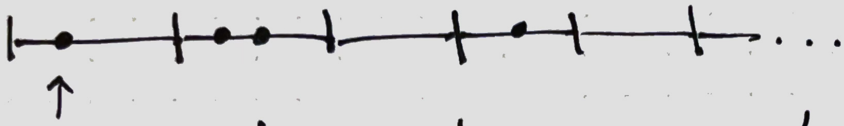
MSAI Statistics HW3 recital

Problem 1

$X_1, \dots, X_n \sim \mathrm{Pois}(\lambda)$



$x = \#$ of events per unit of time $= \{0, 1, 2, \dots\}$

$$p(x) = \frac{\lambda^x}{x!} e^{-\lambda}$$

λ - average # of events per unit of time

$$\sum_{x=0}^{+\infty} \dots = e^{\lambda}$$

1) MLE(λ):

$$\mathcal{L}(\lambda; \underline{X}) = \prod_{i=1}^{n} p(x_i | \lambda) = \frac{\lambda^{x_1 + x_2 + \dots + x_n}}{x_1! x_2! \dots x_n!} e^{-\lambda n}$$

$$\ell = \log \mathcal{L} = \left(\sum_{i=1}^{n} x_i\right) \cdot \log \lambda - \lambda n - \log\left(\prod x_i!\right)$$

$$\frac{\partial \ell}{\partial \lambda} = 0 = \left(\sum_{i=1}^{n} x_i\right) \frac{1}{\lambda} - n = 0 \rightarrow \hat{\lambda}_{MLE} = \frac{1}{n} \sum_{i=1}^{n} x_i$$

2) MOM(λ):

$$\mathbb{E}_{Pois}(x) = \sum_{x=0}^{\infty} x \cdot \frac{\lambda^x}{x!} e^{-\lambda}$$

$$\mathbb{E}_{pois}(x) = e^{-\lambda} \sum_{x=0}^{\infty} x \frac{\lambda^x}{x!} \; = $$

$$(e^{\lambda})'_{\lambda} = \left(\sum_{x=0}^{\infty} \frac{\lambda^x}{x!} \right)'_{\lambda} = \sum_{x=0}^{\infty} \frac{x \overbrace{\lambda^{x-1} \lambda}^{\lambda^x}}{x!} \cdot \frac{1}{\lambda} = \frac{1}{\lambda}(\dots)$$

$$(x^a)'_x = a x^{a-1} \qquad e^{\lambda}$$

$$= \; \cancel{e^{-\lambda}} \cdot \cancel{e^{\lambda}} \cdot \lambda$$

$$\frac{1}{n} \sum_{i=1}^{n} x_i = \hat{\lambda}_{MOM}$$

3) $$I(\lambda) = \overset{\mathbb{V}}{\cancel{\mathbb{E}}} \left(\frac{\partial \log p(x \mid \lambda)}{\partial \lambda} \right)$$

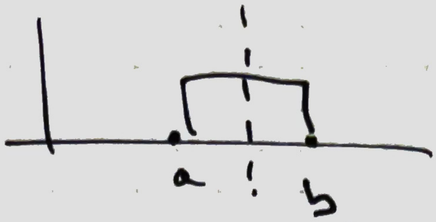
$$I_n(\lambda) \underset{(1)}{=} \mathbb{E}\left(\left(\sum x_i\right) \frac{1}{\lambda} - n \right) =$$

$$= \left(\sum \underbrace{\mathbb{E}(x_i)}_{\lambda} \right) \frac{1}{\lambda} - n = n \lambda \frac{1}{\lambda} - n$$

$$I(\lambda) = \mathbb{V}\left(\cancel{n}\, x \frac{1}{\lambda} - \frac{1}{\cancel{n}} \right) = \mathbb{V}\left(x \cdot \frac{1}{\lambda} \right) =$$

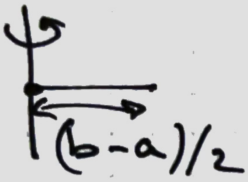
$$= \frac{1}{\lambda^2} \underbrace{\mathbb{V}(x)}_{\lambda} \; \boxed{= \frac{1}{\lambda}}$$

Pr 2. $U(a,b)$



$$Var(U(a,b)) = \mathbb{E}((x-\mu)^2) =$$

$$= \cancel{2}.$$

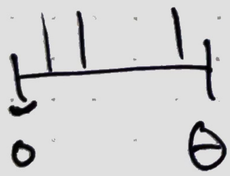


$$\frac{1}{b-a}\int_0^{(b-a)/2} x^2\,dx = \frac{1}{3}x^3\Big|_0^{(b-a)/2} =$$

$$= \frac{1}{3}\cdot\frac{1}{2^{\cancel{3}}}(b-a)^3$$

$$= \frac{1}{\underbrace{3\cdot 4}_{12}}(b-a)^2 = \frac{1}{12}(b-a)^2$$

$$\begin{cases} 1)\ \bar{X} = \dfrac{b+a}{2} \\ \overline{(X-\bar{X})^2} = \dfrac{1}{12}(b-a)^2 \end{cases}$$



$$\hat{\theta}_{MOM} = 2\bar{X}$$

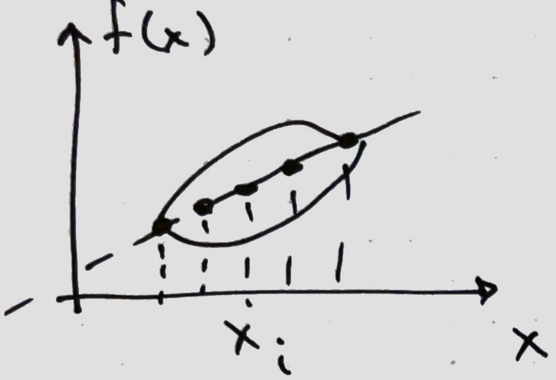
$$\sqrt{V(\bar{X})}\sqrt{12} = \hat{b}-\hat{a}$$

$$\hat{b}+\hat{a} = 2\bar{X}$$

$b =$

Pr. 3

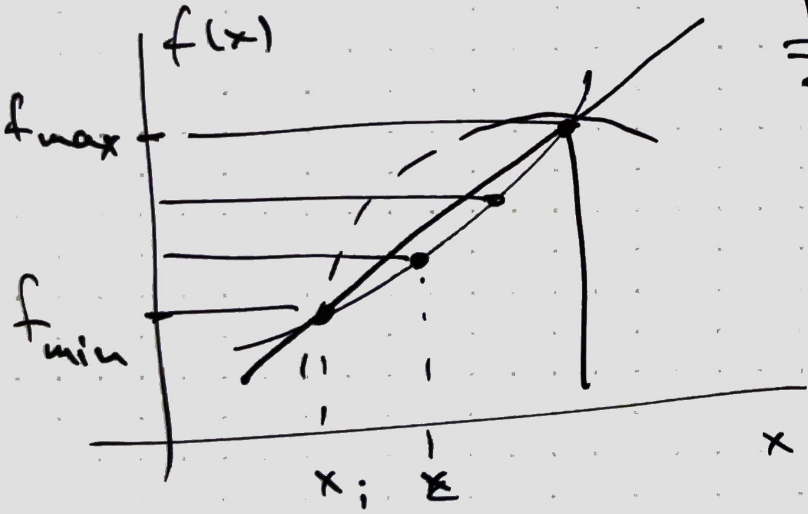
$$KL(p\|q) \geq 0 \quad \forall p, q$$



$$\mathbb{E}(f(x)) \approx \sum_{x_i} p(x_i) \cdot f(x_i)$$

$p(x_i) \geq 0$

$\sum p(x_i) = 1$



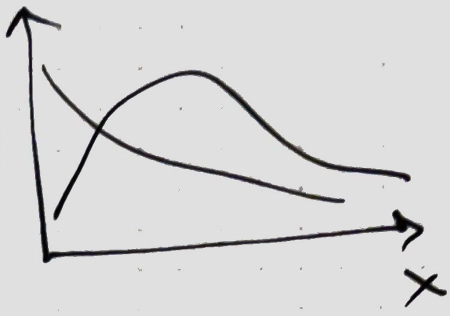
$$KL(p\|q) = \sum_x p(x) \log \frac{p(x)}{q(x)}$$

$$-KL = \sum p \log \frac{q}{p} \leq \log\Big(\underbrace{\cancel{p} \sum \frac{q}{\cancel{p}}}_{1}\Big) =$$

$$KL \geq 0 \qquad\qquad = 0$$

Pr. 4

$x \sim \text{Gamma}($ $\exp(\log x^{k-1})$



$$p(x|k,\theta) = \frac{x^{k-1} e^{-x/\theta}}{\Gamma(k)\,\theta^{k}} =$$

$$p(x|\alpha,\beta) = \frac{x^{\alpha-1} e^{-\beta x}}{\Gamma(\alpha)} \beta^{\alpha}$$

sufficient statistics

$$p(x|\theta) = \frac{\exp\left(\sum T_i(x)\cdot \eta_i(\theta) - A(\eta)\right)}{B(x)} =$$

$$= \frac{1}{\Gamma(k)\theta^{k}} \exp\left(-\frac{1}{\theta}x + (k-1)\log x\right)$$

$$\vec{T}(x) = (x, \log x)$$

$$\vec{\eta}(\vec{\theta}) = (-1/\theta, k-1)$$

$$\vec{\theta} = (k,\theta)$$