Seminar 5

In [1]:
```
import numpy as np
import pandas as pd

import scipy.stats as sts

import IPython.display as dp
import matplotlib.pyplot as plt
import seaborn as sns

dp.set_matplotlib_formats("retina")
sns.set(style="whitegrid", font_scale=1.5)
sns.despine()

%matplotlib inline
```

```
/var/folders/33/j0cl7y453td68qb96j7bqcj4cf41kc/T/ipykernel_1174/310970005
6.py:10: DeprecationWarning: `set_matplotlib_formats` is deprecated since
IPython 7.23, directly use `matplotlib_inline.backend_inline.set_matplotli
b_formats()`
  dp.set_matplotlib_formats("retina")
```

In [2]:
```
plt.rc("figure", figsize=(20, 10))
plt.rc("font", size=13)
```

Recap

A certain company has $n + m$ employees, consisting of n women and m men. The company is deciding which employees to promote.

- Suppose for this part that the company decides to promote t employees, where $1 \leqslant t \leqslant n + m$, by choosing t random employees (with equal probabilities for each set of t employees). What is the distribution of the number of women who get promoted?
- Now suppose that instead of having a predetermined number of promotions to give, the company decides independently for each employee, promoting the employee with probability p. Find the distributions of the number of women who are promoted, the number of women who are not promoted, and the number of employees who are promoted.

Solution

- We are interested in the number of women X in the set of t promoted employees sampled from n women and n men. What is the distribution in question?

- It is hypergeometric distribution $HGeom(n, m, t)$, so we know the answer

$$\mathbb{P}(X = k) = \frac{\binom{n}{k}\binom{m}{t-k}}{\binom{n+m}{t}}$$

- If the company decides independently for each of n women if they will be promoted with equal probabilities p, the number Y of promoted women follows which distribution?

- It is Binomial distribution $Bi(n, p)$, so we know the answer:

$$\mathbb{P}(Y = k) = \binom{n}{k} p^k (1-p)^{n-k}$$

CDFs

Reminder: the cumulative distribution function (CDF) is defined as

$$F_X(x) = \mathbb{P}(X < x)$$

It has the following properties:

- F_X is non-decreasing
- $\lim\limits_{x \to -\infty} F_X(x) = 0$
- $\lim\limits_{x \to +\infty} F_X(x) = 1$
- F_X if right-continuous

Example 1

Draw PMF and CDF of $Bi(3, 0.7)$.

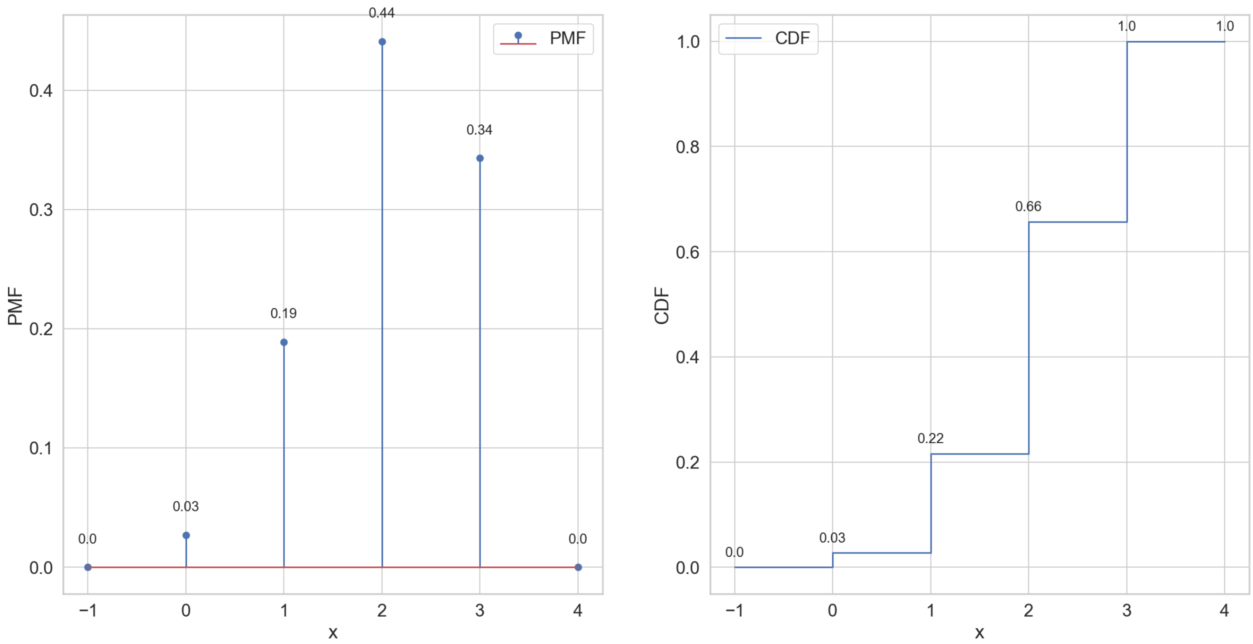
In [3]:
```
my_binomial = sts.binom(n=3, p=0.7) # import scipy.stats as sts
x = np.arange(-1, 5) # import numpy as np
y = my_binomial.pmf(x)

fig, ax = plt.subplots(1,2)
ax[0].stem(x, y, label="PMF")
for xx, yy in zip(x, y):
```

```
    ax[0].text(xx, yy + 0.02, str(round(yy, 2)), horizontalalignment="cen
ax[0].set_xlabel("x")
ax[0].set_ylabel("PMF")
ax[0].legend();

y = my_binomial.cdf(x)

ax[1].step(x, y, where="post", label="CDF")
for xx, yy in zip(x, y):
    ax[1].text(xx, yy + 0.02, str(round(yy, 2)), horizontalalignment="cen
ax[1].set_xlabel("x")
ax[1].set_ylabel("CDF")
ax[1].legend();
```



Example 2

A coin is tossed repeatedly until it lands Heads for the first time. Let X be the number of tosses that landed Tails, and let p be the probability of Heads of the coin. Find the distribution of X and its PMF and CDF.

Solution 2

$$\mathbb{P}(X = k) = \mathbb{P}(k \text{ tails}) \cdot \mathbb{P}(1 \text{ heads}) = (1-p)^k p$$

Geometric distribution $X \sim Geom(p)$

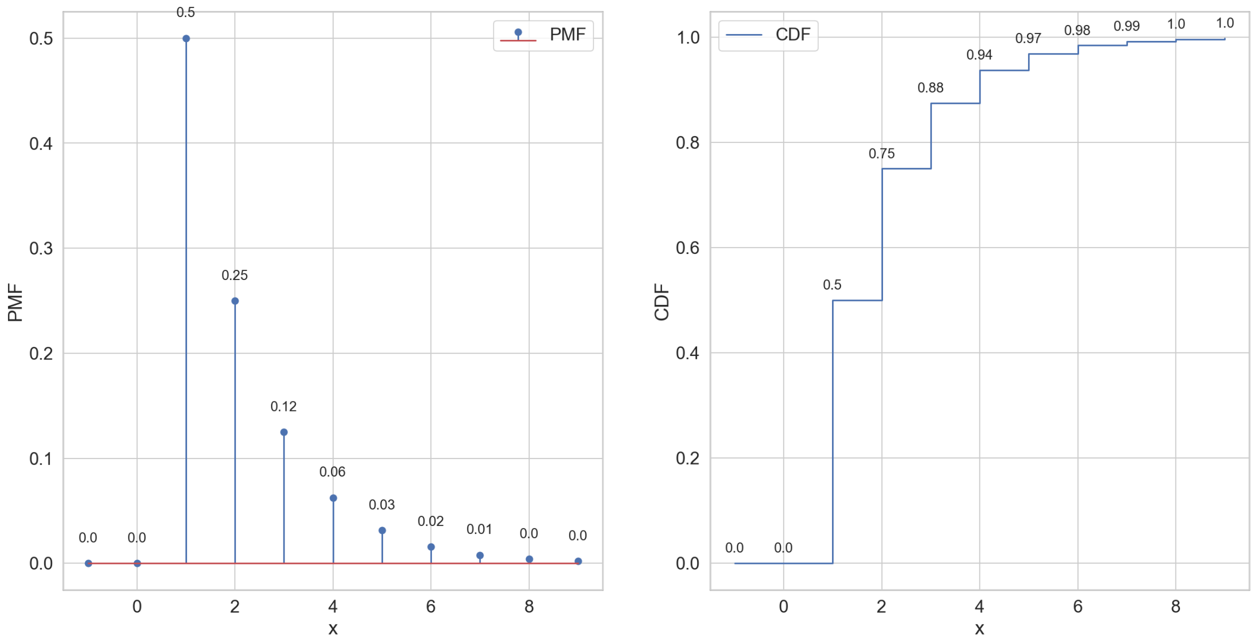
In [10]:
```
my_geometric = sts.geom(p=0.5)
x = np.arange(-1, 10)
y = my_geometric.pmf(x)

fig, ax = plt.subplots(1,2)
```

```
ax[0].stem(x, y, label="PMF")
for xx, yy in zip(x, y):
    ax[0].text(xx, yy + 0.02, str(round(yy, 2)), horizontalalignment="cen
ax[0].set_xlabel("x")
ax[0].set_ylabel("PMF")
ax[0].legend();

y = my_geometric.cdf(x)

ax[1].step(x, y, where="post", label="CDF")
for xx, yy in zip(x, y):
    ax[1].text(xx, yy + 0.02, str(round(yy, 2)), horizontalalignment="cen
ax[1].set_xlabel("x")
ax[1].set_ylabel("CDF")
ax[1].legend();
```



Independence of random variables

Reminder: events A and B were called independent if

$$\mathbb{P}(A \cap B) = \mathbb{P}(A)\mathbb{P}(B)$$

Random variables X and Y are called **independent** if

$$\mathbb{P}(X \leqslant x, Y \leqslant y) = \mathbb{P}(X \leqslant x)\mathbb{P}(Y \leqslant y)$$

Example 3

You roll two dice. X is the random result of first die, Y is the random result of second die, $Z = X + Y$.

How do we find the distribution of Z?

$$\mathbb{P}(Z = k) = \sum_m \mathbb{P}(X = m)\mathbb{P}(Y = k - m)$$

- Are X and Y independent?

Yes

- Are X and Z independent?

No

Transformations of random variables

Random variables transform like functions, i.e. if $Y = \varphi(X)$, then

$$Y(\omega) = \varphi(X(\omega))$$

Examples of transformations

- Linear transformations of random variables and vectors $Y = aX + b$
- Non-linear invertible transformations of random variables $Y = g(X)$
- Sums $Y = X_1 + X_2$

If $g(\cdot)$ is one-to-one, it simplifies to:

$$\mathbb{P}(Y = g(x)) = \mathbb{P}(X = x)$$

The general formula:

$$\mathbb{P}(g(X) = y) = \sum_{x \text{ such that } g(x) = y} \mathbb{P}(X = x)$$

Example 4

Let X be a random variable with CDF F_X. Find CDF of $Y = aX + b$.

Solution 4

If $a > 0$:

$$F_Y(y) = \mathbb{P}(Y \leqslant y) = \mathbb{P}(aX + b \leqslant y) = \mathbb{P}\left(X \leqslant \frac{y-b}{a}\right)$$

Example 5

Let $X \sim Bin(n, p)$. Find the PDF of $Y = \exp(X)$.

Solution 5

So $g(x) = \exp(x)$, it's one-to-one and the inverse is $g^{-1}(x) = \log x$.

$$\mathbb{P}(Y = y) = \mathbb{P}(X = g^{-1}(y)) = \mathbb{P}(X = \log y)$$

Example 6

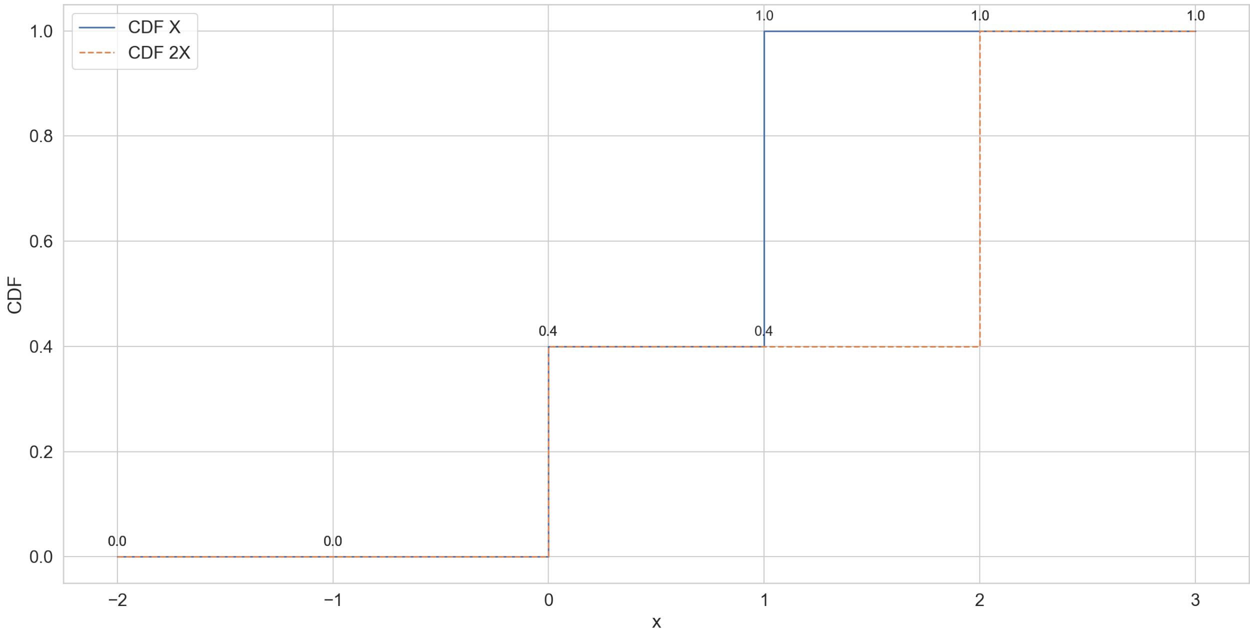
Let $X \sim Be(p)$. Find CDF of $Y = 2X$.

Solution 6

$$F_Y = \mathbb{P}(Y \leqslant y) = \mathbb{P}(2X \leqslant y) = \mathbb{P}\left(X \leqslant \frac{y}{2}\right) = F_X\left(\frac{y}{2}\right)$$

In [5]:
```
my_bernoulli = sts.bernoulli(p=0.6)
x = np.arange(-2, 4)
y = my_bernoulli.cdf(x)
y2 = my_bernoulli.cdf(x / 2)

fig, ax = plt.subplots()
ax.step(x, y, where="post", label="CDF X")
ax.step(x, y2, where="post", ls="--", label="CDF 2X")
for xx, yy, yy2 in zip(x, y, y2):
    ax.text(xx, yy + 0.02, str(round(yy, 2)), horizontalalignment="center
    ax.text(xx, yy2 + 0.02, str(round(yy2, 2)), horizontalalignment="cent
ax.set_xlabel("x")
ax.set_ylabel("CDF")
ax.legend();
```



Example 7

Let $X \sim Be(p)$. Find the distribution of $Y = X^2$.

Solution 7

- $X \sim Be(p)$
- $X \in \{0, 1\}$
- $Y = X^2$
- $Y \in \{0, 1\}$

So we need to define PMF at 0 and 1:

$$\begin{cases} \mathbb{P}(Y=0) = \mathbb{P}(X^2=0) = \mathbb{P}(X=\sqrt{0}) = \mathbb{P}(X=0) = 1-p \\ \mathbb{P}(Y=1) = \mathbb{P}(X^2=1) = \mathbb{P}(X=\pm\sqrt{1}) = \mathbb{P}(X=-1) + \mathbb{P}(X=1) = 0 + p = \end{cases}$$