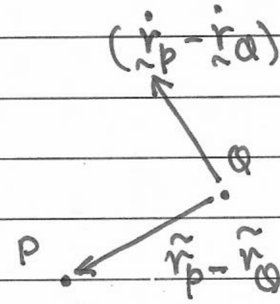
ME 5233 Multibody Dynamics & Applications

Quiz 1 solution

(1)

$$r_P^i = R^i + A^i \bar{u}_P^i \,; \quad r_Q^i = R^i + A^i \bar{u}_Q^i$$



Now
$$\dot{r}_P^i = \dot{R}^i + \dot{\theta}^i A_\theta^i \bar{u}_P^i$$
$$\dot{r}_Q^i = \dot{R}^i + \dot{\theta}^i A_\theta^i \bar{u}_Q^i$$

$$\Rightarrow \quad \dot{r}_P^i - \dot{r}_Q^i = \dot{\theta}^i A_\theta^i (\bar{u}_P^i - \bar{u}_Q^i)$$

$$r_P^i - r_Q^i = A^i (\bar{u}_P^i - \bar{u}_Q^i)$$

$$\therefore \quad (\dot{r}_P^i - \dot{r}_Q^i)^T (r_P^i - r_Q^i)$$

$$= \dot{\theta}^i (\bar{u}_P^i - \bar{u}_Q^i)^T A_\theta^{iT} A^i (\bar{u}_P^i - \bar{u}_Q^i)$$

$$A_\theta^i = \begin{bmatrix} -\sin\theta & -\cos\theta \\ \cos\theta & -\sin\theta \end{bmatrix} ; \quad A^i = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & +\cos\theta \end{bmatrix}$$

$$A_\theta^{iT} A^i = \begin{bmatrix} -\sin\theta & \cos\theta \\ -\cos\theta & -\sin\theta \end{bmatrix} \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \quad \text{(skew-symm)}$$

$$(\bar{u}_P^i - \bar{u}_Q^i)^T \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} (\bar{u}_P^i - \bar{u}_Q^i) = 0$$

The relative velocity is $\perp$ to relative position. For a rigid body there cannot be any relative vel. component along PQ (the distance has to be remain the same)

(2)

$$\underset{\sim}{X}^1 = \begin{Bmatrix} 1 \\ 0 \\ -3 \end{Bmatrix}, \quad Y^1 = \begin{Bmatrix} 3 \\ -3 \\ 1 \end{Bmatrix}, \quad Z^1 = \begin{Bmatrix} -9 \\ -10 \\ -3 \end{Bmatrix}$$

forms an orthogonal triad

Now $\boxed{\alpha_{12} = \underset{\sim}{i}^1 \cdot \underset{\sim}{j}}$; $\boxed{\alpha_{23} = \underset{\sim}{j}^1 \cdot \underset{\sim}{k}}$; $\boxed{\alpha_{13} = \hat{i}^1 \cdot \hat{k}}$

$$\underset{\sim}{i}^1 = \frac{1}{\sqrt{10}}\hat{i} - \frac{3}{\sqrt{10}}\hat{k} \qquad \underset{\sim}{k}^1 = \frac{-9}{\sqrt{190}}\hat{i} \frac{-10}{\sqrt{190}}\hat{j} \frac{-3}{\sqrt{190}}\hat{k}$$

$$\underset{\sim}{j}^1 = \frac{3}{\sqrt{19}}\hat{i} - \frac{3}{\sqrt{19}}\hat{j} + \frac{1}{\sqrt{19}}\hat{k}$$

$$\therefore \boxed{\alpha_{12} = 0} \qquad \boxed{\alpha_{23} = \frac{1}{\sqrt{19}}} \qquad \& \qquad \boxed{\alpha_{13} = \frac{-3}{\sqrt{10}}}$$

(3)

$$\underset{\sim}{b} = \begin{Bmatrix} -3 \\ 2 \\ 4 \end{Bmatrix} \qquad \underset{\sim}{\tilde{b}} = \begin{bmatrix} 0 & -b_3 & b_2 \\ b_3 & 0 & -b_1 \\ -b_2 & b_1 & 0 \end{bmatrix}$$

$$\Rightarrow \boxed{\tilde{b}_{23} = 3 \; ; \; \tilde{b}_{21} = 4}$$

(4)

$$\underset{\sim}{A}_{sym} = \frac{1}{2}(\underset{\sim}{A} + \underset{\sim}{A}^T) \; ; \; \underset{\sim}{A}_{skew-sym} = \frac{1}{2}(\underset{\sim}{A} - \underset{\sim}{A}^T)$$

$$\therefore \quad A_{sym} = \frac{1}{2}\left\{ \begin{bmatrix} -1 & 2 & 0 \\ 3 & 5 & 4 \\ 6 & 2 & -1 \end{bmatrix} + \begin{bmatrix} -1 & 3 & 6 \\ 2 & 5 & 2 \\ 0 & 4 & -1 \end{bmatrix} \right\}$$

$$A_{sym} = \begin{bmatrix} -1 & \frac{5}{2} & 3 \\ \frac{5}{2} & 5 & 3 \\ 3 & 3 & -1 \end{bmatrix}$$

$$A_{skew} = \begin{bmatrix} 0 & -0.5 & -3 \\ 0.5 & 0 & 1 \\ 3 & -1 & 0 \end{bmatrix}$$

5 Line makes angle α, β, γ with resp. to X, Y & Z

What is $\sin^2\frac{\alpha}{2}\cos^2\frac{\alpha}{2} + \sin^2\frac{\beta}{2}\cos^2\frac{\beta}{2} + \sin^2\frac{\gamma}{2}\cos^2\frac{\gamma}{2}$

Now $\sin\frac{\alpha}{2}\cos\frac{\alpha}{2} = \frac{1}{2}\sin\alpha$; $\sin\frac{\beta}{2}\cos\frac{\beta}{2} = \frac{1}{2}\sin\beta$

& $\sin\frac{\gamma}{2}\cos\frac{\gamma}{2} = \frac{1}{2}\sin\gamma$

$\therefore$

$$\sin^2\frac{\alpha}{2}\cos^2\frac{\alpha}{2} + \sin^2\frac{\beta}{2}\cos^2\frac{\beta}{2} + \sin^2\frac{\gamma}{2}\cos^2\frac{\gamma}{2}$$

$$= \frac{1}{4}\left(\sin^2\alpha + \sin^2\beta + \sin^2\gamma\right)$$

$$= \frac{1}{4}\left(1 - \cos^2\alpha + 1 - \cos^2\beta + 1 - \cos^2\gamma\right)$$

$$= \frac{1}{4}\left\{3 - (\cos^2\alpha + \cos^2\beta + \cos^2\gamma)\right\} = \frac{1}{2}$$

as $\cos^2\alpha + \cos^2\beta + \cos^2\gamma = 1$ (sum of dir. cosines' squared $= 1$)