MSAI Probability Home Assignment 6-7
soft deadline: 31/10/2024 19:00 Moscow Time
hard deadline: 31/10/2024 19:00 Moscow Time

As announced earlier, grading for HWs consists of points and bonus points. Solving bonus (indicated with a star) problems is not required, but recommended. Solving all homeworks' normal problems correctly will give you a score of 7, solving all homeworks' bonus problems correctly will give you additional 2 points to the score.

Soft deadline is the intended deadline for this homework. Hard deadline is the date and time of homework discussion webinar, where we will discuss solutions to this homework. After solutions are released, no more homeworks are accepted.

Hand-written solutions are accepted if the handwriting is clear enough and scanned with sufficient quality, but LaTeX is always preferable. This homework includes a python task, which can be solved in Google Colab or in a local Jupyter Notebook. It is thus handy to solve everything (both LaTeX and code) in a single Jupyter Notebook.

Problem 1. (1 points) You have n enumerated letters and n enumerated envelopes. You randomly put letters into envelopes. What is the expected value of the number of coinciding numbers of the letter and its envelope?

Problem 2. (2 points) Consider $X \sim Pois(\lambda_1)$ and $Y \sim Pois(\lambda_2)$ and also $X \perp Y$ (independent). Find the distribution of $Z = X + Y$.

Problem 3. (2 points) Consider $X \sim Pois(\lambda_1)$ and $Y \sim Pois(\lambda_2)$ and also $X \perp Y$ (independent). Find the distribution $X|X+Y$ and its PMF $\mathbb{P}(X = k|X+Y = n)$.

Problem 4. (2 points) Find the expectation and variance of **exponential distribution** $Exp(\lambda)$. Use Python to draw CDF and PDF of $Exp(1)$.

If $X \sim Exp(\lambda)$,

$$F_X(x) = \begin{cases} 1 - e^{-\lambda x}, x \geqslant 0, \\ 0, \text{else} \end{cases}$$

Problem 5. (2 points) Consider independent random variables $X \sim Exp(\lambda_X)$ and $Y \sim Exp(\lambda_Y)$.

1. Find the CDF of $Z = \min\{X, Y\}$.

2. Let $\lambda_X = \lambda_Y$. Find the CDF of $Z = X + Y$. This will be a (very simple case of) **Gamma distribution**. Hint: use the convolution formula.

Problem 6* . (4 bonus points) Compute expectation (2 bonus points) and variance (2 bonus points) of Geometric distribution. Hints:

1. Obtain the series for expectation in terms of p and q.
2. Compare the series with geometric series in q, for which we know that it converges, and we know the limit.
3. Differentiate the geometric series. Because the geometric series converges, we can interchange sum and differential operators. Compare the series for expectation with differentiated geometric series.
4. Make the necessary changes and obtain the expectation.
5. Use LOTUS to get series for $\mathbb{E}\left[X^2\right]$.
6. Compare with differentiated geometric series. Make adjustments, differentiate again.

7. Obtain the variance.

Problem 7^* **.** (1 bonus point) Compute expectation and variance of Negative binomial distribution. Hint: there is a relation between Negative binomial and Geometric distributions.