Probability theory

Lecture 4: Random variables and vectors

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Measurable functions

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Theorem

Let $\mathcal{M} \subset \mathcal{B}$ such that $\sigma(\mathcal{M}) = \mathcal{B}$.

F is $(\mathcal{A}|\mathcal{B})$-measurable

if and only if,

for every $X \in \mathcal{M}$, $f^{-1}(X) \in \mathcal{A}$.

The proof

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$E = \mathbb{R}^n$, $\mathcal{E} = \mathcal{B}(\mathbb{R}^n)$
$\Rightarrow f$ — a random vector.

Why do we need measurability?

Example: an indicator random variable

Let $A \in \mathcal{F}$.

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Indicator of A

$I_A : \Omega \to \{0, 1\}$,
$I_A(\omega) = 1$ if and only if $\omega \in A$.

Functions of random variables

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Theorem

If ξ is an n-dimensional random vector,

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then $f(\xi)$ is a random vector as well.

The proof

Components of a random vector

Theorem

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Theorem

Let $\xi : \Omega \to \mathbb{R}^n$*.*

$\xi = (\xi_1, \dots, \xi_n)$ *is a random vector*

if and only if

$\xi_1, \dots, \xi_n$ *are random variables.*

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Continuous functions

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Corollary

If ξ, η are random variables, then

$\xi + \eta$, $\xi - \eta$, $\xi\eta$, $(\xi/\eta)I(\eta \neq 0)$
are random variables as well.

Limits

Theorem

Let $\xi_1, \xi_2, \ldots$ *be random variables.*

Then $\overline{\lim}_{n\to\infty}\xi_n$, $\underline{\lim}_{n\to\infty}\xi_n$, $\sup_n \xi_n$, $\inf_n \xi_n$ *are random variables as well.*

The proof

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$$\exists\varepsilon > 0\,\forall n$$

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Probability distribution

Theorem

The function

$$\mathsf{P}_\xi : \mathcal{B}(\mathbb{R}^n) \to [0,1],\ \mathsf{P}_\xi(B) = \mathsf{P}(\xi \in B),$$

is a probability on $(\mathbb{R}, \mathcal{B}(\mathbb{R}))$.

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- If P_ξ is absolutely continuous, then its density p_ξ is called density of ξ, ξ is called absolutely continuous.
- If P_ξ is discrete, then ξ is called discrete as well.

Notations

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2. $\mathrm{P}(\{\omega : \xi(\omega) = x\}) =: \mathrm{P}(\xi = x)$,
3. $F_\xi(x) = \mathrm{P}(\{\omega : \xi(\omega) \leq x\}) =: \mathrm{P}(\xi \leq x)$.

Examples

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$$F_\xi(x) = \begin{cases} 0, & x < 0, \\ \frac{1}{4}, & 0 \le x < 1, \\ \frac{1}{2}, & 1 \le x < 2, \\ \frac{3}{4}, & 2 \le x < 3, \\ 1, & x \ge 3. \end{cases}$$

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- $\xi_1, \ldots, \xi_n$ are pairwise independent,
  if, for every i, j, ξ_i, ξ_j are independent.

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- Random vectors from $\{\xi_\alpha\}_{\alpha \in \mathcal{A}}$ are independent, if, for every $n \in \mathbb{N}$ and any $t_1, \ldots, t_n \in \mathcal{A}$, the random vectors $\xi_{t_1}, \ldots, \xi_{t_n}$ are independent.

Independent discrete random variables

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$$\mathsf{P}(\xi_1 = x_1, \ldots, \xi_n = x_n) = \mathsf{P}(\xi_1 = x_1) \ldots \mathsf{P}(\xi_n = x_n).$$

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The same is true for random **vectors**.

Functions of independent vectors

Theorem

Let

- $\xi = (\xi_1, \ldots, \xi_{n_1})$ $\eta = (\eta_1, \ldots, \eta_{n_2})$ *be independent random vectors,*
- $f : \mathbb{R}^{n_1} \to \mathbb{R}^{k_1}$, $g : \mathbb{R}^{n_2} \to \mathbb{R}^{k_2}$ *be Borel functions.*

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Then $f(\xi)$, $g(\eta)$ *are independent.*

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Then $f(\xi)$, $g(\eta)$ *are independent.*

The same is true for **several** vectors.

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$$p_{\xi+\eta}(x) = \int_{\mathbb{R}} p_\xi(x-u)p_\eta(u)du.$$

The proof

An example