Lecture 5:

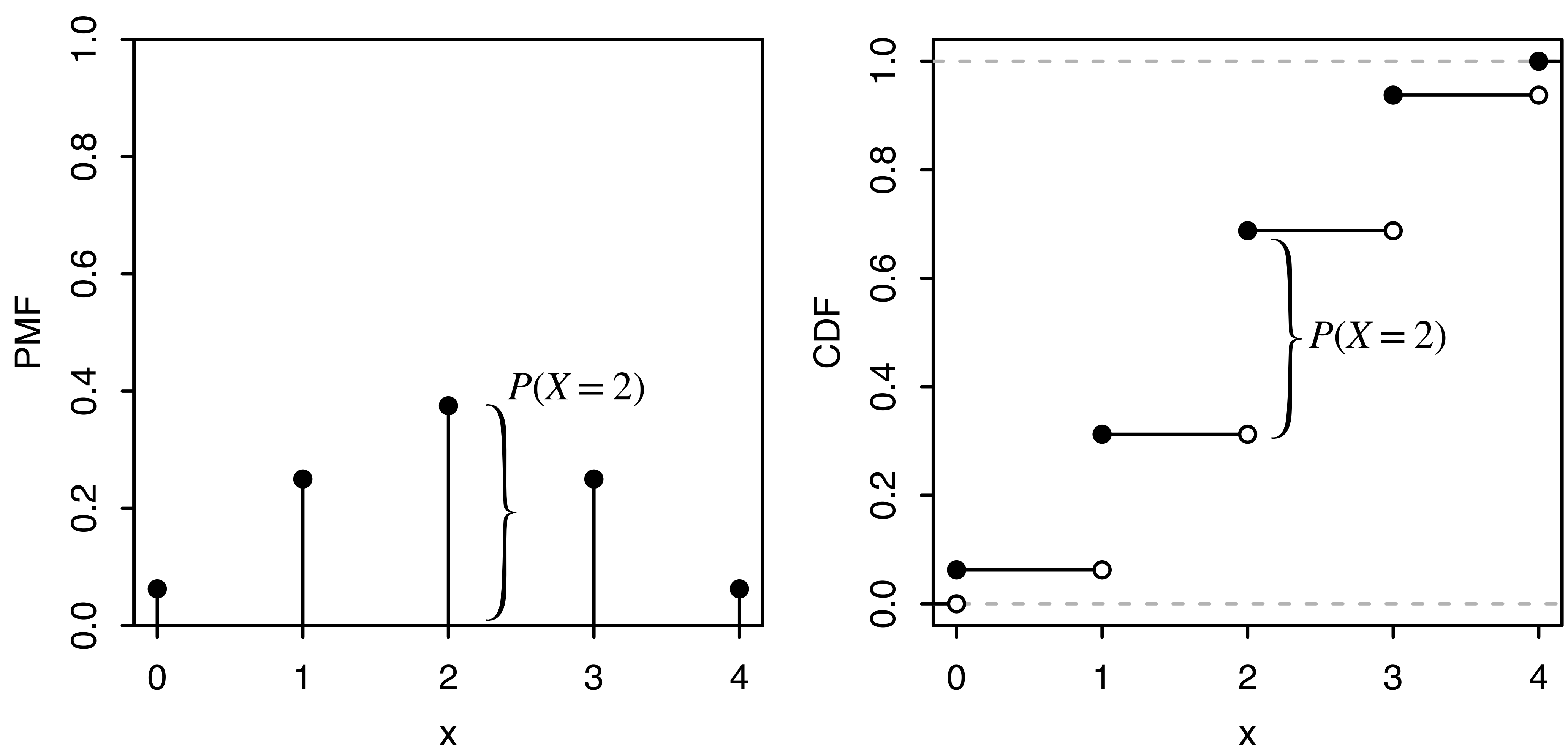
Random variables & their distributions - 2

Cumulative Distribution Functions (CDFs)

Cumulative Distribution Functions (CDFs)

Definition 3.6.1 The **cumulative distribution function** (CDF) of an r.v. X is the function F_X given by $F_X(x) = P(X \leq x)$.
(Only discrete r.v.-s have PMFs, but all r.v.-s have CDFs).

Example 3.6.2 Let $X \sim$ Bin$(4, 1/2)$, here are the PMF and the CDF:



Cumulative Distribution Functions (CDFs)

For discrete r.v.-s, we can freely convert between CDF and PMF:

From PMF to CDF: to find $P(X \leq 1.5)$, we sum the PDF over all values of the support that are ≤ 1.5

$$P(X \leq 1.5) = P(X = 0) + P(X = 1) = \left(\frac{1}{2}\right)^2 + 4\left(\frac{1}{2}\right)^4 = \frac{5}{16}$$

From CDF to PMF: the CDF of a discrete r.v. consists of jumps and flat regions. The (height of a jump at x) = (value of the PMF at x)

(Look at the plot from the example)

Cumulative Distribution Functions (CDFs)

Theorem 3.6.3 (Valid CDFs). Any CDF F has these properties:

1) **Increasing**: If $x_1 \leq x_2$, then $F(x_1) \leq F(x_2)$

2) **Right-continuous**: The CDF is continuous, except for having some jumps. At the point of a jump, the CDF is continuous from the **right** – that is, for any a,

$$F(a) = \lim_{x \to a^+} F(x)$$

3) **Convergence to 0 and 1** in the limits:

$$\lim_{x \to -\infty} F(x) = 0 \quad \text{and} \quad \lim_{x \to \infty} F(x) = 1$$

Functions of random variables

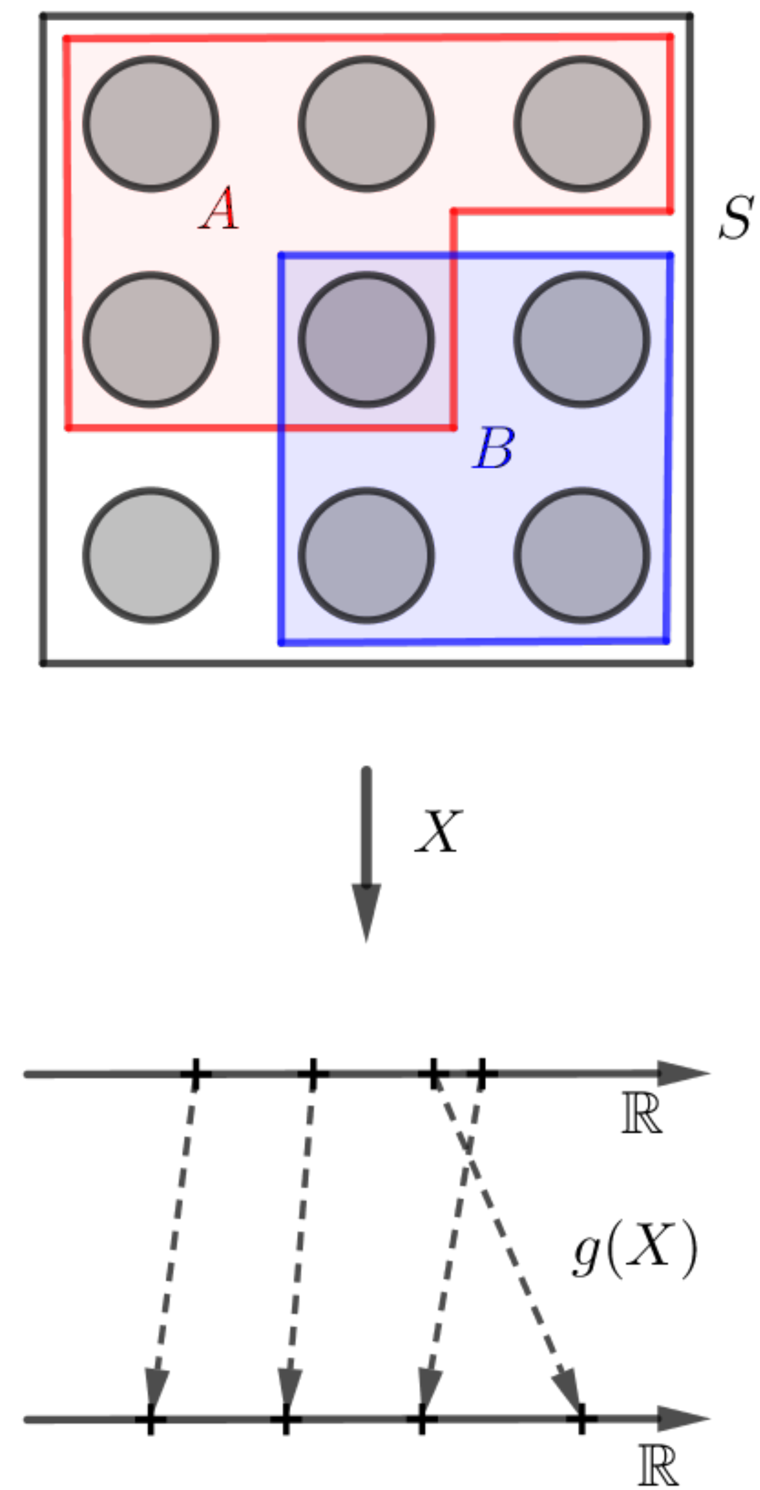
Functions of random variables

What if we take a function of a random variable: if X is an r.v, then $X^2, e^X, \sin(X)$ – are also r.v.-s, as is $g(X)$ for any function $g : \mathbb{R} \to \mathbb{R}$

Definition 3.7.1 (Function of an r.v). For an experiment with sample space S, an r.v. X and a function $g : \mathbb{R} \to \mathbb{R}$, $g(X)$ is the r.v that maps s to $g(X(s))$ for all $s \in S$.

What's the PMF of $Y = g(X)$? If g is a **one-to-one** function, then

$$P(Y = g(x)) = P(X = x)$$



Functions of random variables

In general, if g is **not** one-to-one,

Theorem 3.7.3 (PMF of $g(X)$) Let X – discrete r.v. and $g : \mathbb{R} \to \mathbb{R}$. Then the support of $g(X)$ = $\{y : g(x) = y$ for all $x \in \text{Supp}(X)\}$, and

$$P(g(X) = y) = \sum_{x:g(x)=y} P(X = x)$$

Definition 3.7.5 (Function of 2 r.v.-s). Given an experiment with sample space S, if X and Y are r.v.-s that map $s \in S$ to $X(s)$ and $Y(s)$ resp., then $g(X, Y)$ is the r.v. that maps s to $g(X(s), Y(s))$.

Functions of random variables

Example 3.7.6 (Maximum of two die rolls). We roll two fair 6-sided dice, X = (number on the 1st die), Y = (number on the 2nd).

Part of the S:

s	X	Y	$\max(X, Y)$
(1,2)	1	2	2
(1,6)	1	6	6
(3,1)	3	1	3

So we have →

z	$P(\max(X, Y) = z)$
1	$1/36$
2	$3/36$
3	$5/36$
4	$7/36$
5	$9/36$
6	$11/36$

Independence of random variables

Independence of random variables

Definition 3.8.1 (Independence of two r.v.-s). Random variables X and Y are said to be **independent** if

$$P(X \leq x, Y \leq y) = P(X \leq x)\, P(Y \leq y) \quad \text{for all } x, y \in \mathbb{R}$$

In the discrete case, this is equivalent to the condition:

$$P(X = x, Y = y) = P(X = x)\, P(Y = y)$$

Definition 3.8.2 (Independence of many r.v.-s). Random variables $X_1, \ldots, X_n$ are called **independent** if

$$P(X_1 \leq x_1, \ldots, X_n \leq x_n) = P(X_1 \leq x_1) \ldots P(X_n \leq x_n)$$

for all $x_1, \ldots, x_n \in \mathbb{R}^n$

Independence of random variables

Theorem 3.8.5 (Functions of independent r.v.-s). If X and Y are independent r.v.-s, then **any** function of X is independent of **any** function of Y.

Definition 3.8.6 (i.i.d.). We will often work with r.v.-s that are 1) independent and 2) have the same distribution. We call such r.v.-s **independent and identically distributed**, or **i.i.d.** in short

Example: Let X be the result of a die roll, and let Y be the result of a second, independent die roll. Then X and Y are i.i.d.

Independence of random variables

Theorem 3.8.8 If $X \sim$ Bin(n, p) (the # of successes in n indep. Bernoulli trials with success probability p), then we can write $X = X_1 + \ldots + X_n$, where X_i-s are i.i.d. Bern(p).

Theorem 3.8.9 If $X \sim$ Bin(n, p) and $Y \sim$ Bin(m, p) **and** X is **independent** of Y, then $X + Y \sim$ Bin$(n + m, p)$

Proof: Follows from LOTP by

$$P(X + Y = k) = \sum_{j=0}^{k} P(X + Y = k \,|\, X = j)\, P(X = j) =$$

$$= \sum_{j=0}^{k} P(Y = k - j)\, P(X = j) = \ldots = p^k q^{n+m-k} \binom{n+m}{k}$$

Independence of random variables

Definition 3.8.10 (Conditional independence of r.v.-s). Random variables X and Y are **conditionally independent** given an r.v. Z if for all $x, y \in \mathbb{R}$ and all z in the support of Z,

$$P(X \leq x, Y \leq y \,|\, Z = z) = P(X \leq x \,|\, Z = z)\, P(Y \leq y \,|\, Z = z)$$

(for discrete r.v.-s, it is equivalent to same but with $\leq \rightarrow =$)

Definition 3.8.11 (Conditional PMF). For any discrete r.v.-s X and Z, the function $P(X = x \,|\, Z = z)$ – a function of x for fixed z, is called the **conditional PMF** of X given $Z = z$