Probability theory

Lecture 7: Convergences

Maksim Zhukovskii

MIPT

Almost sure convergence

$\xi_1, \xi_2, \ldots$
converges almost surely (with prob. 1) to ξ,

Almost sure convergence

$\xi_1, \xi_2, \ldots$
converges almost surely (with prob. 1) to ξ,

if

$$P(\xi_n \to \xi) = 1.$$

Almost sure convergence

$\xi_1, \xi_2, \ldots$
converges almost surely (with prob. 1) to ξ,

if

$$P(\xi_n \to \xi) = 1.$$

$$\xi_n \xrightarrow{\text{a.s.}} \xi$$

Convergence in probability

$\xi_1, \xi_2, \ldots$

converges in probability to ξ,

Convergence in probability

$\xi_1, \xi_2, \ldots$
converges in probability to ξ,

if

$$\text{for every } \varepsilon > 0, \lim_{n\to\infty} \mathrm{P}(|\xi_n - \xi| \geq \varepsilon) = 0.$$

Convergence in probability

$\xi_1, \xi_2, \ldots$
converges in probability to ξ,

if

$$\text{for every } \varepsilon > 0, \ \lim_{n \to \infty} \mathrm{P}(|\xi_n - \xi| \geq \varepsilon) = 0.$$

$$\xi_n \xrightarrow{\mathrm{P}} \xi$$

Convergence in probability

$\xi_1, \xi_2, \ldots$

converges in probability to ξ,

if

$$\text{for every } \varepsilon > 0, \ \lim_{n\to\infty} \mathrm{P}(|\xi_n - \xi| \geq \varepsilon) = 0.$$

$$\xi_n \xrightarrow{\mathrm{P}} \xi$$

- the law of large numbers:

$$\frac{S_n - \mathrm{E}S_n}{n} \xrightarrow{\mathrm{P}} 0$$

Convergence in L^p

Let $p > 0$.

Convergence in L^p

Let $p > 0$.

$\xi_1, \xi_2, \ldots$
converges in the L^p-norm to ξ,

Convergence in L^p

Let $p > 0$.

$\xi_1, \xi_2, \ldots$
converges in the L^p-norm to ξ,

if

$$\mathsf{E}\xi^p,\ \mathsf{E}\xi_i^p < \infty, \qquad \lim_{n\to\infty} \mathsf{E}|\xi_n - \xi|^p = 0.$$

Convergence in L^p

Let $p > 0$.

$\xi_1, \xi_2, \ldots$
converges in the L^p-norm to ξ,

if

$$\mathsf{E}\xi^p,\ \mathsf{E}\xi_i^p < \infty, \qquad \lim_{n\to\infty} \mathsf{E}|\xi_n - \xi|^p = 0.$$

$$\xi_n \xrightarrow{L_p} \xi$$

Convergence in L^p

Let $p > 0$.

$\xi_1, \xi_2, \ldots$
converges in the L^p-norm to ξ,

if

$$\mathrm{E}\xi^p,\ \mathrm{E}\xi_i^p < \infty, \qquad \lim_{n\to\infty} \mathrm{E}|\xi_n - \xi|^p = 0.$$

$$\xi_n \xrightarrow{L_p} \xi$$

If $p = 1$, we say that
$\xi_1, \xi_2, \ldots$ converges in mean to ξ.

Convergence in distribution

$\xi_1, \xi_2, \ldots$ converges in distribution (converges weakly) to ξ,

Convergence in distribution

$\xi_1, \xi_2, \ldots$ converges in distribution (converges weakly) to ξ,

if

for every $x \in \mathbb{R}$ at which F_ξ is continuous,

$$\lim_{n\to\infty} F_{\xi_n}(x) = F_\xi(x).$$

Convergence in distribution

$\xi_1, \xi_2, \ldots$ converges in distribution (converges weakly) to ξ,

if

for every $x \in \mathbb{R}$ at which F_ξ is continuous,

$$\lim_{n\to\infty} F_{\xi_n}(x) = F_\xi(x).$$

$$\xi_n \xrightarrow{d} \xi$$

Convergence in distribution

$\xi_1, \xi_2, \ldots$ converges in distribution (converges weakly) to ξ,

if

for every $x \in \mathbb{R}$ at which F_ξ is continuous,

$$\lim_{n\to\infty} F_{\xi_n}(x) = F_\xi(x).$$

$$\xi_n \xrightarrow{d} \xi$$

Equivalent definition:

for every continuous bounded function

$f : \mathbb{R} \to \mathbb{R}$, $\lim_{n\to\infty} \mathsf{E} f(\xi_n) = \mathsf{E} f(\xi)$.

Relations

Theorem

1) $\xi_n \xrightarrow{a.s.} \xi$ *implies* $\xi_n \xrightarrow{\mathrm{P}} \xi$;

Relations

Theorem

1) $\xi_n \xrightarrow{a.s.} \xi$ *implies* $\xi_n \xrightarrow{\mathsf{P}} \xi$;

2) $\xi_n \xrightarrow{\mathsf{P}} \xi$ *implies* $\xi_n \xrightarrow{d} \xi$;

Relations

Theorem

1) $\xi_n \xrightarrow{a.s.} \xi$ *implies* $\xi_n \xrightarrow{\mathsf{P}} \xi$;

2) $\xi_n \xrightarrow{\mathsf{P}} \xi$ *implies* $\xi_n \xrightarrow{d} \xi$;

3) for every $p > 0$, $\xi_n \xrightarrow{L_p} \xi$ *implies* $\xi_n \xrightarrow{\mathsf{P}} \xi$.

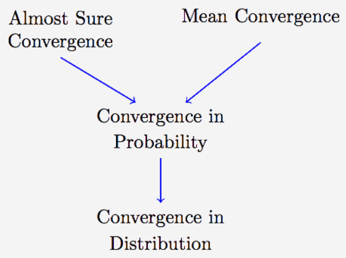
Relations

Theorem

1) $\xi_n \xrightarrow{a.s.} \xi$ implies $\xi_n \xrightarrow{\mathrm{P}} \xi$;

2) $\xi_n \xrightarrow{\mathrm{P}} \xi$ implies $\xi_n \xrightarrow{d} \xi$;

3) for every $p > 0$, $\xi_n \xrightarrow{L_p} \xi$ implies $\xi_n \xrightarrow{\mathrm{P}} \xi$.



The proof of 1)

a.s. $\Rightarrow$ P

The proof of 1)

a.s. $\Rightarrow$ P

Let $\xi_n \xrightarrow{\text{a.s.}} \xi$, $\varepsilon > 0$.

The proof of 1)

a.s. $\Rightarrow$ P

Let $\xi_n \xrightarrow{\text{a.s.}} \xi$, $\varepsilon > 0$.

$$A_n = \{\omega : \ |\xi_n(\omega) - \xi(\omega)| < \varepsilon\},$$

$$B_n = \{\omega : \ \forall k \geq n \ |\xi_k(\omega) - \xi(\omega)| < \varepsilon\} \subset A_n.$$

The proof of 1)

a.s. $\Rightarrow$ P

Let $\xi_n \xrightarrow{\text{a.s.}} \xi$, $\varepsilon > 0$.

$$A_n = \{\omega : \, |\xi_n(\omega) - \xi(\omega)| < \varepsilon\},$$

$$B_n = \{\omega : \, \forall k \geq n \; |\xi_k(\omega) - \xi(\omega)| < \varepsilon\} \subset A_n.$$

It remains to prove that $\mathsf{P}(B_n) \to 1$, $n \to \infty$.

The proof of 1)

a.s. $\Rightarrow$ P

The proof of 1)

a.s. $\Rightarrow$ P

$$A := \{\omega : \exists n \in \mathbb{N}\, \forall k \geq n \; |\xi_k(\omega) - \xi(\omega)| < \varepsilon\},$$

The proof of 1)

a.s. $\Rightarrow$ P

$$A := \{\omega : \exists n \in \mathbb{N}\, \forall k \geq n \, |\xi_k(\omega) - \xi(\omega)| < \varepsilon\},$$

$$\{\xi_n \to \xi\} \subset A,$$

The proof of 1)

a.s. $\Rightarrow$ P

$$A := \{\omega : \exists n \in \mathbb{N} \, \forall k \geq n \; |\xi_k(\omega) - \xi(\omega)| < \varepsilon\},$$

$$\{\xi_n \to \xi\} \subset A,$$

$$\mathrm{P}(A) = 1,$$

The proof of 1)

a.s. $\Rightarrow$ P

$$A := \{\omega : \exists n \in \mathbb{N}\, \forall k \geq n \, |\xi_k(\omega) - \xi(\omega)| < \varepsilon\},$$

$$\{\xi_n \to \xi\} \subset A,$$

$$\mathrm{P}(A) = 1,$$

Clearly, $B_n \uparrow A$.

The proof of 1)

a.s. $\Rightarrow$ P

$$A := \{\omega : \exists n \in \mathbb{N}\, \forall k \geq n\, |\xi_k(\omega) - \xi(\omega)| < \varepsilon\},$$

$$\{\xi_n \to \xi\} \subset A,$$

$$\mathrm{P}(A) = 1,$$

Clearly, $B_n \uparrow A$.

Therefore, $\mathrm{P}(B_n) \to 1,\ n \to \infty$.

The proof of 2)

$$\mathrm{P} \Rightarrow d$$

The proof of 2)

$$\mathrm{P} \Rightarrow d$$

$f : \mathbb{R} \to \mathbb{R}$ — continuous, $|f| \leq C$, $\varepsilon > 0$.

The proof of 2)

$$\mathsf{P} \Rightarrow d$$

$f : \mathbb{R} \to \mathbb{R}$ — continuous, $|f| \leq C$, $\varepsilon > 0$.

Let $\tilde{\xi}_n = \xi I(\xi \leq n)$.

The proof of 2)

$$\mathrm{P} \Rightarrow d$$

$f : \mathbb{R} \to \mathbb{R}$ — continuous, $|f| \leq C$, $\varepsilon > 0$.

Let $\tilde{\xi}_n = \xi I(\xi \leq n)$.

Obviously, $\tilde{\xi}_n \xrightarrow{\text{a.s.}} \xi$.

The proof of 2)

$$\mathsf{P} \Rightarrow d$$

$f : \mathbb{R} \to \mathbb{R}$ — continuous, $|f| \leq C$, $\varepsilon > 0$.

Let $\tilde{\xi}_n = \xi I(\xi \leq n)$.

Obviously, $\tilde{\xi}_n \xrightarrow{\text{a.s.}} \xi$. Therefore, $\tilde{\xi}_n \xrightarrow{\mathsf{P}} \xi$.

The proof of 2)

$$\mathsf{P} \Rightarrow d$$

$f : \mathbb{R} \to \mathbb{R}$ — continuous, $|f| \leq C$, $\varepsilon > 0$.

Let $\tilde{\xi}_n = \xi I(\xi \leq n)$.

Obviously, $\tilde{\xi}_n \xrightarrow{\text{a.s.}} \xi$. Therefore, $\tilde{\xi}_n \xrightarrow{\mathsf{P}} \xi$.

There exists $n_1 \in \mathbb{N}$ such that, for all $n \geq n_1$,

$$\mathsf{P}(|\tilde{\xi}_n - \xi| > 1) < \varepsilon/(6C).$$

The proof of 2)

$$\mathsf{P} \Rightarrow d$$

$f : \mathbb{R} \to \mathbb{R}$ — continuous, $|f| \leq C$, $\varepsilon > 0$.

Let $\tilde{\xi}_n = \xi I(\xi \leq n)$.

Obviously, $\tilde{\xi}_n \xrightarrow{\text{a.s.}} \xi$. Therefore, $\tilde{\xi}_n \xrightarrow{\mathsf{P}} \xi$.

There exists $n_1 \in \mathbb{N}$ such that, for all $n \geq n_1$,

$$\mathsf{P}(|\tilde{\xi}_n - \xi| > 1) < \varepsilon/(6C).$$

Therefore, $\forall n \geq n_1$, $\mathsf{P}(\xi > n) < \varepsilon/(6C)$.

The proof of 2)

$$\mathrm{P} \Rightarrow d$$

The proof of 2)

$$P \Rightarrow d$$

f is uniformly continuous on any $[a, b]$.

The proof of 2)

$$\mathrm{P} \Rightarrow d$$

f is uniformly continuous on any $[a, b]$.
Therefore, there exists a $\delta > 0$ such that,
if $x, y \in [-n_1 - \delta, n_1 + \delta]$, $|x - y| < \delta$,
then $|f(x) - f(y)| < \varepsilon/3$.

The proof of 2)

$$\mathrm{P} \Rightarrow d$$

f is uniformly continuous on any $[a, b]$.
Therefore, there exists a $\delta > 0$ such that,
if $x, y \in [-n_1 - \delta, n_1 + \delta]$, $|x - y| < \delta$,
then $|f(x) - f(y)| < \varepsilon/3$.

Let $n_2 \in \mathbb{N}$ be such that, for all $n \geq n_2$,

$$\mathrm{P}(|\xi_n - \xi| > \delta) < \varepsilon/(6C).$$

The proof of 2)

$$\mathrm{P} \Rightarrow d$$

The proof of 2)

$$\mathrm{P} \Rightarrow d$$

For $n \geq \max\{n_1, n_2\}$,

$$|\mathrm{E}f(\xi_n) - \mathrm{E}f(\xi)| = |\mathrm{E}(f(\xi_n) - f(\xi))| \leq$$

The proof of 2)

$$\mathsf{P} \Rightarrow d$$

For $n \geq \max\{n_1, n_2\}$,

$$|\mathsf{E}f(\xi_n) - \mathsf{E}f(\xi)| = |\mathsf{E}(f(\xi_n) - f(\xi))| \leq$$

$$\mathsf{E}|f(\xi_n) - f(\xi)| =$$

The proof of 2)

$$\mathsf{P} \Rightarrow d$$

For $n \geq \max\{n_1, n_2\}$,

$$|\mathsf{E}f(\xi_n) - \mathsf{E}f(\xi)| = |\mathsf{E}(f(\xi_n) - f(\xi))| \leq$$

$$\mathsf{E}|f(\xi_n) - f(\xi)| = \mathsf{E}\Bigg(|f(\xi_n) - f(\xi)|I(\xi > n_1) +$$

$$|f(\xi_n) - f(\xi)|I(\xi \leq n_1, |\xi - \xi_n| \geq \delta) +$$

$$|f(\xi_n) - f(\xi)|I(\xi \leq n_1, |\xi - \xi_n| < \delta)\Bigg) <$$

The proof of 2)

$$\mathrm{P} \Rightarrow d$$

For $n \geq \max\{n_1, n_2\}$,

$$|\mathrm{E}f(\xi_n) - \mathrm{E}f(\xi)| = |\mathrm{E}(f(\xi_n) - f(\xi))| \leq$$

$$\mathrm{E}|f(\xi_n) - f(\xi)| = \mathrm{E}\Big(|f(\xi_n) - f(\xi)| I(\xi > n_1) +$$

$$|f(\xi_n) - f(\xi)| I(\xi \leq n_1, |\xi - \xi_n| \geq \delta) +$$

$$|f(\xi_n) - f(\xi)| I(\xi \leq n_1, |\xi - \xi_n| < \delta) \Big) <$$

$$2C\mathrm{P}(\xi > n_1) + 2C\mathrm{P}(|\xi - \xi_n| \geq \delta) + \varepsilon/3 < \varepsilon.$$

The proof of 3)

$$L_p \Rightarrow \mathsf{P}$$

The proof of 3)

$L_p \Rightarrow \mathrm{P}$

Let $\varepsilon > 0$.

The proof of 3)

$$L_p \Rightarrow \mathrm{P}$$

Let $\varepsilon > 0$.

$$\mathrm{P}(|\xi_n - \xi| \geq \varepsilon) = \mathrm{P}(|\xi_n - \xi|^p \geq \varepsilon^p) \leq$$

The proof of 3)

$$L_p \Rightarrow \mathrm{P}$$

Let $\varepsilon > 0$.

$$\mathrm{P}(|\xi_n - \xi| \geq \varepsilon) = \mathrm{P}(|\xi_n - \xi|^p \geq \varepsilon^p) \leq$$

$$\frac{\mathrm{E}|\xi_n - \xi|^p}{\varepsilon^p} \to 0.$$

Counterexamples

d does not imply P

Counterexamples

d does not imply P

Let $\xi_1 = \xi_2 = \ldots = -\xi$,

Counterexamples

d does not imply P

Let $\xi_1 = \xi_2 = \ldots = -\xi$, $\xi \sim U(\{-1, 1\})$.

Counterexamples

d does not imply P

Let $\xi_1 = \xi_2 = \ldots = -\xi$, $\xi \sim U(\{-1, 1\})$.
$\xi_n \xrightarrow{d} \xi$, since, for any f, $\mathrm{E}f(\xi_n) = \mathrm{E}f(\xi)$.

Counterexamples

d does not imply P

Let $\xi_1 = \xi_2 = \ldots = -\xi$, $\xi \sim U(\{-1, 1\})$.
$\xi_n \xrightarrow{d} \xi$, since, for any f, $\mathsf{E}f(\xi_n) = \mathsf{E}f(\xi)$.

However, $\mathsf{P}(|\xi - \xi_n| \geq 2) = 1$ for $n \in \mathbb{N}$.

Counterexamples

d does not imply P

Let $\xi_1 = \xi_2 = \ldots = -\xi$, $\xi \sim U(\{-1, 1\})$.
$\xi_n \xrightarrow{d} \xi$, since, for any f, $\mathrm{E}f(\xi_n) = \mathrm{E}f(\xi)$.

However, $\mathrm{P}(|\xi - \xi_n| \geq 2) = 1$ for $n \in \mathbb{N}$.
ξ_n does not converge in probability to ξ.

Counterexamples

P does not imply neither a.s. nor L_p

Counterexamples

P does not imply neither a.s. nor L_p

$\Omega = [0,1]$, $\mathcal{F} = \mathcal{B}[0,1]$, P — uniform

Counterexamples

P does not imply neither a.s. nor L_p

$\Omega = [0, 1]$, $\mathcal{F} = \mathcal{B}[0, 1]$, P — uniform

$\xi_1 = 2^{1/p} I_{[0,1)}$,

Counterexamples

P does not imply neither a.s. nor L_p

$\Omega = [0,1]$, $\mathcal{F} = \mathcal{B}[0,1]$, P — uniform

$\xi_1 = 2^{1/p} I_{[0,1)}$,
$\xi_2 = 2^{2/p} I_{[0,1/2)}$, $\xi_3 = 2^{3/p} I_{[1/2,1)}$,

Counterexamples

P does not imply neither a.s. nor L_p

$\Omega = [0,1]$, $\mathcal{F} = \mathcal{B}[0,1]$, P — uniform

$$\xi_1 = 2^{1/p} I_{[0,1)},$$
$$\xi_2 = 2^{2/p} I_{[0,1/2)},\ \xi_3 = 2^{3/p} I_{[1/2,1)},$$
$$\xi_4 = 2^{4/p} I_{[0,1/4)},\ \xi_5 = 2^{5/p} I_{[1/4,1/2)},$$
$$\xi_6 = 2^{6/p} I_{[1/2,3/4)},\ \xi_7 = 2^{7/p} I_{[3/4,1)},$$

Counterexamples

P does not imply neither a.s. nor L_p

$\Omega = [0,1]$, $\mathcal{F} = \mathcal{B}[0,1]$, P — uniform

$$\xi_1 = 2^{1/p} I_{[0,1)},$$
$$\xi_2 = 2^{2/p} I_{[0,1/2)},\ \xi_3 = 2^{3/p} I_{[1/2,1)},$$
$$\xi_4 = 2^{4/p} I_{[0,1/4)},\ \xi_5 = 2^{5/p} I_{[1/4,1/2)},$$
$$\xi_6 = 2^{6/p} I_{[1/2,3/4)},\ \xi_7 = 2^{7/p} I_{[3/4,1)},\ \dots$$

Counterexamples

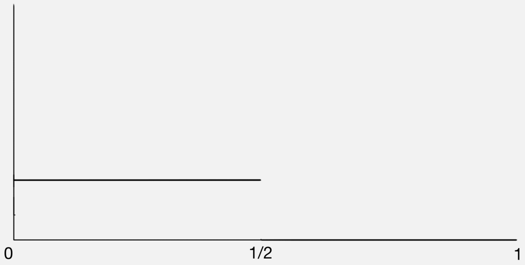
P does not imply neither a.s. nor L_p

Counterexamples

P does not imply neither a.s. nor L_p

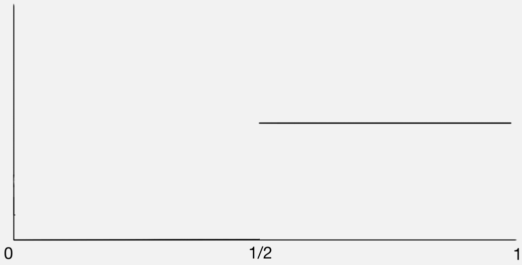
Counterexamples

P does not imply neither a.s. nor L_p



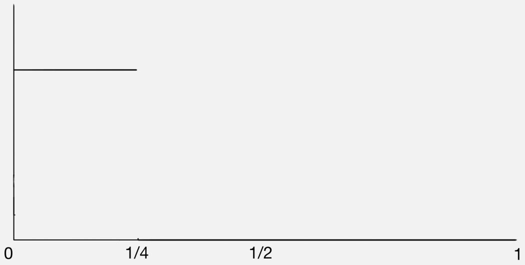
Counterexamples

P does not imply neither a.s. nor L_p



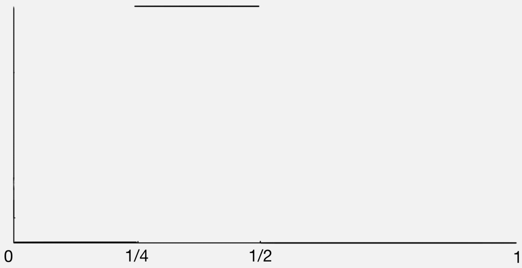
Counterexamples

P does not imply neither a.s. nor L_p



Counterexamples

P does not imply neither a.s. nor L_p



Counterexamples

P does not imply neither a.s. nor L_p

Counterexamples

P does not imply neither a.s. nor L_p

Let $\varepsilon > 0$, $\delta > 0$.

Counterexamples

P does not imply neither a.s. nor L_p

Let $\varepsilon > 0$, $\delta > 0$.
Find k such that $\delta > \frac{1}{2^k}$.

Counterexamples

P does not imply neither a.s. nor L_p

Let $\varepsilon > 0$, $\delta > 0$.
Find k such that $\delta > \frac{1}{2^k}$.
For every $n \geq 2^k$, $\mathrm{P}(|\xi_n| > \varepsilon) \leq \frac{1}{2^k} < \delta$.

Counterexamples

P does not imply neither a.s. nor L_p

Let $\varepsilon > 0$, $\delta > 0$.
Find k such that $\delta > \frac{1}{2^k}$.
For every $n \geq 2^k$, $\mathrm{P}(|\xi_n| > \varepsilon) \leq \frac{1}{2^k} < \delta$.

$$\xi_n \xrightarrow{\mathrm{P}} 0$$

Counterexamples

P does not imply neither a.s. nor L_p

Counterexamples

P does not imply neither a.s. nor L_p

For $\omega \in [0, 1)$, there are infinitely many n such that $\xi_n(\omega) > 1$.

Counterexamples

P does not imply neither a.s. nor L_p

For $\omega \in [0, 1)$, there are infinitely many n such that $\xi_n(\omega) > 1$.

ξ_n does not converge to 0 a.s.

Counterexamples

P does not imply neither a.s. nor L_p

For $\omega \in [0,1)$, there are infinitely many n such that $\xi_n(\omega) > 1$.

ξ_n does not converge to 0 a.s.

$\mathrm{E}|\xi_n|^p \geq 2^{2^k-1}$ for every $n \geq 2^k$.

Counterexamples

P does not imply neither a.s. nor L_p

For $\omega \in [0,1)$, there are infinitely many n such that $\xi_n(\omega) > 1$.

ξ_n does not converge to 0 a.s.

$\mathrm{E}|\xi_n|^p \geq 2^{2^k - 1}$ for every $n \geq 2^k$.
$\mathrm{E}|\xi_n|^p \to \infty$.

Counterexamples

P does not imply neither a.s. nor L_p

For $\omega \in [0,1)$, there are infinitely many n such that $\xi_n(\omega) > 1$.

ξ_n does not converge to 0 a.s.

$\mathsf{E}|\xi_n|^p \geq 2^{2^k-1}$ for every $n \geq 2^k$.
$\mathsf{E}|\xi_n|^p \to \infty$.

ξ_n does not converge to 0 in L_p

Counterexamples

L_p does not imply a.s.

Counterexamples

L_p does not imply a.s.

$\Omega = [0, 1]$, $\mathcal{F} = \mathcal{B}[0, 1]$, P — uniform

Counterexamples

L_p does not imply a.s.

$\Omega = [0,1]$, $\mathcal{F} = \mathcal{B}[0,1]$, P — uniform

$\xi_1 = I_{[0,1)}$,

Counterexamples

L_p does not imply a.s.

$\Omega = [0,1]$, $\mathcal{F} = \mathcal{B}[0,1]$, P — uniform

$\xi_1 = I_{[0,1)}$,

$\xi_2 = I_{[0,1/2)}$, $\xi_3 = I_{[1/2,1)}$,

Counterexamples

L_p does not imply a.s.

$\Omega = [0,1]$, $\mathcal{F} = \mathcal{B}[0,1]$, P — uniform

$\xi_1 = I_{[0,1)}$,

$\xi_2 = I_{[0,1/2)}$, $\xi_3 = I_{[1/2,1)}$,

$\xi_4 = I_{[0,1/4)}$, $\xi_5 = I_{[1/4,1/2)}$,
$\xi_6 = I_{[1/2,3/4)}$, $\xi_7 = I_{[3/4,1)}$,

Counterexamples

L_p does not imply a.s.

$\Omega = [0, 1]$, $\mathcal{F} = \mathcal{B}[0, 1]$, P — uniform

$\xi_1 = I_{[0,1)}$,

$\xi_2 = I_{[0,1/2)}$, $\xi_3 = I_{[1/2,1)}$,

$\xi_4 = I_{[0,1/4)}$, $\xi_5 = I_{[1/4,1/2)}$,
$\xi_6 = I_{[1/2,3/4)}$, $\xi_7 = I_{[3/4,1)}$, $\ldots$

Counterexamples

L_p does not imply a.s.

Counterexamples

L_p does not imply a.s.

For $\omega \in [0, 1)$, there are infinitely many n such that $\xi_n(\omega) > 1$.

Counterexamples

L_p does not imply a.s.

For $\omega \in [0, 1)$, there are infinitely many n such that $\xi_n(\omega) > 1$.

ξ_n does not converge to 0 a.s.

Counterexamples

L_p does not imply a.s.

For $\omega \in [0,1)$, there are infinitely many n such that $\xi_n(\omega) > 1$.

ξ_n does not converge to 0 a.s.

$\mathrm{E}|\xi_n|^p \leq 2^{-k}$ for every $n \geq 2^k$.

Counterexamples

L_p does not imply a.s.

For $\omega \in [0,1)$, there are infinitely many n such that $\xi_n(\omega) > 1$.

ξ_n does not converge to 0 a.s.

$\mathsf{E}|\xi_n|^p \leq 2^{-k}$ for every $n \geq 2^k$.

$$\xi_n \xrightarrow{L_p} 0$$

Counterexamples

a.s. does not imply L_p

Counterexamples

a.s. does not imply L_p

$\Omega = [0, 1]$, $\mathcal{F} = \mathcal{B}[0, 1]$, P — uniform.

Counterexamples

a.s. does not imply L_p

$\Omega = [0,1]$, $\mathcal{F} = \mathcal{B}[0,1]$, P — uniform.

$\xi_n = 2^{(1/p)n} I_{[0,1/n]}$, $n \in \mathbb{N}$.

Counterexamples

a.s. does not imply L_p

$\Omega = [0, 1]$, $\mathcal{F} = \mathcal{B}[0, 1]$, P — uniform.

$\xi_n = 2^{(1/p)n} I_{[0,1/n]}$, $n \in \mathbb{N}$.

For $\omega \in (0, 1]$, $\xi_n(\omega) = 0$ for n large enough.

Counterexamples

a.s. does not imply L_p

$\Omega = [0,1]$, $\mathcal{F} = \mathcal{B}[0,1]$, P — uniform.

$\xi_n = 2^{(1/p)n} I_{[0,1/n]}$, $n \in \mathbb{N}$.

For $\omega \in (0,1]$, $\xi_n(\omega) = 0$ for n large enough.

$$\xi_n \xrightarrow{\text{a.s.}} 0$$

Counterexamples

a.s. does not imply L_p

$\Omega = [0,1]$, $\mathcal{F} = \mathcal{B}[0,1]$, P — uniform.

$\xi_n = 2^{(1/p)n} I_{[0,1/n]}$, $n \in \mathbb{N}$.

For $\omega \in (0,1]$, $\xi_n(\omega) = 0$ for n large enough.

$$\xi_n \xrightarrow{\text{a.s.}} 0$$

$\mathrm{E}|\xi_n|^p = \frac{2^n}{n}$ for every $n \in \mathbb{N}$.

Counterexamples

a.s. does not imply L_p

$\Omega = [0,1]$, $\mathcal{F} = \mathcal{B}[0,1]$, P — uniform.

$\xi_n = 2^{(1/p)n} I_{[0,1/n]}$, $n \in \mathbb{N}$.

For $\omega \in (0,1]$, $\xi_n(\omega) = 0$ for n large enough.

$$\xi_n \xrightarrow{\text{a.s.}} 0$$

$\mathrm{E}|\xi_n|^p = \frac{2^n}{n}$ for every $n \in \mathbb{N}$.

$$\mathrm{E}|\xi_n|^p \to \infty$$

Simple random walk

$\xi_1, \xi_2, \ldots$ — i.i.d. r.v. $\sim U(\{-1, 1\})$.

Simple random walk

$\xi_1, \xi_2, \ldots$ — i.i.d. r.v. $\sim U(\{-1, 1\})$.

$S_0 = 0$, $S_n = \xi_1 + \ldots + \xi_n$
$\{S_n,\ n \in \mathbb{N}\}$ — simple random walk (on $\mathbb{Z}$)

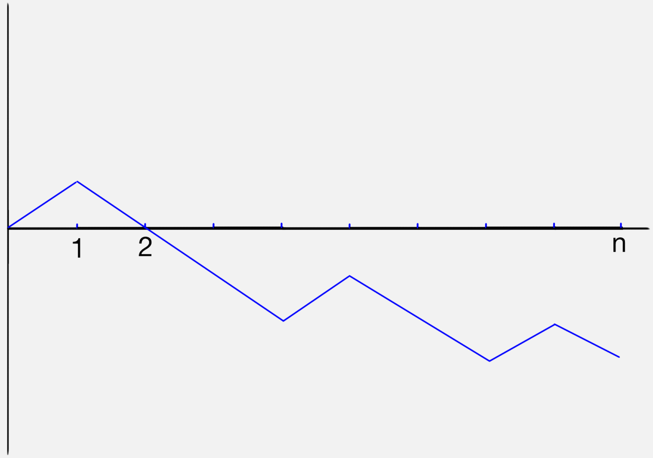
Simple random walk

$\xi_1, \xi_2, \ldots$ — i.i.d. r.v. $\sim U(\{-1, 1\})$.

$S_0 = 0$, $S_n = \xi_1 + \ldots + \xi_n$
$\{S_n,\ n \in \mathbb{N}\}$ — simple random walk (on $\mathbb{Z}$)

For $\omega \in \Omega$,
$\{S_n(\omega), n \in \mathbb{N}\}$ — a trajectory of the random walk.

Trajectory



Counting trajectories

$N(n, x)$ — number of trajectories of $\{S_k,\ k \le n\}$ s.t. $S_n(\omega) = x$.

Counting trajectories

$N(n, x)$ — number of trajectories of $\{S_k,\ k \leq n\}$ s.t. $S_n(\omega) = x$.

- if $|x| > n$,

Counting trajectories

$N(n, x)$ — number of trajectories of $\{S_k,\ k \leq n\}$ s.t. $S_n(\omega) = x$.

- if $|x| > n$, then $N(n, x) = 0$;

Counting trajectories

$N(n, x)$ — number of trajectories of $\{S_k,\ k \leq n\}$ s.t. $S_n(\omega) = x$.

- if $|x| > n$, then $N(n, x) = 0$;
- if $x + n$ is odd,

Counting trajectories

$N(n, x)$ — number of trajectories of $\{S_k,\ k \leq n\}$ s.t. $S_n(\omega) = x$.

- if $|x| > n$, then $N(n, x) = 0$;
- if $x + n$ is odd, then $N(n, x) = 0$;

Counting trajectories

$N(n, x)$ — number of trajectories of $\{S_k,\ k \leq n\}$ s.t. $S_n(\omega) = x$.

- if $|x| > n$, then $N(n, x) = 0$;
- if $x + n$ is odd, then $N(n, x) = 0$;
- otherwise,

Counting trajectories

$N(n, x)$ — number of trajectories of $\{S_k,\ k \le n\}$ s.t. $S_n(\omega) = x$.

- if $|x| > n$, then $N(n, x) = 0$;
- if $x + n$ is odd, then $N(n, x) = 0$;
- otherwise, $N(n, x) = \binom{n}{\frac{n+x}{2}}$.

Counting trajectories

$\tilde{N}(n,x)$ — number of trajectories of $\{S_k,\ k \le n\}$ s.t. $S_n = x,\ S_1, \ldots, S_{n-1} \neq 0$.

Counting trajectories

$\tilde{N}(n,x)$ — number of trajectories of $\{S_k,\ k \le n\}$ s.t. $S_n = x,\ S_1, \dots, S_{n-1} \neq 0$.

Theorem (Reflection principle)

If $x > 0$, then
$\tilde{N}(n,x) = N(n-1, x-1) - N(n-1, -x-1)$.

Counting trajectories

$\tilde{N}(n, x)$ — number of trajectories of $\{S_k,\ k \le n\}$ s.t. $S_n = x,\ S_1, \ldots, S_{n-1} \neq 0$.

Theorem (Reflection principle)

If $x > 0$, then
$\tilde{N}(n, x) = N(n-1, x-1) - N(n-1, -x-1)$.

If $x < 0$, then
$\tilde{N}(n, x) = N(n-1, x+1) - N(n-1, -x+1)$.

The proof

Let $x > 0$.

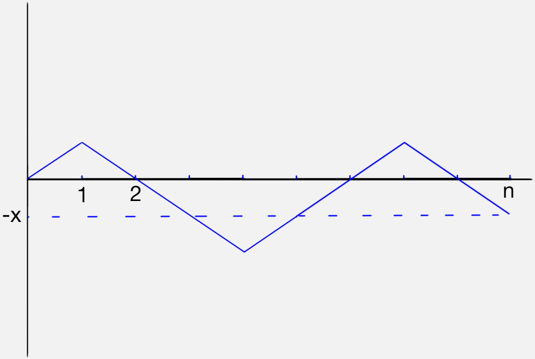
The proof

Let $x > 0$.

There exists a bijection between

$\{$trajectories s.t. $S_1 = 1,\ S_n = -x\}$,

$\{$trajectories s.t. $S_1 = 1,\ S_n = x$, at least one of $S_2, \ldots, S_{n-1}$ equals $0\}$.



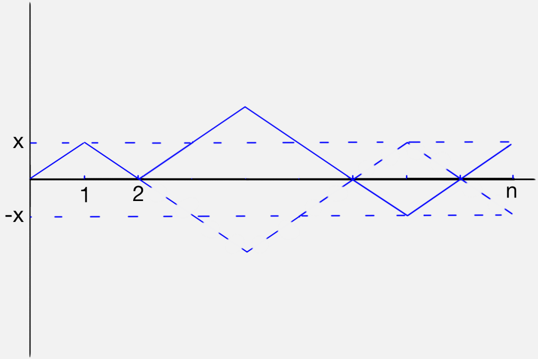
The proof

Let $x > 0$.

There exists a bijection between

$\{$trajectories s.t. $S_1 = 1,\ S_n = -x\}$,

$\{$trajectories s.t. $S_1 = 1,\ S_n = x$, at least one of $S_2, \ldots, S_{n-1}$ equals $0\}$.



The proof

$$\tilde{N}(n, x) =$$

The proof

$\tilde{N}(n,x) =$
the number of trajectories from $(1,1)$ to (n,x) that never meet 0 $=$

The proof

$\tilde{N}(n, x) =$
the number of trajectories from $(1, 1)$ to (n, x) that never meet $0 =$
$N(n-1, x-1)$—the number of trajectories from $(1, 1)$ to (n, x) that meet $0 =$

The proof

$\tilde{N}(n, x) =$
the number of trajectories from $(1, 1)$ to (n, x) that never meet 0 $=$
$N(n-1, x-1)-$the number of trajectories from $(1, 1)$ to (n, x) that meet 0 $=$
$N(n-1, x-1) - N(n-1, -x-1)$.

The proof

$\tilde{N}(n,x) =$
the number of trajectories from $(1,1)$ to (n,x) that never meet $0 =$
$N(n-1,x-1)-$the number of trajectories from $(1,1)$ to (n,x) that meet $0 =$
$N(n-1,x-1) - N(n-1,-x-1)$.

The case $x < 0$ is symmetric.

Counting probabilities

$$P(S_n = x) =$$

Counting probabilities

$$\mathrm{P}(S_n = x) = \frac{1}{2^n} N(n, x) =$$

Counting probabilities

$$P(S_n = x) = \frac{1}{2^n} N(n, x) = \frac{1}{2^n} \binom{n}{\frac{n+x}{2}}$$

Counting probabilities

$$\mathrm{P}(S_n = x) = \frac{1}{2^n} N(n, x) = \frac{1}{2^n} \binom{n}{\frac{n+x}{2}}$$

$$\mathrm{P}(S_1 > 0, \ldots, S_{n-1} > 0, S_n = x) =$$

Counting probabilities

$$\mathrm{P}(S_n = x) = \frac{1}{2^n} N(n,x) = \frac{1}{2^n}\binom{n}{\frac{n+x}{2}}$$

$$\mathrm{P}(S_1 > 0, \ldots, S_{n-1} > 0, S_n = x) = \frac{\tilde{N}(n,x)}{2^n} =$$

Counting probabilities

$$\mathrm{P}(S_n = x) = \frac{1}{2^n} N(n, x) = \frac{1}{2^n} \binom{n}{\frac{n+x}{2}}$$

$$\mathrm{P}(S_1 > 0, \ldots, S_{n-1} > 0, S_n = x) = \frac{\tilde{N}(n, x)}{2^n} =$$

$$\frac{N(n-1, x-1) - N(n-1, -x-1)}{2^n} =$$

Counting probabilities

$$P(S_n = x) = \tfrac{1}{2^n} N(n,x) = \tfrac{1}{2^n}\binom{n}{\frac{n+x}{2}}$$

$$P(S_1 > 0, \ldots, S_{n-1} > 0, S_n = x) = \frac{\tilde{N}(n,x)}{2^n} =$$

$$\frac{N(n-1, x-1) - N(n-1, -x-1)}{2^n} =$$

$$\frac{1}{2^n}\left(\binom{n-1}{\frac{n+x-2}{2}} - \binom{n-1}{\frac{n-x-2}{2}}\right).$$

Returning to 0

Find $\mathrm{P}(S_n \text{ never meets } 0)$.

Returning to 0

Find P(S_n never meets 0).

$$\mathrm{P}(\forall n \geq 1 \ S_n \neq 0) =$$

Returning to 0

Find $\mathrm{P}(S_n \text{ never meets } 0)$.

$$\mathrm{P}(\forall n \geq 1 \; S_n \neq 0) =$$

$$2\mathrm{P}(\forall n \geq 1 \; S_n > 0) =$$

Returning to 0

Find $\mathrm{P}(S_n \text{ never meets } 0)$.

$$\mathrm{P}(\forall n \geq 1 \; S_n \neq 0) =$$

$$2\mathrm{P}(\forall n \geq 1 \; S_n > 0) =$$

$$\lim_{n\to\infty} \sum_{x=1}^{n} \mathrm{P}(S_1 > 0, \ldots, S_{n-1} > 0, S_n = x)$$

Returning to 0

Find $\mathrm{P}(S_n \text{ never meets } 0)$.

$$\mathrm{P}(\forall n \geq 1 \; S_n \neq 0) =$$

$$2\mathrm{P}(\forall n \geq 1 \; S_n > 0) =$$

$$\lim_{n\to\infty} \sum_{x=1}^{n} \mathrm{P}(S_1 > 0, \dots, S_{n-1} > 0, S_n = x) = 0.$$

Law of large numbers

Since $\mathrm{E}S_n = 0$, we get

$$\frac{S_n}{\sqrt{n}f(n)} \xrightarrow{\mathrm{P}} 0$$

for $f(n) \uparrow \infty$.

Law of large numbers

Since $\mathrm{E}S_n = 0$, we get

$$\frac{S_n}{\sqrt{n} f(n)} \xrightarrow{\mathrm{P}} 0$$

for $f(n) \uparrow \infty$.

- What about a.s. convergence?

Law of large numbers

Since $\mathrm{E}S_n = 0$, we get

$$\frac{S_n}{\sqrt{n}f(n)} \xrightarrow{\mathrm{P}} 0$$

for $f(n) \uparrow \infty$.

- What about a.s. convergence?
- Is there a limit of $\frac{S_n}{\sqrt{n}}$?