3D Motion

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3D Motion

- General rigid body motion involves rotation and translation
- Position and orientation of frame B attached to the body relative to frame A BodyCoords
  - $\mathbf{p}_{ab} \in \mathbb{R}^3$ is the position of the origin of frame B from A
  - $\mathbf{R}_{ab} \in SO(3)$ represents orientation of frame B relative to A
- Configuration of the system consists of pair $(\mathbf{p}_{ab}, \mathbf{R}_{ab})$
  - $SE(3) = \{(\mathbf{p}_{ab}, \mathbf{R}_{ab}) : \mathbf{p}_{ab} \in \mathbb{R}^3, \mathbf{R}_{ab} \in SO(3)\}$
  - $SE(3) = \mathbb{R}^3 \times SO(3)$
  - SE stands for Special Euclidean Group
- If $\mathbf{q}_a$, $\mathbf{q}_b$ are coordinates of point q in frames A and B then
  - $\mathbf{q}_a = \mathbf{p}_{ab} + \mathbf{R}_{ab}\mathbf{q}_b$

Homogenous Coordinates

- We denote pair $(\mathbf{p}_{ab}, \mathbf{R}_{ab})$ by $\mathbf{g}_{ab}$ as shorthand
  - We can then simply write $\mathbf{q}_a = \mathbf{g}_{ab}(\mathbf{q}_b)$
- For a vector on body $\mathbf{v} = \mathbf{s} - \mathbf{r}$ we have
  - $\mathbf{g}_*(\mathbf{v}) = \mathbf{g}_{ab}(\mathbf{s}) - \mathbf{g}_{ab}(\mathbf{r}) = \mathbf{R}_{ab}\mathbf{v}$
  - So any vector in frame B is transformed by rotation to frame A
- A new representation for transformation of points & vectors
  - Point representation: $\bar{\mathbf{q}} = \begin{bmatrix} q_1 & q_2 & q_3 & 1 \end{bmatrix}^T$
  - $\bar{\mathbf{q}} \in \mathbb{R}^4$; termed homogenous coordinates
  - Vector representation: $\bar{\mathbf{v}} = \begin{bmatrix} v_1 & v_2 & v_3 & 0 \end{bmatrix}^T$

Rules of Syntax

- Sums and differences of vectors are vectors
- Sum of vector and a point is a point
- Difference between two points is a vector
- Sum of two points is meaningless!!

Homogenous Representation

- Using the homogenous coordinate representation we get

$$\boxed{\bar{\mathbf{q}}_a = \begin{Bmatrix} \mathbf{q}_a \\ 1 \end{Bmatrix} = \begin{bmatrix} \mathbf{R}_{ab} & \mathbf{p}_{ab} \\ \mathbf{0} & 1 \end{bmatrix} \begin{Bmatrix} \mathbf{q}_b \\ 1 \end{Bmatrix} = \bar{\mathbf{g}}_{ab}\bar{\mathbf{q}}_b}$$

- The 4×4 matrix $\bar{\mathbf{g}}_{ab}$ is homogenous representation of $\mathbf{g}_{ab}$
- What is the convenience of such a representation?
  - Let $\mathbf{g}_{bc}$ represent the transformation from frame C to B
  - $\mathbf{g}_{ab}$ represent the transformation from frame B to A
  - Then $\bar{\mathbf{g}}_{ac} = \bar{\mathbf{g}}_{ab}\bar{\mathbf{g}}_{bc}$; simple matrix multiplication

$$\boxed{\bar{\mathbf{g}}_{ac} = \begin{bmatrix} \mathbf{R}_{ab}\mathbf{R}_{bc} & \mathbf{R}_{ab}\mathbf{p}_{bc} + \mathbf{p}_{ab} \\ \mathbf{0} & 1 \end{bmatrix}}$$

Inversion

$$\boxed{\bar{\mathbf{g}}^{-1} = \begin{bmatrix} \mathbf{R}^T & -\mathbf{R}^T\mathbf{p} \\ \mathbf{0} & 1 \end{bmatrix}}$$

- We have then $\mathbf{g}^{-1} = (-\mathbf{R}^T\mathbf{p},\, \mathbf{R}^T)$

Example

- Let us consider rotation of a rigid body about a line in z direction ▸ Rot_About_Z

$$\mathbf{R}_{ab} = \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} ; \mathbf{p}_{ab} = \begin{Bmatrix} 0 \\ l_1 \\ 0 \end{Bmatrix}$$

$$\bar{\mathbf{g}}_{ab} = \begin{bmatrix} \cos\theta & -\sin\theta & 0 & 0 \\ \sin\theta & \cos\theta & 0 & l_1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Exponential Coordinates

- Let us look at a simple one link manipulator ▸ Rot_Joint
  - Axis of rotation $\boldsymbol{\omega} \in \mathbb{R}^3$ and $\|\boldsymbol{\omega}\| = 1$
  - q is a point on the axis of rotation
- The velocity of point p at the tip of the link assuming unit angular velocity
  - $\dot{\mathbf{p}}(t) = \boldsymbol{\omega} \times (\mathbf{p} - \mathbf{q})$
- The above equation can be converted to homogenous coordinates using

$$\tilde{\boldsymbol{\xi}} = \begin{bmatrix} \tilde{\omega} & \mathbf{v} \\ \mathbf{0} & 0 \end{bmatrix}; \mathbf{v} = -\tilde{\omega}\mathbf{q}$$

Exponential Coordinates 2

$$\begin{Bmatrix} \dot{\mathbf{p}} \\ 0 \end{Bmatrix} = \begin{bmatrix} \tilde{\boldsymbol{\omega}} & \mathbf{v} \\ \mathbf{0} & 0 \end{bmatrix} \begin{Bmatrix} \mathbf{p} \end{Bmatrix} ; \dot{\bar{\mathbf{p}}} = \tilde{\boldsymbol{\xi}} \bar{\mathbf{p}}$$

- The solution to the above differential equation is then
  - $\bar{\mathbf{p}}(t) = e^{\tilde{\boldsymbol{\xi}} t} \bar{\mathbf{p}}(0)$; $e^{\tilde{\boldsymbol{\xi}} t}$ represents rotation by t radians

Exponential Coordinates 3

- The velocity of a point attached to prismatic joint [Pri_Joint]
  - $\dot{\mathbf{p}}(t) = \mathbf{v}$
- One can again write $\bar{\mathbf{p}}(t) = e^{\tilde{\boldsymbol{\xi}}t}\bar{\mathbf{p}}(0)$ where

$$\tilde{\boldsymbol{\xi}} = \begin{bmatrix} \mathbf{0} & \mathbf{v} \\ \mathbf{0} & 0 \end{bmatrix}$$

- The $\tilde{\boldsymbol{\xi}}$ matrix is the generalization of $\tilde{\boldsymbol{\omega}} \in so(3)$
- We define the following group
  - $se(3) = \{(\mathbf{v}, \tilde{\boldsymbol{\omega}}) : \mathbf{v} \in \mathbb{R}^3, \tilde{\boldsymbol{\omega}} \in so(3)\}$
  - An element of $se(3)$ is called a twist
- $\boldsymbol{\xi} := (\mathbf{v}, \boldsymbol{\omega})$ represents the twist coordinates with $\xi \in \mathbb{R}^6$

Exponential Coordinates 4

- The exponential transformation $e^{\tilde{\boldsymbol{\xi}}\theta}$ is different from other ones
  - This maps points from their initial coordinates $\mathbf{p}(0) \in \mathbb{R}^3$ to their coordinates after rigid motion is applied
  - $\mathbf{p}(\theta) = e^{\tilde{\boldsymbol{\xi}}\theta}\mathbf{p}(0)$
- This does not map coordinates from one frame to another
- If $\bar{\mathbf{g}}_{ab}(0)$ represents initial configuration of a rigid body relative to frame A then
  - $\bar{\mathbf{g}}_{ab}(\theta) = e^{\tilde{\boldsymbol{\xi}}\theta}\bar{\mathbf{g}}_{ab}(0)$ is the final configuration with respect to frame A
- Exponential map for a twist gives the relative motion of a rigid body

Rigid Body Transformation

- Every rigid transformation can be written as exponential of some twist
  - $\bar{\mathbf{g}} = e^{\tilde{\boldsymbol{\xi}}\theta}$
- The proof of this can be read from Li & Sastry's book
- It will be shown by an example instead Rot_About_Axis

$$\bar{\mathbf{g}} = \begin{bmatrix} \cos\alpha & -\sin\alpha & 0 & -l_2\sin\alpha \\ \sin\alpha & \cos\alpha & 0 & l_1 + l_2\cos\alpha \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Rigid Body Transformation 2

- We have $\boldsymbol{\omega} = [0\ 0\ 1]^T$ and $\theta = \alpha$

$$e^{\tilde{\boldsymbol{\xi}}\theta} = \begin{bmatrix} e^{\tilde{\boldsymbol{\omega}}\theta} & (\mathbf{I} - e^{\tilde{\boldsymbol{\omega}}\theta})(\boldsymbol{\omega} \times \mathbf{v}) + \boldsymbol{\omega}\boldsymbol{\omega}^T\mathbf{v}\theta \\ \mathbf{0} & 1 \end{bmatrix}$$

- We need to solve the following equation to get $\mathbf{v}$

$$\{(\mathbf{I} - e^{\tilde{\boldsymbol{\omega}}\theta})\tilde{\boldsymbol{\omega}} + \boldsymbol{\omega}\boldsymbol{\omega}^T\theta\}\mathbf{v} = \mathbf{p}_{ab}$$

- The above equation becomes

Transformation 3

$$\begin{bmatrix} \sin\alpha & \cos\alpha - 1 & 0 \\ 1-\cos\alpha & \sin\alpha & 0 \\ 0 & 0 & \alpha \end{bmatrix} \mathbf{v} = \begin{Bmatrix} -l_2 \sin\alpha \\ l_1 + l_2\cos\alpha \\ 0 \end{Bmatrix}$$

- The twist coordinates are

$$\xi = \begin{Bmatrix} \frac{l_1 - l_2}{2} \\ \frac{(l_1+l_2)\sin\alpha}{2(1-\cos\alpha)} \\ 0 \\ 0 \\ 0 \\ 1 \end{Bmatrix};\ \alpha \neq 0$$

Comments

- Complicated form for the twist coordinates
  - As this is an absolute transformation from frame B to A
- Suppose we define a new relative transformation
  - $\mathbf{g}(\alpha) = \mathbf{g}_{ab}(\alpha)\mathbf{g}_{ab}^{-1}(0)$
  - $\mathbf{g}_{ab}(0)$ is for $\alpha = 0$ representing pure translation
- Twist coordinates simplify as $\begin{bmatrix} l_1 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}^T$

Velocity of a Rigid Body

- We start by considering pure rotation in $\mathbb{R}^3$
  - $\mathbf{R}_{ab}(t) \in SO(3)$ represents trajectory of frame B rotating relative to frame A
    - ★ Origin of frame B and A are same
    - ★ Frame B is the body coordinate system and A the spatial or reference coordinate system
  - Any point q on the body has a path in the spatial coordinate system
    - ★ $\mathbf{q}_a(t) = \mathbf{R}_{ab}(t)\mathbf{q}_b$
    - ★ Note that $\mathbf{q}_b$ is fixed in frame B
  - Velocity of the point q in frame A is
    - ★ $\mathbf{v}_{q_a}(t) = \frac{d\mathbf{q}_a}{dt} = \dot{\mathbf{R}}_{ab}(t)\mathbf{q}_b$
  - Above representation is inefficient as it requires nine quantities

Rotational Velocity

- Rewrite the velocity equation as below
  - $\mathbf{v}_{q_a}(t) = \dot{\mathbf{R}}_{ab}(t)\mathbf{R}_{ab}^{-1}(t)\mathbf{R}_{ab}(t)\mathbf{q}_b$
- It turns out that $\dot{\mathbf{R}}_{ab}(t)\mathbf{R}_{ab}^{-1}(t)$ is skew-symmetric
  - Differentiate $\mathbf{R}(t)\mathbf{R}(t)^T = \mathbf{I}$ with respect to time
  - This yields $\dot{\mathbf{R}}\mathbf{R}^T + \mathbf{R}\dot{\mathbf{R}}^T = \mathbf{0}$
  - Hence $\dot{\mathbf{R}}\mathbf{R}^T = -\mathbf{R}\dot{\mathbf{R}}^T = -(\dot{\mathbf{R}}\mathbf{R}^T)^T$
- Remember for a rotation matrix $\mathbf{R}^{-1} = \mathbf{R}^T$
- We define the instantaneous spatial angular velocity
  - $\tilde{\boldsymbol{\omega}}_{ab}^s(t) = \dot{\mathbf{R}}_{ab}(t)\mathbf{R}_{ab}^{-1}(t)$; $\boldsymbol{\omega}_{ab}^s(t) \in \mathbb{R}^3$
  - As seen from frame A
- One can define instantaneous body angular velocity
  - $\tilde{\boldsymbol{\omega}}_{ab}^b = \mathbf{R}_{ab}^{-1}(t)\dot{\mathbf{R}}_{ab}(t)$

Other Relations

- We have the following relation
  - $\tilde{\boldsymbol{\omega}}^b_{ab} = \mathbf{R}^{-1}_{ab}\tilde{\boldsymbol{\omega}}^s_{ab}\mathbf{R}_{ab}$; $\boldsymbol{\omega}^b_{ab} = \mathbf{R}^{-1}_{ab}\boldsymbol{\omega}^s_{ab}$
- Hence one can write the velocity of point q as
  - $\mathbf{v}_{q_a} = \tilde{\boldsymbol{\omega}}^s_{ab}\mathbf{R}_{ab}\mathbf{q}_b = \tilde{\boldsymbol{\omega}}^s_{ab}\mathbf{q}_a = \boldsymbol{\omega}^s_{ab} \times \mathbf{q}_a$
- Velocity in body coordinates as
  - $\mathbf{v}_{q_b} = \boldsymbol{\omega}^b_{ab}(t) \times \mathbf{q}_b$

General Motion

- Consider $\mathbf{g}_{ab}(t) \in SE(3)$ to be the trajectory of rigid body
  - Recall this is motion of frame B attached to the body
- As with pure rotation $\dot{\mathbf{g}}_{ab}(t)$ is not directly useful
  - However $\dot{\mathbf{g}}_{ab}\mathbf{g}_{ab}^{-1}$ and $\mathbf{g}_{ab}^{-1}\dot{\mathbf{g}}_{ab}$ have special significance

$$\dot{\mathbf{g}}_{ab}\mathbf{g}_{ab}^{-1} = \begin{bmatrix} \dot{\mathbf{R}}_{ab} & \dot{\mathbf{p}}_{ab} \\ \mathbf{0} & 0 \end{bmatrix} \begin{bmatrix} \mathbf{R}_{ab}^T & -\mathbf{R}_{ab}^T\mathbf{p}_{ab} \\ \mathbf{0} & 1 \end{bmatrix} = \begin{bmatrix} \dot{\mathbf{R}}_{ab}\mathbf{R}_{ab}^T & \dot{\mathbf{p}}_{ab} - \dot{\mathbf{R}}_{ab}\mathbf{R}_{ab}^T\mathbf{p}_{ab} \\ \mathbf{0} & 0 \end{bmatrix}$$

- This has the form of twist

Twist Coordinates

- Analogous to $\tilde{\boldsymbol{\omega}}^s_{ab}$ for rotational velocity
  - $\tilde{\mathbf{V}}^s_{ab} = \dot{\mathbf{g}}_{ab}\mathbf{g}^{-1}_{ab}$; $\mathbf{V}^s_{ab} = \begin{Bmatrix} \mathbf{v}^s_{ab} \\ \boldsymbol{\omega}^s_{ab} \end{Bmatrix}$;
  - $\mathbf{v}^s_{ab} = \dot{\mathbf{p}}_{ab} - \dot{\mathbf{R}}_{ab}\mathbf{R}^T_{ab}\mathbf{p}_{ab}$; $\tilde{\boldsymbol{\omega}}^s_{ab} = \dot{\mathbf{R}}_{ab}\mathbf{R}^T_{ab}$
- The velocity of a point q then is
  - $\mathbf{v}_{q_a} = \dot{\mathbf{g}}_{ab}\mathbf{q}_b = \dot{\mathbf{g}}_{ab}\mathbf{g}^{-1}_{ab}\mathbf{g}_{ab}\mathbf{q}_b = \tilde{\mathbf{V}}^s_{ab}\mathbf{q}_a$
  - $\mathbf{v}_{q_a} = \boldsymbol{\omega}^s_{ab} \times \mathbf{q}_a + \mathbf{v}^s_{ab}$
- $\mathbf{v}^s_{ab}$ not the velocity of origin of body frame B
- Velocity of point on body passing through origin of frame A at time t

In Body Coordinates

- Velocity in body frame

$$\tilde{\mathbf{V}}_{ab}^{b} = \mathbf{g}_{ab}^{-1}\dot{\mathbf{g}}_{ab} = \begin{bmatrix} \mathbf{R}_{ab}^{T}\dot{\mathbf{R}}_{ab} & \mathbf{R}_{ab}^{T}\dot{\mathbf{p}}_{ab} \\ \mathbf{0} & 0 \end{bmatrix}$$

- $\mathbf{v}_{q_b} = \mathbf{g}_{ab}^{-1}\mathbf{v}_{q_a} = \tilde{\mathbf{V}}_{ab}^{b}\mathbf{q}_b = \boldsymbol{\omega}_{ab}^{b} \times \mathbf{q}_b + \mathbf{v}_{ab}^{b}$
- $\mathbf{v}_{ab}^{b}$ is velocity of origin of frame B with respect to A as seen in B
- The spatial and body frame velocities are related as follows
  - $\tilde{\mathbf{V}}_{ab}^{s} = \mathbf{g}_{ab}\tilde{\mathbf{V}}_{ab}^{b}\mathbf{g}_{ab}^{-1}$

$$\mathbf{V}_{ab}^{s} = \begin{Bmatrix} \mathbf{v}_{ab}^{s} \\ \boldsymbol{\omega}_{ab}^{s} \end{Bmatrix} = \begin{bmatrix} \mathbf{R}_{ab} & \tilde{\mathbf{p}}_{ab}\mathbf{R}_{ab} \\ \mathbf{0} & \mathbf{R}_{ab} \end{bmatrix} \begin{Bmatrix} \mathbf{v}_{ab}^{b} \\ \boldsymbol{\omega}_{ab}^{b} \end{Bmatrix}$$

- This transformation is called adjoint transformation with notation $\mathbf{Ad}_g$

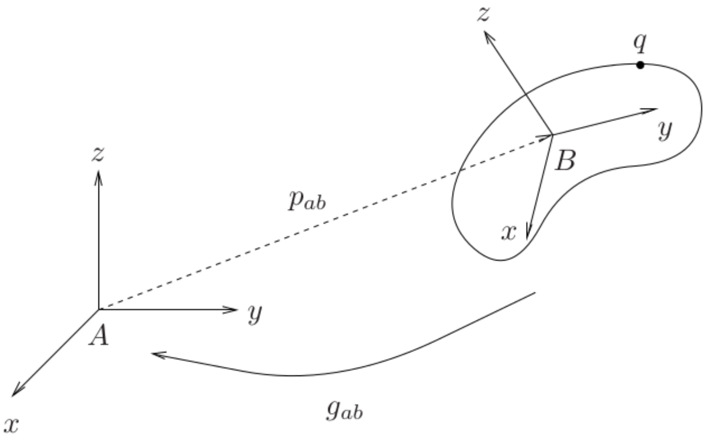
Transformations

- Motion of three coordinate frames A, B & C
  - Spatial velocities relation: $\mathbf{V}^s_{ac} = \mathbf{V}^s_{ab} + \mathbf{Ad}_{g_{ab}} \mathbf{V}^s_{bc}$
- Proof steps
  - We know that $\mathbf{g}_{ac} = \mathbf{g}_{ab}\mathbf{g}_{bc}$
  - Now $\tilde{\mathbf{V}}_{ac} = \dot{\mathbf{g}}_{ac}\mathbf{g}_{ac}^{-1} = (\dot{\mathbf{g}}_{ab}\mathbf{g}_{bc} + \mathbf{g}_{ab}\dot{\mathbf{g}}_{bc})(\mathbf{g}_{bc}^{-1}\mathbf{g}_{ab}^{-1})$
  - $\tilde{\mathbf{V}}_{ac} = \dot{\mathbf{g}}_{ab}\mathbf{g}_{ab}^{-1} + \mathbf{g}_{ab}\tilde{\mathbf{V}}_{bc}\mathbf{g}_{ab}^{-1} = \tilde{\mathbf{V}}_{ab} + \mathbf{g}_{ab}\tilde{\mathbf{V}}_{bc}\mathbf{g}_{ab}^{-1}$
- When converted to twist coordinates $\mathbf{V}^s_{ac} = \mathbf{V}^s_{ab} + \mathbf{Ad}_{g_{ab}} \mathbf{V}^s_{bc}$
- In body coordinates $\mathbf{V}^b_{ac} = \mathbf{Ad}_{g_{bc}^{-1}} \mathbf{V}^b_{ab} + \mathbf{V}^b_{bc}$
  - Note that $\mathbf{Ad}_g^{-1} = \mathbf{Ad}_{g^{-1}}$

Two Link Manipulator

- We look at an example ▸ 2linkMan
  - $$\mathbf{V}^s_{ab} = \begin{Bmatrix} \mathbf{v}_{ab} \\ \boldsymbol{\omega}_{ab} \end{Bmatrix} \dot{\theta}_1;\ \mathbf{v}_{ab} = \begin{Bmatrix} 0 \\ 0 \\ 0 \end{Bmatrix};\boldsymbol{\omega}_{ab} = \begin{Bmatrix} 0 \\ 0 \\ 1 \end{Bmatrix}$$
  - $$\mathbf{V}^s_{bc} = \begin{Bmatrix} \mathbf{v}_{bc} \\ \boldsymbol{\omega}_{bc} \end{Bmatrix} \dot{\theta}_2;\ \mathbf{v}_{bc} = \begin{Bmatrix} l_1 \\ 0 \\ 0 \end{Bmatrix};\boldsymbol{\omega}_{bc} = \begin{Bmatrix} 0 \\ 0 \\ 1 \end{Bmatrix}$$
  - $$\mathbf{V}^s_{ac} = \mathbf{V}^s_{ab} + \mathbf{Ad}_{g_{ab}}\mathbf{V}^s_{bc} = \begin{Bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{Bmatrix} \dot{\theta}_1 + \begin{Bmatrix} l_1 \cos\theta_1 \\ l_1 \sin\theta_1 \\ 0 \\ 0 \\ 0 \\ 1 \end{Bmatrix} \dot{\theta}_2$$

3D Motion



Courtesy: Murray et al., 1994, A Mathematical Introduction to Robotic Manipulation, CRC Press

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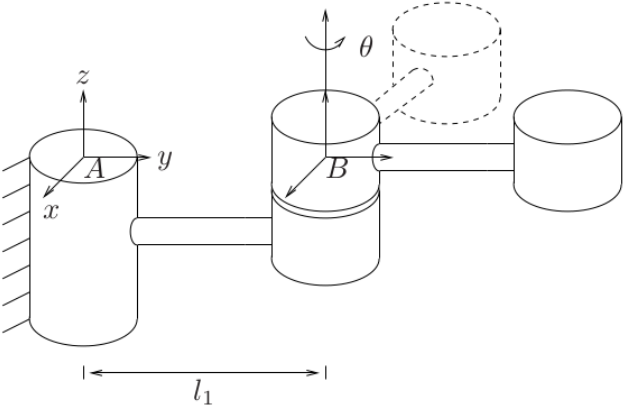

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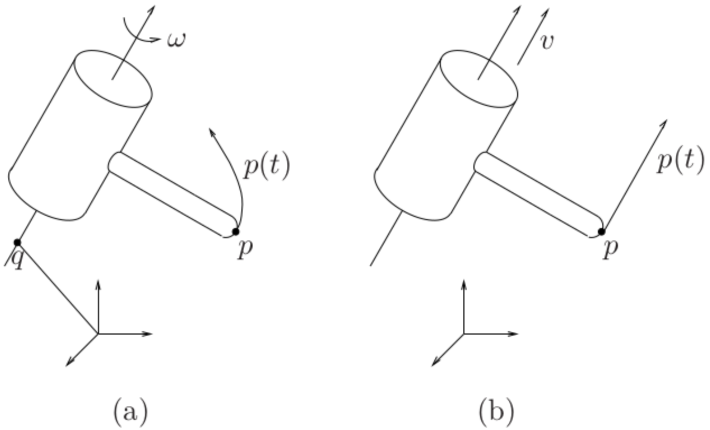
Revolute and Prismatic Joints



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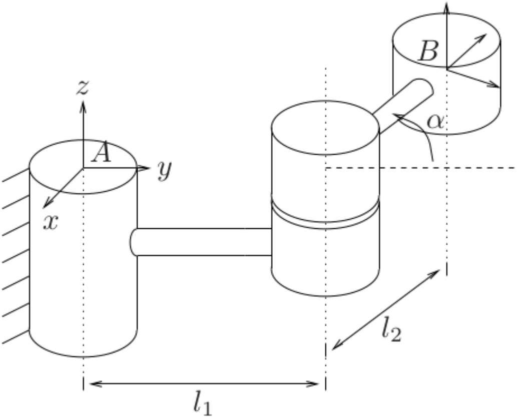
Rotation About Axis



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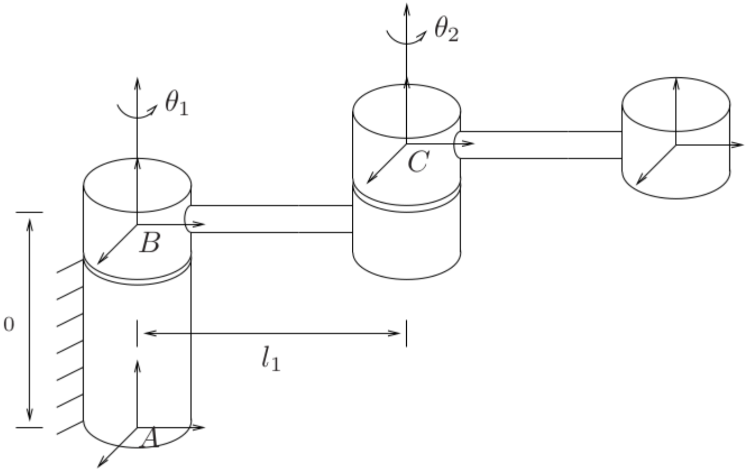
Rotation About Axis



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