(X - \mathbb{E}[X])^\top\right] A^\top = A\Sigma_X A^\top$

So,

- $g(x) = Ax + b$
- $\frac{\partial y}{\partial x} = A$
- $g^{-1}(y) = A^{-1}(y - b)$

Therefore,

$$f_Y(y) = f_X(g^{-1}(y))\left|\det \frac{\partial y}{\partial x}\right|^{-1} = f_X(A^{-1}(y-b))\frac{1}{|\det A|}$$

Example 6

Consider n independent standard normal r.v.s $X_1, \ldots, X_n \sim \mathcal{N}(0,1)$ and vector $X = (X_1, \ldots, X_n)^\top$. We know that

- $\mathbb{E}[X] = 0$
- $\Sigma_X = I$
- $f_X(x) = (2\pi)^{-n/2}\exp(-\frac{1}{2}x^\top x)$

Consider a non-degenerate matrix $A \in \mathbb{R}^{n\times n}$ and $Y = AX + m$. Find its expectation, covariance matrix and distribution.

Solution 6

From the previous example, $\mathbb{E}[Y] = m$ and $\Sigma_Y = AA^\top$

$$\begin{aligned} f_X(x) &= f_X(A^{-1}(y-m))\frac{1}{|\det A|} = \\ &= \frac{1}{(2\pi)^{n/2}|\det A|}\exp\left(-\frac{(y-m)^\top A^{-\top}A^{-1}(y-m)}{2}\right) = \\ &= \frac{1}{(2\pi)^{n/2}\sqrt{\det \Sigma_Y}}\exp\left(-\frac{(y-m)^\top \Sigma_Y^{-1}(y-m)}{2}\right) \end{aligned}$$

What we just did, is we obtained a new random normal vector with controllable parameters from a standard normal random vector. This is a very useful property of a random normal vectors, but also demonstrates the power of linear transforms.

Example 7

Let random vector (X, Y) have PDF $f_{X,Y}(x,y)$. Find the density of $Z = X + Y$.

Solution 7

In order to find this density, consider transform

$$\begin{pmatrix} Z \\ Y \end{pmatrix} = A \begin{pmatrix} X \\ Y \end{pmatrix}$$

So, matrix A is...?

$$A = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$$

Then,

$$f_{Z,Y}(z,y) = f_{X,Y}(A^{-1}(x\ y)^\top, y)|\det A|^{-1} = f_{X,Y}(z-y, y)$$

Finally,

$$f_Z(z) = \int f_{Z,Y}(z,y)dy = \int f_{X,Y}(z-y,y)dy$$

If $X \perp Y$,

$$f_Z(z) = \int f_X(z-y)f_Y(y)dy$$

This is the convolution rule that we studied on Seminar 4.

Convolution rule

Consider independent r.v.s X and Y. Then, their sum $Z = X + Y$ is distributed:

- If they are discrete,

$$\mathbb{P}(Z = n) = \sum_{k=-\infty}^{\infty} \mathbb{P}(X = k)\mathbb{P}(Y = n - k)$$

- If they are continuous,

$$f_Z(z) = \int_{-\infty}^{\infty} f_X(z-y)f_Y(y)dy$$

Example 8

Let $X \sim Bi(n,p)$ and $Y \sim Bi(m,p)$ be independent. What is the distribution of $Z = X + Y$?

Solution 8

$$\begin{aligned}\mathbb{P}(X+Y=k) &= \sum_j \binom{n}{j} p^j(1-p)^{n-j} \binom{m}{k-j} p^{k-j}(1-p)^{m-k+j} = \\ &= p^k(1-p)^{n+m-k}\sum_j \binom{n}{j}\binom{m}{k-j} = \\ &= \binom{n+m}{k} p^k(1-p)^{n+m-k}\end{aligned}$$

$$Z \sim Bi(n+m, p)$$

Example 9

Let $X, Y \sim Exp(1)$ be independent. Find the PDF of $Z = \frac{X}{X+Y}$.

Solution 9

This one will be a non-linear transform. Consider transform $(X, Y) \to (Z, U)$, where $U = X + Y$. The inverse transform is $X = UZ, Y = U - UZ$. Jacobian of the inverse transform is

$$\frac{\partial(x,y)}{\partial(z,u)} = \begin{pmatrix} u & z \\ -u & 1-z \end{pmatrix}$$

The joint density is

$$f_{Z,U}(z,u) = f_{X,Y}(g^{-1}(z,u)) \left|\det \frac{\partial(x,y)}{\partial(z,u)}\right| = f_{X,Y}(uz, u-uz)u = ue^{u}, u > 0, 0 < z \leqslant 1$$

The joint density does not depend on z, it means that marginal density of Z is Uniform with support $[0, 1]$.

Homework problems

1. Compute the moment-generating function of $Geom(p)$. Use it to find expectation and variance.
2. Let X and Y be i.i.d. $Geom(p)$, and $N = X + Y$. Find the joint PMF of X, Y, N. Find the joint PMF of X and N. Find the conditional PMF of X given $N = n$
3. Let $U \sim U[0, \frac{\pi}{2}]$. Find the PDF of $\sin(U)$.
4. Let X and Y be i.i.d. $Exp(\lambda)$, and $T = \log(X/Y)$. Find the CDF and PDF of T.