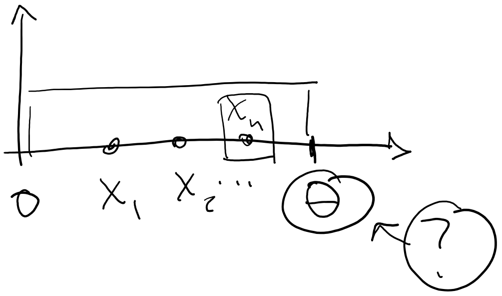
# HW1-2 Recital

15 апреля 2022 г. 19:49

## German tank problem




$X_1, X_2, \dots, X_n \sim U[0, \Theta]$

1) $\hat{\Theta}^{(1)} = \max((X_1, \dots, X_n))$

MLE

2) $\overline{X} = \Theta/2 \rightarrow \hat{\Theta}^{(2)} = \frac{2}{n}\sum_{i=1}^{n} X_i$

MOM estimate

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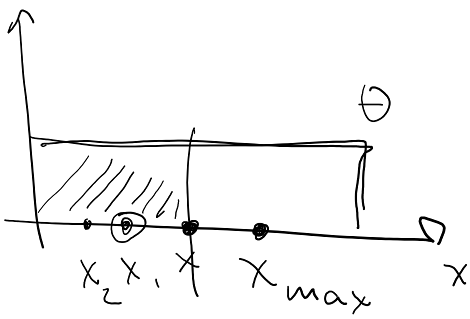
1) a) $\hat{\Theta} = \max(\dots)$

$\text{bias}(\hat{\Theta}) = \underbrace{\mathbb{E}(\hat{\Theta})} - \Theta$

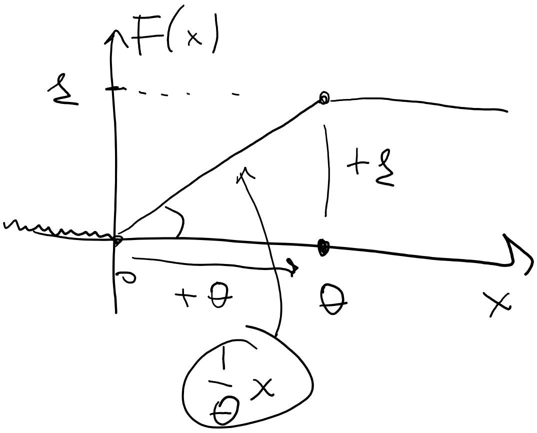



$\Pr(\hat{\Theta} \leq X) =$

$\Pr(\max(X_1, X_2, \dots, X_n) \leq X) =$

$= \underbrace{\underbrace{\Pr(X_1 \leq X)}_{F(x)} \Pr(X_2 \leq X) \dots \Pr(X_n \leq X)}_{n} =$

$F(x)$ CDF of $U[0, \Theta]$




$= \left(\frac{x}{\Theta}\right)^n$ CDF of $\hat{\Theta}$

if $0 \leq x \leq \Theta$

$\mathbb{E}(\hat{\Theta}) \int \dots = \dots \int^{\Theta} \dots n-1 \dots \frac{1}{\dots} \Theta^{n+1}$

$$\mathbb{E}(\hat{\theta}) = \int x \underbrace{dF}_{\frac{dF}{dx}\cdot dx} = \frac{1}{\theta^n}\int_0^\theta \underbrace{x \cdot n x^{n-1}}_{x^n} dx = \frac{1}{\theta^n} n \cdot \frac{\theta^{n+1}}{n+1} =$$

$$\left(\left(\frac{x}{\theta}\right)^n\right)' = \frac{1}{\theta^n} n x^{n-1}$$

$$\int x^n dx = \frac{x^{n+1}}{n+1} + c$$

$$= \frac{n}{n+1}\theta$$

$$\text{bias}(\hat{\theta}) = \frac{n}{n+1}\theta - \theta = -\frac{\theta}{n+1} \xrightarrow[n \to \infty]{} 0$$

$$\frac{n+1-1}{n+1} = 1 - \frac{1}{n+1}$$

b) <u>standard error</u>

$$se(\hat{\theta}) = \sqrt{Var(\hat{\theta})}$$

$$Var(\hat{\theta}) = \mathbb{E}(\hat{\theta}^2) - \mathbb{E}^2(\hat{\theta})$$

$$\int_0^\theta x^2 dF = \int_0^\theta x^2 \cdot \frac{1}{\theta^n} n x^{n-1} dx = \frac{n}{\theta^n}\int_0^\theta \underbrace{x^{n+1}}_{\frac{x^{n+2}}{n+2}\Big|_0^\theta} dx =$$

$$= \frac{n}{\theta^n}\frac{\theta^{n+2}}{n+2} = \frac{n}{n+2}\theta^2$$

$$Var(\hat{\theta}) = \frac{\cancel{n}^{1}}{\cancel{n}+2}\theta^2 - \left(\frac{\cancel{n}^{1}}{\cancel{n}+1}\right)^2\theta^2 \xrightarrow[n \to \infty]{} 0$$

c) $MSE(\hat{\theta}) = \text{bias}^2(\hat{\theta}) + \underbrace{Var(\hat{\theta})}_{= se^2(\hat{\theta})} \xrightarrow[n\to\infty]{} 0$

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2) a) $\hat{\theta} = 2\bar{X} = \frac{2}{n}\sum_{i=1}^{n} X_i$

$\text{bias}(\hat{\theta}) = \underbrace{\mathbb{E}(\hat{\theta})}_{\uparrow} - \theta \boxed{= 0}$

$\underbrace{\mathbb{E}\left(\frac{2}{n}\sum_{i=1}^{n} X_i\right) = \frac{2}{n}\sum_{i=1}^{n} \underbrace{\mathbb{E}(X_i)}_{\frac{\theta}{2}} = \frac{\theta}{n} n = \theta}$

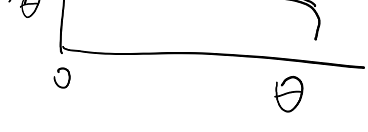
b) $se(\hat{\theta}) = \sqrt{Var(\hat{\theta})}$

$Var\left(\frac{2}{n}\sum_{i=1}^{n} X_i\right) = \mathbb{E}(\hat{\theta}^2) - \mathbb{E}^2(\hat{\theta})$

$\mathbb{E}\left(\frac{4}{n^2}\underbrace{\left(\sum_{i=1}^{n} X_i\right)^2}\right) = \frac{4}{n^2}\left(\sum_{i=1}^{n} \mathbb{E}(X_i^2) + \sum_{i\neq j} \underbrace{\mathbb{E}(X_i \cdot X_j)}_{\overset{\text{indep}}{=} \underbrace{\mathbb{E}(X_i)}_{=\theta/2}\underbrace{\mathbb{E}(X_j)}_{=}}\right)$

$\underbrace{\sum_{i=1}^{n} X_i^2}_{n} + \underbrace{\sum_{i\neq j} X_i X_j}_{\binom{n}{2} = \frac{n(n-1)}{2}}$

$\mathbb{E}(X^2) = \int_0^\theta x^2 \frac{1}{\theta} dx = \frac{1}{\theta}\frac{\theta^3}{3} = \frac{\theta^2}{3}$




$\ldots \to se(\hat{\theta}) = \theta\sqrt{\frac{1}{3n}} \xrightarrow[n\to\infty]{} 0$

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cosistency $(\hat{\Theta})$:

$$\hat{\Theta} \xrightarrow[n \to \infty]{P} \Theta$$

MSE convergence implies convergence in probability

$$\underline{Pr(|\hat{\Theta} - \Theta| > \varepsilon)} \xrightarrow[\underline{n \to \infty}]{} 0$$

strong consistency:

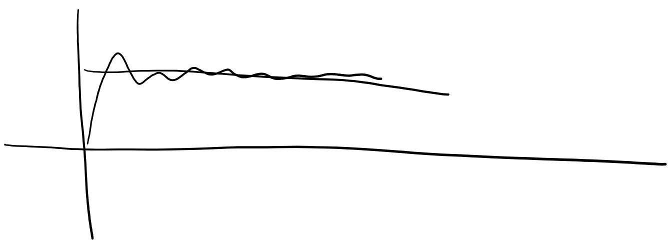
$$\hat{\Theta} \xrightarrow[n \to \infty]{a.s.} \Theta$$

$$Pr(\lim \hat{\Theta}_n \neq \Theta) = 0$$

$$Pr(\lim \hat{\Theta}_n = \Theta) = 1$$

for $\hat{\Theta}_n = 2\bar{X}$

$$\lim \Theta_n = 2 \lim \bar{X}_{(n)}$$



$$\bar{X}_n \xrightarrow{a.s} \mathbb{E}(x)$$

Strong Law of Large Numbers

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From CLT

$$\hat{\Theta}_n - \Theta \underset{n \to \infty}{\sim} \mathcal{N}\left(0, \frac{\sigma}{\sqrt{n}}\right)$$

$\to 0$

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HW2

# HW2

1) $X_1, X_2, \dots, X_n \sim Be(p)$

$Y_1, Y_2, \dots, Y_m \sim Be(q)$

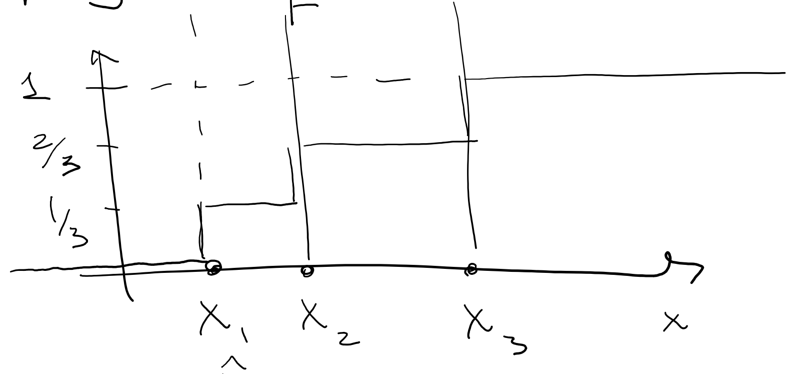
estimator for $p$? $p-q$?

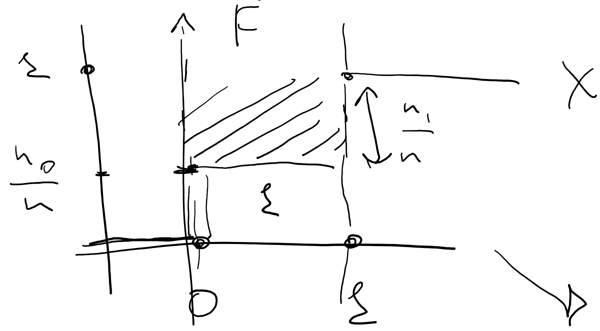
plug-in estimator?

Statistical functionals = functionals of the $F(x)$

$E(X) = \int x \, dF$ Var, ... Median, CDF(x)

plug-in $\hat{F}$ ECDF




$X_1, \dots X_n \to n_0$ 0-s

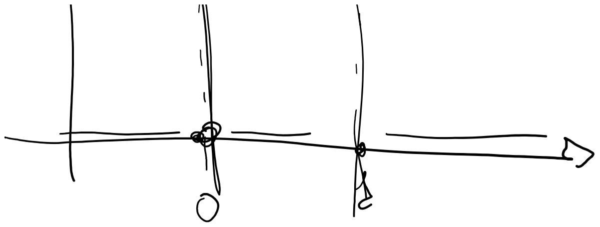
$\to n_1$ 1-s

$n_0 + n_1 = n$

$p = E(Be(p)) = \int x \, dF = \int x \, d\hat{F} =$

$\frac{n_0}{n}\delta_0(x)$ $\frac{n_1}{n}\delta_1(x)$

$= 0 \cdot \frac{n_0}{n} + 1 \cdot \frac{n_1}{n}$




$= 0 \cdot \frac{\;}{n} + 1 \cdot \frac{2}{n}$

$X_1, \dots X_n = 0, 1, 1, \dots 0, 1$

$\bar{X}_{(n)} = \frac{1}{n}\sum_{i=1}^{n} x_i = \frac{n_1}{n}$

$\to \hat{p} = \bar{X}_{(n)}$

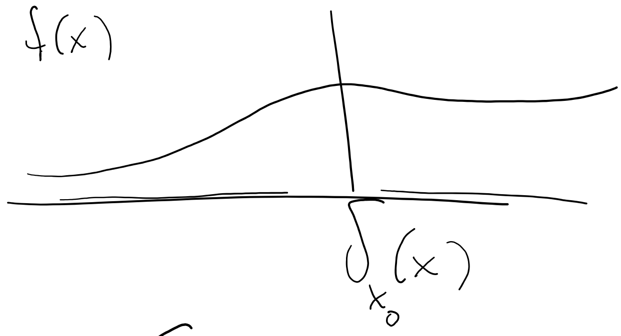
any linear functional

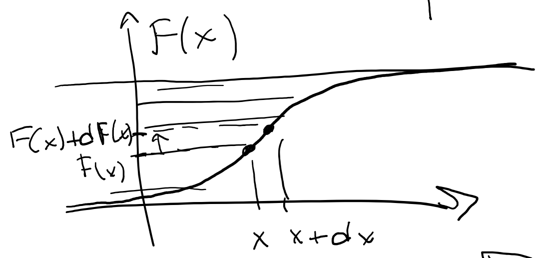
$\int r(x)\,dF = \int r(x)\,d\hat{F} = \frac{1}{n}\sum_{i=1}^{n} r(x_i)$

$\int r(x)\,dF(x)$

$\frac{dF}{dx}\,dx$

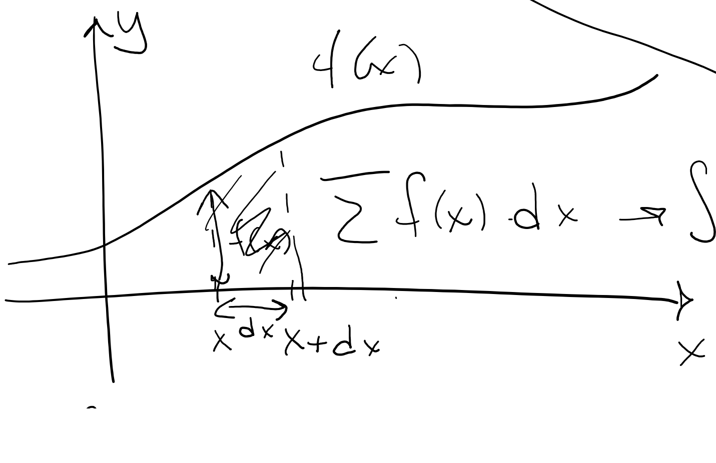
$p(x)$




$\int f(x)\cdot\delta_{x_0}(x)\,dx = f(x_0)$

$\int \boxed{r(x)}\,dF$




$\sum f(x)\,dx \to \int$

$\int f(x)dx$

$$\mathbb{E}(e^X) = \int \underbrace{e^x}_{r(x)} dF$$

plug-in estimator for $e^x$ $= \frac{1}{n}\sum_{i=1}^{n} e^{X_i}$

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$$\hat{p} = \overline{X}_n$$

$$\text{bias}(\hat{p}) = \underbrace{\mathbb{E}(\hat{p})} - p = 0$$

$$\mathbb{E}\left(\frac{1}{n}\sum_i X_i\right) = \frac{1}{n}\sum \underbrace{\mathbb{E}(X_i)}_{p} = \frac{1}{n} p \cdot n = p$$

$Be(p)$

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2) $p - q = \overline{X}_n - \overline{Y}_m$