Probability theory

Lecture 1: Probability space

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MIPT

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   $X/n \approx 1/2$

Sample space and events

Informal definition:

sample space — the set of all possible outcomes of an experiment

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- $A \subseteq \Omega$ — *an event*

Examples

Tossing a coin:

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$A = \{2, 4, 6\}$ — 'outcome is even'

Classical probability

Find the probability that a dice shows an even number.

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- $\Omega = \{\omega_1, \ldots, \omega_n\}$
- $A \subseteq \Omega \Rightarrow \mathrm{P}(A) = \frac{|A|}{n}$.

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P(all these balls are blue) $=?$

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Then $\mathrm{P}(A) = |A|/|\Omega| = \binom{5}{3}/\binom{9}{3} = \frac{5}{42}$.

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- $\mathrm{P} : \mathcal{F} \to [0, 1]$ — *a probability measure* (or, simply, *probability*).

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A is *an event* if $A \in \mathcal{F}$

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- $\mathsf{P}(\Omega) = 1$,
- $A_1, A_2, \ldots \in \mathcal{F}$ are disjoint $\Rightarrow$ $\mathsf{P}(A_1 \sqcup A_2 \sqcup \ldots) = \sum_{i=1}^{\infty} \mathsf{P}(A_i)$.

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  $\mathcal{F}_A := \{\varnothing, \Omega, A, \overline{A}\}$

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- The Borel σ-algebra on $\mathbb{R}$:
  $\mathcal{B}(\mathbb{R}) = \sigma(\{(a, b),\ -\infty \le a < b \le \infty\})$.

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2. Construction of the minimum σ-algebra:

$$\mathcal{F}_* := \bigcap_{\mathcal{F} \supseteq \Sigma \text{ — } \sigma\text{-algebra}} \mathcal{F}$$

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- $A_1, A_2, \ldots \in \mathcal{F}_*$ $\Rightarrow$ $A_1, A_2, \ldots \in \mathcal{F}$ for every $\mathcal{F}$ $\Rightarrow$ $A_1 \cup A_2 \cup \ldots \in \mathcal{F}$ for every $\mathcal{F}$

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4. $\mathcal{F}_*$ is minimum:

If $\mathcal{F} \supseteq \Sigma$ — σ-algebra, then $\mathcal{F}_* \subseteq \mathcal{F}$.

Special probability measures on finite Ω

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- **Bernoulli scheme:**
  $\Omega = \{0, 1\}^n$, $\mathcal{F} = 2^\Omega$,
  $\mathrm{P}(\{(x_1, \dots, x_n)\}) = p^{\sum x_i}(1-p)^{n-\sum x_i}$

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$\mathsf{P}(A) = \frac{\mu(A)}{\mu(\Omega)}$

Chance of meeting in a bar

Alice and Bob decided to meet in a bar on a given day.
Each one has to come in the bar at a random time between 12:00 and 13:00 and wait for another one for 15 minutes.
What is the probability that they will meet?

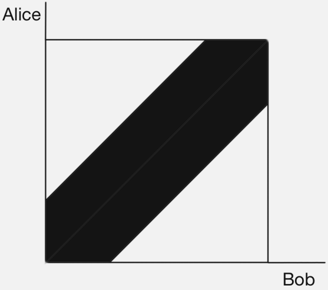
Chance of meeting in a bar

Solution:

$\Omega = [0,1]^2 \subset \mathbb{R}^2$,

A is colored black,

$\mathrm{P}(A) = \frac{\mu(A)}{\mu(\Omega)} = \frac{1-2\cdot\frac{9}{32}}{1} = \frac{7}{16}$.



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Every edge appears *independently* with probability p

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Proof.

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$\varnothing = \varnothing \sqcup \varnothing \Rightarrow$

$\mathsf{P}(\varnothing) = \mathsf{P}(\varnothing) + \mathsf{P}(\varnothing)$

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$B_i = A_i \setminus (A_1 \cup \ldots \cup A_{i-1}) \subseteq A_i$

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$$P(A_1) + P(A_2) + \ldots \qquad \square$$

Continuity of probability

Let $\mathcal{F}$ be a σ-algebra of subsets of Ω, $\mathsf{P} : \mathcal{F} \to [0, 1]$ be such that

- $\mathsf{P}(\Omega) = 1$,
- $\mathsf{P}(A \sqcup B) = \mathsf{P}(A) + \mathsf{P}(B)$ whenever $A, B \in \mathcal{F}$ are disjoint.

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3. for every sequence $A_n \in \mathfrak{F}$ such that $A_n \downarrow A$ ($A_n \supseteq A_{n+1}$, $A = \cap A_n$), $\mathrm{P}(A) = \lim_{n\to\infty} \mathrm{P}(A_n)$;
4. for every sequence $A_n \in \mathfrak{F}$ such that $A_n \downarrow \varnothing$, $\lim_{n\to\infty} \mathrm{P}(A_n) = 0$.

The proof

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$$\lim_{n\to\infty} \sum_{i=1}^{n} (\mathrm{P}(A_i) - \mathrm{P}(A_{i-1})) = \lim_{n\to\infty} \mathrm{P}(A_n).$$

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2)⇒3)

$$\mathrm{P}\left(\bigcap_{n=1}^{\infty} A_n\right) = \mathrm{P}\left(\overline{\bigcap_{n=1}^{\infty} \overline{A}_n}\right) = 1 - \mathrm{P}\left(\bigcup_{n=1}^{\infty} \overline{A}_n\right);$$

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$3) \Rightarrow 4)$ because 4) is special case of 3).

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$$0 = \lim_{n\to\infty} \mathrm{P}(B_n) = \lim_{n\to\infty} \left(\mathrm{P}(A) - \mathrm{P}\left(\bigcup_{i=1}^{n-1} A_i \right) \right)$$

$$= \mathrm{P}(A) - \sum_{i=1}^{\infty} \mathrm{P}(A_i).$$