Lecture 10:

Transformations

Transformations

We'll talk about **transformations** of r.v.s. After applying a function to a r.v. X (or random vector $\mathbf{X}$), the goal is to find the distribution of the transformed variable.
Some examples of transformations:

- **Unit conversion**: In 1D, this is a loc-scale transform: $Y = aX + b$
- **Sums and averages**: $X_1, \ldots, X_n \to \bar{X}_n = (X_1 + \ldots + X_n)/n$
- **Convolutions**: we'll cover these in this lecture
- **Extreme values**: $X_1, \ldots, X_n \to \min / \max(X_1, \ldots, X_n)$, quantiles

Remember, if we just need the expectation of a transformed variable, we have LOTUS. But we want the whole distribution now.

Change of variables

Change of variables

Theorem 8.1.1 (Change of variables in 1D). Let X be a continuous r.v. with PDF f_X and let $Y = g(X)$, where g is 1) differentiable and 2) strictly increasing (/decreasing). Then the PDF of Y is

$$f_Y(y) = f_X(x)\left|\frac{dx}{dy}\right|$$

where $x = g^{-1}(y)$. The support of Y is all $g(x)$ with x in $\text{Supp}(X)$.

8.1.2 When finding the distribution of Y, be sure to:

1) Check the assumptions of the theorem
2) Express the final answer for the PDF of Y as a function of y
3) Specify the support of Y

Change of variables

Example 8.1.3 (Log-Normal PDF). Let $X \sim \mathcal{N}(0,1)$ and $Y = e^X$.

The distr. of Y is called **log-normal**. The PDF of Y is:

$$f_Y(y) = f_X(x)\left|\frac{dx}{dy}\right| = \varphi(x)\frac{1}{e^x} = \varphi(\log y)\frac{1}{y}, \quad y > 0$$

where $\varphi(x) = e^{-x^2/2}/\sqrt{2\pi}$ is the normal PDF.

We can rather work with the CDF, of course:

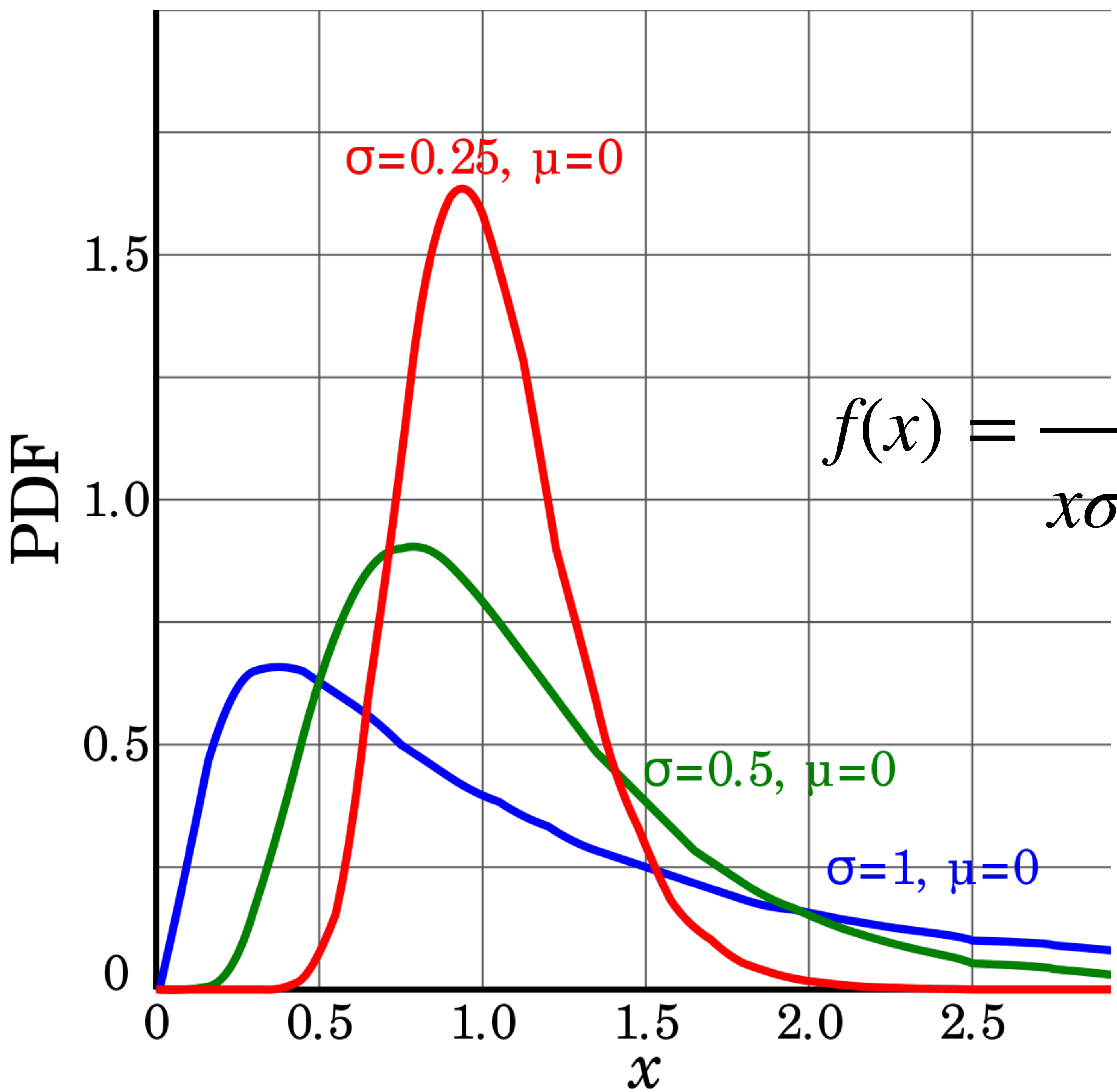
$Y \leq y \ \rightarrow \ e^X \leq y \ \rightarrow \ X \leq \log Y,$ so:

$$F_Y(y) = P(Y \leq y) = P(e^X \leq y) = P(X \leq \log y) = \Phi(\log y)$$

and the PDF is, again, $f_Y(y) = \frac{d}{dy}\Phi(\log y) = \varphi(\log y)\frac{1}{y}, \quad y > 0$

Change of variables

General log-normal: let $X \sim \mathcal{N}(\mu, \sigma)$ and $Y = e^X$.



$$f(x) = \frac{1}{x\sigma\sqrt{2\pi}} \exp\left(-\frac{((\log x) - \mu)^2}{2\sigma^2}\right)$$

Change of variables

Example 8.1.4 (Chi-Squared PDF). Let $X \sim \mathcal{N}(0,1)$ and $Y = X^2$.

The distr. of Y is called **chi-squared**, and we can get it from CDF:

$$X^2 \leq y \quad \rightarrow \quad -\sqrt{y} \leq X \leq \sqrt{y},$$

so $F_Y(y) = P(X^2 \leq y) =$

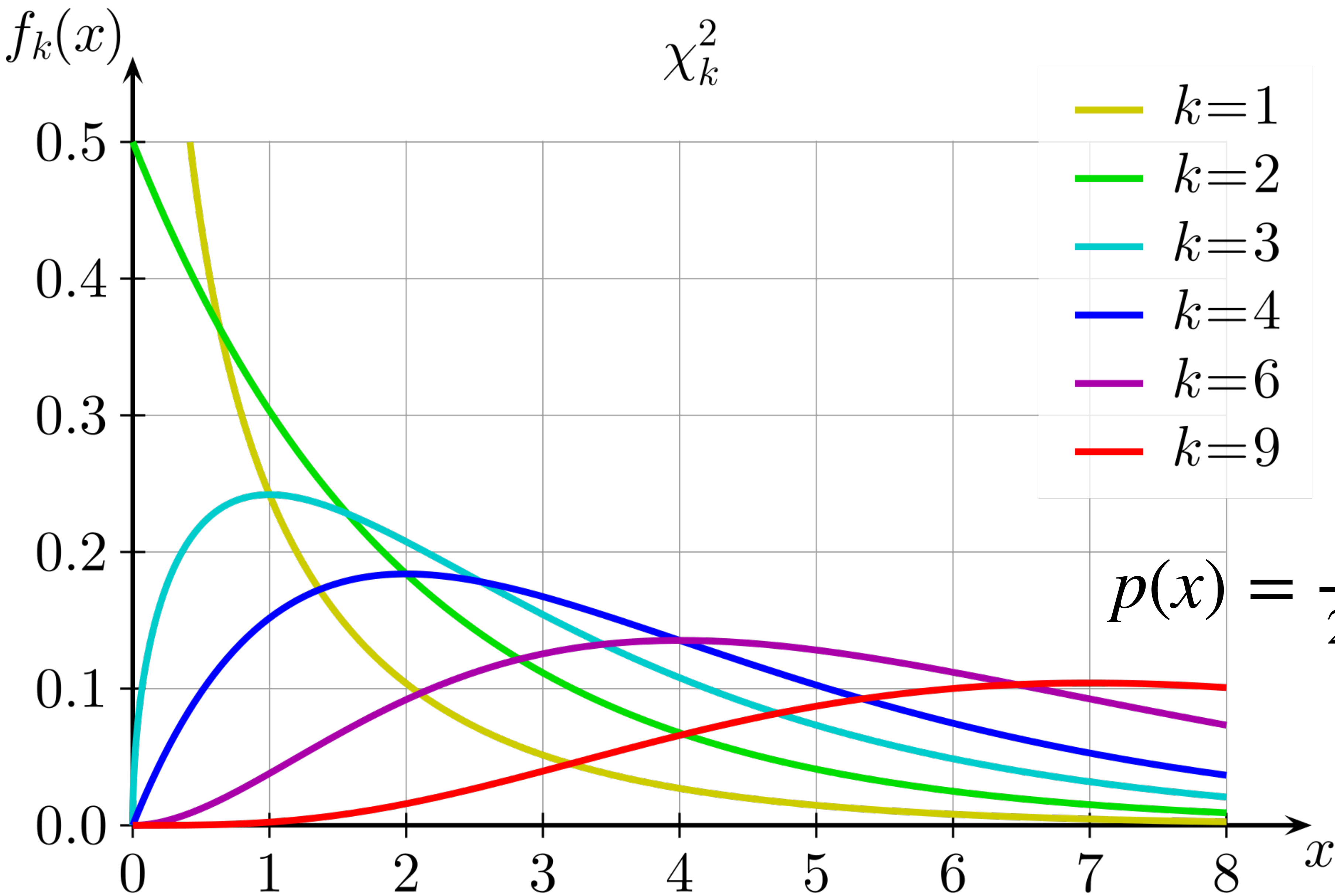
$$= P\left(-\sqrt{y} \leq X \leq \sqrt{y}\right) = \Phi\left(\sqrt{y}\right) - \Phi\left(-\sqrt{y}\right) = 2\Phi\left(\sqrt{y}\right) - 1$$

so $f_Y(y) = 2\varphi\left(\sqrt{y}\right) \cdot \frac{1}{2} y^{-1/2} = \varphi\left(\sqrt{y}\right) y^{-1/2}, \quad y > 0$

Change of variables

General chi-squared: let $Z_i \sim \mathcal{N}(0,1)$, then $Q = \Sigma_{i=1}^{k} Z_i^2 \sim \chi_k^2$

- is said to have k degrees of freedom:



$$p(x) = \frac{1}{2^{k/2}\Gamma(k/2)} x^{k/2-1} e^{-x/2}$$

Change of variables

Theorem 8.1.7 (Change of multiple variables). Let $\mathbf{X} = (X_1, \ldots, X_n)$ be a random vector with joint PDF $f_{\mathbf{X}}$. Let $g : A_0 \to B_0$ be an invertible function where A_0, B_0 are open subsets of $\mathbb{R}^n$, and Supp$(\mathbf{X}) \subset A_0$ and B_0 is the range of g.

Let $\mathbf{Y} = g(\mathbf{X})$, so $\mathbf{y} = g(\mathbf{x})$. Since g is invertible, $\mathbf{X} = g^{-1}(\mathbf{Y})$.

Suppose that all partial derivatives $\dfrac{\partial x_i}{\partial y_i}$ and are continuous, so we can make the **Jacobian matrix:** $\left[\dfrac{\partial \mathbf{x}}{\partial \mathbf{y}}\right]_{ij} = \dfrac{\partial x_i}{\partial y_j}$

Then the joint PDF of $\mathbf{Y}$ is: $f_{\mathbf{Y}}(\mathbf{y}) = f_{\mathbf{X}}(g^{-1}(\mathbf{y})) \cdot |\det \frac{\partial \mathbf{x}}{\partial \mathbf{y}}|$

We have $\det \dfrac{\partial \mathbf{x}}{\partial \mathbf{y}} = \left(\det \dfrac{\partial \mathbf{y}}{\partial \mathbf{x}}\right)^{-1}$, so use the one that is easier to find

Change of variables

Example 8.1.9 (Box-Muller). Let $U \sim$ Unif$(0,2\pi)$ and let $T \sim$ Expo(1) – independent of U. Now define:

$$X = \sqrt{2T}\cos U, \quad Y = \sqrt{2T}\sin U$$

The joint PDF of U, T is $f_{U,T}(u,t) = \frac{1}{2\pi}e^{-t}$, since U, T are independent.

Let's view (X, Y) as a point in the plane, then we have:

$$X^2 + Y^2 = 2T(\cos^2 U + \sin^2 U) = 2T$$

so $(\sqrt{2T}, U)$ is the polar coordinates form of (X, Y). This is an invertible transformation, lets find its Jacobian

Change of variables

Example 8.1.9 (Box-Muller). $U \sim$ Unif$(0{,}2\pi)$, $T \sim$ Expo(1) – indep.
$X = \sqrt{2T}\cos U,\ Y = \sqrt{2T}\sin U,$ PDF of U, T is $f_{U,T}(u,t) = \frac{1}{2\pi}e^{-t}$

$$\frac{\partial(x,y)}{\partial(u,t)} = \begin{pmatrix} -\sqrt{2t}\sin u & \frac{1}{\sqrt{2t}}\cos u \\ \sqrt{2t}\cos u & \frac{1}{\sqrt{2t}}\sin u \end{pmatrix}$$ – the abs. determinant is

$|\det\ldots| = |-\sin^2 u - \cos^2 u| = 1$ is always 1.

So, $f_{X,Y}(x,y) = f_{U,T}(u,t)\cdot|\det\ldots| = \frac{1}{2\pi}e^{-t} = \frac{1}{2\pi}e^{-\frac{1}{2}(x^2+y^2)}$ – a product of two st. normals! (X, Y are i.i.d. $\mathcal{N}(0{,}1)$ r.v.s!)

Hence the Box-Muller method for generating Normal r.v.s!

Convolutions

Convolutions

Theorem 8.2.1 (Convolution). Let X and Y be independent r.v.s and $T = X + Y$. If X, Y are discrete, the PMF of T is

$$P(T = t) = \sum_x P(Y = t - x)\,P(X = x) = \sum_y P(X = t - y)\,P(Y = y)$$

if X, Y are continuous, then the PDF of T is

$$f_T(t) = \int_{-\infty}^{+\infty} f_Y(t - x) f_X(x)\,dx = \int_{-\infty}^{+\infty} f_X(t - y) f_Y(y)\,dy$$

This is called **convolution** of PMF-s/PDF-s. ***Proof*** is straightforward from LOTP.

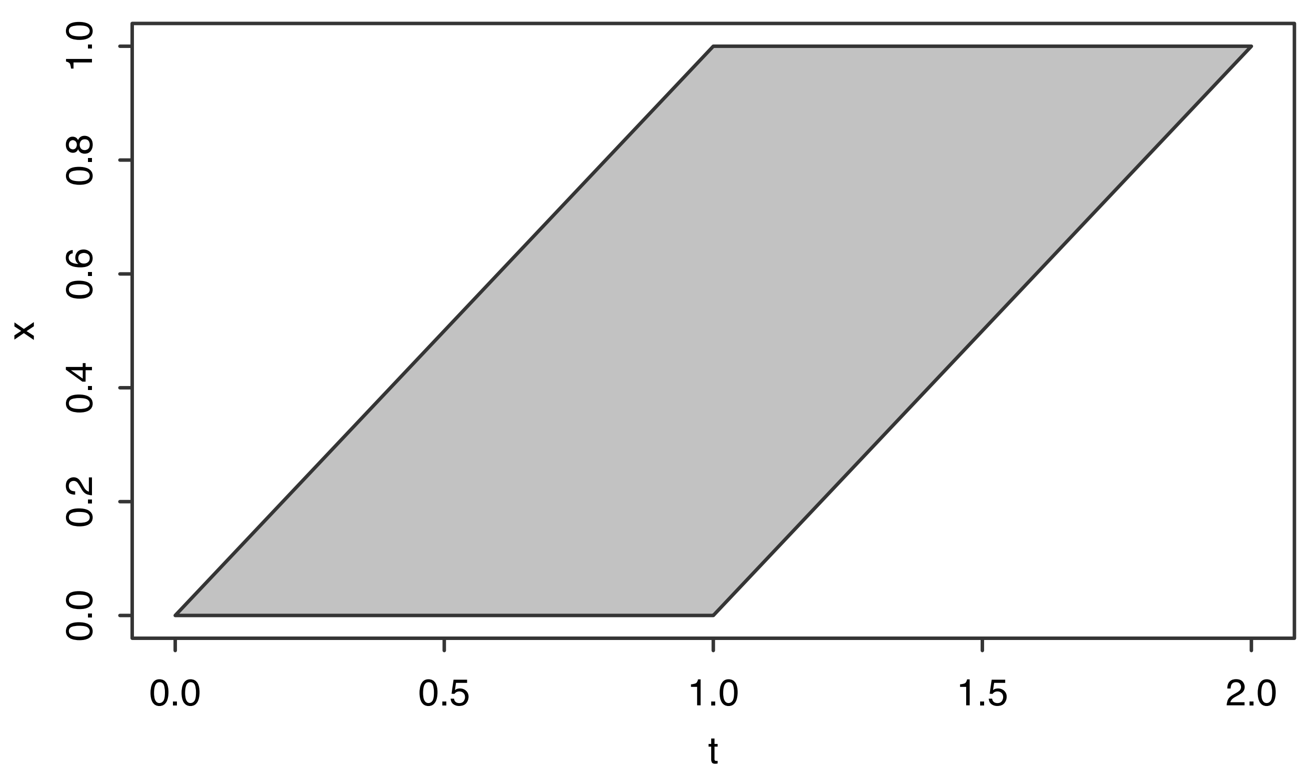
Convolutions

Example 8.2.5 (Uniform convolution). Let $X, Y \overset{i.i.d}{\sim}$ Unif$(0,1)$.

Find the distribution of $T = X + Y$

Solution: The PDF of both X and Y is $g(x) = \begin{cases} 1, & x \in (0,1) \\ 0, & \text{otherwise} \end{cases}$

Convolution: $f_T(t) = \displaystyle\int_{-\infty}^{+\infty} f_Y(t-x) f_X(x)\, dx = \int_{-\infty}^{+\infty} g(t-x)\, g(x)\, dx$



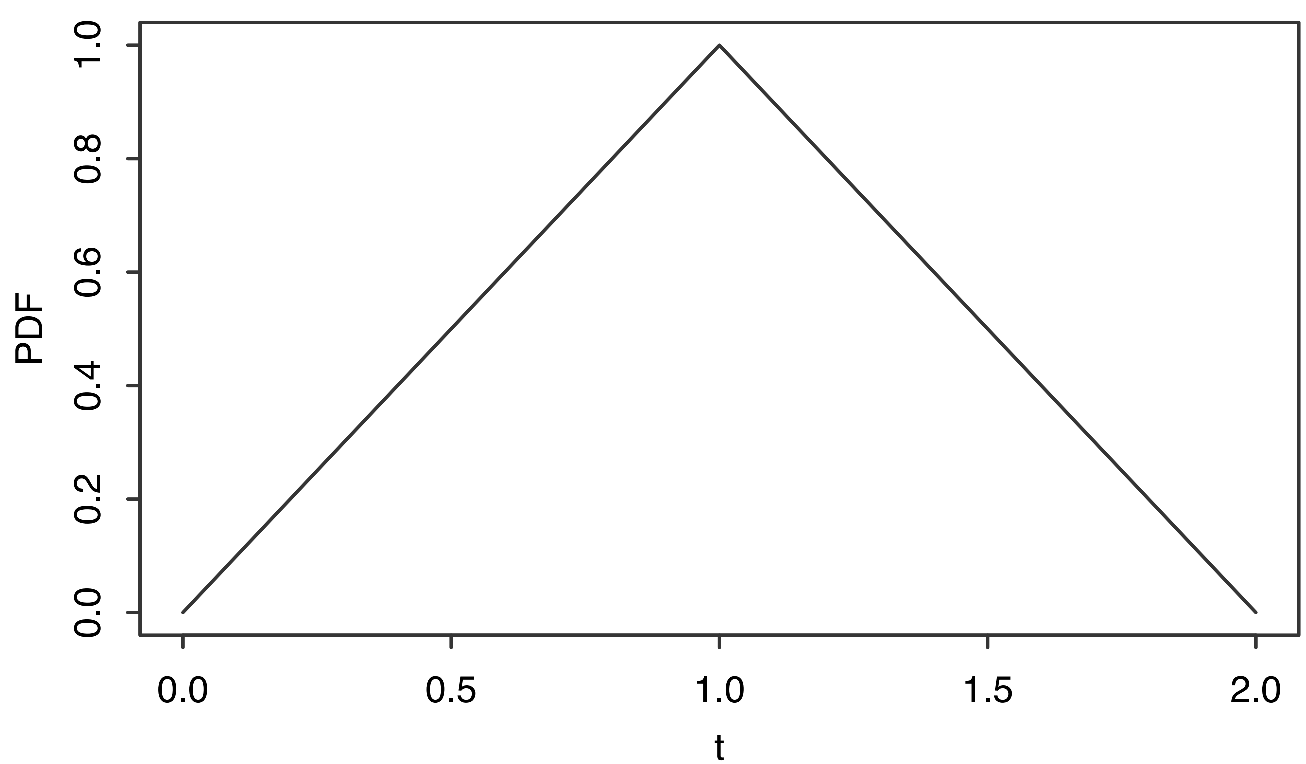
← area where $g(t-x)\, g(x)$ is nonzero

Convolutions

Example 8.2.5 (Uniform convolution). Let $X, Y \overset{i.i.d}{\sim}$ Unif$(0,1)$.

Find the distribution of $T = X + Y$

Solution: So $f_T(t) = \begin{cases} \int_0^t dx = t, & 0 < t < 1, \\ \int_{t-1}^1 dx = 2 - t, & 1 < t < 2 \end{cases}$



← PDF $f_T(t)$

Convolutions

Example 8.2.4 (Exponential convolution). Let $X, Y \overset{i.i.d}{\sim}$ Expo(λ). Find the distribution of $T = X + Y$

Solution: For $t > 0$,

$$f_T(t) = \int_{-\infty}^{+\infty} f_Y(t-x) f_X(x)\, dx = \int_0^t \lambda\, e^{-\lambda(t-x)}\, \lambda\, e^{-\lambda x}\, dx$$

where we restricted the integral to be from 0 to t since both $t - x$ and x should be > 0. That gives

$$f_T(t) = \lambda^2 \int_0^t e^{-\lambda t}\, dx = \lambda^2\, t\, e^{-\lambda t}, \quad t > 0 \quad \text{– this is Gamma}(2,\lambda)$$