Forms of Dynamic Equations

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1. Newton-Euler Form
   - Open Chain vs. Closed Chain Systems

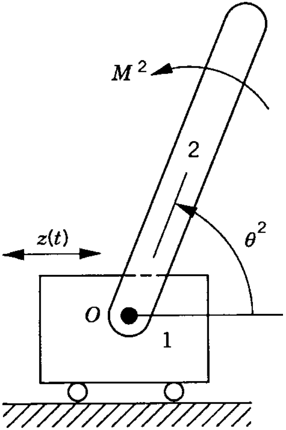
2. Virtual Work
   - Force Elements
   - Workless Constraints
   - Statics
   - Dynamics
   - Lagrange's Equations

Newton-Euler Equations

- Consider a body in unconstrained planar motion
- The motion can be described by three independent co-ordinates
  - Two translations R_x^i and R_y^i of a reference point
  - Rotation θ^i of body with respect to a reference co-ordinate system
- The body is acted on by various forces/moments
- Reference point as centre of mass of body ▸ Dynamic
  - $m^i \ddot{R}_x^i = F_x^i;\ m^i \ddot{R}_y^i = F_y^i;\ J^i \ddot{\theta}^i = M^i$
  - F_x^i and F_y^i are resultant force components at centre of mass
  - M^i is the resultant moment

Example

- Block 1 has specified sliding motion $z(t)$
- Body 2 slender rod with mass m^2 and mass moment of inertia J^2
  - Subject to moment M^2
- Connected to Block 1 by revolute joint at O
- We now derive the Newton-Euler equations



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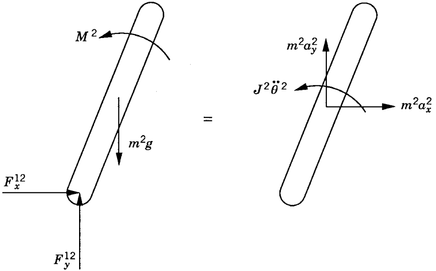
Free-body Diagram of Rod

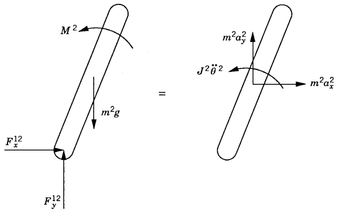
- $m^2\ddot{R}_x^2 = F_x^{12}$
- $m^2\ddot{R}_y^2 = F_y^{12} - m^2 g$
- $J^2\ddot{\theta}^2 = M^2 + F_x^{12}\frac{l}{2}\sin\theta^2 - F_y^{12}\frac{l}{2}\cos\theta^2$
- Since block 1 motion specified only one degree-of-freedom θ^2



Courtesy: A. A. Shabana, 2010, *Computational Dynamics*, Third Edition, John Wiley & Sons.

Acceleration Expressions

- $R_x^2 = z(t) + \frac{l}{2}\cos\theta^2$
- $R_y^2 = \frac{l}{2}\sin\theta^2$
- $\ddot{R}_y^2 = -(\dot{\theta}^2)^2\frac{l}{2}\sin\theta^2 + \ddot{\theta}^2\frac{l}{2}\cos\theta^2$
- $\ddot{R}_x^2 = \ddot{z}(t) - (\dot{\theta}^2)^2\frac{l}{2}\cos\theta^2 - \ddot{\theta}^2\frac{l}{2}\sin\theta^2$



Courtesy: A. A. Shabana, 2010, *Computational Dynamics*, Third Edition, John Wiley & Sons.

Substitution & Elimination

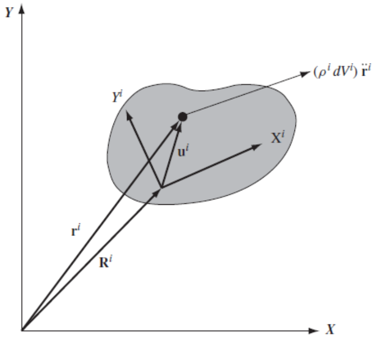
- $m^2\left(\ddot{z}(t) - (\dot{\theta}^2)^2 \frac{l}{2} \cos\theta^2 - \ddot{\theta}^2 \frac{l}{2} \sin\theta^2\right) = F_x^{12}$
- $m^2\left(g - (\dot{\theta}^2)^2 \frac{l}{2} \sin\theta^2 + \ddot{\theta}^2 \frac{l}{2} \cos\theta^2\right) = F_y^{12}$
- Substitute the two in Euler equation

$$\boxed{\left[J^2 + m^2(\frac{l}{2})^2\right]\ddot{\theta}^2 = M^2 - m^2 g \frac{l}{2} \cos\theta^2 + m^2 \ddot{z}(t) \frac{l}{2} \sin\theta^2}$$

- Note that joint forces do not exist in the above equation
  - This is then an embedded formulation

System of Particles

- Rigid body i considered as system of particles
- A particle has mass $\rho^i dV^i$ with dV^i representing infinitesimal volume and ρ^i density of body material
- The inertia force of this particle is then $\rho^i dV^i \ddot{\mathbf{r}}^i$
- Total inertia force $\boxed{\mathbf{F}^i = \int_{V^i} \rho^i \ddot{\mathbf{r}}^i \, dV^i}$



Courtesy: A. A. Shabana, 2010, *Computational Dynamics*, Third Edition, John Wiley & Sons.

Choosing the Reference Point

- The reference point for the body coordinate system can be at any point
  - $\ddot{\mathbf{r}}^i = \ddot{\mathbf{R}}^i + \ddot{\theta}^i \mathbf{A}_\theta^i \bar{\mathbf{u}}^i - (\dot{\theta}^i)^2 \mathbf{A}^i \bar{\mathbf{u}}^i$
- If reference point is chosen as centre of mass of body then
  - $\int_{V^i} \rho^i \mathbf{A}^i \bar{\mathbf{u}}^i \, dV^i = \mathbf{A}^i \int_{V^i} \rho^i \bar{\mathbf{u}}^i \, dV^i = 0$
  - $\int_{V^i} \rho^i \mathbf{A}_\theta^i \bar{\mathbf{u}}^i \, dV^i = \mathbf{A}_\theta^i \int_{V^i} \rho^i \bar{\mathbf{u}}^i \, dV^i = 0$
- Then the inertia force is given by
  - $\boxed{\mathbf{F}^i = \int_{V^i} \rho^i \ddot{\mathbf{R}}^i \, dv^i = m^i \ddot{\mathbf{R}}^i}$
- This is the advantage of choosing the centre of mass as reference point

Euler Equation

- Use D'Alembert principle for moment equilibrium
  - Replace all the forces by the inertia force $\mathbf{F}^i$
  - $\int_{V^i} \rho^i \mathbf{u}^i \times \ddot{\mathbf{r}}^i \, dV^i = \mathbf{M}_R^i = \begin{bmatrix} 0 & 0 & M^i \end{bmatrix}^{\mathrm{T}}$
- We can break LHS into three integrals
  - $\int_{V^i} \rho^i \mathbf{u}^i \times \ddot{\mathbf{R}}^i \, dV^i = \mathbf{0}$; $(\dot{\theta}^i)^2 \int_{V^i} \rho^i \mathbf{u}^i \times \mathbf{u}^i \, dV^i = \mathbf{0}$
  - First zero by definition of centre of mass and second due to cross product of parallel vectors being zero
  - Third term is
    - ⋆ $\int_{V^i} \rho^i \mathbf{u}^i \times (\boldsymbol{\alpha}^i \times \mathbf{u}^i) \, dV^i = \int_{V^i} \rho^i \tilde{\mathbf{u}}^i \tilde{\boldsymbol{\alpha}}^i \mathbf{u}^i \, dV^i$
  - This yields the following equation
    - ⋆ $\boxed{\ddot{\theta}^i \int_{V^i} \rho^i ({x^i}^2 + {y^i}^2) \, dV^i = \ddot{\theta}^i \int_{V^i} \rho^i \left\{ (\bar{x}^i)^2 + (\bar{y}^i)^2 \right\} dV^i = M^i}$
    - ⋆ $J^i \ddot{\theta}^i = M^i$

Constrained Dynamics

- We use a simple example to understand number of constraint forces equals number of dependent coordinates RodDisc
- From the free-body diagrams one can write for body 2 FBDRodDisc

$$
\boxed{
\begin{aligned}
m^2 \ddot{R}^2_x &= F^{12}_x - F^{23}_x \\
m^2 \ddot{R}^2_y &= F^{12}_y - F^{23}_y - m^2 g \\
J^2 \ddot{\theta}^2 &= M^2 + F^{12}_x \frac{l}{2} \sin\theta^2 - F^{12}_y \frac{l}{2} \cos\theta^2 \\
&\quad + F^{23}_x \frac{l}{2} \sin\theta^2 - F^{23}_y \frac{l}{2} \cos\theta^2
\end{aligned}
}
$$

Constrained Dynamics 2

- From free-body diagrams one can write for body 3 ▸ FBDRodDisc

$$\boxed{\begin{aligned} m^3\ddot{R}^3_x &= F^{23}_x \\ m^3\ddot{R}^3_y &= F^{23}_y - m^3 g \\ J^3\ddot{\theta}^3 &= M^3 \end{aligned}}$$

- We have 6 equations in 10 unknowns
- Note that the number of independent co-ordinates is 2
  - Number of dependent co-ordinates is equal to the number of constraints
- The general form of the Newton-Euler equations
  - $\boxed{\mathbf{M}\ddot{\mathbf{q}} = \mathbf{Q}_e + \mathbf{Q}_c}$
  - $\mathbf{Q}_e$ external force and $\mathbf{Q}_c$ constraint force vector

Augmented Formulation

- We augment the 6 Newton-Euler equations with second derivative of constraint equations ▸ Constraints
  - $\mathbf{C_q}\ddot{\mathbf{q}} = \mathbf{Q}_d$ ▸ Jacobian
- This leads to a form

$$\begin{bmatrix} \mathbf{M} & \mathbf{C}_\mathbf{q}^\mathrm{T} \\ \mathbf{C_q} & \mathbf{0} \end{bmatrix} \begin{Bmatrix} \ddot{\mathbf{q}} \\ \boldsymbol{\lambda} \end{Bmatrix} = \begin{Bmatrix} \mathbf{Q}_e \\ \mathbf{Q}_d \end{Bmatrix}; \quad \boldsymbol{\lambda} = \begin{Bmatrix} F_x^{12} \\ F_y^{12} \\ F_x^{23} \\ F_y^{23} \end{Bmatrix}$$

- We then have $\mathbf{Q}_c = -\mathbf{C}_\mathbf{q}^\mathrm{T}\boldsymbol{\lambda}$
- $\boldsymbol{\lambda}$ is the vector of Lagrange Multipliers

Embedded Formulation

- Eliminate the dependent degrees-of-freedom
- Rewrite the constraint equations as
  - $\begin{bmatrix}\mathbf{C}_{\mathbf{q}_i} & \mathbf{C}_{\mathbf{q}_d}\end{bmatrix}\begin{Bmatrix}\ddot{\mathbf{q}}_i \\ \ddot{\mathbf{q}}_d\end{Bmatrix} = \mathbf{Q}_d$
  - $\ddot{\mathbf{q}}_d = -(\mathbf{C}_{\mathbf{q}_d})^{-1}\mathbf{C}_{\mathbf{q}_i}\ddot{\mathbf{q}}_i + (\mathbf{C}_{\mathbf{q}_d})^{-1}\mathbf{Q}_d$
- We now define the following relation
  - $\ddot{\mathbf{q}} = \mathbf{B}_i\ddot{\mathbf{q}}_i + \gamma_i$
  - $\mathbf{B}_i = \begin{bmatrix}\mathbf{I} & -(\mathbf{C}_{\mathbf{q}_d})^{-1}\mathbf{C}_{\mathbf{q}_i}\end{bmatrix}^{\mathrm{T}}; \gamma_i = \begin{bmatrix}\mathbf{0} & (\mathbf{C}_{\mathbf{q}_d})^{-1}\mathbf{Q}_d\end{bmatrix}^{\mathrm{T}}$

Embedded Form 2

- Now we use the new relation in the Newton-Euler equations
  - $\mathbf{M}\ddot{\mathbf{q}} = \mathbf{Q}_e + \mathbf{Q}_c$
  - $\mathbf{M}\mathbf{B}_i\ddot{\mathbf{q}}_i = \mathbf{Q}_e + \mathbf{Q}_c - \mathbf{M}\boldsymbol{\gamma}_i$
- Make it symmetric by multiplying equation with $\mathbf{B}_i^{\mathrm{T}}$
  - $\boxed{\mathbf{B}_i^{\mathrm{T}}\mathbf{M}\mathbf{B}_i\ddot{\mathbf{q}}_i = \mathbf{B}_i^{\mathrm{T}}\mathbf{Q}_e - \mathbf{B}_i^{\mathrm{T}}\mathbf{M}\boldsymbol{\gamma}_i}$
- Note that $\mathbf{B}_i^{\mathrm{T}}\mathbf{Q}_c = \mathbf{0}$ which means no constraint forces in the above equation
  - Left as an exercise to prove
- This is the embedded formulation

Open Chain Systems

- One of the advantages of augmented formulation is that both open and closed chain systems can be treated alike
- For other methods special attention required for closed kinematic chains
- We look at two different modelling approaches to open loop chains first
  - First method where each body dynamics is developed
  - Involves all the reactions
  - Second method involves cuts at specific joints with dynamics for selected sub-systems

Method I

- We revisit the 2-link manipulator
- The free-body diagrams of each link is done ▸ FBD2Link
- From these the Newton-Euler equations for body 2 are

$$
\begin{aligned}
F_x^{12} - F_x^{23} &= m^2 \ddot{R}_x^2 \\
F_y^{12} - m^2 g - F_y^{23} &= m^2 \ddot{R}_y^2 \\
F_x^{12} l_O^2 \sin\theta^2 - F_y^{12} l_O^2 \cos\theta^2 + M^2 + & \\
F_x^{23} l_A^2 \sin\theta^2 - F_y^{23} l_A^2 \cos\theta^2 &= J^2 \ddot{\theta}^2
\end{aligned}
$$

Method I ...

- The free-body diagram for link 3 ▸ FBD2Link
- From these the Newton-Euler equations for body 3 are

$$
\boxed{
\begin{aligned}
F_x^{23} &= m^3 \ddot{R}_x^3 \\
F_y^{23} - m^3 g &= m^3 \ddot{R}_y^3 \\
F_x^{23} l_A^3 \sin\theta^3 - F_y^{23} l_A^3 \cos\theta^3 + M^3 &= J^3 \ddot{\theta}^3
\end{aligned}
}
$$

- We look at inverse dynamics to show difference between open and closed chains
  - Kinematic quantities are known
- Inverse dynamics form ▸ FinalForm
  - Of the form $\mathbf{A}\mathbf{x} = \mathbf{b}$ with $\mathbf{x}$ the reaction forces and external moments and $\mathbf{b}$ the accelerations

Method I ...

- Invert the inverse dynamics equation to get the reaction forces and external moments

$$
\begin{aligned}
F_x^{12} &= m^2 \ddot{R}_x^2 + m^3 \ddot{R}_x^3 \\
F_y^{12} &= m^2 \ddot{R}_y^2 + m^2 g + m^3 \ddot{R}_y^3 + m^3 g \\
F_x^{23} &= m^3 \ddot{R}_x^3 \\
F_y^{23} &= m^3 \ddot{R}_y^3 + m^3 g \\
M^2 &= J^2 \ddot{\theta}^2 - A_{51} m^2 \ddot{R}_x^2 - A_{52}(m^2 \ddot{R}_y^2 + m^2 g) \\
&\quad - (A_{51} + A_{53}) m^3 \ddot{R}_x^3 - (A_{51} + A_{53})(m^3 \ddot{R}_y^3 + m^3 g) \\
M^3 &= J^3 \ddot{\theta}^3 - A_{63} m^3 \ddot{R}_x^3 - A_{64}(m^3 \ddot{R}_y^3 + m^3 g)
\end{aligned}
$$

- Last 2 equations do not involve the reaction forces

$\mathbf{A}$ elements

- The elements of the matrix $\mathbf{A}$ in the previous slide

$$
\boxed{
\begin{aligned}
A_{51} &= l_O^2 \sin\theta^2; & A_{52} &= -l_O^2 \cos\theta^2 \\
A_{53} &= l_A^2 \sin\theta^2; & A_{54} &= -l_A^2 \cos\theta^2 \\
A_{63} &= l_A^3 \sin\theta^2; & A_{64} &= -l_A^3 \cos\theta^2
\end{aligned}
}
$$

- l_O^2 and l_A^2 are the distances of joints at O and A from centre of mass of link 2
- l_A^3 is the distance of joints at A from centre of mass of link 3

Method II

- For open chains the last two equations can be done without finding reactions
- Let us take the first sub-system FBD SubsysFBD
- We take moments about point A for link 3
  - $M_e = M^3 - m^3 g l_A^3 \cos\theta^3$
    - Moments due to external forces/moments
  - $M_i = -m^3 \ddot{R}_x^3 l_A^3 \sin\theta^3 + m^3 \ddot{R}_y^3 l_A^3 \cos\theta^3 + J^3\theta^3$
    - Moments due to inertia forces/moments
- From D'Alembert's principle we have $M_e = M_i$
- $\boxed{M^3 = J^3\ddot{\theta}^3 + (m^3 g + m^3\ddot{R}_y^3) l_A^3 \cos\theta^3 - m^3 \ddot{R}_x^3 l_A^3 \sin\theta^3}$

Method II ...

- Second FBD of sub-system ▸ SubsysFBD
- Moments are now taken about joint at O
- $M_e = M^2 + M^3 - m^2 g l_O^2 \cos\theta^2 - m^3 g(l^2 + l_A^3 \cos\theta^3)$
  - Moments of the external forces/moments
- $M_i = J^2\ddot{\theta}^2 + J^3\ddot{\theta}^3 - m^2\ddot{R}_x^2 l_O^2 \sin\theta^2 + m^2\ddot{R}_y^2 l_O^2 \cos\theta^2 - m^3\ddot{R}_x^3(l^2\sin\theta^2 + l_A^3\sin\theta^3) + m^3\ddot{R}_y^3(l^2\cos\theta^2 + l_A^3\cos\theta^3)$
  - Moments of the inertia forces/moments
- One can equate the two and get an expression for $M^2 + M^3$
- Subtract earlier expression for M^3 from this to get M^2
  - Forms the foundation of the recursive methods

Closed Chain Analysis

- We now look at 4-bar mechanism dynamics 4BarMech
  - Closed loop chain example
- First the dynamics is derived using individual free-body diagrams 4BarFBDR1
- The equations for body 2 are

$$
\begin{aligned}
F_x^{12} - F_x^{23} &= m^2 \ddot{R}_x^2 \\
F_y^{12} - m^2 g - F_y^{23} &= m^2 \ddot{R}_y^2 \\
F_x^{12} l_O^2 \sin\theta^2 - F_y^{12} l_O^2 \cos\theta^2 + M^2 & \\
+F_x^{23} l_A^2 \sin\theta^2 - F_y^{23} l_A^2 \cos\theta^2 &= J^2 \ddot{\theta}^2
\end{aligned}
$$

Closed Chain: Method I

- First the dynamics is derived using individual free-body diagrams
  4BarFBDR2
- The equations for body 3 are

$$
\begin{aligned}
F_x^{23} + F_x^3 - F_x^{34} &= m^3 \ddot{R}_x^3 \\
F_y^{23} - m^3 g + F_y^3 - F_y^{34} &= m^3 \ddot{R}_y^3 \\
F_x^{23} l_A^3 \sin\theta^3 - F_y^{23} l_A^3 \cos\theta^3 + & \\
F_x^{34} l_B^3 \sin\theta^3 - F_y^{34} l_B^3 \cos\theta^3 &= J^3 \ddot{\theta}^3
\end{aligned}
$$

Closed Chain: Method I ...

- First the dynamics is derived using individual free-body diagrams
  ▸ 4BarFBDR3
- The equations for body 4 are

$$
\begin{aligned}
F_x^{34} - F_x^{41} &= m^4 \ddot{R}_x^4 \\
F_y^{41} - m^4 g - F_y^{41} &= m^4 \ddot{R}_y^4 \\
F_x^{34} l_B^4 \sin\theta^4 - F_y^{34} l_B^4 \cos\theta^4 + M^4 & \\
+F_x^{41} l_C^4 \sin\theta^4 - F_y^{41} l_C^4 \cos\theta^4 &= J^4 \ddot{\theta}^4
\end{aligned}
$$

- We have 9 equations for the inverse dynamics which can be solved for 9 unknowns
  - 8 of them are the reactions at the revolute joints
  - Only one of M^2, F_x^3, F_y^3 and M^4 can be unknown

Closed Chain: Method II

- We can now write the equations in the form $\mathbf{Ax} = \mathbf{b}$
  - $\mathbf{x}$ contains the 8 unknown reaction forces and M^4
- Next get reduced set of equations from sub-system approach
- One equation from moments about point A ‣ 4BarBody1
  - $F_x^{12} l^2 \sin\theta^2 - F_y^{12} l^2 \cos\theta^2 + M^2 + m^2 g l_A^2 \cos\theta^2 = J^2\ddot{\theta}^2 + m^2 \ddot{R}_x^2 l_A^2 \sin\theta^2 - m^2 \ddot{R}_y^2 l_A^2 \cos\theta^2$
- Second equation from moments about point B ‣ 4Bar2Bodies
  - $F_x^{12}(l^2 \sin\theta^2 + l^3 \sin\theta^3) - F_y^{12}(l^2 \cos\theta^2 + l^3 \cos\theta^3) + M^2 + m^2 g(l_A^2 \cos\theta^2 + l^3 \cos\theta^3) + (m^3 g - F_y^3) l_B^3 \cos\theta^3 + F_x^3 l_B^3 \sin\theta^3 = J^2\ddot{\theta}^2 + m^2 \ddot{R}_x^2 (l_A^2 \sin\theta^2 + l^3 \sin\theta^3) - m^2 \ddot{R}_y^2 (l_A^2 \cos\theta^2 + l^3 \sin\theta^3) + J^3\ddot{\theta}^3 + m^3 \ddot{R}_x^3 l_B^3 \sin\theta^3 - m^3 \ddot{R}_y^3 l_B^3 \cos\theta^3$

Reducing the Equation for Two Bodies

- The second equation can be reduced using first equation to a form
  - $F_x^{12} l^3 \sin\theta^3 - F_y^{12} l^3 \cos\theta^3 + m^2 g l^3 \cos\theta^3 + (m^3 g - F_y^3) l_B^3 \cos\theta^3 + F_x^3 l_B^3 \sin\theta^3 = m^2 \ddot{R}_x^2 l^3 \sin\theta^3 - m^2 \ddot{R}_y^2 l^3 \sin\theta^3 + J^3 \ddot{\theta}^3 + m^3 \ddot{R}_x^3 l_B^3 \sin\theta^3 - m^3 \ddot{R}_y^3 l_B^3 \cos\theta^3$
- We can derive a moment equation using 3 bodies 4Bar3Bodies
  - The moment equation is first written about point C

Moment from 3 Bodies FBD

$$
\begin{aligned}
-F_y^{12}l^1 + m^2g(l_A^2\cos\theta^2 + l^3\cos\theta^3 + l^4\cos\theta^4) + \\
F_x^3(l_B^3\sin\theta^3 + l^4\sin\theta^4) + (m^3g - F_y^3)(l_B^3\cos\theta^3 + l^4\cos\theta^4) + \\
M^4 + m^4gl_C^4\cos\theta^4 = \\
J^2\ddot{\theta}^2 + m^2\ddot{R}_x^2(l_A^2\sin\theta^2 + l^3\sin\theta^3 + l^4\sin\theta^4) \\
-m^2\ddot{R}_y^2(l_A^2\cos\theta^2 + l^3\cos\theta^3 + l^4\cos\theta^4) + J^3\ddot{\theta}^3 + \\
m^3\ddot{R}_x^3(l_B^3\sin\theta^3 + l^4\sin\theta^4) - m^3\ddot{R}_y^3(l_B^3\cos\theta^3 + l^4\cos\theta^4) + \\
J^4\ddot{\theta}^4 + m^4\ddot{R}_x^4l_C^4\sin\theta^4 - m^4\ddot{R}_y^4l_C^4\cos\theta^4
\end{aligned}
$$

- $l^1 = l^2\cos\theta^2 + l^3\cos\theta^3 + l^4\cos\theta^4$
- Use moment equations about A & B to reduce this equation

Moment from 3 Bodies FBD

$$
\begin{aligned}
F_x^{12} l^4 \sin\theta^4 + (m^2 g - F_y^{12}) l^4 \cos\theta^4 + F_x^3 l^4 \sin\theta^4 + \\
(m^3 g - F_y^3) l^4 \cos\theta^4 + M^4 + m^4 g l_C^4 \cos\theta^4 = \\
+(m^2 \ddot{R}_x^2 + m^3 \ddot{R}_x^3) l^4 \sin\theta^4 - (m^2 \ddot{R}_y^2 + m^3 \ddot{R}_y^3) l^4 \cos\theta^4 + \\
J^4 \ddot{\theta}^4 + m^4 \ddot{R}_x^4 l_C^4 \sin\theta^4 - m^4 \ddot{R}_y^4 l_C^4 \cos\theta^4
\end{aligned}
$$

- Clearly these 3 equations can be solved to get F_x^{12}, F_y^{12} and M^4
- Another set of reduced equations can provide F_x^{23}, F_y^{23} and M^4 or F_x^{34}, F_y^{34} and M^4

Closed Chain Dynamics

- In closed chains reduced equations always will contain reactions
  - Unlike the open chain equations
- These reaction forces can be eliminated by further manipulation
- So one way is to systematically make the closed chain an open chain by cuts at secondary joints
  - Develop the dynamic equations using the open chain
  - At joints write connectivity conditions
  - Leads to a system of differential-algebraic equations

Introduction to Virtual Work

- Unlike Newton-Euler method virtual work principle does not consider the constraint forces
- Requires only a scalar quantity to define static or dynamic equations
- Principle of virtual work leads to number of equations equal to number of independent co-ordinates
  - Systematic procedure for embedding formulation
- One can also derive Lagrange's equations from principle of virtual work

Virtual Displacements

- Generalized Co-ordinates is a term used to describe any set of coordinates describing system configuration
- A virtual displacement is an infinitesimal displacement consistent with kinematic constraints imposed on motion of system
- They are imaginary as time is held fixed when they occur
- Let us go back to the unconstrained body PosVec
  - $\mathbf{r}_P^i = \mathbf{R}^i + \mathbf{u}_P^i = \mathbf{R}^i + \mathbf{A}^i\bar{\mathbf{u}}_P^i$
- A virtual change in the position vector of P^i
  - $\delta\mathbf{r}_P^i = \delta\mathbf{R}^i + \delta(\mathbf{A}^i\bar{\mathbf{u}}_P^i) = \delta\mathbf{R}^i + \mathbf{A}_\theta^i\bar{\mathbf{u}}_P^i\delta\theta^i$

Kinematic Constraints

- We can write the virtual displacement as
  - $\delta \mathbf{r}_P^i = \begin{bmatrix} \mathbf{I} & \mathbf{A}_\theta^i \bar{\mathbf{u}}_P^i \end{bmatrix} \delta \mathbf{q}^i;\ \delta \mathbf{q}^i = \begin{bmatrix} \delta \mathbf{R}^i & \delta \theta^i \end{bmatrix}^{\mathrm{T}}$
- If system not kinematically driven then
  - Number of constraint equations n_c will be less than degrees-of-freedom n
  - There will be n_c dependent co-ordinates $\mathbf{q}_d$ and $n - n_c$ independent co-ordinates $\mathbf{q}_i$
- Slider-crank mechanism constraints SliderCrank
  - $l^2 \cos \theta^2 + l^3 \cos \theta^3 = R_x^4;\ l^2 \sin \theta^2 + l^3 \sin \theta^3 = 0$
- Virtual displacements will lead to form
  - $-l^2 \sin \theta^2 \delta \theta^2 - l^3 \sin \theta^3 \delta \theta^3 - \delta R_x^4 = 0$
  - $l^2 \cos \theta^2 \delta \theta^2 + l^3 \cos \theta^3 \delta \theta^3 = 0$

Dependent and Independent Co-ordinates

- The slider-crank mechanism has one independent co-ordinate
  - We select say θ^2
- Then $\mathbf{q}_d = \begin{bmatrix} \theta^3 & R_x^4 \end{bmatrix}^{\mathrm{T}}$
- The virtual displacement equations can be re-written as

$$\begin{bmatrix} -l^3 \sin\theta^3 & -1 \\ l^3 \cos\theta^3 & 0 \end{bmatrix} \begin{Bmatrix} \delta\theta^3 \\ \delta R_x^4 \end{Bmatrix} = \delta\theta^2 \begin{Bmatrix} l^2 \sin\theta^2 \\ -l^2 \cos\theta^2 \end{Bmatrix}$$

- This can be written in the form

$$\begin{Bmatrix} \delta\theta^3 \\ \delta R_x^4 \end{Bmatrix} = \frac{1}{l^3 \cos\theta^3} \begin{Bmatrix} -l^2 \cos\theta^2 \\ l^2 l^3 \sin(\theta^3 - \theta^2) \end{Bmatrix} \delta\theta^2$$

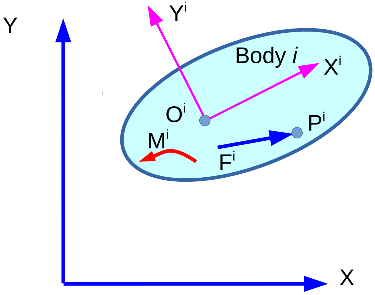
- If $\theta^3 = \frac{\pi}{2}$ or $\frac{3\pi}{2}$ one cannot find $\mathbf{q}_d$

General Form

- The general form of the constraint equations
  - $\mathbf{C}(\mathbf{q}, t) = \mathbf{0}$
- If we apply virtual displacements then we have
  - $\mathbf{C_q}\delta\mathbf{q} = \mathbf{0}$; $\mathbf{C_q}$ is Jacobian matrix
- Jacobian matrix can be partitioned as
  - $\begin{bmatrix}\mathbf{C}_{\mathbf{q}_d} & \mathbf{C}_{\mathbf{q}_i}\end{bmatrix}\begin{Bmatrix}\delta\mathbf{q}_d \\ \delta\mathbf{q}_i\end{Bmatrix} = \mathbf{0}$; $\delta\mathbf{q}_d = -\mathbf{C}_{\mathbf{q}_d}^{-1}\mathbf{C}_{\mathbf{q}_i}\delta\mathbf{q}_i$
- We can then write $\delta\mathbf{q} = \mathbf{B}_i\delta\mathbf{q}_i$ with $\mathbf{B}_i = \begin{bmatrix}-\mathbf{C}_{\mathbf{q}_d}^{-1}\mathbf{C}_{\mathbf{q}_i} & \mathbf{I}\end{bmatrix}^{\mathrm{T}}$

Virtual Work

- The rigid body i acted on by moment M^i and force $\mathbf{F}^i$ at point P^i
- Virtual work for this system of forces
  - $\delta W^i = \mathbf{F}^{i^{\mathrm{T}}} \delta \mathbf{r}^i_P + M^i \delta\theta^i$
- This can be rewritten as
  - $\delta W^i \;=\; \mathbf{F}^{i^{\mathrm{T}}} \delta \mathbf{R}^i + (\mathbf{F}^{i^{\mathrm{T}}} \mathbf{A}^i_\theta \bar{\mathbf{u}}^i_p + M^i)\delta\theta^i$
  - $\delta W^i = \mathbf{Q}^{i\,T}_R \delta \mathbf{R}^i + Q^i_\theta \delta\theta^i$
- $\mathbf{Q}^i_R$ and Q^i_θ are generalized forces



Force Elements

- We now look at the generalized forces commonly encountered in multi-body dynamics
- Gravity
  - The virtual work is $\delta W^i = -m^i g \delta y^i$
  - If the reference point same as centre of mass $\delta y^i = \delta R_y^i$
  - If not $y^i = R_y^i + \bar{u}_x^i \sin\theta^i + \bar{u}_y^i \cos\theta^i$
    - Then $\delta y^i = \delta R_y^i + (\bar{u}_x^i \cos\theta^i - \bar{u}_y^i \sin\theta^i)\delta\theta^i$
  - $\delta W^i = Q_y^i \delta R_y^i + Q_\theta^i \delta\theta^i = -m^i g \delta R_y^i - m_g^i(\bar{u}_x^i \cos\theta^i - \bar{u}_y^i \sin\theta^i)\delta\theta^i$

Spring-Damper-Actuator Element

- Two bodies connected by a spring-damper-actuator ▸ SpDaAc
- The force $f_s = k(l - l_0) + c\dot{l} + f_a$
  - Spring assumed linear with stiffness k
  - Damper force proportional to relative velocity between points P^i and P^j
  - l_0 is undeformed length of spring
- The virtual work can be written as $\delta W = -f_s \delta l$
  - δl is the virtual change in the distance between P^i and P^j
  - f_a could be a non-linear function of time/ body co-ordinates and velocities

Spring-Damper ...

- The distance $\mathbf{r}_P^{ij}$ between P^i and P^j is
  - $\mathbf{r}_P^{ij} = \mathbf{R}^i + \mathbf{A}^i\bar{\mathbf{u}}_P^i - \mathbf{R}^j - \mathbf{A}^j\bar{\mathbf{u}}_P^j;\ l = \sqrt{{\mathbf{r}_P^{ij}}^T \mathbf{r}_P^{ij}}$
- The virtual change $\delta l = \frac{\partial l}{\partial \mathbf{q}}\delta\mathbf{q}$
  - $\mathbf{q} = \begin{bmatrix}\mathbf{R}^i & \theta^i & \mathbf{R}^j & \theta^j\end{bmatrix}^{\mathrm{T}}$
- We have the following form for δl
  - $\delta l = \frac{1}{\sqrt{{\mathbf{r}_P^{ij}}^T \mathbf{r}_P^{ij}}}{\mathbf{r}_P^{ij}}^T \frac{\partial \mathbf{r}_P^{ij}}{\partial \mathbf{q}} = \mathbf{I}_P^T \begin{bmatrix}\frac{\partial \mathbf{r}_P^{ij}}{\partial \mathbf{q}^i} & \frac{\partial \mathbf{r}_P^{ij}}{\partial \mathbf{q}^j}\end{bmatrix} \begin{Bmatrix}\delta\mathbf{q}^i \\ \delta\mathbf{q}^j\end{Bmatrix}$
  - $\mathbf{I}_P$ is the unit vector along P^iP^j
  - $\frac{\partial \mathbf{r}_P^{ij}}{\partial \mathbf{q}^i} = \begin{bmatrix}\mathbf{I} & \mathbf{A}_\theta^i\bar{\mathbf{u}}_P^i\end{bmatrix};\ \frac{\partial \mathbf{r}_P^{ij}}{\partial \mathbf{q}^j} = \begin{bmatrix}-\mathbf{I} & -\mathbf{A}_\theta^j\bar{\mathbf{u}}_P^j\end{bmatrix}$

Rotational Spring-Damper

- The velocity $\dot{l}$ is given by
  - $\dot{l} = \frac{\partial l}{\partial \mathbf{q}}\dot{\mathbf{q}} = \mathbf{I}_P^T \frac{\partial \mathbf{r}_P^{ij}}{\partial \mathbf{q}}\dot{\mathbf{q}}$
- Rotational Spring-Damper
  - $M_s = k_r(\theta - \theta_0) + c_r\dot{\theta}$; $\theta = \theta^i - \theta^j$
- Virtual work done by this combination is
  - $\delta W = -\{k_r(\theta - \theta_0) + c_r\dot{\theta}\}(\delta\theta^i - \delta\theta^j)$

Workless Constraints

- Internal reaction forces between particles of a rigid body do no work
- Distance between two particles i and j remains constant
  - $(\mathbf{r}^i - \mathbf{r}^j)^{\mathrm{T}}(\mathbf{r}^i - \mathbf{r}^j) = c_1$
- For a virtual change in the position of these particles
  - $(\mathbf{r}^i - \mathbf{r}^j)^{\mathrm{T}}(\delta\mathbf{r}^i - \delta\mathbf{r}^j) = 0$
- If $\mathbf{F}_c^{ij}$ is constraint force acting on i then force on j is $\mathbf{F}_c^{ji} = -\mathbf{F}_c^{ij}$
- These forces act along the line joining i and j
  - $\mathbf{F}_c^{ij} = c_2(\mathbf{r}^i - \mathbf{r}^j)$

Virtual Work of Constraint Forces

- The virtual work due to the constraint forces
  - $\delta W = \mathbf{F}_c^{ij\,\mathrm{T}} \delta \mathbf{r}^i + \mathbf{F}_c^{ji\,\mathrm{T}} \delta \mathbf{r}^j = \mathbf{F}_c^{ij\,\mathrm{T}} (\delta \mathbf{r}^i - \delta \mathbf{r}^j)$
- Since the force acts along ij line we have
  - $\delta W = c_2 (\mathbf{r}^i - \mathbf{r}^j)^{\mathrm{T}} (\delta \mathbf{r}^i - \delta \mathbf{r}^j) = 0$
- Similar comments apply to other friction free conenctions
- For the revolute joint between body i and body j RevJoint
  - $\delta W^i = \mathbf{F}_P^{ij\,\mathrm{T}} \delta \mathbf{r}_P$; $\delta W^j = -\mathbf{F}_P^{ij\,\mathrm{T}} \delta \mathbf{r}_P$
  - Net virtual work $\delta W = 0$
- Used to eliminate constraint forces in dynamic equations
  - Leads to number of equations equal to number of independent co-ordinates

Statics: Equipollent System

- Let a force $\mathbf{F}^i$ act at a point P^i in a rigid body i
- The virtual work done is $\delta W = \mathbf{F}^{i^{\mathrm{T}}} \delta \mathbf{r}_P^i$
- With respect to the reference point
  - $\delta \mathbf{r}_P^i = \delta \mathbf{R}^i + \mathbf{A}_\theta^i \bar{\mathbf{u}}_P^i \delta\theta^i$
- Then $\delta W^i = \mathbf{F}^{i^{\mathrm{T}}} \delta \mathbf{R}^i + \mathbf{F}^{i^{\mathrm{T}}} \mathbf{A}_\theta^i \bar{\mathbf{u}}_P^i \delta\theta^i$
- Force $\mathbf{F}^i$ at P^i equipollent to force $\mathbf{F}^i$ at reference point and moment $M^i \hat{\mathbf{k}} = \mathbf{u}_P \times \mathbf{F}^i = \bar{\mathbf{u}}_P^{i^{\mathrm{T}}} \mathbf{A}_\theta^{i^{\mathrm{T}}} \mathbf{F}^i \hat{\mathbf{k}}$
- The virtual work is
  $\delta W_r^i = \mathbf{F}^i \delta \mathbf{R}^i + M^i \delta\theta^i = \mathbf{F}^i \delta \mathbf{R}^i + \bar{\mathbf{u}}_P^{i^{\mathrm{T}}} \mathbf{A}_\theta^{i^{\mathrm{T}}} \mathbf{F}^i \delta\theta^i$
- Comparison shows $\boxed{\delta W^i = \delta W_r^i}$

Virtual Work

- Consider a body i acted by a number of forces and moments as shown ▸ VWStatics
- Virtual work done
  - $\delta W^i = \mathbf{F}_1^{i^T}\delta\mathbf{r}_1^i + \mathbf{F}_2^{i^T}\delta\mathbf{r}_2^i + \ldots + \mathbf{F}_{n_f}^{i^T}\delta\mathbf{r}_{n_f}^i + (M_1^i + M_2^i + \ldots + M_{n_m}^i)\delta\theta^i$
- Compact notation $\delta W^i = \sum_{j=1}^{n_f}\mathbf{F}_j^{i^T}\delta\mathbf{r}_j^i + \Big(\sum_{k=1}^{n_m} M_k^i\Big)\delta\theta^i$
- The virtual work of the equipollent system ▸ VWStatics
  - $\delta W_r^i = \mathbf{F}_e^{i^{\mathrm{T}}}\delta\mathbf{r}_e^i + M_e^i\delta\theta^i$
  - $\mathbf{r}_e^i$ is the position vector of point of application of $\mathbf{F}_e^i$
- From our previous discussion $\boxed{\delta W^i = \delta W_r^i}$

Virtual Work: Statics

- $$\sum_{j=1}^{n_f} \mathbf{F}_j^{i^T} \delta \mathbf{r}_j^i + \Big(\sum_{k=1}^{n_m} M_k^i\Big) \delta\theta^i = \mathbf{F}_e^{i^{\mathrm{T}}} \delta \mathbf{r}_e^i + M_e^i \delta\theta^i$$
- If body i is in static equilibrium
  - $\mathbf{F}_e = \mathbf{0}$; $M_e = 0$
  - $$\sum_{j=1}^{n_f} \mathbf{F}_j^{i^T} \delta \mathbf{r}_j^i + (\sum_{k=1}^{n_m} M_k^i) \delta\theta^i = 0$$
- The above equation is the principle of virtual work in statics
  - For a body in static equilibrium the virtual work of all the forces and moments that act on this body must be equal to zero

System of Connected Bodies

- The virtual work can be split into
  - $\delta W^i = \delta W_e^i + \delta W_c^i$
  - δW_e^i is the virtual work of the external forces and moments
  - δW_c^i is the virtual work of the constraint forces and moments
- If there are n_b bodies one can sum the virtual work
  - $\sum_{i=1}^{n_b} \delta W^i = \sum_{i=1}^{n_b} \delta W_e^i + \sum_{i=1}^{n_b} \delta W_c^i$
- Joint constraint forces equal in magnitude but opposite in direction on the connected bodies
  - $\sum_{i=1}^{n_b} \delta W_c^i = 0$
- For multi-body system the principle of virtual work becomes
  - $\boxed{\delta W_e = \sum_{i=1}^{n_b} \delta W_e^i = 0}$

Generalized Forces

- Let n_f forces act on the multi-body system as well as n_m moments
- Then one can write
  - $\delta W_e = \sum_{j=1}^{n_f} \mathbf{F}_j \delta \mathbf{r}_j + \sum_{j=1}^{n_m} M_j \delta \theta_j$
- Now one can express $\mathbf{r}_j$ and θ_j in terms of the independent co-ordinates $\mathbf{q}_i$
  - $\delta \mathbf{r}_j = \frac{\partial \mathbf{r}_j}{\partial \mathbf{q}_i} \delta \mathbf{q}_i$; $\delta \theta_j = \frac{\partial \theta_j}{\partial \mathbf{q}_i} \delta \mathbf{q}_i$
- $$\delta W_e = \left(\sum_{j=1}^{n_f} \mathbf{F}_j \frac{\partial \mathbf{r}_j}{\partial \mathbf{q}_i} + \sum_{j=1}^{n_m} M_j \frac{\partial \theta_j}{\partial \mathbf{q}_i} \right) \delta \mathbf{q}_i = \mathbf{Q}_e^T \delta \mathbf{q}_i$$
- For a system in static equilibrium $\mathbf{Q}_e = \mathbf{0}$

Example

- Let us consider the slider-crank mechanism in static equilibrium
  ▸ SlidCrank
- Crank external moment M^2 and slider external force F^4
- We now look at each individual link starting with crank
  - $\delta W^2 = \mathbf{F}^{12^{\mathrm{T}}} \delta \mathbf{r}_O^2 - \mathbf{F}^{23^{\mathrm{T}}} \delta \mathbf{r}_A^2 - m^2 g \delta R_y^2 + M^2 \delta \theta^2 = 0$
- For the connecting rod
  - $\delta W^3 = \mathbf{F}^{23^{\mathrm{T}}} \delta \mathbf{r}_A^3 - \mathbf{F}^{34^{\mathrm{T}}} \delta \mathbf{r}_B^3 - m^3 g \delta R_y^3 = 0$
- For the piston we have
  - $\delta W^4 = \mathbf{F}^{34^{\mathrm{T}}} \delta \mathbf{r}_B^4 + (F^{41} - m^4 g) \delta R_y^4 + F^4 \delta R_x^4 = 0$

Example Slide 2

- $\delta \mathbf{r}_O^2 = \mathbf{0}$ as pivot is fixed and $\delta \mathbf{r}_A^2 = \delta \mathbf{r}_A^3$ as well as $\delta \mathbf{r}_B^3 = \delta \mathbf{r}_B^4$
- Also $\delta R_y^4 = 0$ as slider constrained to move horizontally
- Adding the three virtual work equations
  - $\boxed{-m^2 g \delta R_y^2 - m^3 g \delta R_y^3 + M^2 \delta\theta^2 + F^4 \delta R_x^4 = 0}$
- We have the following relations
  - $\delta R_y^2 = l_O^2 \cos\theta^2 \delta\theta^2$; $\delta R_y^3 = l^2 \cos\theta^2 \delta\theta^2 + l_A^3 \cos\theta^3 \delta\theta^3$
  - $\delta R_x^4 = -l^2 \sin\theta^2 \delta\theta^2 - l^3 \sin\theta^3 \delta\theta^3$
- From the fact that $\sin\theta^3 = -\frac{l_2}{l_3} \sin\theta^2$ we have
  - $\delta\theta^3 = -\frac{l^2 \cos\theta^2}{l^3 \cos\theta^3} \delta\theta^2$

Example Continued

- The virtual displacements become
  - $\delta R^2_y = l^2_O \cos\theta^2 \delta\theta^2$
  - $\delta R^3_y = l^2 \cos\theta^2 (1 - \frac{l^3_A}{l^3}) \delta\theta^2$
  - $\delta R^4_y = l^2 (-\sin\theta^2 + \cos\theta^2 \tan\theta^3) \delta\theta^2$
- From this the generalized force Q_e is
  - $$Q_e = -m^2 g l^2_O \cos\theta^2 - m^3 g l^2 \cos\theta^2 (1 - \frac{l^3_A}{l^3}) + M^2 + F^4 l^2 (-\sin\theta^2 + \cos\theta^2 \tan\theta^3) = 0$$

Virtual Work: Dynamics

- For a rigid body i in motion we have
  - $\mathbf{F}^i - m^i\mathbf{a}^i = \mathbf{0}$; $M^i - J^i\ddot{\theta}^i = 0$
  - $\mathbf{F}^i$ is the resultant of all forces and M^i sum of all moments about centre of mass
- We multiply the first equation by $\delta\mathbf{R}^i$ and second by $\delta\theta^i$
  - $(\mathbf{F}^i - m^i\mathbf{a}^i)^T\delta\mathbf{R}^i = 0$; $(M^i - J^i\ddot{\theta}^i)\delta\theta^i = 0$
- Adding the two we have
  - $(\mathbf{F}^i - m^i\mathbf{a}^i)^T\delta\mathbf{R}^i + (M^i - J^i\ddot{\theta}^i)\delta\theta^i = 0$
  - $\mathbf{F}^{i^T}\delta\mathbf{R}^i + M^i\delta\theta^i - m^i\mathbf{a}^{i^T}\delta\mathbf{R}^i - J^i\ddot{\theta}^i\delta\theta^i = \delta W^i - \delta W_i^i = 0$
- The principal of virtual work in dynamics for a body i
  - $\boxed{\delta W_e^i + \delta W_c^i - \delta W_i^i = 0}$

Multi-body System

- For n_b connected bodies the virtual work can be written as
  - $\sum_{i=1}^{n_b}(\delta W_e^i + \delta W_c^i - \delta W_i^i) = 0$
- As before we have $\sum_{i=1}^{n_b} \delta W_c^i = 0$
  - Joint constraint forces acting on two adjacent bodies are equal in magnitude and opposite in direction
- We then have the principle of virtual work in dynamics
  - $\boxed{\sum_{i=1}^{n_b}(\delta W_e^i - \delta W_i^i) = \delta W_e - \delta W_i = 0}$
- In terms of the independent co-ordinates $\mathbf{q}_i$ we have
  - $\delta W_e = \mathbf{Q}_e^{\mathrm{T}} \delta \mathbf{q}_i$; $\delta W_i = \mathbf{Q}_i^{\mathrm{T}} \delta \mathbf{q}_i$
- Hence we have $(\mathbf{Q}_e^{\mathrm{T}} - \mathbf{Q}_i^{\mathrm{T}})\delta \mathbf{q}_i = 0$
  - This implies $\mathbf{Q}_e = \mathbf{Q}_i$ as $\mathbf{q}_i$ are independent

Example Revisited

- We re-visit slider-crank mechanism with inertia forces SCDyn
- Crank external moment M^2 and slider external force F^4
- Recall the generalized external force Q_e
  - $Q_e = -m^2 g l_O^2 \cos\theta^2 - m^3 g l^2 \cos\theta^2 (1 - \frac{l_A^3}{l^3}) + M^2 + F^4 l^2 (-\sin\theta^2 + \cos\theta^2 \tan\theta^3)$
- The virtual work due to inertia forces
  - $\delta W_i = m^2 \ddot{R}_x^2 \delta R_x^2 + m^2 \ddot{R}_y^2 \delta R_y^2 + J^2 \ddot{\theta}^2 \delta\theta^2 + m^3 \ddot{R}_x^3 \delta R_x^3 + m^3 \ddot{R}_y^3 \delta R_y^3 + J^3 \ddot{\theta}^3 \delta\theta^3 + m^4 \ddot{R}_x^4 \delta R_x^4$

Example Continued

- $\delta R_x^2 = -l_O^2 \sin\theta^2 \delta\theta^2$; $\delta R_y^2 = l_O^2 \cos\theta^2 \delta\theta^2$
- $\delta R_x^3 = -l^2\Big(\sin\theta^2 - \frac{l_A^3}{l^3}\cos\theta^2 \tan\theta^3\Big)\delta\theta^2$
- $\delta R_y^3 = l^2 \cos\theta^2\Big(1 - \frac{l_A^3}{l^3}\Big)\delta\theta^2$; $\delta\theta^3 = -\frac{l^2}{l^3}\frac{\cos\theta^2}{\cos\theta^3}\delta\theta^2$
- $\delta R_y^4 = l^2\Big(-\sin\theta^2 + \cos\theta^2 \tan\theta^3\Big)\delta\theta^2$
- Final expression for the generalized inertia force Q_i GenInerForc

Lagrange's Equations

- The virtual work done by inertia forces in body i
  - $\delta W_i^i = \int_{V^i} \rho \ddot{\mathbf{r}}^{i^{\mathrm{T}}} \delta \mathbf{r}^i \, dV^i$
- Now we can write $\delta \mathbf{r}^i = \frac{\partial \mathbf{r}^i}{\partial \mathbf{q}} \delta \mathbf{q}$ and from $\delta W_i^i = \mathbf{Q}_i^i \delta \mathbf{q}$
  - $\mathbf{Q}_i^i = \int_{V^i} \rho \left(\frac{\partial \mathbf{r}^i}{\partial \mathbf{q}} \right)^{\mathrm{T}} \ddot{\mathbf{r}}^i \, dV^i$
- Now the velocity of a particle in body i
  - $\dot{\mathbf{r}}^i = \frac{\partial \mathbf{r}^i}{\partial \mathbf{q}} \dot{\mathbf{q}} + \frac{\partial \mathbf{r}^i}{\partial t}$ which implies $\frac{\partial \dot{\mathbf{r}}^i}{\partial \dot{\mathbf{q}}} = \frac{\partial \mathbf{r}^i}{\partial \mathbf{q}}$
- Using a similar argument we can show $\frac{\partial \ddot{\mathbf{r}}^i}{\partial \ddot{\mathbf{q}}} = \frac{\partial \dot{\mathbf{r}}^i}{\partial \dot{\mathbf{q}}} = \frac{\partial \mathbf{r}^i}{\partial \mathbf{q}}$

Lagrange's Equations 2

- From the relations one can re-write
  - $\mathbf{Q}_i^i = \int_{V^i} \rho \left(\frac{\partial \dot{\mathbf{r}}^i}{\partial \dot{\mathbf{q}}}\right)^{\mathrm{T}} \ddot{\mathbf{r}}^i \, dV^i$
- Now we have
  - $\frac{d}{dt}\left\{\left(\frac{\partial \dot{\mathbf{r}}^i}{\partial \dot{\mathbf{q}}}\right)^{\mathrm{T}} \dot{\mathbf{r}}^i\right\} = \frac{d}{dt}\left\{\left(\frac{\partial \dot{\mathbf{r}}^i}{\partial \dot{\mathbf{q}}}\right)^{\mathrm{T}}\right\} \dot{\mathbf{r}}^i + \left(\frac{\partial \dot{\mathbf{r}}^i}{\partial \dot{\mathbf{q}}}\right)^{\mathrm{T}} \ddot{\mathbf{r}}^i$
  - Or $\left(\frac{\partial \dot{\mathbf{r}}^i}{\partial \dot{\mathbf{q}}}\right)^{\mathrm{T}} \ddot{\mathbf{r}}^i = \frac{d}{dt}\left\{\left(\frac{\partial \dot{\mathbf{r}}^i}{\partial \dot{\mathbf{q}}}\right)^{\mathrm{T}} \dot{\mathbf{r}}^i\right\} - \frac{d}{dt}\left\{\left(\frac{\partial \dot{\mathbf{r}}^i}{\partial \dot{\mathbf{q}}}\right)^{\mathrm{T}}\right\} \dot{\mathbf{r}}^i$
- This can be brought into a form
  - $\left(\frac{\partial \dot{\mathbf{r}}^i}{\partial \dot{\mathbf{q}}}\right)^{\mathrm{T}} \ddot{\mathbf{r}}^i = \frac{d}{dt}\left\{\frac{\partial}{\partial \dot{\mathbf{q}}}\left(\frac{1}{2} \dot{\mathbf{r}}^{i^{\mathrm{T}}} \dot{\mathbf{r}}^i\right)\right\} - \frac{\partial}{\partial \mathbf{q}}\left(\frac{1}{2} \dot{\mathbf{r}}^{i^{\mathrm{T}}} \dot{\mathbf{r}}^i\right)$
- The kinetic energy $T^i = \frac{1}{2} \int_{V^i} \rho \, \dot{\mathbf{r}}^{i^{\mathrm{T}}} \dot{\mathbf{r}}^i \, dV^i$

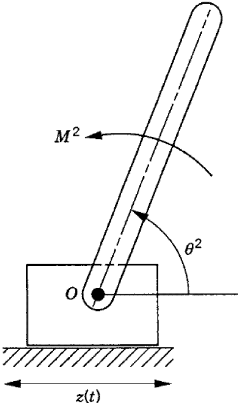
Lagrange's Equations Finally

- So we finally have the generalized inertia force as
  - $$\boxed{\mathbf{Q}_i^i = \frac{d}{dt}\left(\frac{\partial T^i}{\partial \dot{\mathbf{q}}}\right)^{\mathrm{T}} - \left(\frac{\partial T^i}{\partial \mathbf{q}}\right)^{\mathrm{T}}}$$
- For a system with n_b bodies we have
  - $$\boxed{\mathbf{Q}_i = \sum_{i=1}^{n_b}\left\{\frac{d}{dt}\left(\frac{\partial T^i}{\partial \dot{\mathbf{q}}}\right)^{\mathrm{T}} - \left(\frac{\partial T^i}{\partial \mathbf{q}}\right)^{\mathrm{T}}\right\} = \frac{d}{dt}\left(\frac{\partial T}{\partial \dot{\mathbf{q}}}\right)^{\mathrm{T}} - \left(\frac{\partial T}{\partial \mathbf{q}}\right)^{\mathrm{T}}}$$
- From the principle of virtual work in dynamics we have
  - $$\boxed{\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_j}\right) - \frac{\partial T}{\partial q_j} = Q_{ej};\ j = 1, 2, \ldots, n}$$
  - q_j are the independent co-ordinates

Example for Lagrange's Equation

- Rod has mass m^2 and J^2 is mass moment of inertia
- Length l with bottom block having specified $z(t)$
- Independent co-ordinate is θ^2
- Kinetic energy

$$T = \frac{1}{2}m\left\{(\dot{R}_x^2)^2 + (\dot{R}_y^2)^2\right\} + \frac{1}{2}J^2(\dot{\theta}^2)^2$$



Courtesy: A. A. Shabana, 2010, *Computational Dynamics*, Third Edition, John Wiley & Sons.

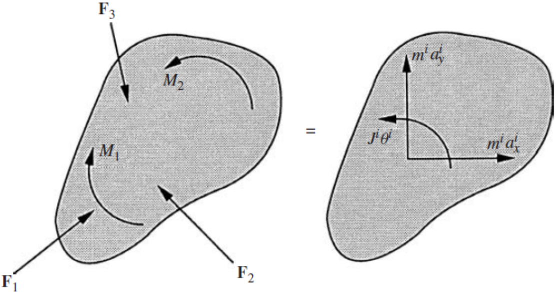
Example 2

- We have $R^2_x = z(t) + \frac{l}{2}\cos\theta^2$ and $R^2_y = \frac{l}{2}\sin\theta^2$
- $\dot{R}^2_x = \dot{z}(t) - \frac{l}{2}\sin\theta^2\dot{\theta}^2$; $\dot{R}^2_y = \frac{l}{2}\cos\theta^2\dot{\theta}^2$
- The kinetic energy then is
  - $T = \frac{1}{2}m^2\{\dot{z}^2 - l\sin\theta^2\dot{z}\dot{\theta}^2 + \frac{l^2}{4}(\dot{\theta}^2)^2\} + \frac{1}{2}J^2(\dot{\theta}^2)^2$
- The virtual work due to external forces
  - $\delta W_e = -m^2 g\delta R^2_y + M^2\delta\theta^2 = (-\frac{1}{2}m^2 gl\cos\theta^2 + M^2)\,\delta\theta^2$
- Lagrange's equation of motion
  - $\frac{d}{dt}(\frac{\partial T}{\partial\dot{\theta}^2}) - \frac{\partial T}{\partial\theta^2} = Q_e$

Final Equation

- $\frac{\partial T}{\partial \dot{\theta}^2} = (J^2 + \frac{1}{4}m^2 l^2)\dot{\theta}^2 - \frac{1}{2}m^2 l \sin\theta^2 \dot{z}$
- $\frac{\partial T}{\partial \theta^2} = -\frac{1}{2}m^2 l \cos\theta^2 \dot{z}\dot{\theta}^2$
- $\frac{d}{dt}(\frac{\partial T}{\partial \dot{\theta}^2}) = (J^2 + \frac{1}{4}m^2 l^2)\ddot{\theta}^2 - \frac{1}{2}m^2 l \sin\theta^2 \ddot{z} - \frac{1}{2}m^2 l \cos\theta^2 \dot{z}\dot{\theta}^2$
- Hence the final form of the equation is
  - $\boxed{J_O\ddot{\theta}^2 + \frac{1}{2}m^2 g l \cos\theta^2 = M^2 + \frac{1}{2}m^2 l \sin\theta^2 \ddot{z}}$
  - $J_O = J^2 + \frac{1}{4}m^2 l^2$

Free-Body Diagram



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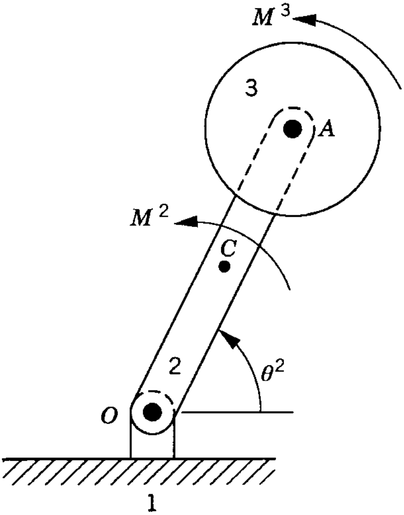
Rod Attached to Disc



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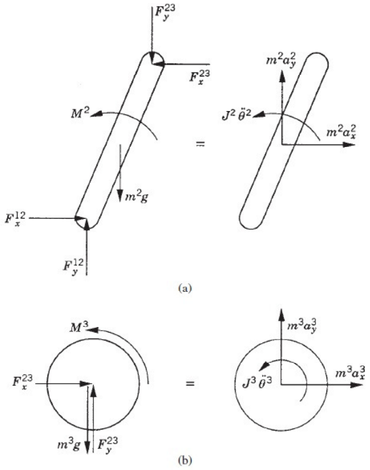
Free-Body Diagram



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Constraint Equations for Rod-Disc Problem

$$
\begin{aligned}
R_x^2 - \frac{l}{2}\cos\theta^2 &= 0 \\
R_y^2 - \frac{l}{2}\sin\theta^2 &= 0 \\
R_x^2 + \frac{l}{2}\cos\theta^2 - R_x^3 &= 0 \\
R_y^2 + \frac{l}{2}\sin\theta^2 - R_y^3 &= 0
\end{aligned}
$$

- We have eliminated the degrees-of-freedom of fixed body 1
- Assumed $R_x^1 = R_y^1 = \theta^1 = 0$

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Constraint Jacobian for Rod-Disc Problem

$$
\mathbf{C_q} = \begin{bmatrix} 1 & 0 & \frac{l}{2}\sin\theta^2 & 0 & 0 & 0 \\ 0 & 1 & -\frac{l}{2}\cos\theta^2 & 0 & 0 & 0 \\ 1 & 0 & -\frac{l}{2}\sin\theta^2 & -1 & 0 & 0 \\ 0 & 1 & \frac{l}{2}\cos\theta^2 & 0 & -1 & 0 \end{bmatrix}
$$

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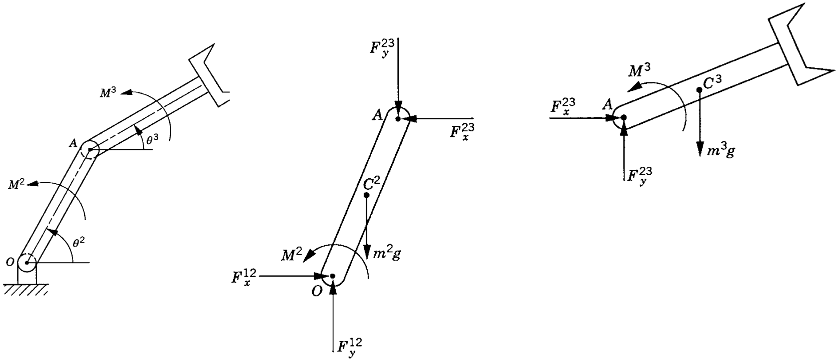
Method I: FBD



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Inverse Dynamics Form

$$\begin{bmatrix} 1 & 0 & -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ l_O^2 \sin\theta^2 & -l_O^2 \cos\theta^2 & l_A^2 \sin\theta^2 & -l_A^2 \cos\theta^2 & 1 & 0 \\ 0 & 0 & l_A^3 \sin\theta^2 & l_A^3 \sin\theta^2 & 0 & 1 \end{bmatrix} \begin{Bmatrix} F_x^{12} \\ F_y^{12} \\ F_x^{23} \\ F_y^{23} \\ M^2 \\ M^3 \end{Bmatrix} = \begin{Bmatrix} m^2 \ddot{R}_x^2 \\ m^2 \ddot{R}_y^2 \\ m^3 \ddot{R}_x^3 \\ m^3 \ddot{R}_y^2 \\ J^2 \ddot{\theta}^2 \\ J^3 \ddot{\theta}^3 \end{Bmatrix}$$

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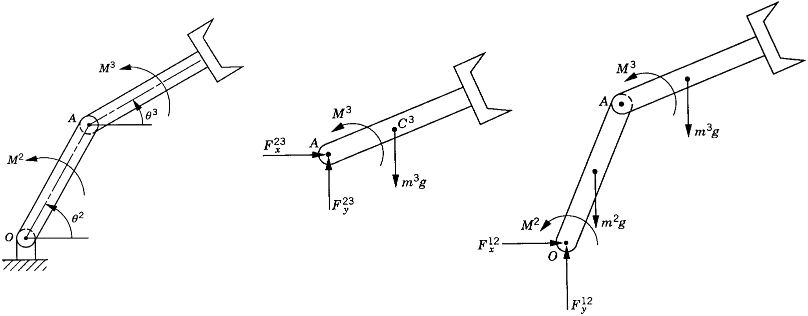
Method II: FBD



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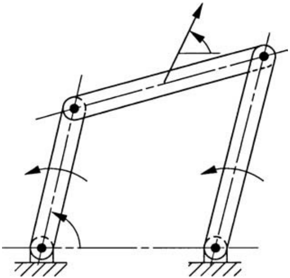
Closed Chain Dynamics Example



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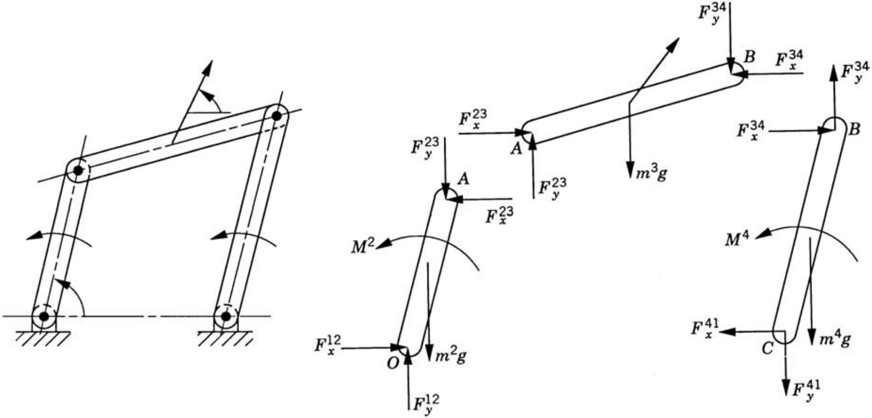
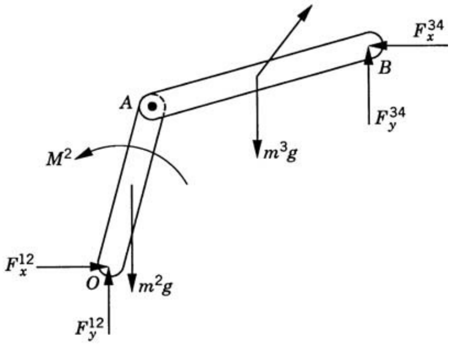
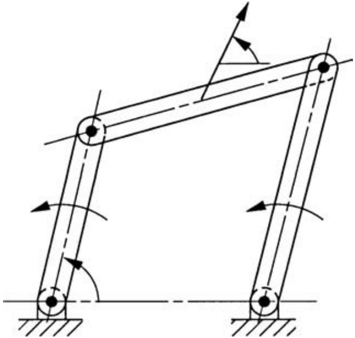
Closed Chain Dynamics: FBD I



Courtesy: A. A. Shabana, 2010, *Computational Dynamics*, 3^{rd} Edition, John Wiley & Sons. R1 R2 R3 R4

Closed Chain Dynamics: Two Bodies





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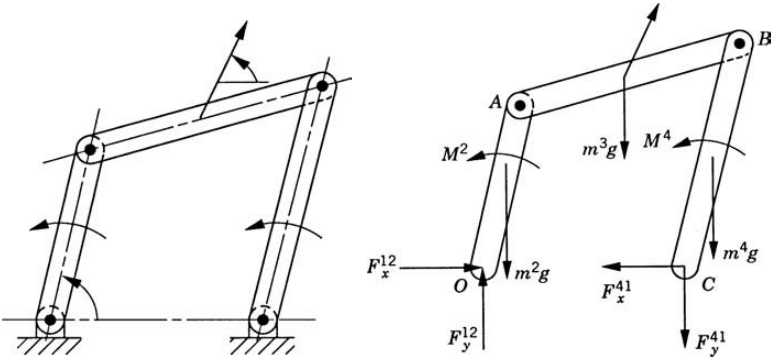
Closed Chain Dynamics: Three Bodies



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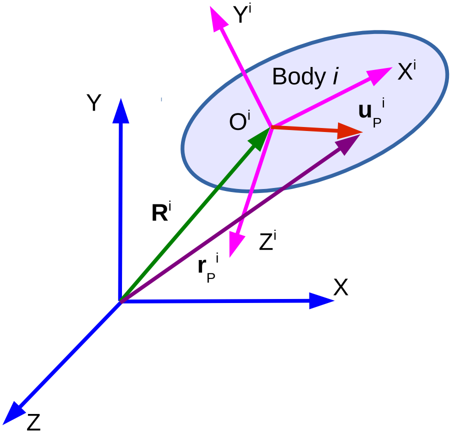
Position Vector: Unconstrained Body



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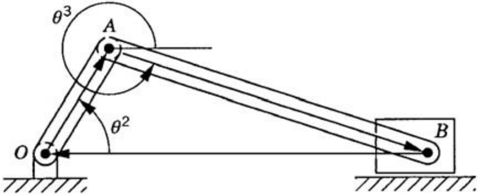
Kinematically Constrained Bodies



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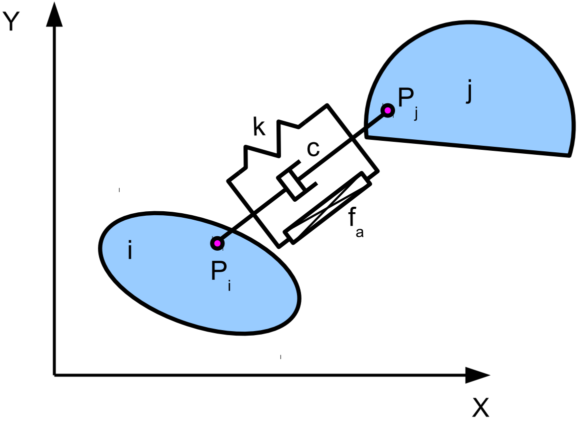
Spring-Damper-Acutuator Element



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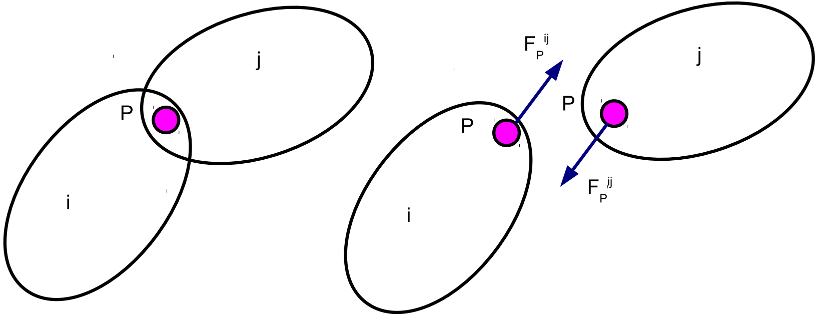
Virtual Work: Revolute Joint



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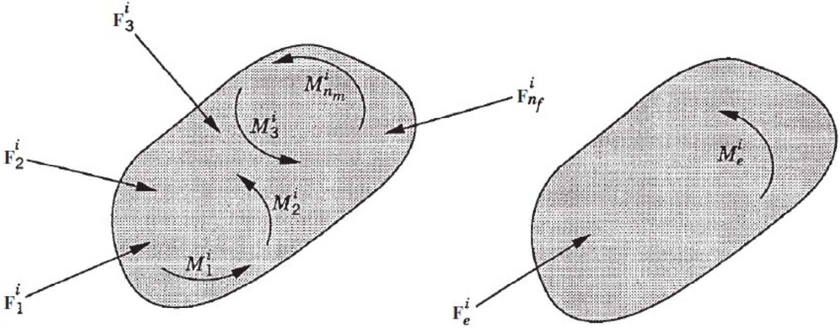
Virtual Work in Statics



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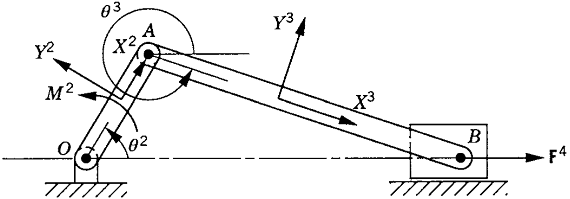
Example Case



Courtesy: A. A. Shabana, 2010, *Computational Dynamics*, 3^{rd} Edition, John Wiley & Sons. ◂ Return ◂ Return2

Generalized Inertia Force

$$
\begin{aligned}
Q_i = -m^2\ddot{R}^2_x l^2_O \sin\theta^2 + m^2\ddot{R}^2_y l^2_O \cos\theta^2 + J^2\ddot{\theta}^2 - \\
m^3\ddot{R}^3_x l^2\left(\sin\theta^2 - \frac{l^3_A}{l^3}\cos\theta^2\tan\theta^3\right) + \\
m^3\ddot{R}^3_y l^2\cos\theta^2\left(1 - \frac{l^3_A}{l^3}\right) - J^3\ddot{\theta}^3\frac{l^2\cos\theta^2}{l^3\cos\theta^3} \\
+ m^4\ddot{R}^4_x l^2\left(-\sin\theta^2 + \cos\theta^2\tan\theta^3\right)
\end{aligned}
$$

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