Probability intro

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Conditional probability

$$p(x|y) = \frac{p(x,y)}{p(y)}$$

Product rule

$$p(x,y) = p(x|y)p(y)$$

Sum rule

$$p(x) = \int p(x|y)p(y)\mathrm{dy} = \int \mathrm{p(x,y)dy}$$

Bayes theorem

$$p(y|x) = \frac{p(x|y)p(y)}{\int p(x|y)p(y)\mathrm{dy}}$$

Bayesian vs Frequentist

Bayesian vs Frequentist

$$X = (X_1, \dots, X_n) \sim p(x|\theta)$$

Frequentist

$$\theta^{MLE} = \arg\max_\theta p(X|\theta)$$

Bayesian

Introduce $p(\theta)$ that encodes our beliefs about θ, then

$$p(\theta|X) = \frac{p(X|\theta)p(\theta)}{\int p(X|\theta)p(\theta)\mathrm{d}\theta}$$

Bayesian vs Frequentist

	Frequentist	Bayesian
Randomness interpretation	Objective randomness	Subjective randomness
Variables	Random and deterministic	All random
Inference	Maximum likelihood	Bayes theorem
Estimate	Point estimate	Posterior distribution
Application	$n \gg d$	$\forall n$

$$\lim_{n \gg d} p(\theta|X) = \delta(\theta - \theta^{MLE})$$

Bayesian advantages

Bayesian advantages

1. Regularization

$$\log p(\theta|X) \propto \log p(X|\theta) + \log p(\theta)$$

2. Compositioning (allows for chaining)

$$p(y|z,x) = \frac{p(z|y)p(y|x)}{\int p(z|y)p(y|x)\mathrm{dy}}$$

3. Streaming data: $p(\theta|X)$ encodes X
4. Latent variable models: $p(\theta|X)$ encodes MORE than X
5. Scalability

Point estimates with posterior

Point estimates with posterior

Posterior mean estimation

$$\bar{\theta} = \int \theta p(\theta|X)\mathrm{d}\theta$$

Maximum a posteriori estimation

$$\theta^{MAP} = \arg\max_\theta p(\theta|X)$$

$$\theta^{MAP} = \arg\max_\theta p(\theta|X) = \arg\max_\theta \log p(\theta|X) = \arg\max_\theta (\log p(X|\theta) + \log p(\theta))$$

Example

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Suppose that you have a sample $X = (X_1, \dots, X_n) \sim p(x|\theta)$, where

$$\begin{aligned} p(x|\theta) &= Be(\theta) \\ p(\theta) &= U[0,1] \end{aligned}$$

Then,

$$p(\theta|X) \propto p(x|\theta)p(\theta) = p^s(1-p)^{n-s}$$

$$p(x|\alpha,\beta) = \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)}x^{\alpha-1}(1-x)^{\beta-1}$$

$$p(\theta|X) = Beta(s+1, n-s+1)$$

Example

$$\begin{aligned} p(x|\theta) &= Be(\theta) \\ p(\theta) &= U[0,1] \\ p(\theta|X) &= Beta(s+1, n-s+1) \end{aligned}$$

$$\bar{\theta} = \mathbb{E}[Beta(s+1, n-s+1)]$$

$$\bar{\theta} = \mathbb{E}[Beta(s+1, n-s+1)] = \frac{\alpha}{\alpha+\beta} = \frac{s+1}{n-1}$$

$$\bar{\theta} = \lambda_n \theta^{MSE} + (1-\lambda_n)\tilde{\theta}$$

where $\tilde{\theta} = \frac{1}{2}$ is prior mean and $\lambda_n = \frac{n}{n+2}$

Example

Suppose that you have a sample $X = (X_1, \dots, X_n) \sim p(x|\theta)$, where

$$\begin{aligned} p(x|\theta) &= Be(\theta) \\ p(\theta) &= Beta(\alpha, \beta) \end{aligned}$$

Then,

$$p(\theta|X) = Beta(\alpha+s, \beta+n-s)$$

And

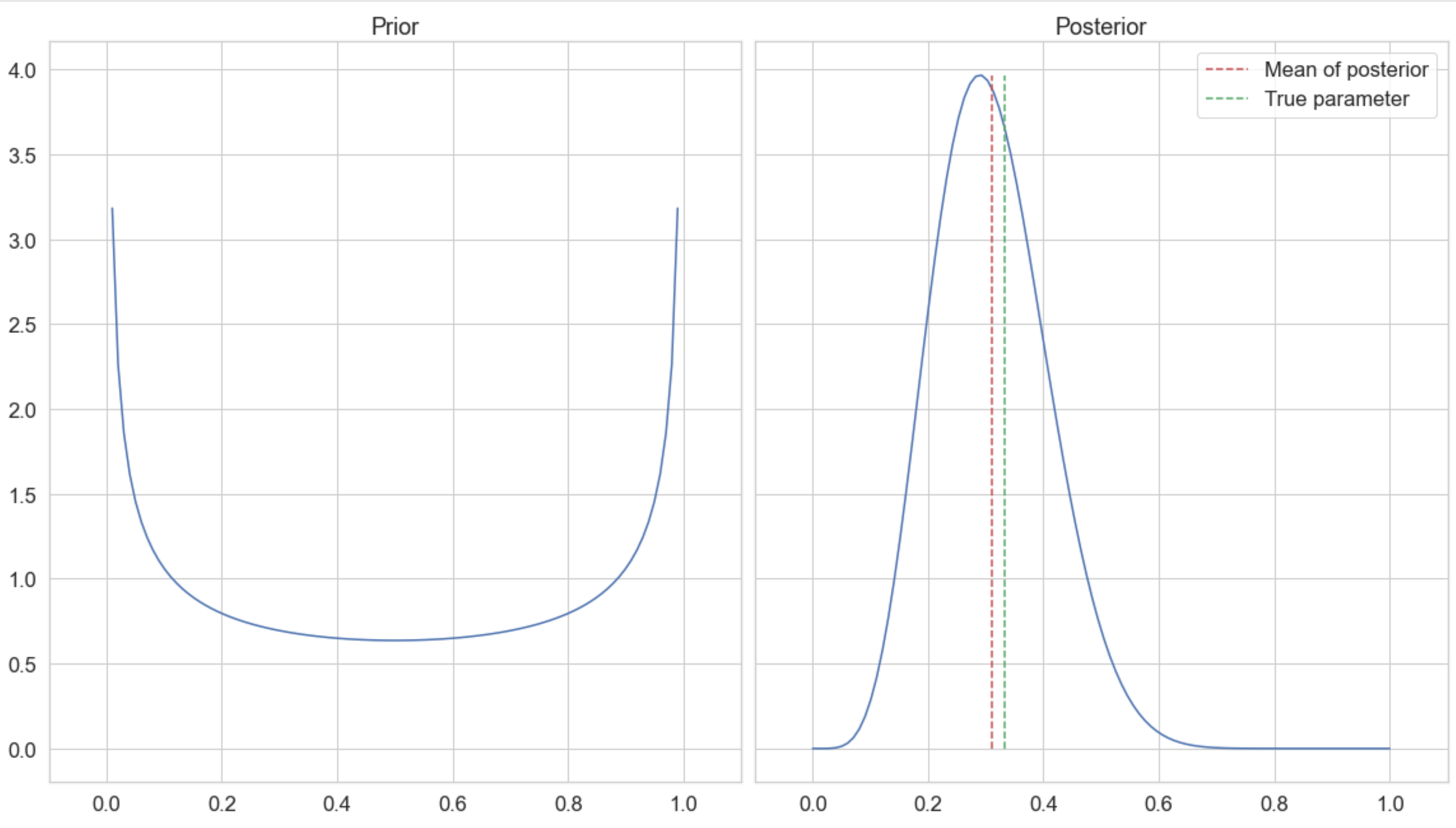
$$\bar{\theta} = \mathbb{E}[Beta(\alpha+s, \beta+n-s)] = \frac{\alpha}{\alpha+\beta} = \frac{\alpha+s}{\alpha+\beta+n} = \frac{n}{\alpha+\beta+n}\theta^{MLE} + \frac{\alpha+\beta}{\alpha+\beta+n}\tilde{\theta}$$

Code

In [1]:
```
import numpy as np
import scipy.stats as sts

import matplotlib.pyplot as plt
import seaborn as sns

sns.set(style="whitegrid", font_scale=1.5)
sns.despine()

%matplotlib inline
```

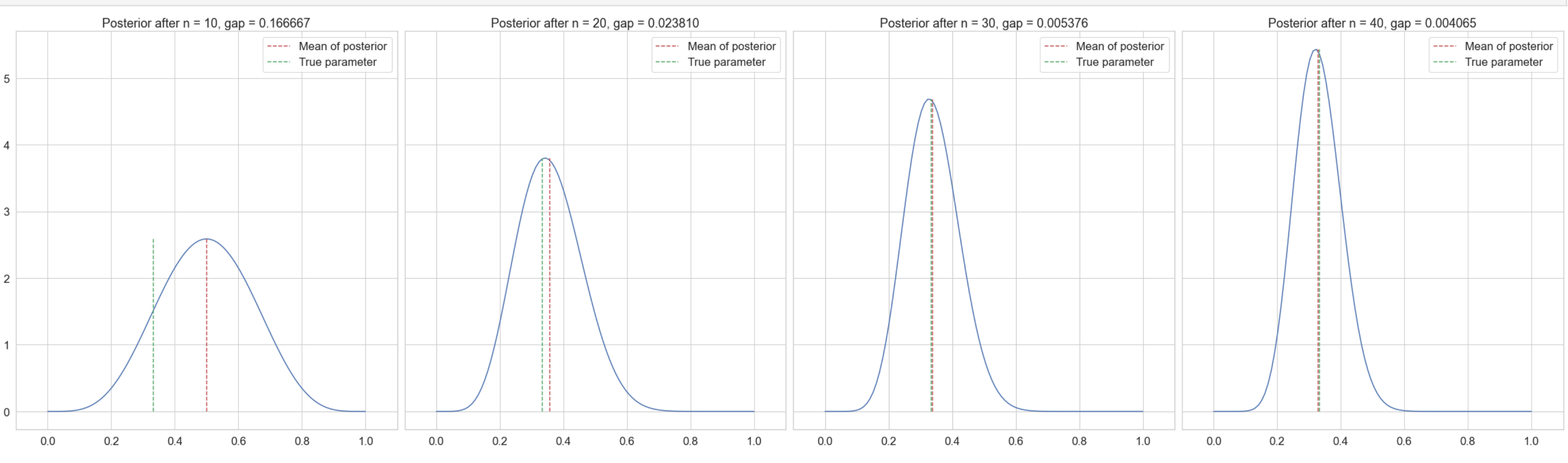
In [2]:
```
bernoulli = sts.bernoulli(1/3)

n = 20
X = bernoulli.rvs(n)
s = X.sum()

alpha = 0.5 # 2
beta = 0.5 # 5

# prior = sts.uniform()
prior = sts.beta(alpha, beta)

# true_posterior = sts.beta(s+1, n-s+1)
true_posterior = sts.beta(alpha+s, beta+n-s)

xx = np.linspace(0, 1, 100)
```

In [3]:
```
fig, ax = plt.subplots(1, 2, figsize=(16,9), sharey=True)
ax[0].plot(xx, prior.pdf(xx))
ax[0].set_xlim((-0.1, 1.1))
ax[0].set_title("Prior")
ax[1].plot(xx, true_posterior.pdf(xx))
ax[1].set_xlim((-0.1, 1.1))
ax[1].set_title("Posterior")
ax[1].vlines((alpha + s) / (alpha + beta + n), 0, true_posterior.pdf(xx).max(), ls='--', color='r', label="Mean of posterior")
ax[1].vlines(1/3, 0, true_posterior.pdf(xx).max(), ls='--', color='g', label="True parameter")
ax[1].legend()
fig.tight_layout()
```



In [4]:
```
bernoulli = sts.bernoulli(1/3)

n = 10

alpha = 0.5 # 4.0
beta = 0.5 # 10.0

# prior = sts.uniform()
prior = sts.beta(alpha, beta)

# true_posterior = sts.beta(s+1, n-s+1)
true_posterior = sts.beta(alpha+s, beta+n-s)

xx = np.linspace(0, 1, 100)
```

In [5]:
```
fig, ax = plt.subplots(1, 4, figsize=(32,9), sharey=True)
cur_alpha = alpha
cur_beta = beta
for i, ax in zip(range(1, 5), ax.flatten()):
    sample = bernoulli.rvs(n, random_state=1234+i)
    s = sample.sum()
    true_posterior = sts.beta(cur_alpha+s, cur_beta+n-s)
    ax.plot(xx, true_posterior.pdf(xx))
    ax.set_xlim((-0.1, 1.1))
    ax.set_title(
        f"Posterior after n = {n * i}, gap = {np.abs((cur_alpha + s) / (cur_alpha + cur_beta + n) - 1/3):3f}"
    )
    ax.vlines(
        (cur_alpha + s) / (cur_alpha + cur_beta + n),
        0, true_posterior.pdf(xx).max(),
        ls='--', color='r', label="Mean of posterior"
    )
    ax.vlines(1/3, 0, true_posterior.pdf(xx).max(), ls='--', color='g', label="True parameter")
    ax.legend()
    cur_alpha = cur_alpha + s
    cur_beta = cur_beta + n - s
fig.tight_layout()
```



Priors

1. Improper (not a distribution)
2. Flat (like in the example)
3. Jeffreys (transformation invariant)

$$p(\theta) = \sqrt{|I(\theta)|}$$

For Bernoulli:

$$p(\theta) = Beta\left(\frac{1}{2}, \frac{1}{2}\right)$$

Bayesian testing: MAP test

Suppose that we have sample $X = (X_1, \dots, X_n) \sim p(x|\theta)$ and we have prior $p(\theta)$. We'd like to test:

$$\begin{aligned} H_0 &: \theta = \theta_0 \\ H_1 &: \theta = \theta_1 \end{aligned}$$

The test statistic is given as follows:

$$\Lambda(X) = \frac{L(X,\theta_1)}{L(X,\theta_0)}$$

We reject the null hypothesis, if

$$\Lambda(X) \geqslant \frac{\mathbb{P}(H_0)}{\mathbb{P}(H_1)}$$

Statistical Decision Theory

How to compare different decision rules?

Denote:

- decision rule $\delta(X)$
- loss function $\ell(\theta, \delta)$
- risk function $R(\theta, \delta) = \mathbb{E}_X \ell(\theta, \delta(X))$

The estimate is given by

$$\hat{\theta} = \arg\min_\delta R(\theta, \delta)$$

With prior $p(\theta)$ and posterior $p(\theta|X)$, we have Bayesian rule for a given loss function:

$$\delta^B = \arg\min_\delta \int \ell(\theta, \delta(X)) p(\theta|X)\mathrm{d}\theta$$