3D Rotation

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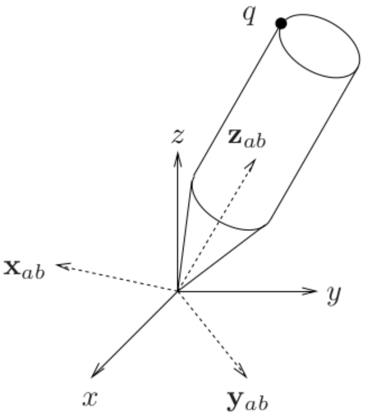
3D Rotational Motion

- Orientation of body
  - Relative orientation between body coordinate frame B and fixed frame A
- Define $\mathbf{x}_{ab}$, $\mathbf{y}_{ab}$ and $\mathbf{z}_{ab} \in \mathbb{R}^3$ as coordinates of axes of B with respect to A
- Stack these next to each other as follows
  - $\mathbf{R} = [\mathbf{x}_{ab}\ \mathbf{y}_{ab}\ \mathbf{z}_{ab}]$
- This is a rotation matrix



Courtesy: Murray et al., 1994, A Mathematical Introduction to Robotic Manipulation, CRC Press

Properties

- A rotation matrix has two key properties
  - Follows from its construction
- Let $\mathbf{r}_1$, $\mathbf{r}_2$ and $\mathbf{r}_3 \in \mathbb{R}^3$ be the columns of $\mathbf{R}$
  - We have $\mathbf{r}_i^T \mathbf{r}_j = \begin{cases} 0, & i \neq j \\ 1, & i = j \end{cases}$
  - Orthonormal coordinates
- From this it follows $\boxed{\mathbf{R}^T\mathbf{R} = \mathbf{R}\mathbf{R}^T = \mathbf{I}}$
- We can also then deduce that $\det \mathbf{R} = \pm 1$
- To determine the sign recall that $\det \mathbf{R} = \mathbf{r}_1^T(\mathbf{r_2} \times \mathbf{r}_3)$
  - For right handed coordinate system $\mathbf{r_2} \times \mathbf{r}_3 = \mathbf{r}_1$
  - Hence $\det \mathbf{R} = 1$
- Set of all 3×3 matrices satisfying above two properties
  - $SO(3)$ (Special Orthogonal)

Transformation

- The rotation matrix $\mathbf{R} \in SO(3)$ serves as a transformation
  - Coordinates of a point from one frame to another
- Let us revisit the figure shown earlier RotFig
- The co-ordinates of point $q = (x_b,\ y_b,\ z_b)$ relative to frame B
- This can be transformed to frame A since
  - $\mathbf{q}_a = x_b\mathbf{x}_{ab} + y_b\mathbf{y}_{ab} + z_b\mathbf{z}_{ab}$
  - This can be written as $\mathbf{q}_a = \mathbf{R}\mathbf{q}_b$
- $\mathbf{R}$ considered a map from $\mathbb{R}^3$ to $\mathbb{R}^3$
  - Rotates the coordinates of a point in frame B to frame A
- Use to define effect on vector $\mathbf{v}_b = \mathbf{q}_b - \mathbf{p}_b$
  - $\mathbf{R}\mathbf{v}_b = \mathbf{R}(\mathbf{q}_b - \mathbf{p}_b) = \mathbf{q}_a - \mathbf{p}_a = \mathbf{v}_a$

Transformation 2

- Rotation matrices can be combined to form new rotation matrices
  - Using matrix multiplication
- Frame C has an orientation $\mathbf{R}_{bc}$ relative to B
- B has a relative orientation $\mathbf{R}_{ab}$ relative to A
  - Orientation of C with respect to A: $\mathbf{R}_{ac} = \mathbf{R}_{ab}\mathbf{R}_{bc}$
- Rotates the coordinates of a point from frame C to A by first rotating from C to B and then from B to A

Exponential Coordinates for Rotation

- A common motion encountered is rotation of a link about an axis ‣ AxisRot
- Axis of rotation represented by $\boldsymbol{\omega} \in \mathbb{R}^3$ and angle of rotation $\theta \in \mathbb{R}$
- Every rotation of link corresponds to some $\mathbf{R} \in SO(3)$
  - Express as function of $\boldsymbol{\omega}$ and θ
- If link rotates with unit angular velocity about $\boldsymbol{\omega}$ velocity of point q ‣ AxisRot
  - $\dot{\mathbf{q}}(t) = \boldsymbol{\omega} \times \mathbf{q}(t) = \tilde{\boldsymbol{\omega}}\mathbf{q}(t)$
- Above is ODE with solution $\mathbf{q}(t) = e^{\tilde{\boldsymbol{\omega}}t}\mathbf{q}(0)$
  - $\mathbf{q}(0)$ is the initial position at $t = 0$

Exponential Coordinates 2

- We have $e^{\tilde{\omega}t} = \mathbf{I} + \tilde{\boldsymbol{\omega}}t + \frac{(\tilde{\omega}t)^2}{2!} + \frac{(\tilde{\omega}t)^3}{3!} + \cdots$
- If we rotate for θ units of time then
  - $\mathbf{R}(\boldsymbol{\omega}, \theta) = e^{\tilde{\omega}\theta}$
- $\tilde{\boldsymbol{\omega}}$ is a skew-symmetric matrix belonging to space $so(3)$
  - $so(3) = \{S \in \mathbb{R}^{3\times 3} : S^T = -S\}$
- For $\tilde{\boldsymbol{a}} \in so(3)$ following relations hold
  - $\tilde{\boldsymbol{a}}^2 = \boldsymbol{a}\boldsymbol{a}^T - \|\boldsymbol{a}\|^2\mathbf{I}$; $\tilde{\boldsymbol{a}}^3 = -\|\boldsymbol{a}\|^2\tilde{\boldsymbol{a}}$
  - Other higher order terms can be found recursively
- If we let $\boldsymbol{a} = \boldsymbol{\omega}\theta$ with $\|\boldsymbol{\omega}\| = 1$ then we have
  - $e^{\tilde{\omega}\theta} = \mathbf{I} + (\theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} - \cdots)\tilde{\boldsymbol{\omega}} + (\frac{\theta^2}{2!} - \frac{\theta^4}{4!} + \frac{\theta^6}{6!} - \cdots)\tilde{\boldsymbol{\omega}}^2$
  - $\boxed{e^{\tilde{\omega}\theta} = \mathbf{I} + \tilde{\boldsymbol{\omega}}\sin\theta + \tilde{\boldsymbol{\omega}}^2(1 - \cos\theta)}$; Rodrigues' Formula

Rotation Matrix

$$e^{\tilde{\omega}\theta} = \begin{bmatrix} \omega_1^2 v_\theta + c_\theta & \omega_1\omega_2 v_\theta - \omega_3 s_\theta & \omega_1\omega_3 v_\theta + \omega_2 s_\theta \\ \omega_1\omega_2 v_\theta + \omega_3 s_\theta & \omega_2^2 v_\theta + c_\theta & \omega_2\omega_3 v_\theta - \omega_1 s_\theta \\ \omega_1\omega_3 v_\theta - \omega_2 s_\theta & \omega_2\omega_3 v_\theta + \omega_1 s_\theta & \omega_3^2 v_\theta + c_\theta \end{bmatrix}$$

- In the above $v_\theta = (1 - \cos\theta)$, $s_\theta = \sin\theta$ and $c_\theta = \cos\theta$
- It is a 4 parameter representation of the rotation matrix
  - Axis represented by $\omega_1, \omega_2, \omega_3$ and rotation about this axis by angle θ
- Exponential coordinates are called canonical coordinates of the rotation group

Rotation about Z

- Suppose we look at rotation about Z axis
- $\omega_1 = \omega_2 = 0$ and $\omega_3 = 1$
- Rotation matrix is

$$\boxed{e^{\tilde{\boldsymbol{\omega}}\theta} = \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}}$$

Other Coordinates

- Connect other representations with the exponential coordinates
- Euler Angles
  - Let us start with two frames A & B coincident
  - Now rotate frame B about its z axis by angle α
  - Then we rotate about the new y axis by angle β
  - Lastly one more rotation about new z axis by γ
- This yields a net orientation $\mathbf{R}_{ab}(\alpha,\ \beta,\ \gamma)$
  - The angles $(\alpha,\ \beta,\ \gamma)$ are ZYZ Euler Angles
- $\boxed{\mathbf{R}_{ab} = \mathbf{R}_z(\alpha)\mathbf{R}_y(\beta)\mathbf{R}_z(\gamma)}$
  - The final form of the matrix is shown next

Euler Angle Rotation Matrix

$$\mathbf{R}_{ab} = \begin{bmatrix} c_\alpha c_\beta c_\gamma - s_\alpha s_\gamma & -c_\alpha c_\beta s_\gamma - s_\alpha c_\gamma & c_\alpha s_\beta \\ s_\alpha c_\beta c_\gamma + c_\alpha s_\gamma & -s_\alpha c_\beta s_\gamma + c_\alpha c_\gamma & s_\alpha s_\beta \\ -s_\beta c_\gamma & s_\beta s_\gamma & c_\beta \end{bmatrix}$$

- c_α implies $\cos\alpha$ while s_β is shorthand for $\sin\beta$
- Singularity happens when $\mathbf{R}_{ab} = \mathbf{I}$ for this set of Euler angles
  - For instance when $(\alpha,\, \beta,\, \gamma) = (\alpha,\, 0,\, -\alpha)$
- Other choices include YZX called Helmholtz angles and ZYX called Fick angles
  - ZYX has a singularity when rotation about Y is $-\frac{\pi}{2}$

Quaternions

- They generalize complex numbers and used to represent rotation
- A complex number on a unit circle can represent planar rotation
- Formal representation is as follows
  - $\boxed{\mathbf{Q} = q_0 + q_1\mathbf{i} + q_2\mathbf{j} + q_3\mathbf{k};\ q_i \in \mathbb{R},\ i = 0, 1, 2, 3}$
  - q_0 is the scalar component of $\mathbf{Q}$ and $\mathbf{q} = (q_1,\ q_2,\ q_3)$ is the vector component
- Quaternion multiplication denoted by $\cdot$ is distributive and associative but not commutative
  - $\mathbf{i}\cdot\mathbf{i} = \mathbf{j}\cdot\mathbf{j} = \mathbf{k}\cdot\mathbf{k} = \mathbf{i}\cdot\mathbf{j}\cdot\mathbf{k} = -1$
  - $\mathbf{i}\cdot\mathbf{j} = -\mathbf{j}\cdot\mathbf{i} = \mathbf{k};\ \mathbf{j}\cdot\mathbf{k} = -\mathbf{k}\cdot\mathbf{j} = \mathbf{i};\ \mathbf{k}\cdot\mathbf{i} = -\mathbf{i}\cdot\mathbf{k} = \mathbf{j}$
  - $a\mathbf{i} = \mathbf{i}a;\ a\mathbf{j} = \mathbf{j}a;\ a\mathbf{k} = \mathbf{k}a;\ a \in \mathbb{R}$

Qauternions 2

- The conjugate of a quaternion $\mathbf{Q} = (q_0, \mathbf{q})$ is $\mathbf{Q}^* = (q_0, -\mathbf{q})$
- The magnitude of the quaternion is
  - $\|\mathbf{Q}\|^2 = \mathbf{Q} \cdot \mathbf{Q}^* = q_0^2 + q_1^2 + q_2^2 + q_3^2$
- Product of two quaternions $\mathbf{Q}$ and $\mathbf{P}$ is
  - $\boxed{\mathbf{Q} \cdot \mathbf{P} = (q_0 p_0 - \mathbf{q} \cdot \mathbf{p},\ q_0 \mathbf{p} + p_0 \mathbf{q} + \mathbf{q} \times \mathbf{p})}$
- A unit quaternion is one with $\|\mathbf{Q}\| = 1$
- Given a rotation matrix $\mathbf{R} = e^{\tilde{\omega}\theta}$ then
  - $\mathbf{Q} = (\cos\frac{\theta}{2}, \sin\frac{\theta}{2}\boldsymbol{\omega})$
  - $\boldsymbol{\omega} \in \mathbb{R}^3$ represents normalized axis of rotation
- The quaternion representation do not suffer from singularities

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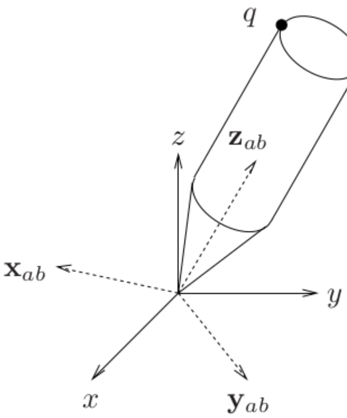

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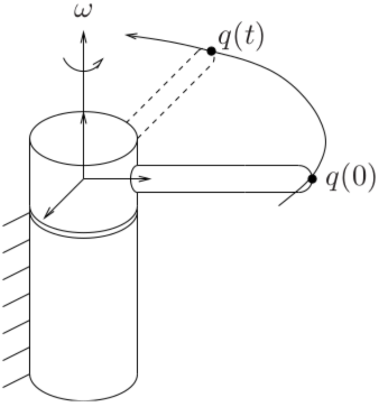
Rotation about Axis



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