- Let G be the group of reals under addition and G' the group of non-zero reals under multiplication.
  Define $\phi : G \to G'$ s.t.
  $\phi(x) = 2^x$
  $\phi(x + y) = 2^{x+y} = 2^x \cdot 2^y$

- $\phi : \mathbb{Z} \to \mathbb{Z}_3$
  $\phi(x) = x \pmod 3$
  $\phi(x + y) = (x + y) \pmod 3$
  $= (x \pmod 3 + y \pmod 3) \pmod 3$

- $\phi : G \to G/N$, N is a normal subgroup of G.
  $\phi(x) = Nx$
  $\phi(x * y) = N_{x*y} = N_x N_y = \phi(x)\phi(y)$

Def$^{\mathbf{n}}$: Let $\phi : G \to G'$ be a group homomorphism and let Ker $\phi = \{x \in G : \phi(x) = e'\}$, where e' is the identity of G'. Call Ker ϕ as the **KERNEL** of ϕ.

Theorem: Let $\phi : G \to G'$ be a homomorphism, then

i) $\phi(e) = e'$

ii) For every $x \in G$, $\phi(x^{-1}) = [\phi(x)]^{-1}$

Proof:

i) $\phi(e) = \phi(ee) = \phi(e)\phi(e)$
   $\phi(e) = e'\phi(e) \to e'$ is the identity in G'
   $\phi(e)\phi(e) = e'\phi(e)$
   $\Rightarrow \phi(e) = e'$ (by right cancellation)

ii) $\phi(xx^{-1}) = \phi(x)\phi(x^{-1})$ — (1)
   $= \phi(e) = e' = \phi(x)[\phi(x)]^{-1}$ — (2)
   $\phi(x)\phi(x^{-1}) = \phi(x)[\phi(x)]^{-1}$
   By left cancellation
   $\phi(x^{-1}) = [\phi(x)]^{-1}$

Theorem: Let $\phi : G \to G'$ be a homomorphism. Ker ϕ is a normal subgroup of G.
Proof: Let $x, y \in$ Ker ϕ, hence $\phi(x) = \phi(y) = e'$.
$\phi(xy) = \phi(x)\phi(y) = e'e' = e'$
$\Rightarrow xy \in$ Ker ϕ
Let $x \in$ Ker ϕ, hence $\phi(x) = e'$.
Now, $\phi(x^{-1}) = [\phi(x)]^{-1} = [e']^{-1} = e'$
$\Rightarrow x^{-1} \in$ Ker ϕ
Thus Ker ϕ is a subgroup of G.

To show that Ker ϕ is a normal subgroup, we need to show that for every $g \in G$ and every $k \in$ Ker ϕ, $gkg^{-1} \in$ Ker ϕ.
$\phi(gkg^{-1}) = \phi(g)\phi(kg^{-1})$
$= \phi(g)\phi(k)\phi(g^{-1})$
$= \phi(g)e'\phi(g^{-1})$
$= \phi(g)\phi(g^{-1})$
$= \phi(g)[\phi(g)]^{-1}$
$= e'$
Hence $gkg^{-1} \in$ Ker ϕ for every $g \in G$ and every $k \in$ Ker ϕ.

Let $g' \in G'$, we say that $g \in G$ is an **inverse image** of g' if $\phi(g) = g'$.

Theorem: Let $\phi : G \to G'$ be a homomorphism with kernel K. For $g' \in G'$, the set of all inverse images of g' is given by the set Kx where x is any particular inverse image of g'.
Proof: Let $g' \in G'$ and $x \in G$ be such that $\phi(x) = g'$.
Now, for any $k \in K$, let $y = kx$.
$\phi(kx) = \phi(k)\phi(x) = e'\phi(x) = g'$