MSAI Statistics & Probability – Week 5 Seminar & HW

Problem 1: Consider an Erdős-Rényi random graph $G(n,p)$ on n vertices (that is, for any two vertices, there is an edge between them with probability p and no edge with probability $1-p$; all edges are independent of each other). Find the expected number of triangles in $G(n,p)$ (triples of vertices all pairwise connected with an edge). Provide an answer for $n=7,\ p=\frac{1}{7}$.

Problem 2: Let $\xi \sim \mathcal{N}(0,\sigma^2)$ (a random variable distributed normally with zero mean and dispersion σ^2). Find $\mathbb{E}\xi^k$ (expectation of k-th power of ξ). Provide an answer for $\sigma^2 = 2,\ k=4$.

Problem 3: Let $\xi \sim \mathrm{Pois}(\lambda)$ (a random variable with Poisson distribution with rate λ). For some natural $k \in \mathbb{N}$, find $\mathbb{E}\binom{\xi}{k}$ (expected $\binom{\xi}{k}$ - binomial coefficient). Provide an answer for $\lambda = 3,\ k=4$.

Problem 4: Find $\mathbb{E}\xi$ if ξ have the following distribution function:

$$F(x) = \begin{cases} 0, & x < -2, \\ 1/5, & -2 \leq x < 1 \\ x^2/4, & 1 \leq x < 2 \\ 1, & x \geq 2 \end{cases}$$

Problem 5: (Conditional expectation). You have two fair dice (six-sided cubes, $1, 2, ..., 6$ on the faces, each face equally probable). You throw them and read the sum of their values, $z = x+y$. Find the expected $\mathbb{E}(x|z)$.

To do this, consider the following functional

$$J(\alpha) = \sum_{x,z} (x - \alpha z)^2 p(x,z) = \sum_{x,y} (x - \alpha(x+y))^2 p(x,y)$$

- it only depends on α. You know all possible x outcomes and all possible $z = x+y$ outcomes and their probabilities! Use any mathematical software to compute the expression of $J(\alpha)$ as a function of α, and then minimize it with respect to α.