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1 Lecture 1

The notion of measure is going to generalize the real line's notion of **distance**. Recall that $\mathbb{Q}$ can be constructed from the integers as follows, defining $\mathbb{N}_+ = \{1, 2, 3, \ldots\}$ and $\mathbb{Z} = \{\ldots, -2, -1, 0, 1, 2, \ldots\}$ and we can write $\mathbb{Q} = \{\frac{m}{n} : m \in \mathbb{Z}, n \in \mathbb{N}_+\}$. Recall also that $\mathbb{Q}$ is a countable set.

Recall that $\mathbb{Q}$ is a dense subset of the real line, which we will revisit. First, we define the notion of a **distance** (or **metric**) between two rational numbers, a function $d : \mathbb{Q} \times \mathbb{Q} \to [0, \infty)$:

$$d\left(\frac{m_1}{n_1}, \frac{m_2}{n_2}\right) = \frac{|m_1 n_2 - n_1 m_2|}{n_1 n_2}$$

the distance is also a rational number. However, not all Cauchy sequences in the rationals **converge** to a rational number, the metric space is not complete.

Definition 1.1
A **Cauchy sequence** is a sequence $\{x_n\}$ such that for all $\epsilon > 0$, there exists a threshold $n_0 \in \mathbb{N}$ such that if we have $n, m \geq n_0$, $d(x_n, x_m) < \epsilon$.

We can construct the reals by filling in absences in the rationals. To see these holes, we will represent real numbers with their decimal representation. Now, everyone has a unique representation, except, $0.999\cdots = 1.000\ldots$. There are only countably many "awkward" points here (terminating decimals are a subset of the rationals), so it's not a big issue. We will just ban the 9s version, i.e. $1/2 = 0.5000$. If I select values for the decimal places at random, then with probability 0 I get a repeating decimal (rational number). This means the rationals are very slim among the reals.

Let's take π and write it as a Cauchy sequence of rationals $(3, 3.1, 3.14, 3.141, \ldots)$. Since π is not rational, we have that the Cauchy sequence doesn't converge to a rational number. This seems to be a way to construct real numbers; why don't we identify π with this Cauchy sequence? But this isn't the only Cauchy sequence that converges to π. Also from infinite series we know that $\frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \ldots$; multiplying by 4 and taking partial sums forms another Cauchy sequences. However, defining the relation between two convergent Cauchy sequences (in the real numbers) x and y that $x \sim y$ if x and y have the same limit. One can check this is an equivalence relation. This implies:

Definition 1.2
The set of real numbers, $\mathbb{R}$ is the collection of equivalence classes of Cauchy sequences of rationals where for two sequences x, y, $x \sim y$ if $d(x_n, y_n) \to 0$.

Thus, $\mathbb{R}$ is the **completion** of the rational numbers $\mathbb{Q}$. Let's look at its properties

1. The distance function is as follows. Take $(x_n) \in X$, $(y_n) \in Y$

$$d : \mathbb{R} \times \mathbb{R} \to [0, \infty)$$
$$d(X, Y) = \lim_n d(x_n, y_n)$$

   This is also the distance function for any completion of a metric space (where on the right, the distance function is inherited from the original space). It turns out it doesn't matter what representation we use (we are modding out by everything in the equivalence classes).

2 Lecture 3

2.1 Weak Law of Large Numbers

Last time we discussed the weak law of large numbers, which states that for the experiment of flipping ∞ coins with the ith coin flip given as $B_i = \mathbf{1}\{i\text{th flip is H}\}$, that for all $\epsilon > 0$,

$$\lim_n \mathrm{P}\left(|\frac{1}{n}\sum_{i=1}^{n} B_i - \frac{1}{2}| > \epsilon\right) = 0$$

This means that looking at the sum of the first n terms, if we take big enough n, the sum of intervals that are bad (bijecting real binary sequences with a sequence of heads and tails) takes up an arbitrarily small portion of the real line.

2.2 Strong Law of Large Numbers

Now let us formulate the SLLN. Consider the sequence of functions $b_1 = \mathbf{1}_{[1/2,1)}$, $b_2 = \mathbf{1}_{[1/4,1/2)} + \mathbf{1}_{[3/4,1)}, \dots$ such that for $x \in [0,1]$ we have $x = 0.b_1(x)b_2(x)\dots$.

Definition 2.1
Call the set N of normal numbers is

$$N = \{x \in [0,1] : \lim_n \frac{1}{n}\sum_{i=1}^{n} b_i(x) = \frac{1}{2}\}$$

The informal strong law is thus if U is picked uniformly at random from the interval $[0,1]$, $\mathbb{P}[U \in N] = 1$, i.e. $m(N) = 1$ for a Lebesgue measure m (not defined yet). For sets that have Lebesgue measure 0 or 1, we can go for a more direct formulation.

Definition 2.2
$A \subseteq [0,1]$ is negligible if for all $\epsilon > 0$, we can find open intervals $O_i \subseteq \mathbb{R}$ for $i \in \mathbb{N}$ such that they form an open cover of A ($A \subseteq \bigcup_{i=1}^{\infty} O_i$) with $\sum_{i=1}^{\infty} \ell(O_i) < \epsilon$.

The negligible sets will be those with $m(A) = 0$. We claim $\mathbb{Q} \cap [0,1)$. Then taking an open interval around each rational $\frac{p}{q}$ of ϵe^{-q} suffices. Since the rationals are countable, this makes a countable open cover. In fact, any countable set is negligible; you can just order them as $a_1, a_2, \dots$ and we can pick $\ell(O_i) = \epsilon/2^i$ to surround a_i.

Theorem 2.1
N^c is negligible.

We can also write, calling $\beta_k(x) = \frac{1}{k}\sum_{i=1}^{k} b_i(k)$:

$$N = \bigcap_{n=1}^{\infty}\bigcup_{j=1}^{\infty}\bigcap_{j}^{\infty}\{x \in [0,1] : |\beta_k(x) - \frac{1}{2}| < \frac{1}{n}\}$$

quantifiers can instead be replaced by union and interaction. $\bigcap$ is roughly for all, and $\bigcup$ is there exists.

Theorem 2.2
A countable union of negligible sets is negligible.

Proof 2.1
Consider negligible sets N_i. We wish to cover $N = \bigcup_i N_i$. This means we can choose covers for any ϵ we want, in particular,

$$\left\{ O_{n,i} : i \in C_n, N_n \subseteq \ bigcup_{i=1}^{\infty} O_{n,i}, \sum_{i=1}^{\infty} \ell(O_{n,i}) < \frac{\epsilon}{2^n} \right\}$$

But then $N \subseteq \bigcup_n \bigcup_i O_{n,i}$. But also $\sum_n \sum_i \ell(O_{n,i}) < \epsilon$.

But is N negligible? But this would imply $N \cup N^c = [0, 1]$ is negligible. This seems not possble. How do we prove it? Consider a finite number of intervals $O_i = (a_i, b_i)$ of length $d_i = b_i - a_i$. If $d = \sum d_i < \epsilon$, then clearly it should not be possible to cover the entire interval. But $\mathbb{Q}$ is dense in $[0, 1)$, yet we could still cover $\mathbb{Q} \subseteq [0, 1)$. The difference is $[0, 1)$ is compact.