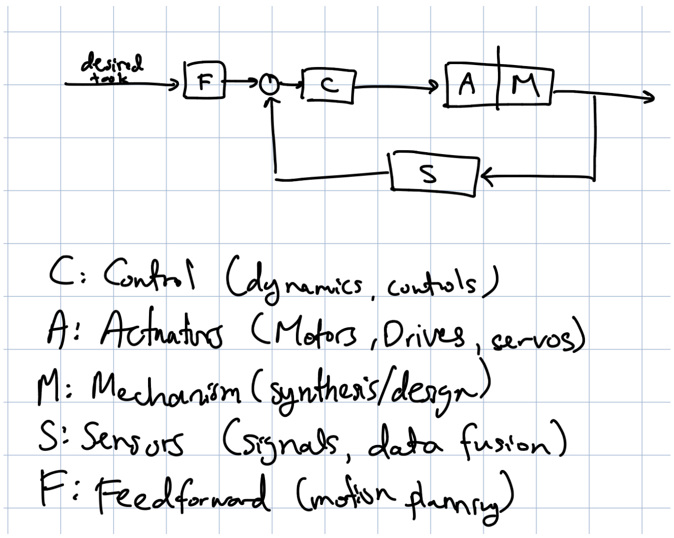
Contents

1 Lecture 1

1.1 Robotics History

Definition 1.1 (Robot, Robotics)
A robot is a "mechanical device" that sometimes resembles a human and is capable of performing a variety of often complex human taks.
Robotics is the science and building of robots.

The CAMSF function block model:



In ancient history (the Egyptians, Greeks, Romans), simple machines such as inclined planes, levers, pullies, capstands and cranks. These mechanisms relied on multiplying force using some kind of mechanical ratios (whether by trading off distance, energy, etc).

2 Lecture 2

2.1 Rigid Body Transformations

We begin with some basic definitions.

Definition 2.1 (Point)
In this class, a point $p \in \mathbb{R}^3$ is represented as:

$$p = \begin{bmatrix} p_x \\ p_y \\ p_z \end{bmatrix}$$

where x, y, z are some orthonormal axes that make a coordinate frame. In $\mathbb{R}^n$ we will write:

$$p = \begin{bmatrix} p_1 \\ p_2 \\ \vdots \\ p_n \end{bmatrix}$$

The distance of the point from the origin is:

$$\|p\| = \sqrt{p_1^2 + \cdots + p_n^2}$$

Definition 2.2 (Vector)
A vector $v \in \mathbb{R}^n$ is defined as the displacement between two points:

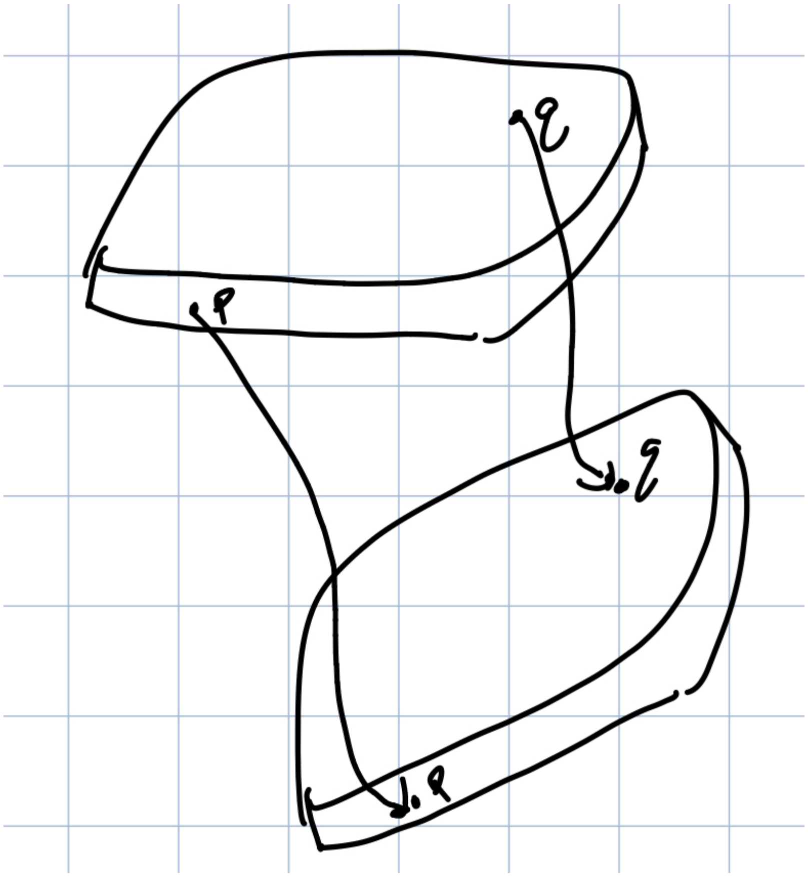
$$v = p - q = \begin{bmatrix} p_1 - q_1 \\ p_2 - q_2 \\ \vdots \\ p_n - q_n \end{bmatrix} = \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix}$$

The norm of the vector is defined the same way.

Definition 2.3 (Matrix)
A matrix $A \in \mathbb{R}^{n \times m}$ is a n-by-m array of real numbers.

Now we define how point-mass motion functions. First, there is some $p(0) = \begin{bmatrix} x(0) & y(0) & z(0) \end{bmatrix}^T$. Then, from this, we can get $p(t)$, its position at time t. We define the "trajectory" of an object as this map from time to position.

Think about a "rigid body" like your cell-phone.



As you move this rigid body, if you take two points on the rigid body, their distance does not change.

$$\|p(t) - q(t)\| = \|p(0) - q(0)\| = \text{constant}$$

i.e. the size of the vector between them stays the same. Furthermore, orientations stay the same. This leads to the following definition:

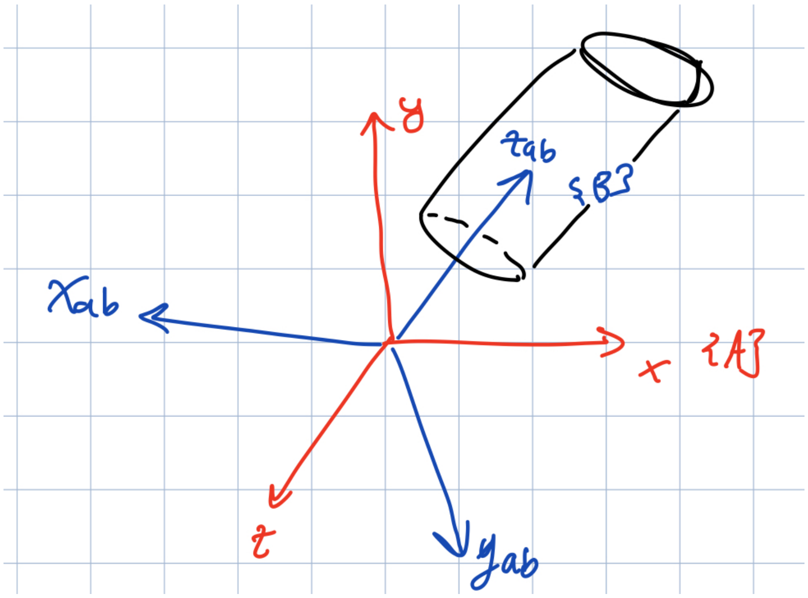
Definition 2.4 (Rigid Body Transformation)
A rigid body transformation is a function g that has its domain as the set of all vectors in R^3 such that:

- Length (of vectors) is preserved.
$$\|g(p) - g(q)\| = \|p - q\|$$
- Orientation is preserved
$$g_*(v \times w) = g_*(v) \times g_*(w)$$

We use $*$ to denote g operating on a vector rather than a point.

2.2 Rotational Motion in $\mathbb{R}^3$

So the first step of modeling rotation is to choose a spatial reference frame A. Then, we must attach a frame B to the body.



where $x_{ab} \in \mathbb{R}^3$ is the coordinates of x_b in frame A.

Definition 2.5 (Rotation Matrix)
The rotation (or orientation) matrix of B with respect to A is defined as:

$$R_{ab} = \begin{bmatrix} x_{ab} & y_{ab} & z_{ab} \end{bmatrix}$$

For any rotation matrix, the columns are orthonormal, i.e.

$$r_i^T \cdot r_j = \begin{cases} 0 & i \neq j \\ 1 & i = j \end{cases}$$

which means R is orthogonal, meaning it is in the set $O^{(3)}$.

$$R \in O(3) \implies R^T R = RR^T = I$$

What is the determinant of R?

$$\det\left(R^T R\right) = \det R^T \cdot \det R = (\det R)^2 = 1 \implies \det R = \pm 1$$

However, if R is a rotation matrix, $\det(R)$ must be 1. Why?

$$det R = r_1^T (r_2 \times r_3) = 1$$

Since $r_2 \times r_3 = r_1$ in a right-handed coordinate system.

Definition 2.6 (SO(n))
The special orthogonal group SO(3) is

$$R \in \mathbb{R}^{3\times 3} \mid RR^T = R^T R = I, \det(R) = 1$$

This can be extrapolated to n dimensions, SO(n):

$$R \in \mathbb{R}^{n\times n} \mid RR^T = R^T R = I, \det(R) = 1$$

We used this term group, which is a mathematical object.

Definition 2.7 (Groups)
A group is a set G requipped with by a binary operator $\cdot$ which satisfies

- The binary operation maps back onto the set itself (closure).
- The binary operation is associative $(g_1 \cdot (g_2 \cdot g_3) = (g_1 \cdot g_2) \cdot g_3)$
- There is a unique identity element such that $g \cdot e = e \cdot g = g$
- There is a unique inverse for every element such that $g \cdot g^{-1} = e$

Here are some examples of groups.

Example 2.1
Here are some groups and non groups:

- $(\mathbb{R}^3, +)$
- $(\{0, 1\}, + \mod 2)$
- $(\mathbb{R}, \times)$ is NOT a group
- $(SO(3), \cdot)$

It turns out $SO(3)$ (equipped with matrix multiplication) IS a group! Let's prove this.

Proof 2.1
Consdier $R_1, R_2 \in SO(3)$. Then

$$(R_1R_2)^T(R_1R_2) = R_2^T R_1^T R_1 R_2 = I$$

and

$$\det(R_1R_2) = \det(R_1)\det(R_2) = 1 \cdot 1 = 1$$

so therefore $R_1 \cdot R_2$. The identity element is the identity matrix, to take inverses you just transpose, and associativity comes for free from matrix multiplication properties. Thus, $SO(3)$ is a group.

Now we are ready to dive into how rotations actually function. Consider a point q and its coordinates in B: $q_b = \begin{bmatrix} x_b & y_b & z_b \end{bmatrix}^T$. Now to derive its coordinates in a, we simply realize:

$$q_a = x_{ab}x_b + y_{ab}y_b + z_{ab}z_b = R_{ab}q_b$$

Finally, let us show rotations are a rigid body transformation. First we need to define the skew symmetric part of a vector.

Definition 2.8 (Skew Symmetry)
A matrix A is skew symmetric if $A^T = -A$. We denote the set of all skew symmetric matrices in $\mathbb{R}^3$ is $so(3)$.

Definition 2.9 (Skew Symmetric Part)
The skew-symmetric part of a vector $a = \begin{bmatrix} a_1 & a_2 & a_3 \end{bmatrix}^T$ is:

$$\hat{a} = \begin{bmatrix} 0 & -a_3 & a_2 \\ a_3 & 0 & -a_1 \\ -a_2 & a_1 & 0 \end{bmatrix}$$

Now we can see the following:

$$a \times b = \hat{a}b$$

Note that also:

Theorem 2.1
We have the following cross product identities:

- Scalar triple product
$$a \cdot (b \times c) = c \cdot (a \times b) = b \cdot (c \times a)$$
- Vector triple product
$$a \times (b \times c) = (a \cdot c)b - (a \cdot b)c$$
- Jacobi Identity
$$(a \times b) \times c + (c \times a) \times b + (b \times c) \times a = 0$$

These convenient facts come in handy, now we can prove that R preserves distance and orientation, showing it is a rigid transformation:

Proof 2.2
Consider two points p_1, p_2,

$$\begin{aligned}
\|Rp_1 - Rp_2\|^2 &= \|R(p_1 - p_2)\|^2 \\
\|Rp_1 - Rp_2\|^2 &= (p_1 - p_2)^T R^T R(p_1 - p_2) \\
\|Rp_1 - Rp_2\|^2 &= (p_1 - p_2)^T (p_1 - p_2) \\
\|Rp_1 - Rp_2\|^2 &= \|p_1 - p_2\|^2
\end{aligned}$$

so the norm does not change.
Now consider two vectors v and w:

$$\begin{aligned}
R(v \times w) &= R\hat{v}w \\
&= R\hat{v}R^T Rw
\end{aligned}$$

We claim $\hat{Rv} = R\hat{v}R^T$, which will complete the proof. To show this, suppose the rows of R are r_i^T. Then:

$$\begin{aligned}
\hat{Rv} &= \widehat{\begin{bmatrix} r_1^T \\ r_2^T \\ r_3^T \end{bmatrix} v} \\
&= \widehat{\begin{bmatrix} r_1^T v \\ r_2^T v \\ r_3^T v \end{bmatrix}}
\end{aligned}$$

But the other side is:

$$R\hat{v}R^T = R \begin{bmatrix} \hat{v}r_1 \\ \hat{v}r_2 \\ \hat{v}r_3 \end{bmatrix}$$
$$= \begin{bmatrix} r_1^T \hat{v}r_1 & r_1^T \hat{v}r_2 & r_1^T \hat{v}r_3 \\ r_2^T \hat{v}r_1 & r_2^T \hat{v}r_2 & r_2^T \hat{v}r_3 \\ r_3^T \hat{v}r_1 & r_3^T \hat{v}r_2 & r_3^T \hat{v}r_3 \end{bmatrix}$$

Where the i, j component is:

$$r_i \cdot (v \times r_j) = v \cdot (r_i \times r_j) = \begin{cases} 0 & i = j \\ \text{the other dimension} & i < j \\ \text{negative the other dimension} & i > j \end{cases}$$

Note this is exactly the entries of the RHS, so we are done.

3 Lecture 3

3.1 Exponential Map

Recall that for two dimensions, we can parameterize angles like this:

$$e^{i\phi} = \cos\phi + i\sin\phi$$

These form the set $S^1 = \{z \in \mathbb{C} \mid |z| = 1\}$ To do this, we need a few notions first.

Recall that the differential equation:

$$\begin{aligned} \dot{x}(t) &= ax(t) \\ x(0) &= x_0 \end{aligned}$$

has the soution $x(t) = e^{at}x_0$. The same solution generalizes quite easily to a vector system, with a matrix exponential instead of a scalar one.

Consider some matrix $R \in SO(3)$. Then, there are 6 orthogonality constraints:

$$r_i \cdot r_j = \begin{bmatrix} 0 & i \neq j \\ 1 & i = j \end{bmatrix}$$

which means 3 independent degrees of freedom I can pick.

Now, consider the motion of a point q on a rotating link. The motion is given by the differential equation:

$$\begin{aligned} \dot{q}(t) = \omega \times q(t) &= \hat{\omega} q(t) \\ q(0) &= q_0 \end{aligned}$$

where ω points in the direction of the axis we are rotating about. We can solve this is the same way as before!

$$q(t) = e^{\hat{\omega}t} q_0$$

where of course, the matrix exponential is:

$$e^{\hat{\omega}t} = I + om\hat{e}gat + \frac{(om\hat{e}gat)^2}{2!} + \dots$$

We can then consider the following map, supposing $\hat{\omega}$ has unit norm and now we are rotating by some angle θ instead of for some time t (hence why we need normalization to rotate at unit speed).

Definition 3.1 (Exponential Map)

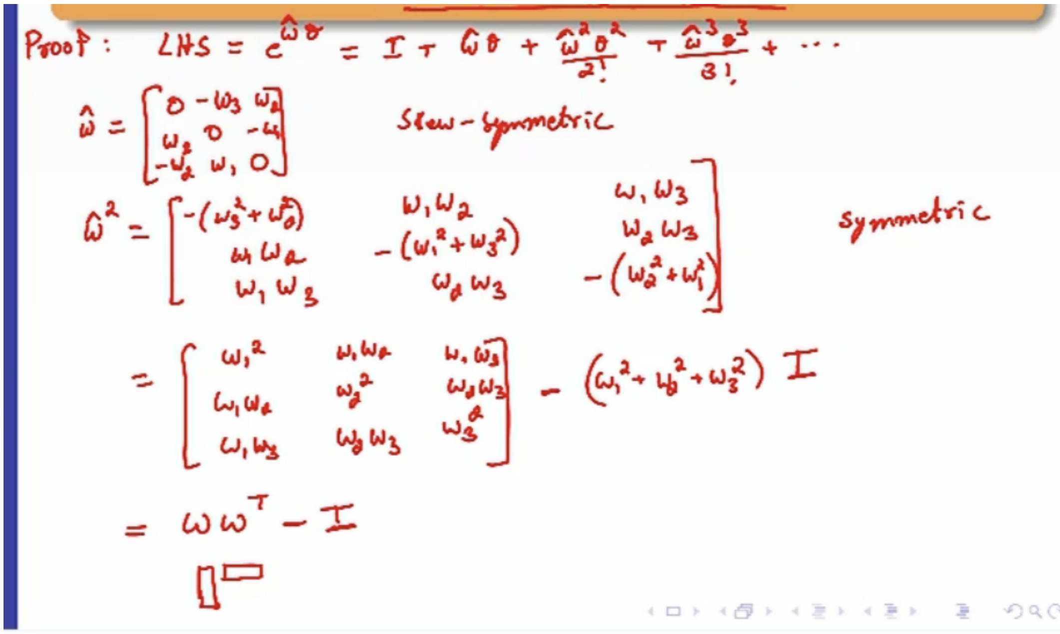
$$\exp : so(3) \to SO(3), \hat{\omega}\theta \mapsto e^{\hat{\omega}\theta}$$

i.e. we are claiming that $e^{\hat{\omega}\theta}$ is a rotation matrix. We will see this later.

It turns out we can write this infinite summation as a finite one, using Rodrigues' formula (again with unit norm assumption).

$$e^{\hat{w}\omega} = I + \hat{\omega}\sin\theta + \hat{\omega}^2(1 - \cos\theta)$$

The first part of the proof is pretty bashy. I have attached a screenshot to save my fingers.



Continuing on, we have:

$$
\begin{aligned}
\hat{\omega}^3 &= \hat{\omega}(\omega\omega^T - I) = \hat{\omega}\omega\omega^T - \hat{\omega} = -\hat{\omega} \\
\hat{\omega}^4 &= \hat{\omega}(-\hat{\omega}) = -\hat{\omega}^2 \\
\hat{\omega}^5 &= \hat{\omega} \\
&\vdots
\end{aligned}
$$

Then we have:

$$e^{\hat{\omega}\theta} = I + \hat{\omega}\left[\theta - \frac{\theta^3}{3!} + \dots\right] + \hat{\omega^2}\left[\frac{\theta^2}{2!} - \frac{\theta^4}{4!} + \dots\right]$$

which is exactly the claim (these are the power series of sin and cos).

Without the unit norm assumption, the full formula is:

Theorem 3.1 (Rodrigues' Formula)
For a vector ω and scalar θ we have:

$$e^{\hat{\omega}\theta} = I + \frac{\hat{\omega}}{\|\omega\|}\sin\|\omega\|\theta + \frac{\hat{\omega}^2}{\|\omega\|^2}(1 - \cos\|\omega\|\theta)$$

We will now show that this exponential map is a rotation.

Proof 3.1
Let $R = e^{\hat{\omega}\theta}$. Then:

$$
\begin{aligned}
R^{-1} &= e^{-\hat{\omega}\theta} \\
&= e^{\hat{\omega}^T\theta} \\
&= \left(e^{\hat{\omega}\theta}\right)^T \\
&= R^T
\end{aligned}
$$

Let us also see that the determinant is 1. We know from orthogonality of R that it must have determinant either -1 or 1. However,

$$\det \exp(0) = 1$$

and the determinant is continuous with respect to θ, so it cannot jump to -1. Thus, the determinant must always be 1 and we have $e^{\hat{\omega}\theta} \in SO(3)$.

Furthermore, the exponential map is onto.

Proof 3.2
Given a rotation $R \in SO(3)$, there is some ω with unit norm and θ such that: $R = e^{\hat{\omega}\theta}$. Let

$$R = \begin{bmatrix} r_{11} & r_{12} & r_{13} \\ r_{21} & r_{22} & r_{23} \\ r_{31} & r_{32} & r_{33} \end{bmatrix}$$

and $v_\theta = 1 - \cos\theta, c_\theta = \cos\theta, s_\theta = \sin\theta$ By Rodrigues' formula (again I'm lazy):

$$e^{\hat{\omega}\theta} = \begin{bmatrix} \omega_1^2 v_\theta + c_\theta & \omega_1\omega_2 v_\theta - \omega_3 s_\theta & \omega_1\omega_3 v_\theta + \omega_2 s_\theta \\ \omega_1\omega_2 v_\theta + \omega_3 s_\theta & \omega_2^2 v_\theta + c_\theta & \omega_2\omega_3 v_\theta - \omega_1 s_\theta \\ \omega_1\omega_3 v_\theta - \omega_2 s_\theta & \omega_2\omega_3 v_\theta + \omega_1 s_\theta & \omega_3^2 v_\theta + c_\theta \end{bmatrix}$$

Now we want to recover ω and θ to conclude the proof. Consider the trace of this matrix:

$$\begin{aligned} \mathrm{tr}(R) &= \omega_1^2 v_\theta + c_\theta + \omega_2^2 v_\theta + c_\theta + \omega_3^2 v_\theta + c_\theta \\ &= 1 + 2c_\theta \\ &= \sum_{i=1}^{3} \lambda_i \end{aligned}$$

Now we have a few cases (which then we can figure out stuff by adding and subtracting equations).

- Case 1: $\mathrm{tr}(R) = 3$ or $R = I, \omega = 0 \implies \omega\theta = 0$.
- Case 2: $-1 < \mathrm{tr}(R) < 3$

$$\theta = \arccos\frac{\mathrm{tr}(R) - 1}{2} \implies \omega = \frac{1}{2s_\theta}\begin{bmatrix} r_{32} - r_{23} \\ r_{13} - r_{31} \\ r_{21} - r_{12} \end{bmatrix}$$

- Case 3: $\mathrm{tr}(R) = -1 \implies \cos\theta = -1 \implies \theta = \pm\pi$

Also note that if $\omega\theta$ is a solution, then $\omega(\theta \pm n\pi)$ is a solution.

Definition 3.2 (Exponential Coordinate)
The $\omega\theta \in \mathbb{R}^3$ with $e^{\hat{\omega}\theta} = R$ is called the exponential coordinates of R.

The exponential map is one-to-one when restricted to an open ball in $\mathbb{R}^3$ of radius π (as we saw in the proof). Thus, $SO(3)$ can be visualized as such a ball; since we can turn any ω here into a rotation (i.e. $\exp\left(\frac{\omega}{\|\omega\|}\|\omega\|\right)$).

Theorem 3.2 (Euler)
Any orientation is equivalent to a rotation about a fixed axis $\omega \in \mathbb{R}^3$ through an angle $\theta \in [-\pi, \pi]$.

3.2 Euler Angles

We can parameterize $SO(3)$, the rotation group, in other ways as well.

One way is to define roll pitch and yaw angles (φ, θ, ψ) about the standard coordinate axes.

We can also rotate frames; generally we use ZYX euler angles, where we rotate about z, then y', then x''. We use α, β, γ for these, but now the axes themselves you're rotating about themselves are changing!

$$R_{ab} = R_z(\alpha) R_{y'}(\beta) R_{x''}(\gamma)$$

The problem is with Euler angles is there are some combinations of angles that create you cannot move it to anywhere you want.

4 Lecture 4

4.1 Quaternions

We want a representation of rotations without the annoying singularity. Recall that a complex number z on the unit circle can be represented as $e^{i\theta}$, i.e. it can represent one angle. The basis of this is 1 and i.

Now let's extend this to three angles. We call such an object a quaternion Q.

$$Q = (q_0, \mathbf{q}) = q_0 + iq_1 + jq_2 + kq_3$$

where

$$i^2 = j^2 = k^2 = ijk = -1, ij = k, jk = i, ki = j$$

Now how do you encode the idea of rotation about an axis ω by angle θ?

$$Q = \left(\cos\frac{\theta}{2}, \omega\sin\frac{\theta}{2}\right)$$

we see that:

$$||Q|| = \sqrt{q_0^2 + q_1^2 + q_2^2 + q_3^2} = 1$$

Definition 4.1 (Conjugate, Product)
The conjugate $Q^* = (q_0, q)^* = (q_0, -q)$. Note that this means $||Q||^2 = QQ^*$.
The product of two quaternions $Q = (q_0, q)$, $P = (p_0, p)$ is:

$$QP = (q_0p_0 - q \cdot p, q_0p + p_0q + q \times p)$$

Theorem 4.1 (Quaternion Properties)
For quaternions we have:

- The set of all unit quaternions forms a group.
- If $R = e^{\hat{\omega}\theta}$, then $Q = \left(\cos\frac{\theta}{2}, \omega\sin\frac{\theta}{2}\right)$.
- Q acts on $x \in \mathbb{R}^3$ by QXQ^*, where $X = (0, x)$.

It turns out quaternions are pretty good at representing rotations. In fact, they double cover the space of rotations, but this is bad because you might (unwind a lot when doing a small change in angle.

4.2 $SE(3)$

$SE(3)$ represents all the configurations of a rigid body.

$$SE(3) = \{(p, R) \mid p \in \mathbb{R}^3, R \in SO(3)$$

where p is the translations (p_ab is the coordinates of the origin of B in frame A). Each of these elements is a transformation:

$$g_{ab} = (p_{ab}, R_{ab}) : \mathbb{R}^3 \to \mathbb{R}^3$$
$$q_b \mapsto q_a = p_{ab} + R_{ab}q_b$$

This motivates another way to represent $\mathbb{R}^3$ "points" homogenously. Suppose you have a point

$$q = \begin{bmatrix} q_1 \\ q_2 \\ q_3 \end{bmatrix}$$

Then its homogenous representation is:

$$\bar{q} = \begin{bmatrix} q_1 \\ q_2 \\ q_3 \\ 1 \end{bmatrix}$$

Then for a vector:

$$v = p - q = \begin{bmatrix} p_1 - q_1 \\ p_2 - q_2 \\ p_3 - q_3 \end{bmatrix}$$

Then its homogenous representation is:

$$\bar{v} = \begin{bmatrix} p_1 - q_1 \\ p_2 - q_2 \\ p_3 - q_3 \\ 0 \end{bmatrix}$$

So any vector ends with 0 and any point ends with 1. Note the following:

- point - point = vector
- vector + point = point
- vector + vector = vector
- point + point = meaningless

This makes rigid motion always linear! Consider going from reference frame A to B.

$$q_a = p_{ab} + R_{ab} q_b$$
$$\begin{bmatrix} q_a \\ 1 \end{bmatrix} = \begin{bmatrix} R_{ab} & p_{ab} \\ 0 & 1 \end{bmatrix} \begin{bmatrix} q_b \\ 1 \end{bmatrix}$$
$$\bar{q}_a = \bar{g}_{ab} \cdot \bar{q}_b$$

where

$$\bar{g}_{ab} = \begin{bmatrix} R_{ab} & p_{ab} \\ 0 & 1 \end{bmatrix}$$

Note that for a vector, instead the transformation g ONLY does a rotation and not a translation (the translations between its endpoints "cancel out").

We call the set of these the Special Euclidean Group.

$$SE(3) = \{\begin{bmatrix} R & p \\ 0 & 1 \end{bmatrix} \in \mathbb{R}^{4\times 4} \mid p \in \mathbb{R}^3, R \in SO(3)\}$$

It also turns out $SE(3)$ is also a group (over matrix multiplication).

Proof

First note that multiplying any elements of $SE(3)$ yields:

$$\bar{g}_{ab}\bar{g}_{bc} = \begin{bmatrix} R_{ab} & p_{ab} \\ 0 & 1 \end{bmatrix} \begin{bmatrix} R_{bc} & p_{bc} \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} R_{ab}R_{bc} & R_{ab}p_{bc} + p_{ab} \\ 0 & 1 \end{bmatrix}$$

which is also a member of $SE(3)$.

Next, see that the identity matrix is indeed an identity and associativity follows from matrix multiplication.

One can confirm that the inverse of (p, R) is $(R^T p, R^T)$.

Also, elements of $SE(3)$ are all rigid body transformations.

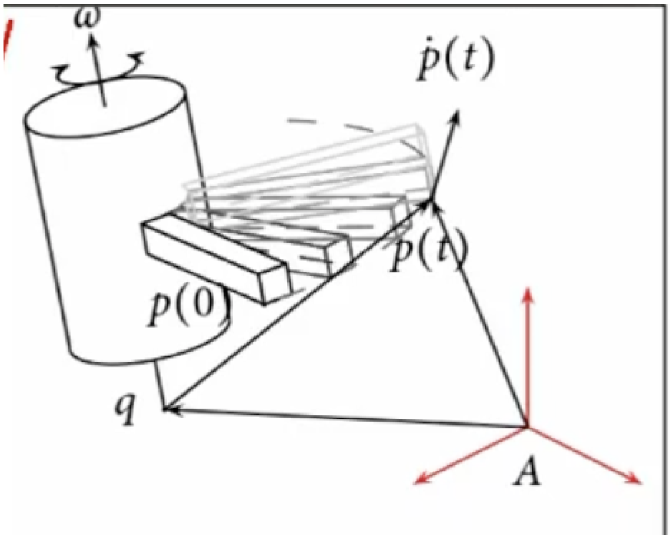
Proof
Clearly the rotation part preserves length. Let's look at the translation:

$$\|(q_1 + p_{ab}) - (q_2 + p_{ab})\| = \|q_1 - q_2\|$$

Thus the transform preserves length.

Now consider the cross product. Again, the rotation clearly preserves it, and vectors are NOT translated by a g, since g_* doesn't apply the translation.

We can also parametrize $SE(3)$ with exponential coordinates. Consider the following setup:



It satisfies this differential equation:

$$\dot{p}(t) = \omega \times (p(t) - q) = \hat{\omega} p - \omega \times q$$

Which can be written as:

$$\begin{bmatrix} \dot{p} \\ 0 \end{bmatrix} = \begin{bmatrix} \hat{\omega} & -\omega \times q \\ 0 & 0 \end{bmatrix} \begin{bmatrix} p \\ 1 \end{bmatrix}$$

Which is then:

$$\dot{\bar{p}} = \hat{\xi} \bar{p}(t) \implies \bar{p}(t) = e^{\hat{\xi} t} \bar{p}(0)$$