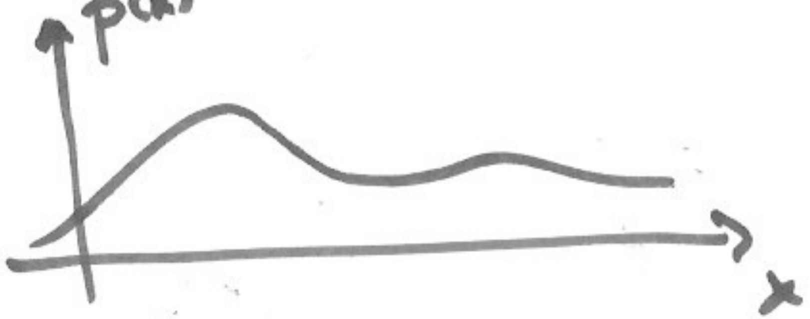
MSAI - Probability - 9

$t \to it$

$x \sim p(x)$

$M_x(t) = \mathbb{E}_{p(x)}\left(e^{tx}\right)$



$\mathbb{E}(x) \quad \mathbb{E}(x^2) \quad \mathbb{E}(x^3)$

$\mathbb{E}(x^2) - (\mathbb{E}(x))^2 = Var(x)$

$$\frac{d}{dt}\mathbb{E}\left(e^{tx}\right) = \mathbb{E}\left(\frac{d}{dt}e^{tx}\right)$$

$$\underbrace{\mathbb{E}\left(x\, e^{tx}\right)\Big|_{t \to 0}}_{\mathbb{E}(x)}$$

Problem: $x \sim \mathrm{Bin}(n, \tilde{p})$

$$\underbrace{H/T \quad H/T \quad \dots}_{n} \quad \underbrace{{}^{0}/_{1}}_{\xi_i} \qquad 0 \leftarrow 1-\tilde{p}, \quad 1 \leftarrow \tilde{p}$$

$$x = \sum_{i=1}^{n} \xi_i$$

$$M_{x|\tilde{p}}(t) = (1-\tilde{p}+\tilde{p}\cdot e^{t})^{n}$$

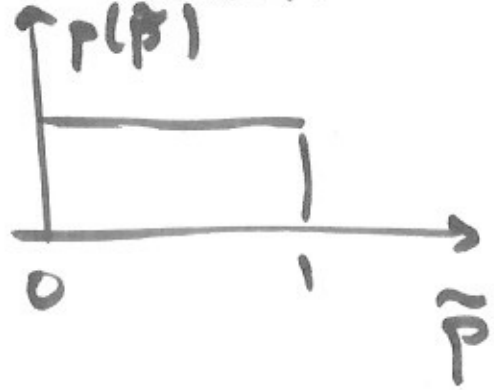
n-fixed

$$\mathrm{Bin}(n,p) = p(x \mid n, \tilde{p})$$

$p(x)$ – ?

$$p(x,\tilde{p}) = p(x \mid \tilde{p})\, p(\tilde{p})$$

Uniform



$$p(x) = \int p(x \mid \tilde{p})\, 1 \cdot d\tilde{p}$$

$$M_x(t) = \int_0^1 (1-\tilde{p}+\tilde{p}e^{t})^{n}\, d\tilde{p} = \frac{1}{n+1}\,\frac{e^{t(n+1)}-1}{e^{t}-1}$$

$$M_x(t) = \frac{1}{n+1}\,\frac{e^{t(n+1)} - 1}{e^t - 1}$$

$$1 + q + q^2 + \dots + q^n = \frac{q^{n+1} - 1}{q - 1}$$

$$= \frac{1}{n+1}\left(1 + e^t + e^{2t} + e^{3t} + \dots + e^{nt}\right)$$

$$\mathbb{E}(x) = \frac{1 + 2 + 3 + \dots + n}{n+1}$$

$$\mathbb{E}(x^2) = \frac{1^2 + 2^2 + 3^2 + \dots + n^2}{n+1}$$

$$x \sim \mathrm{Uni}(\{0, 1, 2, \dots\})$$

$\cancel{X_1, X_2 \sim M_{X_1}}$

$X \sim M_X(t), \quad Y \sim M_Y(t)$

$p(x,y) = p(x)\,p(y)$

x, y - indep (!)

$z = x + y$

$$\boxed{M_z(t)} = \mathbb{E}\Big(\underbrace{e^{t(x+y)}}_{e^{tx}\cdot e^{ty}}\Big) = \mathbb{E}\left(e^{tx}\cdot e^{ty}\right) =$$

$$= \mathbb{E}(e^{tx})\cdot\mathbb{E}(e^{ty}) = \boxed{M_x(t)\cdot M_y(t)}$$

$\boxed{x, y \text{ - indep!}}$