Trig Derivatives

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Someone recently asked me, can you explain why the trig functions have the derivatives they do?

I asked them what a trig function was and they said sin and cos, but couldn't define them. How can one differentiate a function without a definition of the function?

Let us first define the trig functions as a differential equations, then we'll consider the implications of assuming the fundamental trig identities without defining the functions, next we'll define the trig functions as Taylor Series, and finally as linear combinations of complex exponentials.

1 Differential Equations

Definition 1.

$$\frac{\mathrm{d}^2 y}{\mathrm{d}\theta^2}(\theta) = -y(\theta). \tag{1}$$

Let $y = \sin(\theta)$ be the solution to Eq. 1 with initial values $y(0) = 0$, $\frac{\mathrm{d}y}{\mathrm{d}\theta}(0) = 1$. Let $y = \cos(\theta)$ be the solution to Eq. 1 with initial values $y(0) = 1$, $\frac{\mathrm{d}y}{\mathrm{d}\theta}(0) = 0$.

By the Fundamental Theorem of Differential Equations these functions are unique and well-defined.

Theorem 1.

$$\frac{\mathrm{d}\sin}{\mathrm{d}\theta}(\theta) = \cos(\theta) \textit{ and } \frac{\mathrm{d}\cos}{\mathrm{d}\theta}(\theta) = -\sin(\theta)$$

Proof. Differentiating both sides of Eq. 1 we obtain

$$\frac{\mathrm{d}^3 y}{\mathrm{d}\theta^3}(\theta) = -\frac{\mathrm{d}y}{\mathrm{d}\theta}(\theta).$$

Substituting $z = \frac{\mathrm{d}y}{\mathrm{d}\theta}(\theta)$ we discover that $\frac{\mathrm{d}\sin}{\mathrm{d}\theta}$ and $\frac{\mathrm{d}\cos}{\mathrm{d}\theta}$ both fulfill Eq. 1,

$$\frac{\mathrm{d}^2 z}{\mathrm{d}\theta^2}(\theta) = -z(\theta).$$

Now we look at our initial bounds and find that for $\frac{\mathrm{d}\cos}{\mathrm{d}\theta}$, $z(0) = \frac{\mathrm{d}\cos}{\mathrm{d}\theta}(0) = 0$ and $\frac{\mathrm{d}z}{\mathrm{d}\theta}(0) = \frac{\mathrm{d}^2\cos}{\mathrm{d}\theta^2}(0) = -\cos(0) = -1$. It is now apparent that $-\sin(\theta)$ is the solution to the differential equation that defines $\frac{\mathrm{d}\cos}{\mathrm{d}\theta}(\theta)$. $\square$

The proof of $\frac{\mathrm{d}\sin}{\mathrm{d}\theta}(\theta)$ is left as an exercise for the reader.

2 Trigonometric Identities

Definition 2. *We assume the following properties of* sin *and* cos *but do not define them.*

$$\sin^2(\theta) + \cos^2(\theta) = 1 \tag{2}$$

$$\sin(\alpha + \beta) = \sin(\alpha)\cos(\beta) + \sin(\beta)\cos(\alpha) \tag{3}$$

$$\cos(\alpha + \beta) = \cos(\alpha)\cos(\beta) - \sin(\alpha)\sin(\beta) \tag{4}$$

It should be trivial to see that rescaling the input to the functions doesn't change any of these properties and thus, there is a set of functions that fulfill these equations with cardinality $|\mathbb{R}|$.

Theorem 2.

$$\frac{\mathrm{d}\sin}{\mathrm{d}\theta}(\theta) = \cos(\theta)\frac{\mathrm{d}\sin}{\mathrm{d}\theta}(0) \text{ and } \frac{\mathrm{d}\cos}{\mathrm{d}\theta}(\theta) = -\sin(\theta)\frac{\mathrm{d}\sin}{\mathrm{d}\theta}(0)$$

Proof.

$$\begin{aligned}\frac{\mathrm{d}\sin}{\mathrm{d}\theta}(\theta) &= \lim_{h\to 0}\frac{\sin(\theta+h)-\sin(\theta)}{h}\\ &= \lim_{h\to 0}\frac{\sin(\theta)\cos(h)+\sin(h)\cos(\theta)-\sin(\theta)}{h}\\ \frac{\mathrm{d}\sin}{\mathrm{d}\theta}(\theta) &= \sin(\theta)\lim_{h\to 0}\frac{\cos(h)-1}{h} + \cos(\theta)\lim_{h\to 0}\frac{\sin(h)}{h}\end{aligned}$$

Let us try and find the value of those limits. First let's find the value of $\sin(0)$ and $\cos(0)$. Plugging into Eq. 2, 3, and 4, we get the equations $\sin^2(0)+\cos^2(0) = 1$ $\sin(0) = 2\sin(0)\cos(0)$, and $\cos(0) = \cos^2(0) - \sin^2(0)$ (for simplicity we let $x = \sin(0)$ and $y = \cos(0)$) and then solve.

$$x^2 + y^2 = 1 \land x = 2xy \land y = y^2 - x^2$$

$$0 = x(1-2y) \land x^2 = y^2 - y$$

$$x^2 = 1 - y^2 \land (0 = x \lor 1 = 2y) \land 1 - y^2 = y^2 - y$$

$$x^2 = 1 - y^2 \land (0 = x \lor 1 = 2y) \land 0 = 2y^2 - y - 1 = (2y+1)(y-1)$$

$$x^2 + y^2 = 1 \land (0 = x \lor 1 = 2y) \land (-1 = 2y \lor 1 = y)$$

Clearly $1 = 2y$, $-1 = 2y$, and $1 = y$ are inconsistent with each other

$$x^2 + y^2 = 1 \land (0 = x \land (-1 = 2y \lor 1 = y))$$

$$(x^2 + y^2 = 1 \land 0 = x \land -1 = 2y) \lor (x^2 + y^2 = 1 \land 0 = x \land 1 = y)$$

Since the left expression is clearly false, we are left with

$$x^2 + y^2 = 1 \land 0 = x \land 1 = y$$

$$\sin(0) = 0 \land \cos(0) = 1$$

We now return to the previous limits

$$\lim_{h\to 0}\frac{\sin(h)}{h} = \lim_{h\to 0}\frac{\sin(0+h)-0}{h} = \lim_{h\to 0}\frac{\sin(0+h)-\sin(0)}{h} = \frac{\mathrm{d}\sin}{\mathrm{d}\theta}(0)$$

$$\cos(h)-1 = 1-2\sin^2(h/2)-1 = 2\sin^2(h/2)$$

$$\frac{\cos(h)-1}{h} = \frac{\sin^2(h/2)}{h/2}$$

$$r = h/2, h\to 0 \implies r\to 0$$

$$\lim_{h\to 0}\frac{\cos(h)-1}{h} = \lim_{r\to 0}\sin(r)\frac{\sin(r)}{r} = \sin(0)\frac{\mathrm{d}\sin}{\mathrm{d}\theta}(0) = 0$$

Putting this together we obtain

$$\frac{\mathrm{d}\sin}{\mathrm{d}\theta}(\theta) = \sin(\theta)\lim_{h\to 0}\frac{\cos(h)-1}{h} + \cos(\theta)\lim_{h\to 0}\frac{\sin(h)}{h} = \cos(\theta)\frac{\mathrm{d}\sin}{\mathrm{d}\theta}(0)$$

$\square$

The proof of $\frac{\mathrm{d}\cos}{\mathrm{d}\theta}(\theta)$ is left as an exercise for the reader. The selection of a preferred value for $\frac{\mathrm{d}\sin}{\mathrm{d}\theta}(0)$ will inherently pick a unique sin and cos from the family of functions described above.

3 Taylor Series

Let

$$\sin(\theta) = \sum_{i=0}^{\infty}(-1)^i\frac{\theta^{2i+1}}{(2i+1)!}, \qquad \cos(\theta) = \sum_{i=0}^{\infty}(-1)^i\frac{\theta^{2i}}{(2i)!}$$

Let us consider the derivative of $\sin(\theta)$ given by the above equation,

$$\begin{aligned}\frac{\mathrm{d}\sin}{\mathrm{d}\theta}(\theta) &= \frac{\mathrm{d}}{\mathrm{d}\theta}\sum_{i=0}^{\infty}(-1)^i\frac{\theta^{2i+1}}{(2i+1)!} = \sum_{i=0}^{\infty}(-1)^i\frac{\mathrm{d}}{\mathrm{d}\theta}\frac{\theta^{2i+1}}{(2i+1)!}\\ &= \sum_{i=0}^{\infty}(-1)^i(2i+1)\frac{\theta^{2i}}{(2i+1)!} = \sum_{i=0}^{\infty}(-1)^i\frac{\theta^{2i}}{(2i)!} = \cos(\theta)\end{aligned}$$

As usual, differentiating $\cos(\theta)$ is left as an exercise.

4 Complex Exponentials

Finally, for those familiar with complex exponentials, let

$$\sin(\theta) = \frac{e^{i\theta}-e^{-i\theta}}{2i}, \qquad \cos(\theta) = \frac{e^{i\theta}+e^{-i\theta}}{2}.$$

Again we find the derivative of $\sin(\theta)$ and leave the rest of $\cos(\theta)$

$$\frac{\mathrm{d}\sin}{\mathrm{d}\theta}(\theta) == \frac{\mathrm{d}}{\mathrm{d}\theta}\frac{e^{i\theta}-e^{-i\theta}}{2i} = \frac{ie^{i\theta}-(-i)e^{-i\theta}}{2i} = i\frac{e^{i\theta}+e^{-i\theta}}{2i} = \frac{e^{i\theta}+e^{-i\theta}}{2} = \cos(\theta)$$

About these Articles

These are all written to answer questions that people ask me which I think deserve a thorough answer. They are not edited. I rarely read them after drafting them. They may contain egregious errors. Hopefully someone finds them useful and hopefully they inspire someone to study more math then they originally planned to.

If you have a question which requires an answer feel free to shoot me an email at `parthnobel@berkeley.edu`. Realize I prefer questions that ask me to explain math I've studied recently in terms of much simpler math, like how this paper skips all of field, ring, group, and number theory but hopefully still catches an interesting topic in algebra, but you can ask for anything. No promises I'll reply. I'm also happy to do any CS or EE stuff I know, but, again, no promises. In any case I write these only when I have free-time, and I normally give up if I have to reference a text more than three times, so don't expect too much.

In any case, to anyone who finds this, I hope you read it and enjoy.