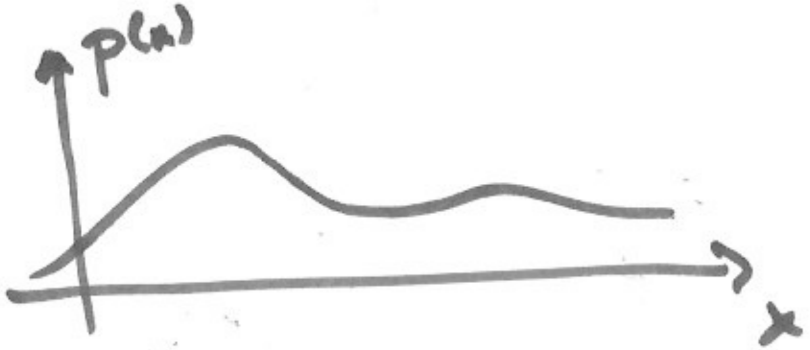
# MSAI - Probability - 9

$t \to it$

$x \sim p(x)$

$M_x(t) = \mathbb{E}_{p(x)}\left(e^{tx}\right)$




$\mathbb{E}(x) \quad \mathbb{E}(x^2) \quad \mathbb{E}(x^3)$

$\mathbb{E}(x^2) - (\mathbb{E}(x))^2 = Var(x)$

$$\frac{d}{dt}\mathbb{E}\left(e^{tx}\right) = \mathbb{E}\left(\frac{d}{dt}e^{tx}\right)$$

$$\mathbb{E}\left(x e^{tx}\right)\Big|_{t \to 0}$$

$$\mathbb{E}(x).$$

**Problem:** $x \sim \text{Bin}(n, \tilde{p})$

$\underbrace{H/T \quad H/T \quad \ldots}_{n} \quad \underbrace{0/1}_{\xi_i}$ $\quad 0 \leftarrow 1-\tilde{p}$, $\quad 1 \leftarrow \tilde{p}$ $\qquad x = \sum_{i=1}^{n} \xi_i$

$$M_{x|\tilde{p}}(t) = (1 - \tilde{p} + \tilde{p} \cdot e^t)^n$$

$$\text{Bin}(n, p) = p(x \mid n, \tilde{p}) \qquad n\text{-fixed}$$

$p(x)$ - ? $\qquad p(x, \tilde{p}) = p(x \mid \tilde{p})\, p(\tilde{p})$

$$p(x) = \int p(x \mid \tilde{p})\, 1 \cdot d\tilde{p}$$


$$M_x(t) = \int_0^1 (1 - \tilde{p} + \tilde{p} e^t)^n \, d\tilde{p} = \frac{1}{n+1} \frac{e^{t(n+1)} - 1}{e^t - 1}$$

$$M_x(t) = \frac{1}{n+1} \frac{e^{t(n+1)} - 1}{e^t - 1} = \frac{1}{n+1}\left(1 + e^t + e^{2t} + e^{3t} + \dots + e^{nt}\right)$$

$$1 + q + q^2 + \dots + q^n = \frac{q^{n+1} - 1}{q - 1}$$

$$\mathbb{E}(x) = \frac{1 + 2 + 3 + \dots + n}{n+1}$$

$$\mathbb{E}(x^2) = \frac{1^2 + 2^2 + 3^2 + \dots + n^2}{n+1}$$

$$x \sim \mathrm{Uni}(\{0, 1, 2, \dots\})$$

$\cancel{x_1, \ x_2 \sim M_{x_1}}$

$x \sim M_x(t), \quad y \sim M_y(t)$

$p(x,y) = p(x)\,p(y)$

$x, y$ - indep (!)

$z = x + y$

$$\boxed{M_z(t)} = \mathbb{E}\big(\underbrace{e^{t(x+y)}}_{e^{tx}\cdot e^{ty}}\big) = \mathbb{E}\left(e^{tx}\cdot e^{ty}\right) =$$

$$= \mathbb{E}(e^{tx})\cdot\mathbb{E}(e^{ty}) = \boxed{M_x(t)\cdot M_y(t)}$$

$\boxed{x, y \text{ - indep!}}$