Seminar 6

Recap of random variables

A **random variable** is a function from sample space to the real numbers $X : S \to \mathbb{R}$.

Recap of distributions

For a random variable $X : S \to \mathbb{R}$, its distribution acts on numbers in $\mathbb{R}$ in the same way as probability function P acts on outcomes.

Functions describing distributions

- For any distribution we have **cumulative distribution function** (CDF) $F_X(x) = \mathbb{P}(X \leqslant x)$
- For discrete distributions we have **probability mass function** (PMF) $\mathbb{P}_X(x) = \mathbb{P}(X = x)$
- For continuous distributions we have **probability density function** (PDF) $f_X(x) = F'_X(x)$

Location-scale transformation

Random variables transform like functions, i.e. if $Y = \varphi(X)$, then $Y(\omega) = \varphi(X(\omega))$.

For a $\varphi(x) = ax + b$ and $a > 0$, we have

$$F_Y(y) = \mathbb{P}(Y \leqslant y) = \mathbb{P}(aX + b \leqslant y) = \mathbb{P}\left(X \leqslant \frac{y-b}{a}\right) = F_X\left(\frac{y-b}{a}\right)$$

Mathematical expectation

Mathematical expectation generalizes the concept of mean. Consider probability space $(S, \mathbb{P})$ and discrete random variable $X : S \to \mathbb{R}$. Then expected value of X is then

$$\mathbb{E}\left[X\right] = \sum_k x_k \mathbb{P}(X = x_k)$$

It may be the case that $\mathbb{E}\left[X\right] = \pm\infty$ or even does not exist.

Example 1

We roll a die and r.v. X is the score of a roll. What is $\mathbb{E}\left[X\right]$?

Solution 1

$$\mathbb{E}\left[X\right] = \sum_{k=1}^{6} k \cdot \mathbb{P}(X = k) = \frac{1}{6}\sum_{k=1}^{6} k = \frac{7}{2}$$

Example 2

We flip a non-symmetric coin and X is the r.v. for heads, $X \sim Be(p)$. What is $\mathbb{E}\left[X\right]$?

Solution 2

$$\mathbb{E}\left[X\right] = 0 \cdot \mathbb{P}(X = 0) + 1 \cdot \mathbb{P}(X = 1) = p$$

Example 3

Consider discrete r.v. X with distribution $\mathbb{P}(X = 2^n) = 2^{-n}$. What is $\mathbb{E}\left[X\right]$?

Solution 3

$$\mathbb{E}\left[X\right] = \sum_{n} 2^n 2^{-n} = \infty$$

Example 4

Consider discrete r.v. X with distribution $\mathbb{P}(X = 2^n) = \mathbb{P}(X = -2^n) = 2^{-n-1}$.
What is $\mathbb{E}\left[X\right]$?

Solution 4

Expectation of r.v. X exists if and only if $\mathbb{E}\left[|X|\right] < \infty$

Example 5

Consider X with **Poisson distribution** $X \sim Pois(\lambda)$:

$$\mathbb{P}(X = k) = \frac{\lambda^k}{k!}e^{-\lambda}$$

What is $\mathbb{E}\left[X\right]$?

Solution 5

$$\mathbb{E}\left[X\right] = \sum_{k=0}^{\infty} k\frac{\lambda^k}{k!}e^{-\lambda} = e^{-\lambda}\sum_{k=0}^{\infty} k\frac{\lambda^k}{k!} = e^{-\lambda}\sum_{k=0}^{\infty}\frac{\lambda^k}{(k-1)!} =$$
$$= e^{-\lambda}\sum_{k=0}^{\infty}\frac{\lambda^{k-1}\lambda}{(k-1)!} = \lambda e^{-\lambda}\sum_{j=0}^{\infty}\frac{\lambda^j}{j!} = \lambda e^{-\lambda}e^{\lambda} = \lambda$$

Properties of expectation

Consider r.v.s X and Y with finite expectations. Then,

1. For any constants a and b it holds $\mathbb{E}\left[aX + b\right] = a\mathbb{E}\left[X\right] + b$
2. $\mathbb{E}\left[X + Y\right] = \mathbb{E}\left[X\right] + \mathbb{E}\left[Y\right]$
3. If $X \leqslant Y$ a.s., then $\mathbb{E}\left[X\right] \leqslant \mathbb{E}\left[Y\right]$ ($X \leqslant Y$ a.s. $\Leftrightarrow \mathbb{P}((x, y) : x > y) = 0$)
4. If $X \perp Y$, then $\mathbb{E}\left[XY\right] = \mathbb{E}\left[X\right]\mathbb{E}\left[Y\right]$

Example 6

Consider X with binomial distribution $X \sim Bi(n, p)$. What is $\mathbb{E}\left[X\right]$?

Solution 6

- We know that $X = \sum_{k=1}^{n} X_k$, where $X_k \sim Be(p)$
- We know that $\mathbb{E}\left[X_k\right] = p$
- Then, $\mathbb{E}\left[X\right] = \sum_{k=1}^{n}\mathbb{E}\left[X_k\right] = np$

In [1]:
```
import numpy as np
import scipy.stats as sts

import IPython.display as dp
import matplotlib.pyplot as plt
import seaborn as sns
```

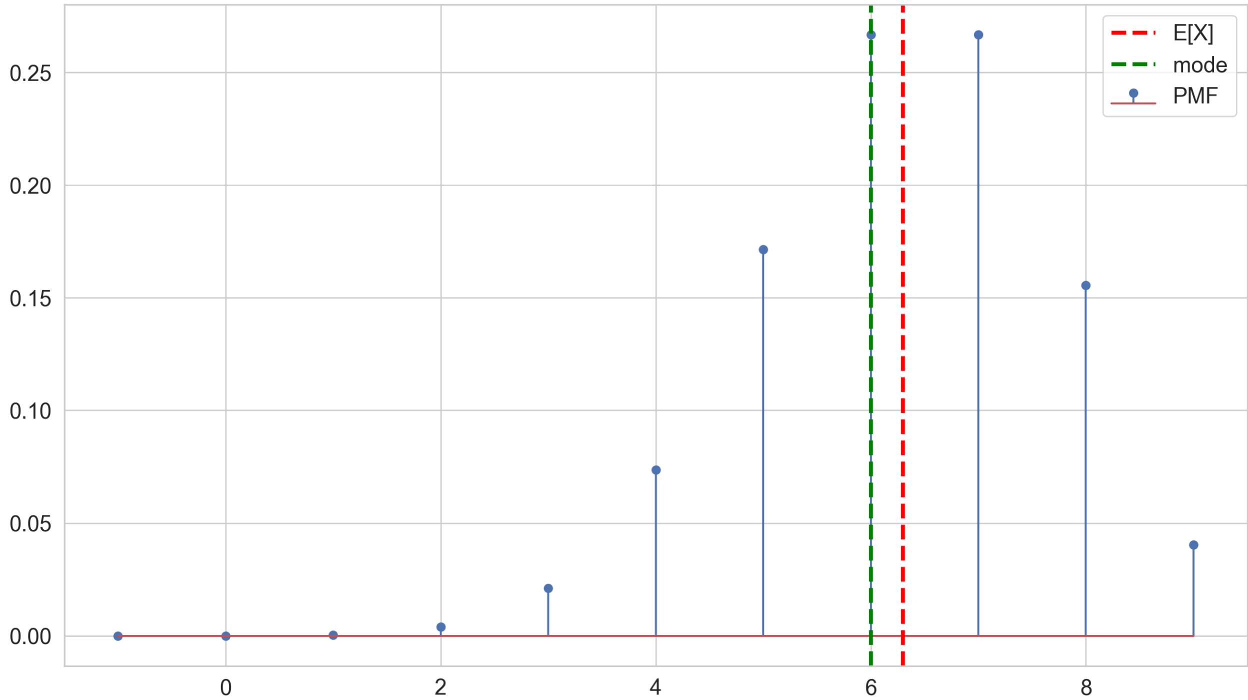
```
dp.set_matplotlib_formats("retina")
sns.set(style="whitegrid", font_scale=1.5)
sns.despine()

%matplotlib inline
```

```
/var/folders/33/j0cl7y453td68qb96j7bqcj4cf41kc/T/ipykernel_28394/260074960
0.py:8: DeprecationWarning: `set_matplotlib_formats` is deprecated since I
Python 7.23, directly use `matplotlib_inline.backend_inline.set_matplotlib
_formats()`
  dp.set_matplotlib_formats("retina")
<Figure size 640x480 with 0 Axes>
```

In [2]:
```
n, p = 9, 0.7
x = np.arange(-1, 10)
y = sts.binom(n, p).pmf(x)

fig, ax = plt.subplots(figsize=(16,9))
ax.stem(x, y, label="PMF")
ax.axvline(n * p, ls="--", linewidth=3, color="red", label="E[X]")
ax.axvline(x[np.argmax(y)], ls="--", linewidth=3, color="green", label="m
ax.legend();
```



Expectation of a function of a random variable (LOTUS)

Consider discrete r.v. X and $Y = \varphi(X)$, then expectation of Y is

$$\mathbb{E}\left[Y\right] = \sum_n \varphi(n)\mathbb{P}\left(X = n\right)$$

Variance

We call **variance** the following quantity of a r.v. X with finite expectation:

$$\mathbb{V}\mathrm{ar}(X) = \mathbb{E}\left[(X - \mathbb{E}[X])^2\right]$$

Example 7

We flip a non-symmetric coin and X is the r.v. for heads, $X \sim Be(p)$. What is $\mathbb{V}\mathrm{ar}\left(X\right)$?

Solution 7

1. We know the formula

$$\mathbb{V}\mathrm{ar}\left(X\right) = \mathbb{E}\left[(X - \mathbb{E}\left[X\right])^2\right]$$

2. We know $\mathbb{E}\left[X\right]$

$$\mathbb{V}\mathrm{ar}\left(X\right) = \mathbb{E}\left[(X - p)^2\right] = \mathbb{E}\left[X^2 - 2pX + p^2\right]$$

3. We know that expectation is linear

$$\mathbb{V}\mathrm{ar}\left(X\right) = \mathbb{E}\left[X^2\right] - 2p\mathbb{E}\left[X\right] + p^2 = \mathbb{E}\left[X^2\right] - p^2$$

4. For $Y = X^2$ we can compute

$$\mathbb{E}\left[Y\right] = 0 \cdot \mathbb{P}(Y = 0) + 1 \cdot \mathbb{P}(Y = 1) = \mathbb{P}(Y = 1) = \mathbb{P}(X^2 = 1) = \mathbb{P}(X = 1) =$$

5. Finally,

$$\mathbb{V}\mathrm{ar}\left(X\right) = p - p^2 = p(1 - p)$$

Properties of variance

1. $\mathbb{V}\mathrm{ar}\left(X\right) \geqslant 0$ and $\mathbb{V}\mathrm{ar}\left(X\right) = 0$ if and only if $X = const$ a.s.
2. If holds

$$\mathbb{V}\mathrm{ar}\left(X\right) = \mathbb{E}\left[X^2\right] - \left(\mathbb{E}\left[X\right]\right)^2$$

3. It holds

$$\mathbb{V}\mathrm{ar}\,(aX + b) = a^2 \mathbb{V}\mathrm{ar}\,(X)$$

4. If $X \perp Y$, it holds

$$\mathbb{V}\mathrm{ar}\,(X + Y) = \mathbb{V}\mathrm{ar}\,(X) + \mathbb{V}\mathrm{ar}\,(Y)$$

Example 8

Consider X with binomial distribution $X \sim Bi(n, p)$. What is $\mathbb{V}\mathrm{ar}\,(X)$?

Solution 8

- We know that $X = \sum_{k=1}^{n} X_k$, where $X_k \sim Be(p)$
- We know that $\mathbb{V}\mathrm{ar}\,(X_k) = p(1-p)$
- Then, $\mathbb{V}\mathrm{ar}\,(X) = \mathbb{V}\mathrm{ar}\,\left(\sum_{k=1}^{n} X_k\right) = \sum_{k=1}^{n} \mathbb{V}\mathrm{ar}\,(X_k) = np(1-p)$

In [3]:
```
n, p = 9, 0.7
x = np.arange(-1, 10)
y = sts.binom(n, p).pmf(x)

fig, ax = plt.subplots(figsize=(16,9))
ax.stem(x, y, label="PMF")
ax.axvline(n * p, ls="--", color="k", label="E[X]")
ax.axvline(n * p ** 2, ls="--", linewidth=3, color="red", label="E[X] - V
ax.axvline(n * p * (2 - p), ls="--", linewidth=3, color="green", label="E
ax.legend();
```