Probability theory

Lecture 5: Expectation

Maksim Zhukovskii

MIPT

Expectation of a discrete r.v.

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If Ω is finite or countably infinite,
then $E\xi = \sum_{\omega \in \Omega} \xi(\omega) P(\{\omega\})$.

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- If $X = \mathbb{Z} \setminus \{0\}$, $\mathrm{P}(\xi = k) = \frac{1}{2\zeta(2)k^2}$, then $\mathrm{E}\xi$ does not exist.

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   $\int_{-\infty}^\infty \frac{x-a+a}{\sqrt{2\pi\sigma^2}} e^{-\frac{(x-a)^2}{2\sigma^2}} dx =$
   $\sqrt{\frac{\sigma^2}{2\pi}} \int_{-\infty}^\infty e^{-\frac{(x-a)^2}{2\sigma^2}} d\left(\frac{(x-a)^2}{2\sigma^2}\right) + a = a$.

General case

$\mathsf{E}\xi = \int_\Omega \xi \, d\mathsf{P}$, if the Lebesgue integral exists.

Simple random variables

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Lebesgue integral

$\int_\Omega \xi dP = \sum_{i=1}^n x_i P(A_i)$

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- *for every* $\omega \in \Omega$, $\xi_n(\omega) \to \xi(\omega)$ *as* $n \to \infty$,
- $\xi_{n+1}(\omega) \geq \xi_n(\omega)$ *for every* $n \in \mathbb{N}$ *and* $\omega \in \Omega$.

The proof

$$\xi_n = \sum_{k=1}^{n2^n} \frac{k-1}{2^n} I\left(\frac{k-1}{2^n} \le \xi < \frac{k}{2^n}\right)$$

The integral of a non-negative r.v.

Lebesgue integral

$\int_\Omega \xi dP = \lim_{n\to\infty} \int_\Omega \xi_n dP$

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$$\int_\Omega \xi dP = \int_\Omega \xi^+ dP - \int_\Omega \xi^- dP$$

The expectation does not exist, if both $\int_\Omega \xi^+ dP$, $\int_\Omega \xi^- dP$ are infinite.

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6) $\mathsf{E}(\xi + \eta) = \mathsf{E}\xi + \mathsf{E}\eta$.

Counting expectation of a binomial r.v.

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$$\mathsf{E}\xi = \mathsf{E}(\xi_1 + \ldots + \xi_n) = np$$

Lebesgue's dominated convergence th.

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Integration by substitution

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If $\mathsf{E}\varphi(\xi)$ exists, then

$$\mathsf{E}\varphi(\xi) = \int_{\mathbb{R}} \varphi \, d\mathsf{P}_{\xi}$$

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If $0 \leq f_n \uparrow f$ and $\lim_{n\to\infty} \int f_n d\mu < \infty$,

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Examples

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$$\mathsf{E}\xi^2 = \int\limits_{-\infty}^{\infty} x^2 \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}}\, dx = \dots$$

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   $= 1.$

2. $\xi \sim Pois(\lambda)$.
   $\mathsf{E}e^{\xi} = \sum_{k=0}^{\infty} e^k \frac{\lambda^k e^{-\lambda}}{k!} = e^{\lambda(e-1)}.$

Expectation of a product of two r.v.

Theorem

Let ξ, η *be independent random variables,* $\mathsf{E}|\xi\eta| < \infty$. *Then* $\mathsf{E}\xi\eta = \mathsf{E}\xi\mathsf{E}\eta$.

Expectation of a product of two r.v.

Theorem

Let ξ, η be independent random variables, $\mathsf{E}|\xi\eta| < \infty$. *Then* $\mathsf{E}\xi\eta = \mathsf{E}\xi\mathsf{E}\eta$.

$$\mathsf{E}\xi\eta = \int_{\mathbb{R}^2} xy d\mathsf{P}_{(\xi,\eta)} =$$

$$\int_{\mathbb{R}} x d\mathsf{P}_{\xi} \times \int_{\mathbb{R}} y d\mathsf{P}_{\eta} = \mathsf{E}\xi\mathsf{E}\eta.$$

The proof in the discrete case

Let ξ, η be independent discrete random variables taking values from sets X, Y respectively.

The proof in the uniform case

Let ξ, η be independent absolutely continuous random variables with densities p_ξ, p_η respectively.