MSAI Probability Home Assignment 1-3
soft deadline: 13/10/2024 23:59 Moscow Time
hard deadline: 24/10/2024 18:00 Moscow Time

As announced earlier, grading for HWs consists of points and bonus points. Solving bonus (indicated with a star) problems is not required, but recommended. Solving all homeworks' normal problems correctly will give you a score of 7, solving all homeworks' bonus problems correctly will give you additional 2 points to the score.

Soft deadline is the intended deadline for this homework. Hard deadline is the date and time of homework discussion webinar, where we will discuss solutions to this homework. After solutions are released, no more homeworks are accepted.

Hand-written solutions are accepted if the handwriting is clear enough and scanned with sufficient quality, but LaTeX is always preferable. This homework includes a python task, which can be solved in Google Colab or in a local Jupyter Notebook. It is thus handy to solve everything (both LaTeX and code) in a single Jupyter Notebook.

Problem 1. (2 points) Find the probability that in a class of n students, at least two share the same birthday. Use counting rules. Find a numerical answer for $n = 50$.

Problem 2. (1 point) Prove that

$$\sum_{j=0}^{k} \binom{n}{j} \binom{m}{k-j} = \binom{m+n}{k}$$

A story proof can be accepted as well as mathematical derivation.

Problem 3. (2 points) A fair die is rolled n times. What is the probability that at least 1 of the 6 values never appears? Give a formula answer with derivation. Hint: use inclusion-exclusion formula.

Problem 4. (3 points) There are two baskets. The first basket contains one white ball, the second basket contains one black ball. One basket is chosen randomly and a white ball is put into the chosen basket. The balls in this basket are shuffled. Then one ball is extracted from this basket. This ball turns out to be white. What is the posterior probability that the second ball drawn from this basket is also white?

Problem 5. (2 points) If you get a positive result on a COVID test that only gives a false positive with probability 0.001 (true positive with probability 0.999), what's the chance that you've actually got COVID, if

1. (1 point) The prior probability that a person has COVID is 0.01
2. (1 point) The prior probability that a person has COVID is 0.0001

Problem 6* . (2 bonus point) Use the result of Problem 1 and Python to create a plot of this probability, and note after which n the probability becomes more than 50%.

Problem 7* . (2 bonus point) Use the result of Problem 3 and Python to create a plot of this probability, and note after which n the probability becomes less than 1%.

Problem 8* . (2 bonus points) The cloakroom of a theater has randomly permuted all n visitors' hats. Find the probability that at least one visitor gets his hat. Give a formula answer with derivation. Given $n = 4$, give a number answer. Hint: use inclusion-exclusion formula.

Problem 9* . (2 bonus points) You are the contestant on the Monty Hall show. Monty is trying out a new version of his game, with rules as follows. You get to choose one of three doors. One door has a car behind it, another has a computer, and the other door has a goat (with all permutations equally likely). Monty, who knows which prize is behind each door, will open a door (but not the one you chose) and then let you choose whether to switch from your current choice to the other unopened door.

Suppose that Monty always opens the door that reveals your less preferred prize out of the two alternatives, e.g., if he is faced with the choice between revealing the goat or the computer, he will reveal the goat. Monty opens a door, revealing a goat. Given this information, should you switch? If you do switch, what is your probability of success in getting the car?