

A Short Information-Theoretic Analysis of Linear Auto-Regressive Learning

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Abstract

In this note, we give a short information-theoretic proof of the consistency of the Gaussian maximum likelihood estimator in linear auto-regressive models. Our proof yields nearly optimal non-asymptotic rates for parameter recovery and works without any invocation of stability in the case of finite hypothesis classes.

1. Introduction

Learning the dynamics of a linear dynamical system is a classical problem in for instance signal processing, system identification and econometrics. It is also arguably one of the simplest examples of an auto-regressive learning problem, thereby rendering it an instance of self-supervised learning. Understanding the sample complexity—and which quantities are of relevance for it—of such learning problems is key in the current era of large language models.

The traditional approach for analyzing sequential (self-) supervised learning problems operates via comparison of the empirical and population excess risk functionals. Recently, [Jeon and Van Roy \(2024\)](#) provided an information-theoretic proof approach for learning from dependent data eschewing any such direct comparison. However, their results only apply to the Bayesian setting. Nevertheless, this eschewing of reasoning of the lower tail of the empirical risk is highly desirable as directly proving (anti-)concentration inequalities relating these quantities typically comes with significant technical overhead even in the linear setting ([Simchowitz et al., 2018](#)). The situation is even worse in the nonlinear setting, where most known extensions require various mixing (stochastic stability) notions that seem excessive ([Ziemann et al., 2024](#)). Moreover, there is no reason to believe that many relevant time-series applications, such as natural language, are mixing stochastic processes.

Inspired by the recent information-theoretic Bayesian analysis of [Jeon and Van Roy \(2024\)](#) we point out that tools from information theory can also be used in the frequentist setting to establish parameter recovery bounds for linear system identification. Moreover, the advantages of these ideas, notably sidestepping control of the lower tail, also extend. We present our illustration of this below.

Theorem 1

Fix $W_{1:n} \sim N(0, I)$ and let P_{A_\star} be such that the $Z_{1:n}$ satisfy $Z_k = A_\star Z_{k-1} + W_k$ for $k = 2, \dots, n$ and $Z_1 = W_1$. The maximum likelihood estimator \hat{A} (defined in Section 3) over any hypothesis class of the form $\mathcal{P} = \{P_A : A \text{ varies}\}$ containing $P_\star = P_{A_\star}$ achieves:

$$\mathbf{E} \operatorname{tr} \left(\left(A_\star - \hat{A} \right)^\top \left(A_\star - \hat{A} \right) \frac{1}{n} \sum_{i=1}^n \sum_{k=1}^i A_\star^{k-1} A_\star^{\top, k-1} \right) \leq \frac{10^4 \times I(\hat{P} \parallel Z_{1:n})}{n}. \quad (1)$$

We remark that 1) $I(\hat{P} \parallel Z_{1:n}) \leq \log |\mathcal{P}|$ for any finite hypothesis class \mathcal{P} and so the right hand side of (1) admits control decaying with n —the estimator \hat{A} is consistent and converges at a so-called fast rate; 2) whenever A_\star is sufficiently stable, the result can be extended to parametric hypothesis classes isometric to compact subsets of Euclidean space via a standard discretization argument; 3) the rather large constant (2×10^4) is a consequence of instantiating a result of Devroye et al. (2018, Theorem 1.1) and there has been no attempt in the literature to optimize the corresponding constant in their result; and 4) while we have side-stepped control of the lower tail of the empirical risk functional, we have instead relied on the approximately closed form of the Gaussian total variation distance—a luxury we do not have for general learning problems. However, one might instead hope to exploit the fact that f -divergences in parametric families are locally a quadratic.

2. Information-Theoretic Preliminaries

For two probability measures P, Q defined on the same probability space we denote their KL-divergence $d_{\text{KL}}(P \parallel Q) \triangleq \int \log \frac{dP}{dQ} dP$, their total variation distance by $d_{\text{TV}}(P \parallel Q) \triangleq \frac{1}{2} \int \left| \frac{dP}{d\lambda} - \frac{dQ}{d\lambda} \right| d\lambda$ and their squared Hellinger distance by $d_{\text{H}}^2(P \parallel Q) \triangleq \frac{1}{2} \int \left(\sqrt{\frac{dP}{d\lambda}} - \sqrt{\frac{dQ}{d\lambda}} \right)^2 d\lambda$ where λ is a joint dominating measure. If $(X, Y) \sim P_{X,Y}$ we denote their mutual information by $I(X \parallel Y) \triangleq d_{\text{KL}}(P_{X,Y} \parallel P_X \otimes P_Y)$.

Lemma 2 (Donsker and Varadhan (1975)) *Fix two probability measures P and Q on a common measure space (Ω, \mathcal{F}) with $P \ll Q$. Then:*

$$d_{\text{KL}}(P \parallel Q) = \sup_F \left\{ \int_{\Omega} F(\omega) dP(\omega) - \log \int_{\Omega} e^{F(\omega)} dQ(\omega) \right\} \quad (2)$$

where the supremum is taken over \mathcal{F} -measurable and P -exponentially-integrable $F : \Omega \rightarrow \mathbb{R}$.

Lemma 3 *Fix two probability measures P and Q and let λ be a joint dominating measure. We have that:*

$$d_{\text{H}}^2(P \parallel Q) = 1 - \int \exp \left(\frac{1}{2} \log \frac{dQ}{d\lambda} - \frac{1}{2} \log \frac{dP}{d\lambda} \right) dP. \quad (3)$$

Proof Elementary algebraic manipulations give us that:

$$\begin{aligned} d_{\text{H}}^2(P \parallel Q) &= \frac{1}{2} \int \left(\sqrt{\frac{dP}{d\lambda}} - \sqrt{\frac{dQ}{d\lambda}} \right)^2 d\lambda = 1 - \int \sqrt{\frac{dP}{d\lambda} \frac{dQ}{d\lambda}} d\lambda \\ &= 1 - \int \sqrt{\frac{dQ}{dP}} dP = 1 - \int \exp \left(\frac{1}{2} \log \frac{dQ}{dP} - \frac{1}{2} \log \frac{dP}{d\lambda} \right) dP \end{aligned} \quad (4)$$

where we note that the integral is 0 if $\frac{dP}{d\lambda} = 0$ justifying the fraction following the third equality. ■

3. Learning Generative Models in Hellinger Distance

Let $Z_{1:n}$ be a sequence drawn according to P_\star and let \hat{P} be the maximum likelihood estimator (MLE) over a class \mathcal{P} with joint dominating measure λ . Recall that a distribution \hat{P} is a MLE over \mathcal{P} if $-\log \frac{d\hat{P}}{d\lambda}(Z_{1:n}) + \log \frac{dP}{d\lambda}(Z_{1:n}) \leq 0, \forall P \in \mathcal{P}$. We begin by a variation of the information-theoretic analysis of [Zhang \(2006\)](#) inspired by the mutual-information decoupling of [Xu and Raginsky \(2017\)](#).

Theorem 4 *Let \hat{P} be the maximum likelihood estimator over \mathcal{P} . If $P_\star \in \mathcal{P}$ we have that:*

$$\mathbf{E} \left[d_H^2(\hat{P} \parallel P_\star) \right] \leq -\log \left(1 - \mathbf{E}_{\hat{P}} d_H^2(\hat{P} \parallel P_\star) \right) \leq I(\hat{P} \parallel Z_{1:n}). \quad (5)$$

The key observation is that the right hand side of (5) does not necessarily grow with n , whereas the left measures the distance between distributions of an increasing number of variables.

Proof

Let λ be a joint dominating measure for P_\star and \hat{P} (e.g. their mixture). We have that

$$\begin{aligned} & -\log \mathbf{E}_{Z \otimes \hat{P}} \exp \left(\frac{1}{2} \log \frac{d\hat{P}}{d\lambda} - \frac{1}{2} \log \frac{dP_\star}{d\lambda} \right) \\ &= -\log \mathbf{E}_{\hat{P}} \mathbf{E}_Z \exp \left(\frac{1}{2} \log \frac{d\hat{P}}{d\lambda} - \frac{1}{2} \log \frac{dP_\star}{d\lambda} \right) \quad (\text{Fubini}) \\ &= -\log \left(1 - \mathbf{E}_{\hat{P}} d_H^2(\hat{P} \parallel P_\star) \right) \quad (Z \sim P_\star \text{ and Theorem 3}) \\ &\geq \mathbf{E}_{\hat{P}} d_H^2(\hat{P} \parallel P_\star). \quad (x \in [0, 1] \Rightarrow x \leq -\log(1-x)) \end{aligned} \quad (6)$$

To finish we instantiate Donsker-Varadhan with $Q = P_Z \otimes P_{\hat{P}}$ and $P = P_{Z, \hat{P}}$:

$$\begin{aligned} -\log \mathbf{E}_{Z \otimes \hat{P}} \exp \left(\frac{1}{2} \log \frac{d\hat{P}}{d\lambda} - \frac{1}{2} \log \frac{dP_\star}{d\lambda} \right) &\leq -\frac{1}{2} \mathbf{E}_{Z, \hat{P}} \log \frac{d\hat{P}}{dP_\star} + I(\hat{P} \parallel Z_{1:n}) \quad (\text{Donsker-Varadhan}) \\ &\leq I(\hat{P} \parallel Z_{1:n}). \quad (\text{Optimality of MLE}) \end{aligned} \quad (7)$$

By combining (6) with (7) the result follows. ■

4. Proof of the Main Result

Let us write that P_A for the distribution under which $Z_{1:n} = \mathbf{L}_A W_{1:n}$ where

$$\mathbf{L}_A = \begin{bmatrix} I & 0 & 0 & \cdots & 0 \\ A & I & 0 & \cdots & 0 \\ A^2 & A & I & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ A^{n-1} & A^{n-2} & \cdots & \cdots & I \end{bmatrix} \quad \text{and} \quad \mathbf{L}_A^{-1} = \begin{bmatrix} I & 0 & 0 & \cdots & 0 \\ -A & I & 0 & \cdots & 0 \\ 0 & -A & I & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & 0 & \cdots & -A & I \end{bmatrix}. \quad (8)$$

Let $\Sigma_A = \mathbf{L}_A \mathbf{L}_A^\top$ be the covariance matrix of the joint distribution under P_A . We define \mathbf{L}_{A_\star} and Σ_{A_\star} analogously. By [Devroye et al. \(2018, Theorem 1.1\)](#) it suffices to control $\text{tr}(\Sigma_{A_\star} \Sigma_A^{-1} - I)$. Namely, we have that (where the second inequality follows by Cauchy-Schwarz):

$$\text{tr}(\Sigma_{A_\star} \Sigma_A^{-1} - I) \leq 10^4 d_{\text{TV}}^2(P_A \parallel P_\star) \leq 10^4 d_{\text{H}}^2(P_A \parallel P_\star). \quad (9)$$

and use the trace cyclic property to observe that

$$\text{tr} \Sigma_{A_\star} \Sigma_A^{-1} = \text{tr} \mathbf{L}_{A_\star} \mathbf{L}_{A_\star}^\top (\mathbf{L}_A \mathbf{L}_A^\top)^{-1} = \text{tr} \mathbf{L}_A^{-1} \mathbf{L}_{A_\star} \mathbf{L}_{A_\star}^\top \mathbf{L}_A^{\top, -1}. \quad (10)$$

Straightforward calculation now yields that

$$\mathbf{L}_A^{-1} \mathbf{L}_{A_\star} = \begin{bmatrix} I & 0 & 0 & 0 & \cdots & 0 \\ A_\star - A & I & 0 & 0 & \cdots & 0 \\ A_\star^2 - AA_\star & A_\star - A & I & 0 & \cdots & 0 \\ \vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\ A_\star^{n-1} - AA_\star^{n-2} & A_\star^{n-2} - AA_\star^{n-3} & \cdots & \cdots & A_\star - A & I \end{bmatrix}. \quad (11)$$

And hence we may write for the diagonal elements

$$[\mathbf{L}_{A_\star} \mathbf{L}_A^{-1} (\mathbf{L}_{A_\star} \mathbf{L}_A^{-1})^\top]_{ii} = I + \sum_{k=1}^i (A_\star^k - AA_\star^{k-1}) ((A_\star^k - AA_\star^{k-1}))^\top. \quad (12)$$

In other words, by repeated use of the trace cyclic property:

$$\begin{aligned} \text{tr}(\Sigma_A^{-1} \Sigma_{A_\star} - I) &= \text{tr}(\mathbf{L}_A^{-1} \mathbf{L}_{A_\star} (\mathbf{L}_A^{-1} \mathbf{L}_{A_\star})^\top - I) \\ &= \text{tr} \sum_{i=1}^n \sum_{k=1}^i (A_\star^k - AA_\star^{k-1}) (A_\star^k - AA_\star^{k-1})^\top \\ &= \text{tr} \left((A_\star - A)^\top (A_\star - A) \sum_{i=1}^n \sum_{k=1}^i A_\star^{k-1} A_\star^{\top, k-1} \right). \end{aligned} \quad (13)$$

The result follows by instantiating the above with $P_A = P_{\hat{A}}$. ■

References

- Luc Devroye, Abbas Mehrabian, and Tommy Reddad. The total variation distance between high-dimensional gaussians with the same mean. *arXiv preprint arXiv:1810.08693*, 2018.
- Monroe D Donsker and SR Srinivasa Varadhan. Asymptotic evaluation of certain markov process expectations for large time, i. *Communications on Pure and Applied Mathematics*, 28(1):1–47, 1975.
- Hong Jun Jeon and Benjamin Van Roy. Information-theoretic foundations for machine learning. *arXiv preprint arXiv:2407.12288*, 2024.

- Max Simchowitz, Horia Mania, Stephen Tu, Michael I. Jordan, and Benjamin Recht. Learning without mixing: Towards a sharp analysis of linear system identification. In *Conference On Learning Theory*, pages 439–473. PMLR, 2018.
- Aolin Xu and Maxim Raginsky. Information-theoretic analysis of generalization capability of learning algorithms. *Advances in neural information processing systems*, 30, 2017.
- Tong Zhang. From ε -entropy to kl-entropy: Analysis of minimum information complexity density estimation. *The Annals of Statistics*, pages 2180–2210, 2006.
- Ingvar Ziemann, Stephen Tu, George J Pappas, and Nikolai Matni. Sharp rates in dependent learning theory: Avoiding sample size deflation for the square loss. In *Forty-first International Conference on Machine Learning*, 2024.