Contents

1 Lecture 1

We have seen a limit before in Calculus. Intuitively, when we say $\lim_{x\to c} f(x) = L$, this means that $f(x) \approx L$ for x's close to c, but NOT at c (there can be a hole at c). To rigorize this:

Definition 1.1 (Limit)
For a function $f : [a, b] \to \mathbb{R}$, $c \in (a, b)$ and $L \in \mathbb{R}$
If for every $\epsilon > 0$ we have some $\delta > 0$ such that

$$0 < |x - c| < \delta \implies |f(x) - L| < \epsilon$$

then the limit as x approaches c of $f(x)$ is L.
We denote this by

$$\lim_{x\to c} f(x) = L$$

The upper bound in this definition gives a window between $c - \delta$ and $c + \delta$ (but not including c), and we guarantee that the function lies within ϵ of the limiting value in this windowl.

For example, we know that

$$\lim_{x\to 0} \frac{\sin(x)}{x} = 1$$

This means for small x, $\frac{\sin x}{x} \approx 1$. Or one could say $\sin(x) \approx x$.

Definition 1.2 (Continuity)
A function f is continuous at a point c if

$$\lim_{x\to c} f(x) = f(c)$$

Continuity gives us some nice properties.

Theorem 1.1 (Intermediate Value Theorem)
Let $f : [a, b] \to \mathbb{R}$ be continuous on $[a, b]$.
If L is some value between $f(a)$ and $f(b)$ then there exists some $c \in (a, b)$ such that

$$f(c) = L$$

That is, f takes on every value between $f(a)$ and $f(b)$ at least once over the interval $[a, b]$.

As a corollary, if a continuous function f changes sign in an interval, it must have a zero in that interval.

Now, let's think about tangent lines. As usual, we can form the slope of a secant line between two points x and c as $\frac{f(x)-f(x)}{x-c}$.

Definition 1.3 (Differentiability)
A function $f : [a, b] \to \mathbb{R}$ is differentiable at c if:

$$f'(c) = \lim_{x\to c} \frac{f(x) - f(c)}{x - c}$$

The function f' is termed the **derivative** of f.

This gives us a natural way to approximate f as a line:

$$f(x) \approx f'(c)(x - c) + f(c)$$

We have a nice theorem, namely that on a closed interval $[a, b]$, there is a tangent line with the same slope as the secant line between a and b.

Theorem 1.2 (Mean Value Theorem)
Let $f : [a, b] \to \mathbb{R}$ be continuous on $[a, b]$ and differentiable on (a, b). Then there exists $\xi \in (a, b)$ such that

$$f'(\epsilon) = \frac{f(b) - f(a)}{b - a}$$

The closed part of the interval only really enforces continuity at the endpoints, since:

Theorem 1.3
If f is differentiable at c, then f is continuous at c.

We also define the following sets:

Definition 1.4

$$\begin{aligned} C(\mathbb{R}) &= \{f : f \text{ is continuous}\} \\ C^1(\mathbb{R}) &= \{f : f' \text{ exists}, f' \in C(\mathbb{R})\} \\ C^n(\mathbb{R}) &= \{f : f', f'', \ldots, f^{(n)} \text{ exists}, f', f'', \ldots, f^{(n)} \in C(\mathbb{R})\} \\ C^\infty(\mathbb{R}) &= \{f : f^{(n)} \text{ exists}\} \end{aligned}$$

Where clearly for $m > n$

$$C^m \subset C^n \subset C$$

Let's look at some counter-examples:

$$\frac{\mathrm{d}x^{1/2}}{\mathrm{d}x} = \frac{1}{2}x^{-1/2}$$

This derivative is not continuous at $x = 0$, but the original function is continuous at $x = 0$. We can atttempt to get rid of this by restricting the domain, but we have to now introduce the GSI team.