Pre-images of PRE Shuffle Sorts

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Abstract

The Prefix-preserving Shuffle: (PRE) is simultaneously a shuffling and sorting algorithm. In every iteration, an image is separated into a left and right bound. Afterwards, both sides are interleaved together. This paper describes the preimages alongside examples.

1 Introduction

This is written in response to the paper "Sorting permutations via shuffles" by Lara Pudwell and Rebecca Smith, in particular, the Prefix-preserving Shuffle (PRE) algorithm, which provides further instructions on this algorithm.

2 Background Context

Additional information, such as terminology, and additional Context behind some of the of the terminology for this algorithm is necessarry to be able to understand the methods for finding its pre-images.

2.1 Relationship between Images and Levels

A *level* refers to the number of parent images between that image and the root image. The root would be labelled as level 0, and the child would be at level 1. There exists a very clear correlation between parent/child relationship between images. In the context of this paper, a *Child* image is the output of a *Parent* image for a single iteration of PRE. If a child image is found on level 2, then the parent image will be found 1 level before the child. For example, looking at figure 1, $(1, 3, 4, 2)$ is the parent of the child $(1, 2, 3, 4)$. There can be several parents to a single child, and in the example, $(1, 2, 3, 4)$ resides on *level* 0, the level on which the final sorted list, from least to greatest, is present. The parents of $(1, 2, 3, 4)$ reside on the lower level, *level* 1. The parents on level 1 are children to the parents on level 2 and so on.

The total number of levels, can be calculated with the formula:

$a_n - 1 = a_m$, where $n > 0$, is the total number of elements in a given set.

2.2 Example

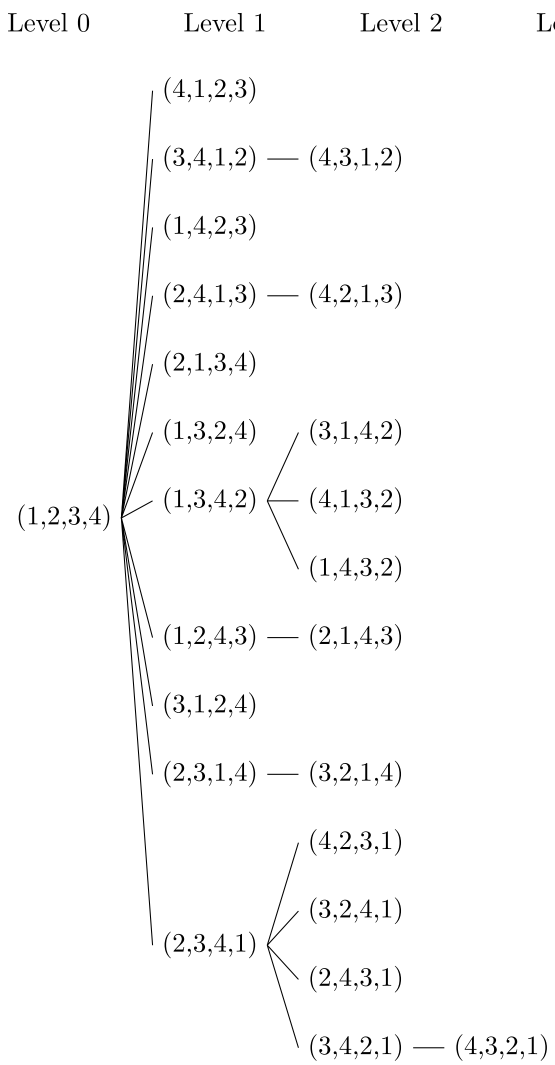

Figure 1: All images of (1,2,3,4) using PRE

2.3 Layers and Descents

There exists a relationship between the number of descents in an image, and which level the image is found in. A descent in the context of this paper is defined as the decrease in value in a set, from a left-bound element to the right-bound element. A descent can also be defined as:

$$A = a_1, a_2, a_3, a_{i-1}, a_i, \ldots, a_n \text{ where } i \text{ is the descent } a_{i-1} > a_i.$$

Using the PRE algorithm, each set found on a level has the same number of descents. For every iteration of PRE, there is 1 less descent. For example, Level 0 has 0 descents, Level 1 has 1, Level 2 has 2, and so on.

The explanation for this is that in the PRE algorithm, when either π or π' run out of entries during the sorting process, the remaining entries get pushed to the output. The remaining entries that get pushed remain in the same place for the next iteration. In Proposition 1, it was determined that the location of these outputs given by PRE are determined by:

$$A = a_1, a_2, a_3, a_{i-1}, a_i, a_{k-1}, a_k \text{ where } k-1 \text{ is the second descent } a_{k-1} > a_k.$$

In Proposition 1, everything from a_1 to a_{k-1} gets sorted from least to greatest. Because the first descent a_{i-1} is included in this sorting process, it is removed in the next image. As a result, the next child and level will always have 1 less descent compared to the parent.

The fixed number of descents per layer can be used to determine the layer in which a set is currently in. For every set, there exists a maximum of $a_n - 1 = a'_m$, where a_n is the total number of possible elements in a set. Subtracting a'_m from our current number of descents gives us the number of total layers beneath the current set.

3 Method

The following section provides a method to find the preimages for a permutation iterated with PRE:

Theorem 1. *Let the set $B = \pi_1 \ldots \pi_n$ be a permutation of length n, where the highest possible number of descents d, is $d < n - 1$.*

Let i be the location of the first descent, such that $\pi_{i-1} > \pi_i$. Next, let $B' = \pi'_1 \ldots \pi'_{i-1} \pi_i \ldots \pi_n$, where $\pi'_1 \ldots \pi'_{i-1}$ is a permutation of $\pi_1 \ldots \pi_{i-1}$ which has exactly one decent and $\pi'_{i-1} > \pi_i$.

1. *For every permutation from $\pi_1 \ldots \pi_{i-1}$, If $\pi'_{i-1} > \pi_{i-1}$ and $\pi'_1 \ldots a'_{i-1}$ has exactly one descent, replace $\pi_1 ... \pi_{i-1}$ in set B with $\pi'_1 ... \pi'_{i-1}$, and push the set as an output.*

2. *If no permutation is found which satisfies these conditions, then the set has no pre-images, or parents.*

Proof. Using Proposition 2.4, We know that $\pi' = \pi_1 \ldots \pi_{i-1}$ is increasing and $\pi_{i-1} > \pi_i$. The first descent π'' (if it exists) can be decomposed into $\pi''^{\star} = \pi_i \ldots \pi_{j-1}$. Every value after π_{j-1} can be defined with $\pi''' = \pi_j \ldots \pi_n$, where $\pi_{j-1} > \pi_j$. Otherwise, $\pi''^{*} = \pi$ if π'' has no descents. With every iteration of PRE, Steps 1 and 2 interleave π' and $\pi''^{\star}$ in increasing order, therefore, eliminating the descent between π_{i-1} and π_i. In addition to Proposition 2.4, let the preimage of the given permutation be defined as $\pi_1^{\star\star} \ldots \pi_n^{\star\star}$. Therefore, every permutation where PRE is applied can be defined as $\pi' \pi''^{\star} \pi'''^{\star\star}$ or $\pi' \pi''^{\star} \pi'''$. Because π''' begins with the second descent π_j, for π''' to exist, the first descent between π_{i-1} and π_i must also exist. Therefore, the preimage needs to have exactly 1 additional descent before π_j, such that $\pi'^{\star\star} \pi''^{\star\star} \pi'''^{\star\star}$. If no permutation from π' to π'' exists, such that there's only 1 additional descent and $\pi_{j-1}^{\star\star} > \pi_j$, then the given permutation doesn't have a preimage. If we implement PRE on a preimage of the permutation, then $\pi'^{\star\star}$ and $\pi''^{\star\star}$ will be sorted, the descent between π_{i-1} and π_i will be removed, outputting the original given permutation $\pi' \pi'' \pi'''^{\star\star}$.

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