Seminar 5

In [1]:
```
import numpy as np
import pandas as pd

import scipy.stats as sts

import IPython.display as dp
import matplotlib.pyplot as plt
import seaborn as sns

dp.set_matplotlib_formats("retina")
sns.set(style="whitegrid", font_scale=1.5)
sns.despine()

%matplotlib inline
```

```
/var/folders/33/j0cl7y453td68qb96j7bqcj4cf41kc/T/ipykernel_1174/310970005
6.py:10: DeprecationWarning: `set_matplotlib_formats` is deprecated since
IPython 7.23, directly use `matplotlib_inline.backend_inline.set_matplotl
ib_formats()`
  dp.set_matplotlib_formats("retina")
```

In [2]:
```
plt.rc("figure", figsize=(20, 10))
plt.rc("font", size=13)
```

CDFs

Reminder: the cumulative distribution function (CDF) is defined as

$$F_X(x) = \mathbb{P}(X \leqslant x)$$

It has the following properties:

- F_X is non-decreasing
- $\lim\limits_{x \to -\infty} F_X(x) = 0$
- $\lim\limits_{x \to +\infty} F_X(x) = 1$
- F_X if right-continuous

Example 1

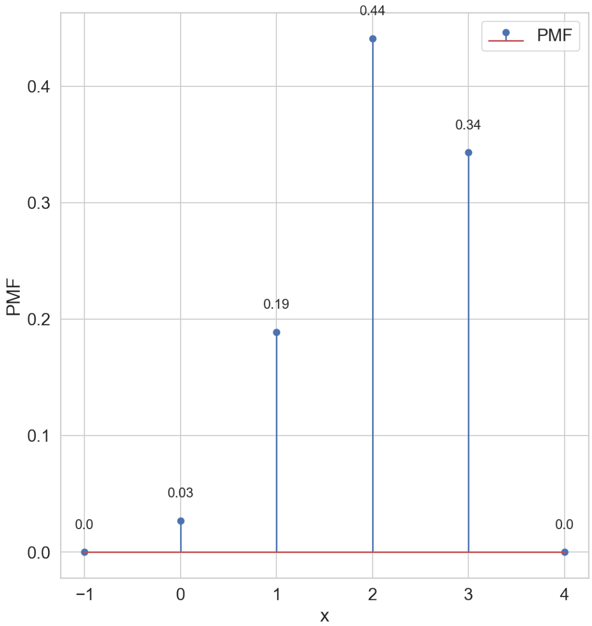
Draw PMF and CDF of $Bi(3, 0.7)$.

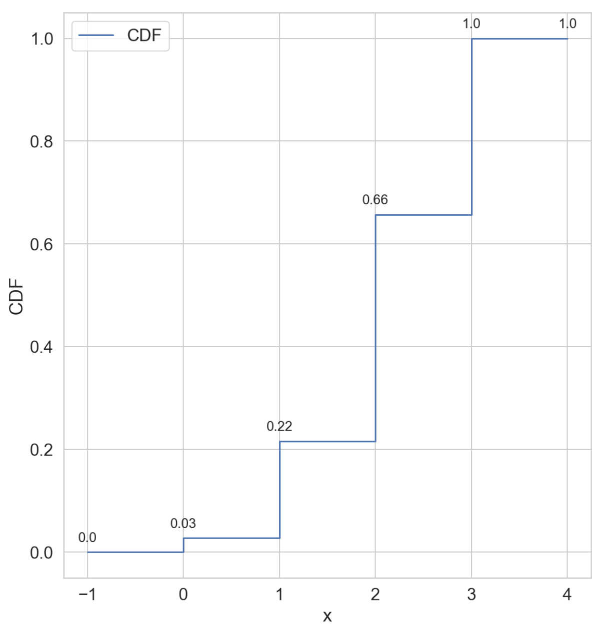
In [3]:
```
my_binomial = sts.binom(n=3, p=0.7) # import scipy.stats as sts
x = np.arange(-1, 5) # import numpy as np
y = my_binomial.pmf(x)

fig, ax = plt.subplots(1,2)
ax[0].stem(x, y, label="PMF")
for xx, yy in zip(x, y):
    ax[0].text(xx, yy + 0.02, str(round(yy, 2)), horizontalalignment="cen
ax[0].set_xlabel("x")
ax[0].set_ylabel("PMF")
ax[0].legend();

y = my_binomial.cdf(x)

ax[1].step(x, y, where="post", label="CDF")
for xx, yy in zip(x, y):
    ax[1].text(xx, yy + 0.02, str(round(yy, 2)), horizontalalignment="cen
ax[1].set_xlabel("x")
ax[1].set_ylabel("CDF")
ax[1].legend();
```





Example 2

A coin is tossed repeatedly until it lands Heads for the first time. Let X be the number of tosses that landed Tails, and let p be the probability of Heads of the coin. Find the distribution of X and its PMF and CDF.

Solution 2

$$\mathbb{P}(X = k) = \mathbb{P}(k \text{ tails}) \cdot \mathbb{P}(1 \text{ heads}) = (1 - p)^k p$$

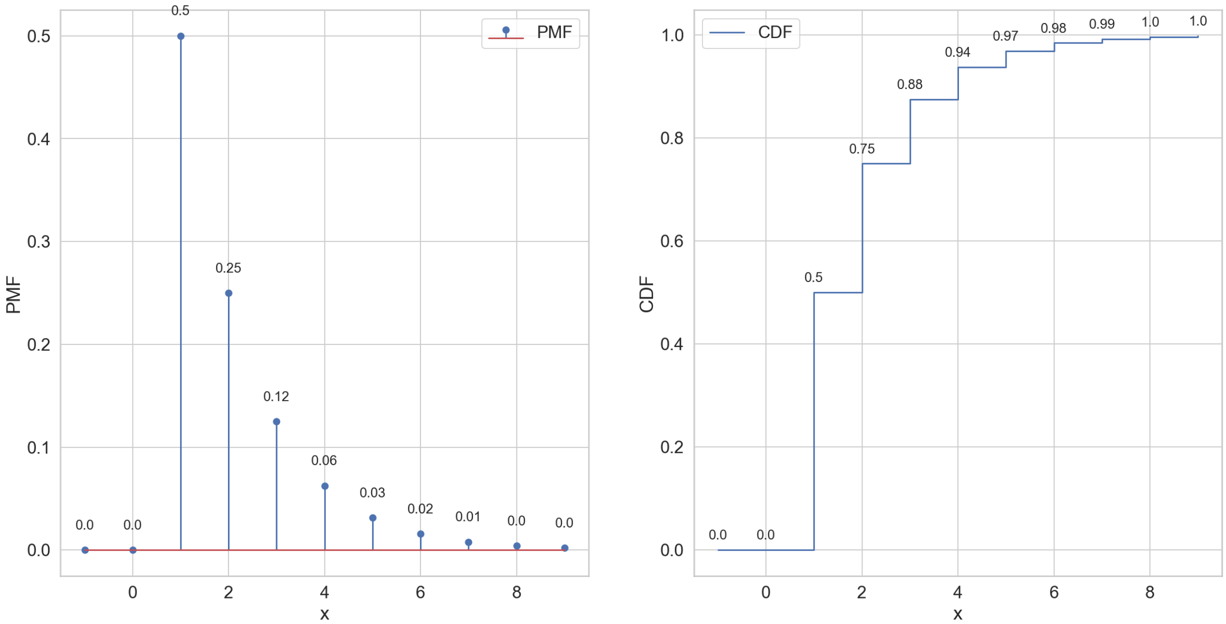
Geometric distribution $X \sim Geom(p)$

In [10]:
```
my_geometric = sts.geom(p=0.5)
x = np.arange(-1, 10)
y = my_geometric.pmf(x)

fig, ax = plt.subplots(1,2)
ax[0].stem(x, y, label="PMF")
for xx, yy in zip(x, y):
    ax[0].text(xx, yy + 0.02, str(round(yy, 2)), horizontalalignment="cen
ax[0].set_xlabel("x")
ax[0].set_ylabel("PMF")
ax[0].legend();

y = my_geometric.cdf(x)

ax[1].step(x, y, where="post", label="CDF")
for xx, yy in zip(x, y):
    ax[1].text(xx, yy + 0.02, str(round(yy, 2)), horizontalalignment="cen
ax[1].set_xlabel("x")
ax[1].set_ylabel("CDF")
ax[1].legend();
```



Independence of random variables

Reminder: events A and B were called independent if

$$\mathbb{P}(A \cap B) = \mathbb{P}(A)\mathbb{P}(B)$$

Random variables X and Y are called **independent** if

$$\mathbb{P}(X \leqslant x, Y \leqslant y) = \mathbb{P}(X \leqslant x)\mathbb{P}(Y \leqslant y)$$

Example 3

You roll two dice. X is the random result of first die, Y is the random result of second die, $Z = X + Y$.

How do we find the distribution of Z?

$$\mathbb{P}(Z = k) = \sum_m \mathbb{P}(X = m)\mathbb{P}(Y = k - m)$$

- Are X and Y independent?

Yes

- Are X and Z independent?

No

Transformations of random variables

Random variables transform like functions, i.e. if $Y = \varphi(X)$, then

$$Y(\omega) = \varphi(X(\omega))$$

Example 4

Let X be a random variable with CDF F_X. Find CDF of $Y = aX + b$.

Solution 4

If $a > 0$:

$$F_Y(y) = \mathbb{P}(Y \leqslant y) = \mathbb{P}(aX + b \leqslant y) = \mathbb{P}\left(X \leqslant \frac{y-b}{a}\right)$$

Example 5

Let $X \sim Be(p)$. Find CDF of $Y = 2X$.

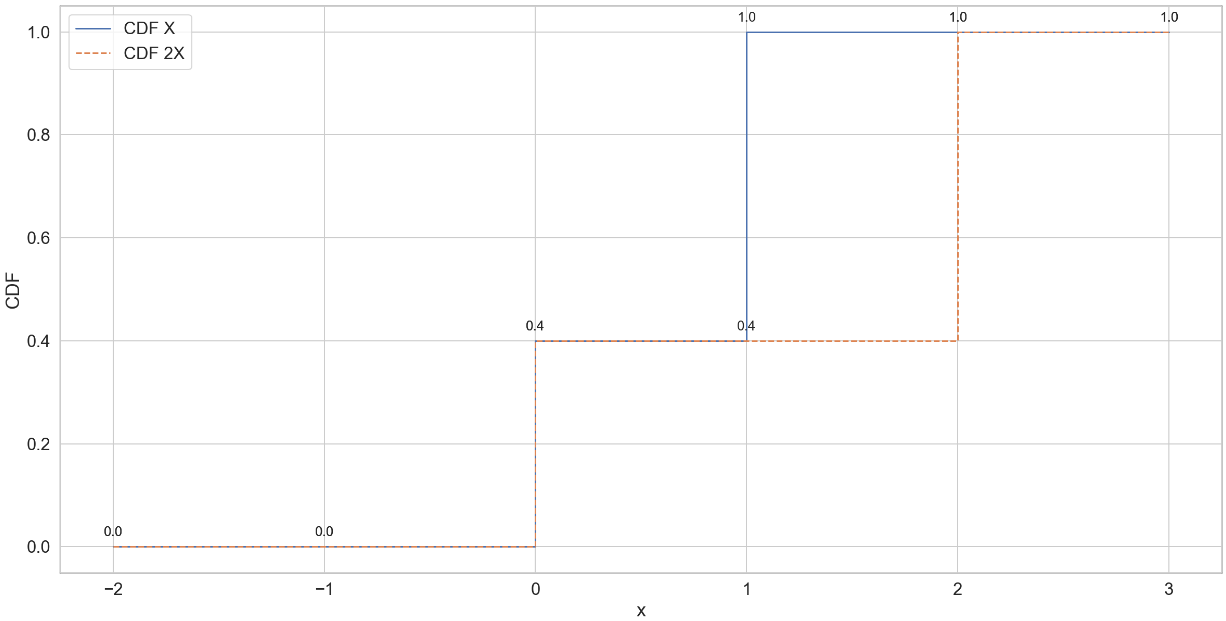
Solution 5

$$F_Y = \mathbb{P}(Y \leqslant y) = \mathbb{P}(2X \leqslant y) = \mathbb{P}\left(X \leqslant \frac{y}{2}\right) = F_X\left(\frac{y}{2}\right)$$

In [5]:

```
my_bernoulli = sts.bernoulli(p=0.6)
x = np.arange(-2, 4)
y = my_bernoulli.cdf(x)
y2 = my_bernoulli.cdf(x / 2)

fig, ax = plt.subplots()
ax.step(x, y, where="post", label="CDF X")
ax.step(x, y2, where="post", ls="--", label="CDF 2X")
for xx, yy, yy2 in zip(x, y, y2):
    ax.text(xx, yy + 0.02, str(round(yy, 2)), horizontalalignment="center
    ax.text(xx, yy2 + 0.02, str(round(yy2, 2)), horizontalalignment="cent
ax.set_xlabel("x")
ax.set_ylabel("CDF")
ax.legend();
```



Example 6

Let $X \sim Be(p)$. Find the distribution of $Y = X^2$.

Solution 6

- $X \sim Be(p)$
- $X \in \{0, 1\}$
- $Y = X^2$
- $Y \in \{0, 1\}$

So we need to define PMF at 0 and 1:

$$\begin{cases} \mathbb{P}(Y=0) = \mathbb{P}(X^2=0) = \mathbb{P}(X=\sqrt{0}) = \mathbb{P}(X=0) = 1-p \\ \mathbb{P}(Y=1) = \mathbb{P}(X^2=1) = \mathbb{P}(X=\pm\sqrt{1}) = \mathbb{P}(X=-1) + \mathbb{P}(X=1) = 0 + p = \end{cases}$$