# Problem Solutions

## B.1 Problem Solving

Most of the problems in this text can be completed without reference to additional material than is in the text. In some cases, new information is introduced in the problem statement to extend the readers knowledge.

In some cases data files may be needed. These are available on the web site of the Wind Energy Center at the University of Massachusetts (http://www.umass.edu/windenergy/). This site also contains the Wind Engineering MiniCodes that have been developed at the University of Massachusetts at Amherst. A number of these codes may be useful in solving problems and, in some cases, may be needed to solve problems. The Wind Engineering MiniCodes are a set of short computer codes for examining wind energy related issues, especially in the context of an academic setting.

A number of files are used in the problems. In addition, a number of the problems have solutions which are illustrated in spreadsheet files. Lists of these files are provided at the end of this document.

It has been discovered that some problems have errors in the problem statements. This obviously makes it difficult to solve the problem. A full list of errata and corrected problem statements can also be found at the same web site mentioned above (http://www.umass.edu/windenergy). The problem statements included with these solutions have already been corrected.

## B.2 Chapter 2 Problems

**2.1** Based on average speed data only, estimate the annual energy production from a horizontal axis wind turbine with a 12 m diameter operating in a wind regime with an average wind speed of 8 m/s. Assume that the wind turbine is operating under standard atmospheric conditions ($\rho$ = 1.225 kg/m$^3$). Assume a turbine efficiency of 0.4.

**SOLUTION**

The annual energy production (in kWh) can be determined from:

Annual energy production (kWh) = (efficiency) ($P$) (8760 hrs)

Where $P$ is determined from Equation 2.7, $P = \pi R^2 \frac{1}{2} \rho U^3$. For an average wind speed of 8 m/s:

$$P = \pi (6)^2 \frac{1}{2} (1.225)(8)^3 = 35.46\ kW$$

Therefore:

Annual energy production = (0.4)(35.46)(8760) = 124,300 kWh

Note that future consideration of the variability of the wind regime and the characteristics of the wind machine will greatly change the output of an actual machine operating in a real wind regime.

**2.2** A 40 m diameter, three bladed wind turbine produces 700 kW at a wind speed (hub height) of 14 m/s. The air density is 1.225 kg/m3. Find:

a) The rotational speed (rpm) of the rotor at a tip-speed ratio of 5.0.
b) What is the tip-speed (m/s)?
c) If the generator turns at 1800 rpm, what gear ratio is needed to match the rotor speed to the generator speed.
d) What is the efficiency of the wind turbine system (including blades, transmission, shafts, and generator) under these conditions?

**SOLUTION**

a) $\lambda = 5$

$\lambda = \dfrac{\Omega R}{U}$ (Equation 2.76)

$$\Omega = \frac{\lambda U}{R} = \frac{(5.0)(14\, m/s)}{20\, m} = 3.5\, rad/s$$

$$N_{rotor} = \frac{(\Omega\, rad/s)(60\, s/\min)}{2\pi\, rad/rev} = 33.42\, rpm$$

b) $U_{tip} = \Omega R = (20\, m)(3.5\, rad/s) = 70\, m/s$

c) $N_{gen}$ =1800 rpm; $N_{rotor}$ = 33.42 rpm

gearbox ratio $= N_{gen} / N_{rotor} = 53.86$

(d) $P = \dfrac{1}{2}\rho\pi R^2 C_p \eta U^3$ ; assume Betz limit: $C_P$ =16/27

$$\eta = \frac{P}{\frac{1}{2}\rho \pi R^2 C_p U^3} = 0.56$$

**2.3** a) Determine the wind speed at a height of 40 m over surface terrain with a few trees, if the wind speed at a height of 10 m is known to be 5 m/s. For your estimate use two different wind speed estimation methods.

b) Using the same methods as part a), determine the wind speed at 40 m if the trees were all removed from the terrain.

**SOLUTION**

a) First Method: Assume a simple logarithmic wind profile. The wind speed at a height of 40 m can be calculated via Equation 2.34

$$\frac{U(z)}{U(z_r)} = \frac{\ln(z/z_0)}{\ln(z_r/z_0)}$$

From Table 2.2 $z_o$ = 100 mm = 0.1 m

Thus

$$U(40) = (5)\frac{\ln(40/0.1)}{\ln(10/0.1)} = 6.51\ m/s$$

Second Method: Assume a power law wind speed profile. The wind speed at a height of 40 m is determined via Equation 2.36

$$\frac{U(z)}{U(z_r)} = \left(\frac{z}{z_r}\right)^{\alpha}$$

Let's assume $\alpha = 1/7 = 0.142$, then:

$$U(40) = (5)\left(\frac{40}{10}\right)^{0.142} = 6.09\ m/s$$

b) First Method: With the trees cut down, assume a rough pasture surface terrain. Using the data from Table 2.2, $z_o$ = 10 mm = 0.01 m. Thus:

$$U(40) = (5)\frac{\ln\left(\frac{40}{0.01}\right)}{\ln\left(\frac{10}{0.01}\right)} = 6.0\ m/s$$

Second Method: If we still assume $\alpha = 1/7$, the same estimate as part a) holds.

**2.4** A 30 m diameter wind turbine is placed on a 50 m tower in terrain with a power law coefficient (α) of 0.2. Find the ratio of available power in the wind at the highest point the rotor reaches to its lowest point.

**SOLUTION**

D = 30 m, hub height = 50 m, α = 0.2
$z_{low}$ = 50 - 15 = 35 m
$z_{high}$ = 50 + 15 = 65 m
The ratio of the wind speeds is:

$$\frac{U(z_{high})}{U(z_{low})} = \left(\frac{z_{high}}{z_{low}}\right)^{\alpha} = \left(\frac{65}{35}\right)^{\alpha} = 1.13$$

The ratio of the powers is the cube of the ratio of the wind speeds:

$$\frac{P(z_{high})}{P(z_{low})} = \left[\frac{U(z_{high})}{U(z_{low})}\right]^3 = 1.13^3 = 1.44$$

**2.5** Find the size of a wind turbine rotor (diameter in m) that will generate 100 kW of electrical power in a steady wind (hub height) of 7.5 m/s. Assume that the air density is $\rho$ = 1.225 kg/m$^3$, $C_p$ = 16/27 and $\eta$ = 1.

**SOLUTION**

a) $P$ = 100 kW; $U$ = 7.5 m/s, $\rho$ = 1.225 kg/m$^3$, $C_P$ = 16/27, $\eta$ = 1

$P = \frac{1}{2}\rho\pi R^2 C_p \eta U^3$ ; therefore:

$$D = 2\sqrt{\frac{P}{\frac{1}{2}\rho\pi C_p \eta U^3}} = 28.8\ m$$

**2.6** From an analysis of wind speed data (hourly interval average, taken over a one year period), the Weibull parameters are determined to be $c$ = 6 m/s and $k$ = 1.8.

a) What is the average velocity at this site?

b) Estimate the number of hours per year that the wind speed will be between 6.5 and 7.5 m/s during the year.

c) Estimate the number of hours per year that the wind speed is above 16 m/s.

**SOLUTION**

a) From Equation 2.62

$$\overline{U} = \int_0^\infty U\,p(U)dU = c\Gamma\left(1+\frac{1}{k}\right)$$

Therefore

$$\overline{U} = c\Gamma\left(1+\frac{1}{k}\right) = (6)\Gamma\left(1+\frac{1}{1.8}\right) = (6)(0.8893) = 5.34\ m/s$$

b) The number of hours that the wind speed will be between 6.5 and 7.5 m/s during the year is equal to the probability that the wind speed will be between 6.5 and 7.5 m/s times the number of hours in the year. It is most conveniently found using the cumulative distribution function, Equation 2.61

$$p(6.5 < U < 7.5) = F(7.5) - F(6.5) = \exp\left(-\left(\frac{6.5}{c}\right)^k\right) - \exp\left(-\left(\frac{7.5}{c}\right)^k\right) = 0.9198$$

Thus, the number of hours = 0.09198 x 8760 = 806 hr

c) Similar to part b), the probability that the wind speed will be equal to or greater than 16 m/s is given by:

$$\mathrm{p}(U \geq 16) = \int_{16}^{\infty} \mathrm{p(U)dU} = 1 - F(16) = \exp(-\left(\frac{16}{6}\right)^{1.8}) = 0.002896$$

Thus, the number of hours = 0.002896 x 8760 = 25 hr

**2.7** Analysis of time series data for a given site has yielded an average velocity of 6 m/s. It is determined that a Rayleigh wind speed distribution gives a good fit to the wind data.

a) Based on a Rayleigh wind speed distribution, estimate the number of hours that the wind speed will be between 9.5 and 10.5 m/s during the year.

b) Using a Rayleigh wind speed distribution, estimate the number of hours per year that the wind speed is equal to or above 16 m/s.

**SOLUTION**

Parts a) and b) could be solved as the previous problem assuming a Weibull distribution with $k$ = 2, which corresponds to the Rayleigh distribution, but the most direct method is to use the exact equation for the Rayleigh distribution.

Thus, for a)

$$\mathrm{p}(9.5 < \mathrm{U} < 10.5) = \int_{10\text{-}0.5}^{10+0.5} \mathrm{p(U)dU} = F(10.5) - F(9.5)$$

$$= \exp\left(-\frac{\pi}{4}\left(\frac{9.5}{6}\right)^2\right) - \exp\left(-\frac{\pi}{4}\left(\frac{10.5}{6}\right)^2\right) = 0.0494$$

Thus, the number of hours = 0.0492 x 8760 = 433 hr

b) Using Equation 2.59

$$\mathrm{p}(U \geq 16) = \text{1-F(16)} = \exp\left(-\frac{\pi}{4}\left(\frac{16}{6}\right)^2\right) = 0.00375$$

Thus, the number of hours = 0.00375 x 8760 = 33 hr

**2.8** Estimate the annual production of a 12 m diameter horizontal axis wind turbine operating at standard atmospheric conditions ( $\rho$ = 1.225 kg/m$^3$) in a 8 m/s average wind speed regime. You are to assume that the site wind speed probability density is given by the Rayleigh density distribution.

**SOLUTION**

The average power for this machine can be found from Equation 2.82:

$$\overline{\mathrm{P}} = \rho\left(\frac{2}{3}D\right)^2 \overline{U}^3$$

Thus

$$\overline{\mathrm{P}} = 1.225\left(\frac{2}{3}x12\right)^2 8^3 = 40.2\,kW$$

The annual energy production would be (40.2) x (8760) kWh = 352,000 kWh

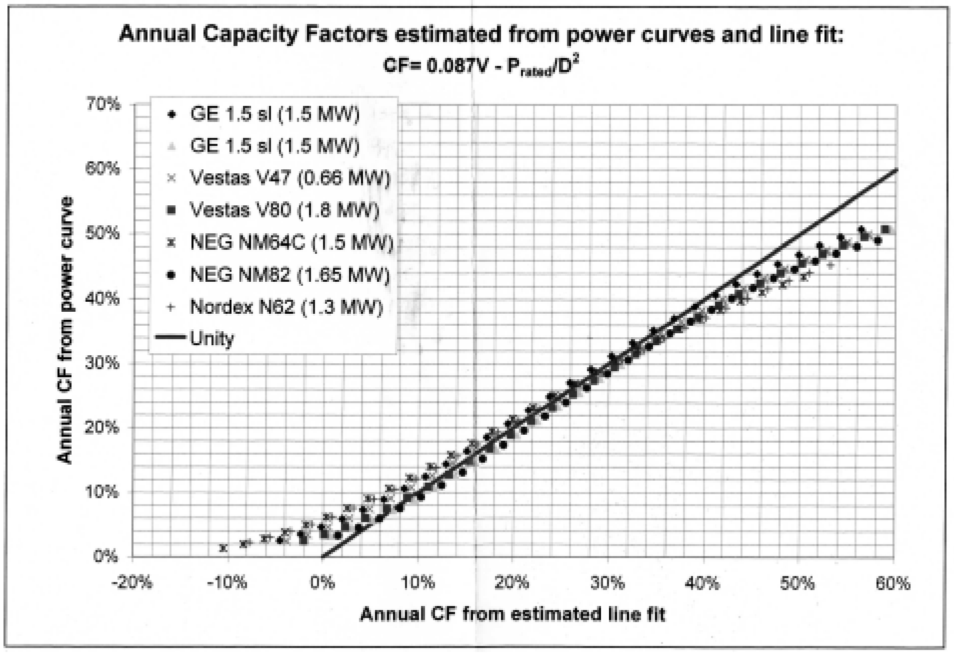
**2.9** Assuming a Rayleigh distribution, a researcher (see Masters (2004) *Renewable and Efficient Electric Power Systems.* Wiley) has proposed the following simple relationship for the capacity factor (CF) of a wind turbine:

$$CF = 0.087\overline{U} - \frac{P_R}{D^2}$$

where $\overline{U}$ is the average wind speed at the hub, $P_R$ is the rated power (kW) and $D$ is the rotor diameter (m). Furthermore, this researcher claims that this equation is accurate to within 10% for capacity factors between 0.2 to 0.5. Your problem is to check out this equation for a selection (say 5) commercial wind turbines.

**SOLUTION**

Based on wind turbine manufacturers' data from the web, the following graph was obtained for a number of wind turbines operating at various average hub height wind velocities. Thus, it looks like this prediction method gives a reasonable first estimate for capacity factor when a minimum of wind data and turbine data are available.




**2.10** A wind turbine with a rotor diameter of 55 m is rated at 1 MW at a hub height wind speed of 14 m/s. It has a cut-in speed of 4 m/s and a cut-out speed of 25 m/s. Assume that this machine is located at a site where the mean wind speed is 10 m/s and that a Rayleigh wind speed distribution can be used. Calculate the following:

a) The number of hours per year that the wind is below the cut-in speed.
b) The number of hours per year that the machine will be shut down due to wind speeds above the cut-out velocity.
c) The energy production (kWh/year) when the wind turbine is running at rated power.

**SOLUTION**

a) The probability that the wind speed is smaller than or equal to a given wind speed is given by Equation 2.59 using the Rayleigh probability density function.

This gives:

$$F\left(U_{cut-in}\right) = 1 - \exp\left(-\frac{\pi}{4}\left(\frac{U_{cut-in}}{\overline{U}}\right)^2\right) = 0.1181$$

Thus, the number of hours per year = (0.118) x (8760 hrs) = 1035 hr/yr

b) The number of hours when the wind is higher than the cut-out velocity is given by:

$$\left[1 - F\left(U_{cut-out}\right)\right](8760) = \exp\left(-\frac{\pi}{4}\left(\frac{U_{cut-out}}{\overline{U}}\right)^2\right)(8760) = (0.0074)(8760) = 64.7\, hr/yr$$

c) The number of hours that this occurs is the number of hours that the wind speed is greater than rated speed minus the number of hours that the machine is shut down (part B). The number of hours when the wind is higher than 14m/s is determined from:

$$\left[1 - F\left(U_{rated}\right)\right](8760) = \exp\left(-\frac{\pi}{4}\left(\frac{U_{rated}}{\overline{U}}\right)^2\right)(8760) = (0.2145)(8760) = 1879\, hr/yr$$

Thus, the number of hr/yr at rated power = 1879 – 64.7 = 1814.3 hr/yr

And, the delivered energy is 1000 x 1814 kWh/yr = 1.814 x $10^6$ kWh/yr

**2.11** Based on the spreadsheet (*MtTomData.xls*) which contains one month of data (mph) from Holoyke, MA, determine:

a) The average wind speed for the month

b) The standard deviation

c) A histogram of the velocity data (via the method of bins–suggested bin width of 2 mph)

d) From the histogram data develop a velocity–duration curve

e) From above develop a power–duration curve for a given 25 kW Turbine at the Holyoke site.

For the wind turbine, assume:

| | |
|---|---|
| $P = 0$ kW | $0 < U \le 6$ (mph) |
| $P = U^3/625$ kW | $6 < U \le 25$ (mph) |
| $P = 25$ kW | $25 < U \le 50$ (mph) |
| $P = 0$ kW | $50 < U$ (mph) |

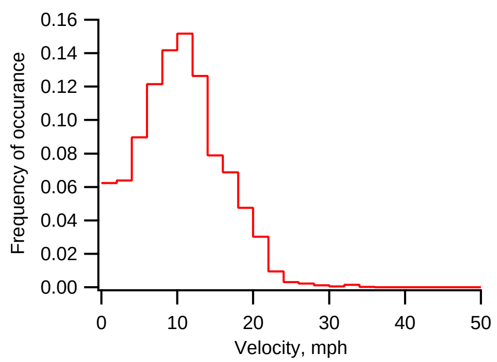
f) From the power duration curve, determine the energy that would be produced during this month in kWh.

**SOLUTION:**

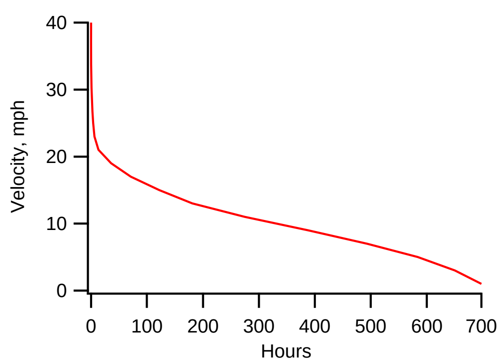
a) Using given data: Average wind speed = 10.4 mph

b) Using given data: Standard deviation = 5.45 mph

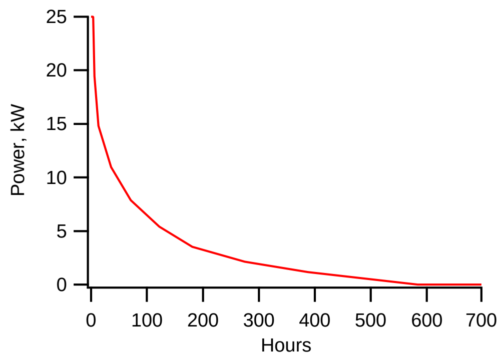
c) Using bin reduced data, the following graph results:




d) From the histogram data, the following velocity duration curve results:




e) The following power duration curve results:




f) The total energy produced can be determined from integrating the product of the turbine power and the numbers of hours of operation at that power level, yielding an annual energy production of 2474 kWh

**2.12** Using results from Problem 2.11, carry out the following:

a) Determine Weibull and Rayleigh velocity distribution curves and normalize them appropriately. Superimpose them on the histogram of Problem 2.11.

b) Determine the Weibull and Rayleigh velocity duration and power duration curves and superimpose them on the ones obtained from the histogram.

c) Using the Weibull distribution, determine the energy that would be produced by the 25 kW machine at the Holyoke site.

d) Suppose the control system of the 25 kW machine were modified so that it operated as shown in Figure B.1 (and as detailed in Table B.1) How much less energy would be produced at Mt. Tom with the modified machine? Find a fourth order polynomial fit to the power curve. Use the Weibull distribution to calculate the productivity (in any manner you choose). Plot the power duration curve using the modified power curve and the cubic power curve from Problem 2.11e).

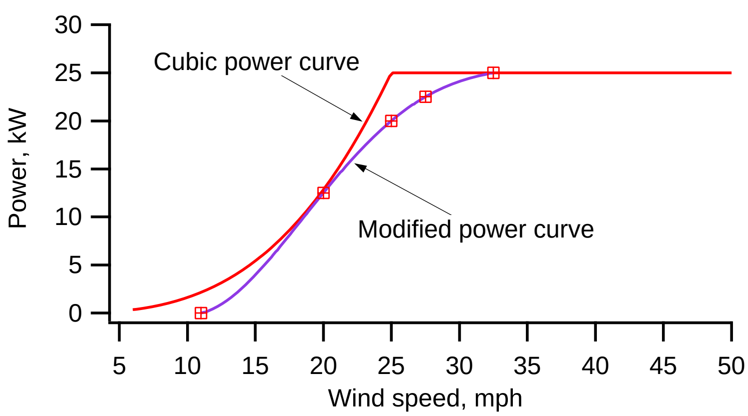


**Figure B.1** Power curves for Problems 2.11 and 2.12

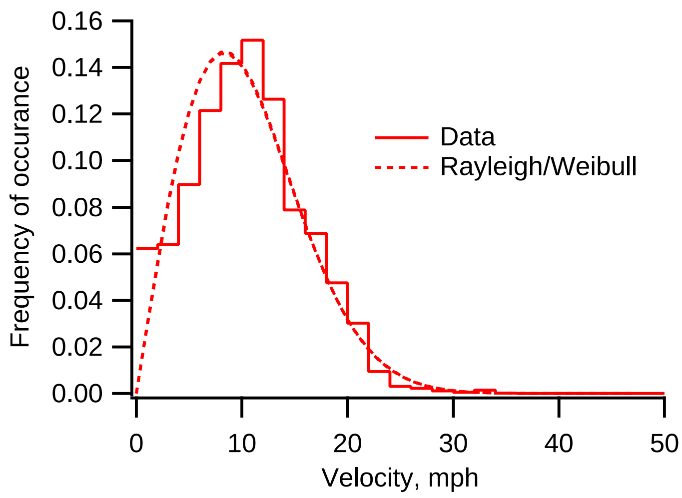
**Table B.1** Power curve below rated power for Problem 2.12d

| Wind speed (mph) | Power (kW) |
|---|---|
| 11 | 0 |
| 20 | 12.5 |
| 25 | 20 |
| 27.5 | 22.5 |
| 32.5 | 25 |

**SOLUTION**

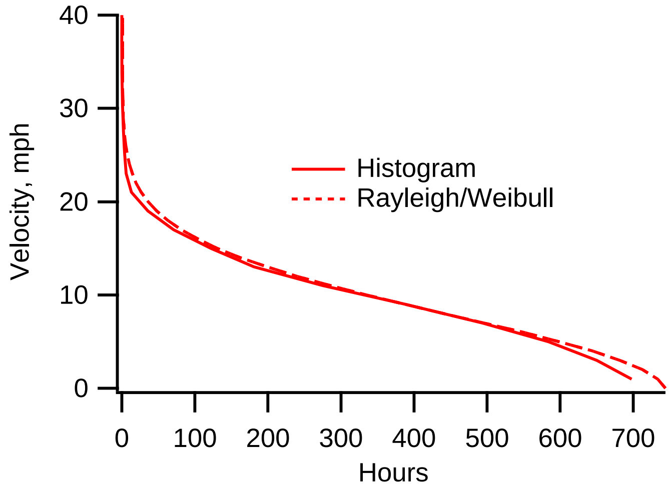
a) For the Rayleigh distribution, one just needs the average velocity. For the Weibull distribution, one needs to calculate $k$ and $c$. The resulting values should be $k = 2.016$ and $c = 11.74$ mph. The distributions are shown below. Because $k = 2.016$ the Weibull and Rayleigh distributions are indistinguishable.

Note that the histogram of Problem 2.11 shows the probability that the wind is in a wind speed bin that is 2 mph wide. The Rayleigh and Weibull probability density distributions have units of probability per mph. For comparison with the histogram, the Weibull and Rayleigh distributions in the graphs have been scaled by a factor of 2 to represent probability per 2 mph.

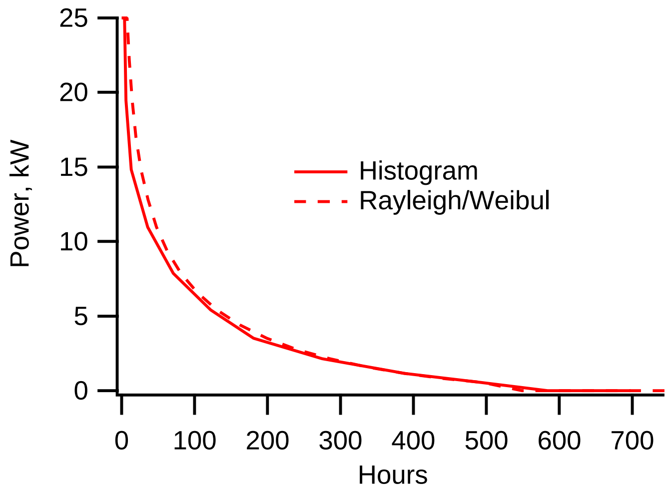



b) The velocity duration and power duration curves are:

Velocity duration curve:

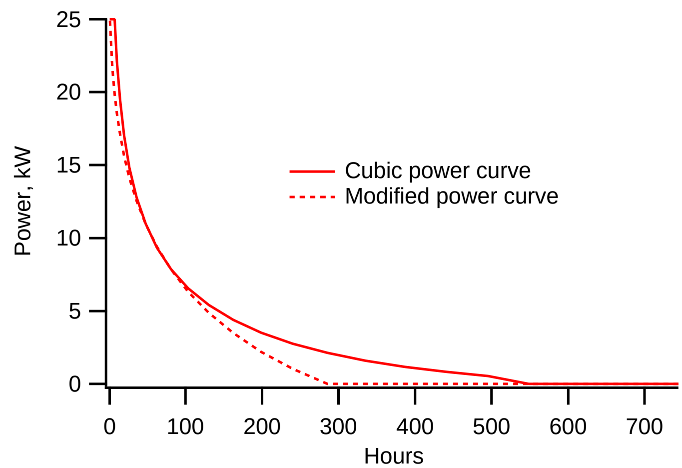



Power duration curve:




c) The total energy produced can be determined from integrating the product of the turbine power and the numbers of hours of operation at that power level, yielding an annual energy production of 2474 kWh using the histogram data and 2454 kWh using the Weibull distribution.

d) Once a polynomial curve is fit to the data, the result look like:




The power output from the modified turbine can be determined a number of ways, the easiest involves determination of the area under the power duration curve. In all cases, the resulting answer should be about 1848 kWh.

**2.13** Similar to Equation 2.27 in the text, the following empirical expression has been used to determine the power spectral density (psd) of the wind speed at a wind turbine site with a hub height of $z$. The frequency is $f$ (Hz), and $n$ ($n = fz/U$) is a non-dimensional frequency.

$$\frac{f\,S(f)}{(2.5u^*)^2} = \frac{11.40n}{1+192.4n^{5/3}}$$

where

$$u^* = \frac{0.4[U(z)]}{\ln\left(\frac{z}{z_0}\right)}$$

Determine the power spectral density of the wind at a site where the surface roughness is 0.05 m ($z_0$) and the hub height is 30 m, and the mean wind speed is 7.5 m/s.

**SOLUTION**

First solve for the friction velocity, $u^*$ and $f\ S(n) = \dfrac{41.84}{1+n^{5/3}}$ from:

$$u^* = \frac{\overline{u(z)}(0.4)}{\ln(\frac{z}{z_o})} = \frac{7.5(0.4)}{\ln(\frac{30}{0.05})} = 0.469\ m/s$$

Substituting the expression for $n$ as a function of $f$ into the definition of the spectrum:

$$\frac{f\,S(f)}{(2.5u^*)^2} = \frac{11.40(fz/U)}{1+192.4(fz/U)^{5/3}}$$

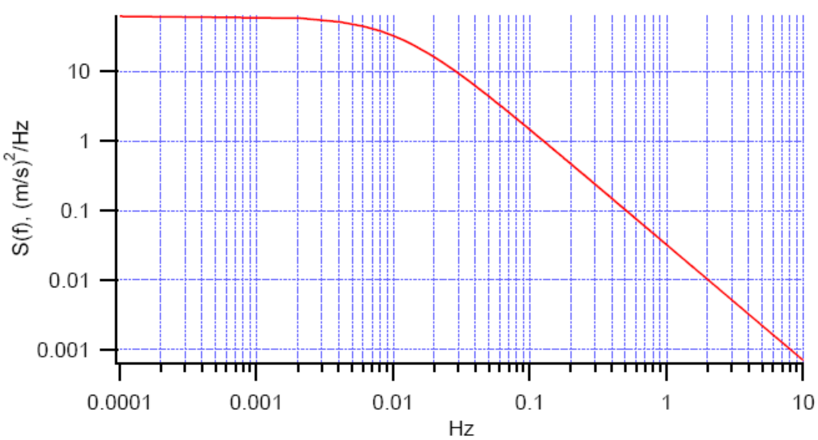
Then $S(f)$ is:

$$S(f) = \frac{11.40(z/U)(2.5u^*)^2}{1+192.4(z/U)^{5/3} f^{5/3}}$$

Substituting in values of $z$ = 30 m and $U$ = 7.5 m/s:

$$S(f) = \frac{62.69}{1+1939.3f^{5/3}}$$

A graph of the power spectral density (psd) is shown below:




**2.14** This problem uses the power spectral density (psd) to examine variance in the wind. A time series of hourly wind speeds (mph) from Mt. Tom for approximately 1 year is included in the data *MtTomWindUM.txt*. Routines to perform psd analysis are included with the UMass Wind Engineering MiniCodes. When the psd is graphed vs. the frequency it is hard to see features of interest. For this reason it is common to graph $f\,S(f)$ on the $y$ axis vs. $\ln(f)$ on the $x$ axis. When doing this the area under the curve between any two frequencies is proportional to the total variance associated with the corresponding range of frequencies.

a) Use the MiniCodes to calculate the psd for the Mt. Tom wind data. Focus on the variations in wind over time periods of less than one month by using a segment length of 512.

b) Show from the results that the total variance as given by the integral of the psd (i.e. $S(f)$ vs. $f$) is approximately the same as what would be obtained in the normal way.

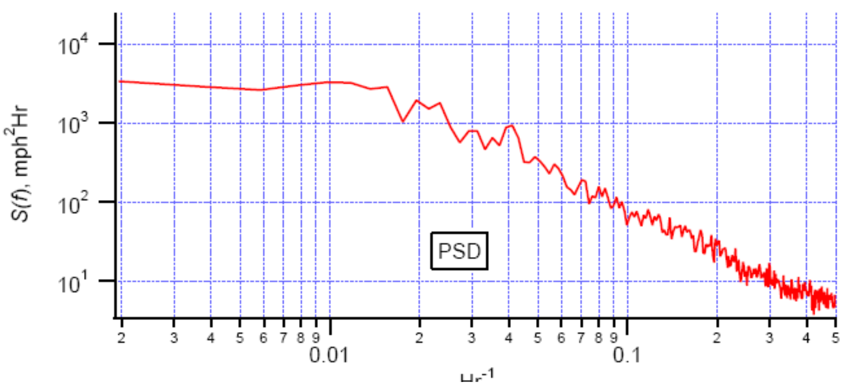
c) Show by equations that the area under the curve in a plot of $fS(f)$ vs. $\ln(f)$ is the same as it would be for a plot of $S(f)$ vs. $f$.

d) Plot $f\,S(f)$ vs. $\ln(f)$

e) Find the amount of variance associated with diurnal fluctuations. Use frequencies corresponding to cycle times from 22 hours to 27 hours. How much variance is associated with higher frequency variations and how much with lower frequency variations?

**<u>SOLUTION</u>**

a) The psd is shown below:

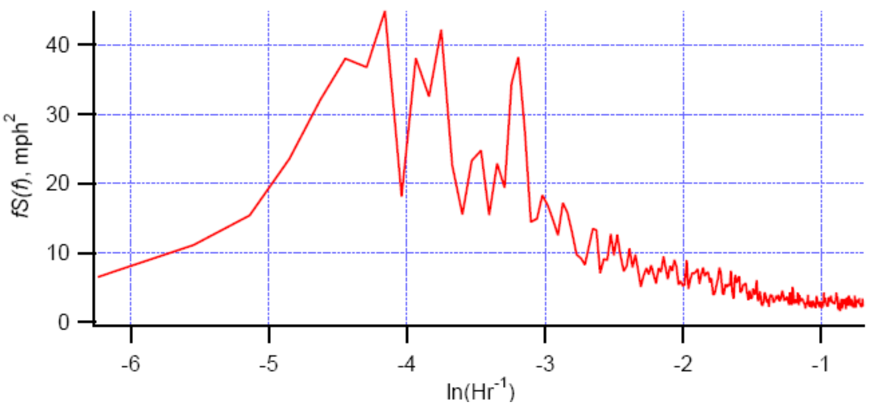



b) The variance may be found from $\sigma^2 = \int_0^\infty S(f)df \approx \sum_{i=1}^{N-1}\big(S(f_i)+S(f_{i+1})\big)\big[f_{i+1}-f_i\big]$. The value is 110.5 (mph)$^2$. This is very close to the actual value, which is 118.2 (mph)$^2$.

c) The result can be shown by referring to the defining equation for the natural log:

$$\int_0^\infty f\,S(f)d(\ln(f)) = \int_0^\infty f\,S(f)d\left(\int \frac{df}{f}\right) = \int_0^\infty \frac{f}{f}S(f)df = \int_0^\infty S(f)df = \sigma^2$$

d) The graph is shown below.




e) The variance associated with frequencies between 22 and 27 hours (0.045 to 0.037 $Hr^{-1}$) can be found from integrating the PSD over the correct range of frequencies. The result is about 6.6 mph$^2$. Almost all of the variance in this data set of hourly averages is in frequencies corresponding to time periods of longer than 20 hours.

**2.15** A variety of techniques are available for creating data sets that have characteristics similar to that of real data. The Wind Engineering MiniCodes include a few of these methods. In the ARMA technique (see Appendix C) the user must input long-term mean, standard deviation, and autocorrelation at a specified lag. The code will return a time series with values that are close to the desired values. (Note: a random number generator is used in the data synthesis routines, so any given time series will not be exactly the same as any other.)

a) Find the mean, standard deviation, and autocorrelation for the Mt. Tom data: *MtTom7Hzms.txt*. This data, in m/s, is collected at a 25 m height, with a sampling frequency of 7.4 Hz. Plot a time series of the data. Determine the autocorrelation for a lag of up to 2000 points.

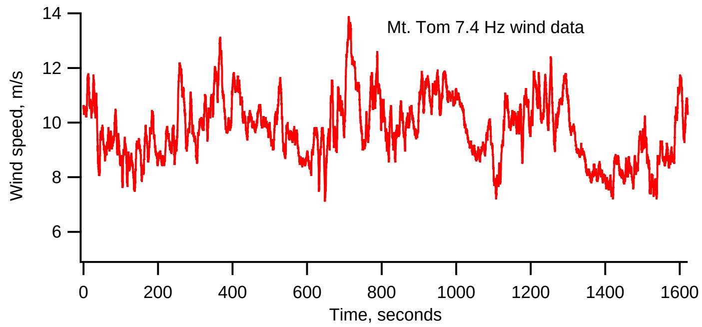
b) Using the ARMA code with the autocorrelation at a lag of one time step, synthesize and plot a time series of 10,000 data points with equivalent statistics to those found in part a. Show a time series graph of the synthesized data.

c) Find the autocorrelation for a lag of up to 2000 points for the synthetic data and plot the autocorrelations of both the real and the synthesized on the same graph.

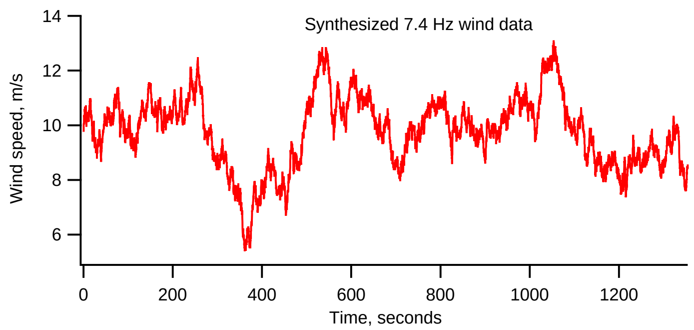
d) Comment on any similarities or differences between the two plots.

**SOLUTION**

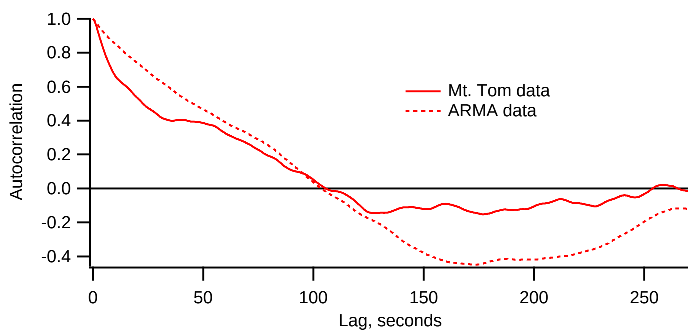
a) For the real data: mean = 9.76 m/s, sd. Dev =1.59 m/s, autocorrelation = 0.997 at a lag of one time step. The data are shown below:




b) A sample synthetic time series is shown below:




c) The autocorrelations are shown below:




d) Both autocorrelations look similar, with the wind showing some degree of correlation over periods of 100 seconds (about 740 lags). The synthesized data is much more negatively correlated than the real data over time periods longer than 100 seconds. The synthesized data is also more correlated than the real data over shorter time periods (up to 50 seconds).

## B.3 Chapter 3 Problems

**3.1** The blades of a wind turbine are ready to be installed on a turbine on top of a ridge. The horizontal blades are supported at each end by saw horses, when a storm front arrives. The turbine crew huddles in their truck as the rain starts and the wind picks up, increasing eventually to 26.82 m/s (60 mph). Realizing that the wind coming up the western slope of the ridge roughly follows the 10 degree slope, the field engineer performs a quick calculation and drives his truck upwind of the blades to disrupt the airflow around the blades, preventing them from being lifted by the wind and damaged.

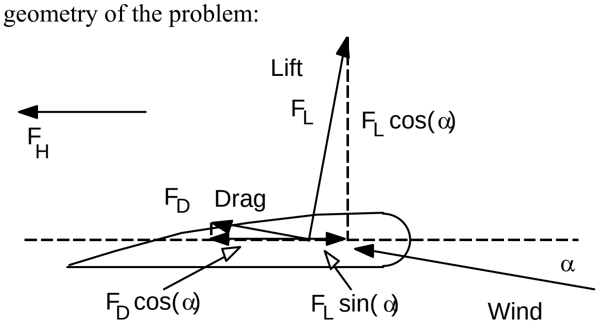
The blades are 4.57 m (15 feet) long, 0.61 m (2 feet) wide, and have a mass of 45.36 kg (100 lbm). As the front arrives the temperature drops to 21.2°C (70°F). Assume that the blades are approximately symmetric airfoils (the engineer remembered that potential flow theory predicts that, prior to stall, the lift coefficient of a symmetric airfoil is approximately: $C_l = 2\pi \sin\alpha$ ). Assume that the center of both the lift and the drag force is concentrated over the center of mass of the blade and that the leading edge is facing into the direction of the wind. Assume the air density is 1.20 kg/m$^3$.

a) Was there a reason to be concerned? At what wind speed will the blades be lifted by the wind, assuming that there is no drag?

b) If they are lifted by a 26.82 m/s (60 mph) wind, how fast will they be accelerated horizontally, if the blade's lift to drag ratio, $C_d / C_l$ , is 0.03?

**SOLUTION**

a) From the geometry of the problem:




The planform area of the blades is 2.79 m$^2$ (30 ft$^2$). From the definition of the lift coefficient:

$$\mathrm{F_L} = \frac{1}{2} C_l \rho U^2 A = \frac{1}{2}(2\pi \sin\alpha)\rho U^2 A$$

Assuming that there is no drag, the criteria for the blades being lifted by the wind is:

$$\mathrm{W} = \mathrm{F_L} \cos\alpha = (\pi \sin\alpha)\rho U^2 A \cos\alpha$$

where $W$ is the blade weight. Therefore, the wind speed at which the blades will move is:

$$U = \sqrt{\frac{\mathrm{W}}{(\pi \sin\alpha \cos\alpha)\rho A}}$$

Here $\alpha = 10^\circ$ = 0.175 radians, $\rho$ = 1.20 kg/m$^3$, so:

$$U = \sqrt{\frac{45.36}{(\pi(0.173)(0.985))(1.2)(2.79)}} = 15.76 \text{ m/s} = 51.7 \text{ ft/s} = 35.2 \text{ mph}\cdot$$

The engineer did the correct thing and got a pat on the back from his boss.

b) Again, from the geometry of the problem:

$$\mathrm{F_H} = (-C_l \sin\alpha + C_d \cos\alpha)\frac{1}{2}\rho U^2 A$$

or

$$\mathrm{F_H} = \mathrm{C_l}\left(-\sin\alpha + {}^{C_d}\!/_{C_l} \cos\alpha\right)\frac{1}{2}\rho U^2 A$$

so

$$\mathrm{F_H} = (\pi \sin\alpha)\left(-\sin\alpha + {}^{C_d}\!/_{C_l} \cos\alpha\right)\rho U^2 A\cdot$$

If U= 26.82 m/s (60 mph, 88 ft/s), so:

$$\mathrm{F_H} = (\pi 0.174)(-0.174 + 0.03(0.985))1.2(26.82)^2\, 2.79 = -190.2N = -42.7 \text{ lbf}$$

And the horizontal acceleration, $a_H$ , of the blades is $F_H/m$:

$$a_H = F_H/m = -190.2/45.36 = -4.19 \text{ m/sec}^2 = -13.7 \text{ ft/sec}^2$$

The blade would have taken off INTO the wind, but for the quick thinking of the field engineer!

**3.2** An inventor proposes to use a rotating cylinder to produce lift in a new wind energy device. The cylinder will be $D$ = 0.75 m in diameter and will be $H$ = 7.5 m high. It will rotate with a speed of n = 60 rpm.

a) Recall that circulation around a cylinder is the integral of the tangential velocity about its perimeter. Show that the circulation is given by:

$$\Gamma = \frac{\pi^2 D^2 n}{60}$$

Hint: this is easiest done by using polar coordinates.
b) Find an expression for the lift per unit height around a rotating cylinder in terms of the free stream wind velocity, $U$ (m/s), the rotational speed, $n$ (rpm) and the diameter, $D$ (m), of the cylinder
c) Find the lift force produced by the cylinder in the inventor's device in a 10 m/s wind.

**SOLUTION**
a) The circulation is given by:
The air velocity at the perimeter of the cylinder is:

$$U_{\tan} = (2\pi)\left(\frac{n}{60}\right)\left(\frac{D}{2}\right) = \frac{\pi n D}{60} m/s$$

$$\Gamma = \int_0^{2\pi} \frac{D}{2} U_{\tan} d\theta = \int_0^{2\pi} \frac{D}{2}\left(\frac{\pi D n}{60}\right) d\theta$$

$$= \frac{D^2 n\pi}{180}\int_0^{2\pi} d\theta = \frac{D^2 n\pi}{180}(2\pi)$$

$$= \frac{\pi^2 D^2 n}{60} \quad q.e.d.$$

b) Lift force per unit length is given by:

$$\tilde{L} = \rho U \Gamma = \rho U \pi^2 D^2 n / 60$$

c) Find the lift force produced by the cylinder in the inventor's device in a 10 m/s wind. Use: $D$ = 0.75 m, $H$ = 7.5 m high, n = 60 rpm.

$$\tilde{L} = \rho U \Gamma = \rho U \pi^2 D^2 n / 60$$

$$= (1.225\, kg/m^3)(10\, m/s)(0.75^2\, m^2)(\pi^2)\left(\frac{60\, rev/\min}{60\, s/\min}\right)$$

$$= 68\, N/m$$

The total lift force is thus (68 N/m)(7.5 m) = 510 N.

**3.3** The operating conditions found at two different points of a blade on a wind turbine are (Table B.2):

**Table B.2**

| Location r/R | Wind velocity at blade (m/s) | Wind velocity at blade (ft/s) | Chord (m) | Chord (ft) | Angle of attack (degrees) |
|---|---|---|---|---|---|
| 0.15 | 16.14 | 52.94 | 1.41 | 4.61 | 4.99 |
| 0.95 | 75.08 | 246.32 | 0.35 | 1.15 | 7.63 |

These conditions were determined at 0°C (32°F), for which the kinematic viscosity is $1.33 \times 10^{-5}$ m$^2$/s. What are the Reynolds numbers found at each blade section?

**SOLUTION**

The Reynolds number is defined as:

$$\mathrm{Re} = Uc/\nu$$

where: $\nu$ in the kinematic viscosity of air, $U$ is the velocity of airflow at airfoil and $c$ is the chord length. At 0°C (32°F) the kinematic viscosity of air is $1.33 \times 10^{-5}$ m²/s. The operating Reynolds numbers are, then:

| Location, r/R | Re |
|---|---|
| 0.15 | 1.71e6 |
| 0.95 | 1.98e6 |

**3.4** a) Find $\theta$, $\theta_P$, $\theta_T$, and $c$ for one blade section from $r/R$ = 0.45 to $r/R$ = 0.55 (centered on $r/R$ = 0.50) for an ideal blade (assume $C_d$ = 0, $a'$ = 0). Assume $\lambda$ =7, $B$=3, $R$=5 m, and $C_l$ =1.0 and the minimum $C_d/C_l$ occurs at $\alpha$ = 7.

b) Assume that $C_d/C_l$ actually equals 0.02 for the above blade section and that the free stream wind speed, $U$, equals 10 m/s. Find $U_{rel}, \mathrm{d}F_{L1}, \mathrm{d}F_{D1}, \mathrm{d}F_{N1}, \mathrm{d}F_{T1}, \mathrm{d}Q_1$ for the blade section. Don't forget to consider that the wind velocity is slowed down at the rotor. Use $a$ = 1/3, $a'$ = 0. Assume the air density is 1.24 kg/m³ (20C).

c) For the same blade section find $C_l$, $\alpha$ and $a$ using the general strip theory method (including angular momentum). Also find $C_l$, $\alpha$ and $a$ if the rpm is increased such that $\lambda$ = 8. Ignore drag and tip loss. Use a graphical approach. Assume that the empirical lift curve is $C_l = 0.1143\alpha + 0.2$ ($\alpha$ in degrees): i.e. $C_l$ = 0.2 at $\alpha$ = 0 degrees, $C_l$ = 1.0 at $\alpha$ = 7 degrees.

**SOLUTION**

a) For $r/R$ = 0.50, the section radius, $r$, angle of relative wind, $\varphi$, chord, $c$, section pitch, $\theta_P$, and twist, $\theta_T$ can be determined from:

$$r = (r/R)R$$
$$\lambda_r = (r/R)\lambda$$
$$\varphi = \tan^{-1}(2/(3\lambda_r))$$
$$c = 8\pi r \sin\varphi/(3BC_l\lambda_r)$$
$$\theta_P = \varphi - \alpha$$
$$\theta_T = \theta_P - \theta_{P,0}$$

where $\theta_{P,0}$ is the blade pitch angle at the tip. The results of the calculations appear below.

| $r/R$ | **r,** m | $\varphi$, deg | $\theta_P$, deg | $\theta_T$, deg | **c,** m |
|---|---|---|---|---|---|
| 0.50 | 2.50 | 10.78 | 3.78 | 5.34 | .373 |

b) Using:

$$U_{rel} = U(1-a)/\sin\varphi \tag{3.64}$$

$$\mathrm{d}F_L = C_l \tfrac{1}{2} \rho U_{rel}^{\;2} c\, \mathrm{d}r \tag{3.65}$$

$$\mathrm{d}F_D = C_d \tfrac{1}{2} \rho U_{rel}^{\;2} c\, \mathrm{d}r \tag{3.66}$$

$$\mathrm{d}F_N = \mathrm{d}F_L \cos\varphi + \mathrm{d}F_D \sin\varphi \tag{3.67}$$

$$\mathrm{d}F_T = \mathrm{d}F_L \sin\varphi - \mathrm{d}F_D \cos\varphi \tag{3.68}$$

$$\mathrm{d}Q = B\, r\, \mathrm{d}F_T \tag{3.70}$$

in which d$r$ = 0.10, the various forces and torque at $r/R$ = 0.50 can be determined. If the air density is assumed to be $\rho$ = 1.145 kg/m$^3$ then:

| Relative wind, $\varphi$ degrees | d$F_L$ N | d$F_D$ N | d$F_N$ N | d$F_T$ N | d$Q$ N |
|---|---|---|---|---|---|
| 35.63 | 135.6 | 2.71 | 133.7 | 22.71 | 170.3 |

These forces and torques will be proportionally different if a different air density is assumed.

c) Strip theory requires that, for any given blade geometry:

$$C_l = 4\sin\varphi \frac{(\cos\varphi - \lambda_r \sin\varphi)}{\sigma'(\sin\varphi + \lambda_r \cos\varphi)}$$

in which $\sigma' = Bc/2\pi r$, $\lambda_r = (r/R)\lambda$ and $\varphi = \alpha + \theta_P$.

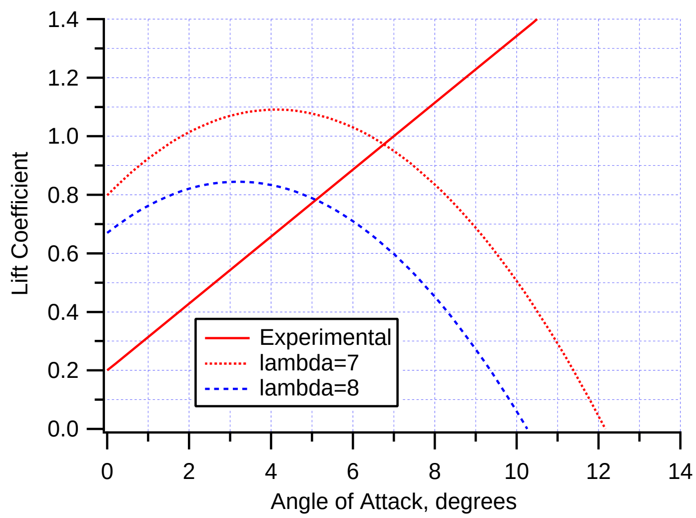
But the experimentally determined blade performance fixes the lift coefficient as a function of angle of attack:

$$C_l = 0.1143\alpha + 0.2$$

The geometry of the blade has been determined in part a). This defines $\theta_P$, the section pitch. The tip speed ratio is also a prescribed value. Thus there are two equations with two unknowns: $C_l$ and $\alpha$. These can be solved graphically or numerically. Below is a table of the values of the experimentally determined lift coefficients and those from the strip theory equation for $\lambda = 7$ and for $\lambda = 8$. The angle of attack at which the experimentally determined lift coefficient equals that from strip theory is the angle attack found in turbine operation.

| $\alpha$ degrees | $C_l$ experiment | $C_l$ $\lambda=7$ | $C_l$ $\lambda=8$ |
|---|---|---|---|
| 2 | 0.429 | 1.014 | 0.821 |
| 2.2 | 0.451 | 1.028 | 0.828 |
| 2.4 | 0.474 | 1.040 | 0.834 |
| 2.6 | 0.497 | 1.051 | 0.839 |
| 2.8 | 0.520 | 1.061 | 0.842 |
| 3 | 0.543 | 1.070 | 0.844 |
| 3.2 | 0.566 | 1.077 | 0.845 |
| 3.4 | 0.589 | 1.082 | 0.844 |
| 3.6 | 0.611 | 1.086 | 0.842 |
| 3.8 | 0.634 | 1.089 | 0.838 |
| 4 | 0.657 | 1.091 | 0.833 |
| 4.2 | 0.680 | 1.091 | 0.827 |
| 4.4 | 0.703 | 1.089 | 0.819 |
| 4.6 | 0.726 | 1.087 | 0.810 |
| 4.8 | 0.749 | 1.083 | 0.800 |
| 5 | 0.771 | 1.077 | 0.788 |
| 5.2 | 0.794 | 1.071 | 0.775 |
| 5.4 | 0.817 | 1.063 | 0.761 |
| 5.6 | 0.840 | 1.053 | 0.745 |
| 5.8 | 0.863 | 1.042 | 0.728 |
| 6 | 0.886 | 1.030 | 0.710 |
| 6.2 | 0.909 | 1.017 | 0.690 |
| 6.4 | 0.931 | 1.002 | 0.669 |
| 6.6 | 0.954 | 0.986 | 0.646 |
| 6.8 | 0.977 | 0.968 | 0.623 |
| 7 | 1.000 | 0.949 | 0.598 |
| 7.2 | 1.023 | 0.929 | 0.571 |
| 7.4 | 1.046 | 0.908 | 0.543 |
| 7.6 | 1.069 | 0.885 | 0.514 |
| 7.8 | 1.091 | 0.860 | 0.483 |
| 8 | 1.114 | 0.835 | 0.452 |

The operating point can also be found graphically:




Either method gives:

For $\lambda = 7$, $C_l = .972$, at $\alpha = 6.75$ degrees.
For $\lambda = 8$, $C_l = 0.782$, and $\alpha = 5.10$ degrees.

The axial induction factor, $a$, can be determined from:

$$a = 1/\left[1 + 4\sin^2\varphi/(\sigma' C_L \cos\varphi)\right]$$

This gives:

For $\lambda = 7$, $a = 0.338$.
For $\lambda = 8$, $a = 0.366$.

Thus, the ideal blade section would have a slightly lower angle of attack than the 7 degrees used in the ideal analysis, a slightly lower lift coefficient and a higher axial induction factor.

At the higher tip speed ratio, the angle of attack and lift coefficient decrease compared to the design conditions and the axial induction factor increases.

**3.5** a) Find $\theta$, $\theta_p$, $\theta_T$, and $c$ at all 10 locations (r/R = 0.10, 0.20, …, 1.0) for the Betz optimum blade. Assume $\lambda = 7$, $B = 3$, $R = 5$ m, and $C_l = 1.0$ and the minimum $C_d/C_l$ occurs at $\alpha = 7$.

b) Sketch the shape (planform) of the blade, assuming that all the quarter chords lie on a straight line.

c) Illustrate the blade twist by drawing plausible airfoils with properly proportioned chord lengths, centered at the quarter chord chords for $r/R$ = 0.10, 0.50, 1.0. Be sure to show where the wind is coming from and what the direction of rotation is.

**SOLUTION**

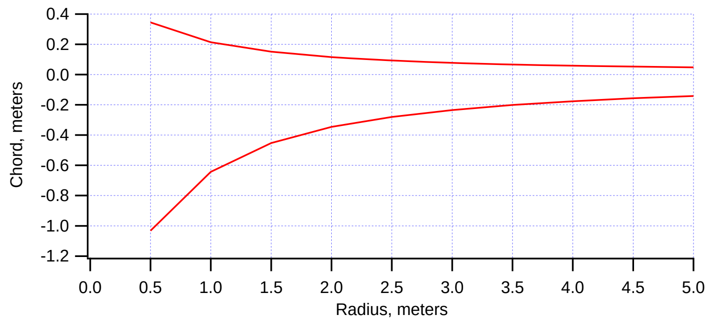
a) For r/R = 0.10, 0.20, ............., 1.0, the section radius, *r*, angle of relative wind, $\varphi$, chord, *c*, section pitch, $\theta_P$, and twist, $\theta_T$, can be determined from:

$$r = (r/R)R$$
$$\lambda_r = (r/R)\lambda$$
$$\varphi = \tan^{-1}\left(2/(3\lambda_r)\right)$$
$$c = 8\pi r \sin\varphi/(3BC_l\lambda_r)$$
$$\theta_P = \varphi - \alpha$$
$$\theta_T = \theta_P - \theta_{P,0}$$

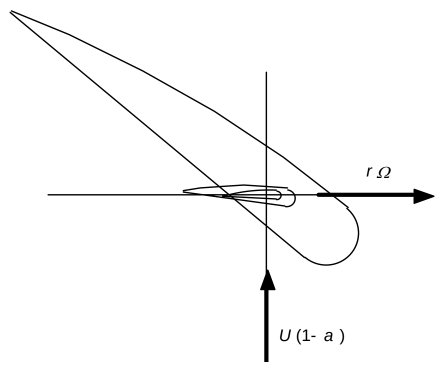
where $\theta_{P,0}$ is the blade pitch angle at the tip. The results of the calculations appear below.

| Section radius *r/R* | Section radius, *r* (m) | Angle of relative wind, $\varphi$ (degrees) | Section pitch, $\theta_p$ degrees | Section twist, $\theta_T$ (degrees) | Chord, c (m) |
|---|---|---|---|---|---|
| 0.10 | 0.50 | 43.60 | 36.60 | 38.16 | 1.376 |
| 0.20 | 1.00 | 25.46 | 18.46 | 20.02 | 0.858 |
| 0.30 | 1.50 | 17.61 | 10.61 | 12.17 | 0.604 |
| 0.40 | 2.00 | 13.39 | 6.39 | 7.95 | 0.462 |
| 0.50 | 2.50 | 10.78 | 3.78 | 5.34 | 0.373 |
| 0.60 | 3.00 | 9.02 | 2.02 | 3.58 | 0.313 |
| 0.70 | 3.50 | 7.75 | 0.75 | 2.31 | 0.269 |
| 0.80 | 4.00 | 6.79 | -0.21 | 1.35 | 0.236 |
| 0.90 | 4.50 | 6.04 | -0.96 | 0.60 | 0.210 |
| 1.00 | 5.00 | 5.44 | -1.56 | 0.00 | 0.189 |

b) The resulting plan form, with the quarter chords lying in a straight line appears below:




b) The blade twist, with the quarter chords lying in a straight line appears below:




Blade Twist Illustration

**3.6** Blades for a two-bladed wind turbine with a 24 m diameter have been designed for a tip speed ratio of 10. The 12-meter blades have the geometric and operational parameters listed in Table B.3 for operation at the design tip speed ratio. The rotor was designed assuming $C_l = 1.0$, $a' = 0$, no drag, and $a = 1/3$ using the methods outlined in the text for the design of an ideal rotor.

We want to know the rotor power coefficient for two assumed conditions: $C_d = 0$ and $C_d = 0.02$. Note that the two equations that have been derived do not serve our purpose here. Equation 3.90 requires a non-zero value for $a'$ and Equation 3.91 has also been derived using relationships between $a$ and $a'$ that require non-zero values of a'.

**Table B.3**

| Section radius $r/R$ | Section radius (m) | Section pitch, $\theta_p$ degrees | Angle of relative wind, $\varphi$ (degrees) | Section twist (degrees) | Chord, c (m) |
|---|---|---|---|---|---|

| | | | | | |
|---|---|---|---|---|---|
| 0.05 | 0.60 | 46.13 | 53.13 | 49.32 | 4.02 |
| 0.15 | 1.80 | 16.96 | 23.96 | 20.15 | 2.04 |
| 0.25 | 3.00 | 7.93 | 14.93 | 11.12 | 1.30 |
| 0.35 | 4.20 | 3.78 | 10.78 | 6.97 | 0.94 |
| 0.45 | 5.40 | 1.43 | 8.43 | 4.61 | 0.74 |
| 0.55 | 6.60 | -0.09 | 6.91 | 3.10 | 0.60 |
| 0.65 | 7.80 | -1.14 | 5.86 | 2.04 | 0.51 |
| 0.75 | 9.00 | -1.92 | 5.08 | 1.27 | 0.45 |
| 0.85 | 10.20 | -2.52 | 4.48 | 0.67 | 0.39 |
| 0.95 | 11.40 | -2.99 | 4.01 | 0.20 | 0.35 |

a) Starting with the definitions of the blade forces and the definition of $C_P$:

$$C_P = P/P_{wind} = \frac{\int_{r_h}^{R} \Omega \, \mathrm{d}Q}{\frac{1}{2}\rho\pi R^2 U^3}$$

derive as simple an equation as you can for the power coefficient, $C_P$, of an ideal Betz limit rotor. The equation should include both lift and drag coefficients and tip speed ratio, and should assume that $a = 1/3$. Ignore tip losses

b) Using the above equation find the rotor $C_P$ at the design tip speed ratio assuming that there is no drag ($C_d = 0$). How does this compare with the Betz limit?

c) For a first approximation of the effect of drag on rotor performance, find the Cp for the same rotor at the design tip speed ratio assuming the more realistic conditions that $C_d/C_l$ actually equals 0.02. Assume that the drag has no effect on the aerodynamics and that the operating conditions assumed for the ideal rotor without drag apply. What effect does drag have on the rotor $C_P$, compared to the $C_P$ assuming that $C_d = 0$?

**SOLUTION**

a) The power coefficient is defined as:

$$C_P = P/P_{wind} = \frac{\int_{r_h}^{R} \Omega \, \mathrm{dQ}}{\frac{1}{2}\rho\pi R^2 U^3}$$

where:

$$dQ = \mathrm{B\ r\ dF}_{T1}$$

or

$$dQ = B \tfrac{1}{2} \rho U_{rel}^{2} \left(C_l \sin\varphi - C_d \cos\varphi\right) \mathrm{c\ r\ dr} \tag{3.71}$$

Inserting this into the definition for $C_P$ and simplifying:

$$C_P = \left(\frac{\Omega\, B}{\pi R^2 U^3}\right)\int_{r_h}^{R} U_{rel}{}^2 \left(C_l \sin\varphi - C_d \cos\varphi\right) \mathrm{c\ r\ dr}$$

Given that: $$U_{rel} = U(1-a)/\sin\varphi \tag{3.64}$$

and $$\lambda = \Omega R/U$$

$$C_P = \left(\frac{\lambda\, B}{\pi R^3}\right)\int_{r_h}^{R} \frac{(1-a)^2 \left(C_l - C_d \cot\varphi\right) \mathrm{c\ r\ dr}}{\sin\varphi}$$

Because *a* is assumed to be 1/3:

$$C_P = \left(\tfrac{4}{9}\right)\left(\frac{\lambda\, B}{\pi R^3}\right)\int_{r_h}^{R} \left(C_l - C_d \cot\varphi\right) \mathrm{c\ r\ dr}$$

This can be approximated by a summation over *i* blade sections:

$$C_P = \left(\tfrac{4}{9}\right)\left(\frac{\lambda\, B}{\pi R^3}\right)\sum_i \frac{\left(C_l - C_d \cot\theta_i\right) \mathrm{c_i\ r_i\ dr}}{\sin\varphi}$$

b) Using the equation derived in part a) with $C_d = 0$:

| Section radius $r/R$ | Local tip speed ratio $\lambda_r$ | Angle of rel. wind $\varphi$ | Local $C_P$ Contribution $\mathrm{d}C_P$ |
|---|---|---|---|
| 0.05 | 0.5 | 53.13 | 0.006 |
| 0.15 | 1.5 | 23.96 | 0.018 |
| 0.25 | 2.5 | 14.93 | 0.030 |
| 0.35 | 3.5 | 10.78 | 0.041 |
| 0.45 | 4.5 | 8.43 | 0.053 |
| 0.55 | 5.5 | 6.91 | 0.065 |
| 0.65 | 6.5 | 5.86 | 0.077 |
| 0.75 | 7.5 | 5.08 | 0.089 |
| 0.85 | 8.5 | 4.48 | 0.101 |
| 0.95 | 9.5 | 4.01 | 0.113 |
| | | Total $C_P$: | 0.593 |

Thus, the $C_p$ for this rotor at the design tip speed, if one assumes no drag, is equal to the Betz limit of 16/27 = 0.593.

b) Using the equation derived in part a) with $C_d = 0.02$:

| Section radius $r/R$ | Local tip speed ratio $\lambda_r$ | Angle of rel. wind $\varphi$ | Local $C_P$ Contribution $dC_P$ |
|---|---|---|---|
| 0.05 | 0.5 | 53.13 | 0.006 |
| 0.15 | 1.5 | 23.96 | 0.017 |
| 0.25 | 2.5 | 14.93 | 0.027 |
| 0.35 | 3.5 | 10.78 | 0.037 |
| 0.45 | 4.5 | 8.43 | 0.046 |
| 0.55 | 5.5 | 6.91 | 0.054 |
| 0.65 | 6.5 | 5.86 | 0.062 |
| 0.75 | 7.5 | 5.08 | 0.069 |
| 0.85 | 8.5 | 4.48 | 0.075 |
| 0.95 | 9.5 | 4.01 | 0.081 |
| | | Total $C_P$ : | 0.474 |

For $C_d = 0.02$ the $C_P$ is 0.474, or 80% of the $C_P$ determined without drag. Thus, for this simple approximation, 20% of the rotor power is dissipated in viscous drag with $C_d/C_l = 0.02$.

**3.7** The Better Wind Turbine Company wants to start marketing wind turbines. The plans call for a 20 meter in diameter, three-bladed, wind turbine. The rotor is to have its peak power coefficient at a tip speed ratio of 6.5. The airfoil to be used has a lift coefficient of 1.0 and a minimum drag to lift ratio at an angle of attack of 7 degrees.

You, as the new blade designer, are to come up with two blade shapes as a starting point for the blade design. One shape assumes that there are no losses and that there is no wake rotation. The second design is based on the optimum rotor shape assuming that there is wake rotation (but still no losses).

Find the chord length, pitch, and twist at 10 stations of the blade, assuming that the blade extends right to the center of the rotor. How do the chord lengths and the twists compare at the tip and at the inner three blade stations?

**SOLUTION**

a) For $r/R$ = 0.05, 0.15, 0.25, ............, 0.95, the section radius, $r$, angle of relative wind, $\varphi$, chord, $c$, section pitch, $\theta_p$, and twist, $\theta_T$ for the Betz limit rotor without wake rotation can be determined from:

$$r = (r/R)R$$
$$\lambda_r = (r/R)\lambda$$
$$\varphi = \tan^{-1}\left(2/(3\lambda_r)\right)$$
$$c = 8\pi r \sin\varphi/(3BC_l\lambda_r)$$
$$\theta_P = \varphi - \alpha$$
$$\theta_T = \theta_P - \theta_{P,0}$$

where $\theta_{P,0}$ is the blade pitch angle at the tip. The angle of the relative wind, the chord, and the pitch can be determined for each blade location. The twist can only be determined once the pitch is determined at the very end of the blade.

For the rotor with wake rotation the same formulas apply except that:

$$\varphi = \left(\tfrac{2}{3}\right)\tan^{-1}(1/\lambda_r)$$

$$c = \frac{8\pi r}{BC_l}(1-\cos\varphi)$$

The results of the calculations for the Betz rotor without wake rotation are:

| Section radius $r/R$ | Section radius $r$ (m) | Chord $c$ (m) | Section pitch, $\theta_p$ (degrees) | Angle of relative wind, $\varphi$ (degrees) | Section twist, $\theta_T$ (degrees) |
|---|---|---|---|---|---|
| 0.05 | 0.5 | 3.86 | 57.01 | 64.01 | 58.15 |
| 0.15 | 1.5 | 2.42 | 27.36 | 34.36 | 28.51 |
| 0.25 | 2.5 | 1.63 | 15.31 | 22.31 | 16.45 |
| 0.35 | 3.5 | 1.21 | 9.33 | 16.33 | 10.48 |
| 0.45 | 4.5 | 0.95 | 5.84 | 12.84 | 6.98 |
| 0.55 | 5.5 | 0.79 | 3.56 | 10.56 | 4.71 |
| 0.65 | 6.5 | 0.67 | 1.97 | 8.97 | 3.11 |
| 0.75 | 7.5 | 0.58 | 0.79 | 7.79 | 1.93 |
| 0.85 | 8.5 | 0.51 | -0.12 | 6.88 | 1.02 |
| 0.95 | 9.5 | 0.46 | -0.84 | 6.16 | 0.31 |
| 1 | 10 | 0.44 | -1.14 | 5.86 | 0.00 |

The results of the calculations for the Betz rotor with wake rotation are:

| Section radius $r/R$ | Section radius $r$ (m) | Chord $c$ (m) | Section pitch, $\theta_p$ (degrees) | Angle of relative wind, $\varphi$ (degrees) | Section twist, $\theta_T$ (degrees) |
|---|---|---|---|---|---|
| 0.05 | 0.5 | 1.39 | 41.00 | 48.00 | 42.17 |
| 0.15 | 1.5 | 1.74 | 23.48 | 30.48 | 24.65 |

| | | | | | |
|---|---|---|---|---|---|
| 0.25 | 2.5 | 1.40 | 14.07 | 21.07 | 15.24 |
| 0.35 | 3.5 | 1.11 | 8.82 | 15.82 | 9.99 |
| 0.45 | 4.5 | 0.91 | 5.58 | 12.58 | 6.75 |
| 0.55 | 5.5 | 0.76 | 3.42 | 10.42 | 4.59 |
| 0.65 | 6.5 | 0.65 | 1.88 | 8.88 | 3.05 |
| 0.75 | 7.5 | 0.57 | 0.73 | 7.73 | 1.90 |
| 0.85 | 8.5 | 0.51 | -0.16 | 6.84 | 1.01 |
| 0.95 | 9.5 | 0.46 | -0.87 | 6.13 | 0.30 |
| 1 | 10 | 0.43 | -1.17 | 5.83 | 0.00 |

At the tip the chord lengths and twist are almost the same. At the inner three blade sections the twist and chord of the Betz limit blade increase rapidly as the radius decreases. The blade design that assumes wake rotation has a maximum chord length at $r/R$ = 0.15 that is much less than that of the other blade. It also has increasing twist at the radius decreases, but the twist is less than the Betz limit blade.

**3.8** The Better Wind Turbine Company wants to start marketing wind turbines. Their plans call for a turbine that produces 100 kW in a 12 m/s wind at a cold site (-22.8°C, -9°F) with an air density of 1.41 kg/m$^3$. They have decided on a 20 meter in diameter, three-bladed, wind turbine. The rotor is to have its peak power coefficient at a tip speed ratio of 7 in a 12 m/s wind. The airfoil to be used has a lift coefficient of 1.0 and a minimum drag to lift ratio at an angle of attack of 7 degrees.

a) You, as the new blade designer, are to come up with the blade shape as a starting point for the blade design. The design is to be based on the optimum rotor shape assuming that there is wake rotation (but no drag or tip losses). Find the chord length, pitch, and twist at 9 stations of the blade (each 1 m long), assuming that the hub occupies the inner tenth of the rotor.

b) Determine the rotor $C_P$ assuming $C_d$ = 0. Again determine the power coefficient assuming that the drag coefficient is 0.02, and that the aerodynamics are the same as the condition without any drag. How much power is lost due to drag? Which part of the blade produces the most power?

c) Does it look like the chosen design is adequate to provide the power that the Better Wind Turbine Company wants?

**SOLUTION**

a) For $r/R$ = 0.15, 0.25, ............., 0.95, the section radius, $r$, angle of relative wind, $\varphi$, chord, $c$, section pitch, $\theta_p$, and twist, $\theta_T$ for the optimum rotor with wake rotation can be determined from:

$$r = (r/R)R$$
$$\lambda_r = (r/R)\lambda$$
$$\varphi = \left(\tfrac{2}{3}\right)\tan^{-1}(1/\lambda_r)$$
$$c = \frac{8\pi r}{BC_l}(1 - \cos\varphi)$$
$$\theta_P = \varphi - \alpha$$
$$\theta_T = \theta_P - \theta_{P,0}$$

where $\theta_{P,0}$ is the blade pitch angle at the tip. The angle of the relative wind, the chord, and the pitch can be determined for each blade location. The twist can only be determined once the pitch is determined at the very end of the blade. The results of the calculations appear below.

| Section radius $r/R$ | Section radius $r$ (m) | Chord $c$ (m) | Section pitch, $\theta_p$ (degrees) | Angle of relative wind, $\varphi$ (degrees) | Section twist, $\theta_T$ (degrees) |
|---|---|---|---|---|---|
| 0.15 | 1.5 | 1.34 | 22.07 | 29.07 | 23.65 |
| 0.25 | 2.5 | 1.58 | 12.83 | 19.83 | 14.41 |
| 0.35 | 3.5 | 1.24 | 7.80 | 14.80 | 9.38 |
| 0.45 | 4.5 | 0.97 | 4.74 | 11.74 | 6.32 |
| 0.55 | 5.5 | 0.79 | 2.71 | 9.71 | 4.29 |
| 0.65 | 6.5 | 0.66 | 1.26 | 8.26 | 2.84 |
| 0.75 | 7.5 | 0.57 | 0.19 | 7.19 | 1.77 |
| 0.85 | 8.5 | 0.49 | -0.64 | 6.36 | 0.94 |
| 0.95 | 9.5 | 0.44 | -1.30 | 5.70 | 0.28 |
| 1 | 10 | 0.39 | -1.58 | 5.42 | 0.00 |

b) The power coefficient can be determined from:

$$C_P = \left(8/\lambda^2\right)\int_{\lambda_h}^{\lambda} \sin^2\varphi(\cos\varphi - \lambda_r \sin\varphi)(\sin\varphi + \lambda_r \cos\varphi)\left[1 - (C_d/C_l)\cot\varphi\right]\lambda_r^{\,2} d\lambda_r$$

where $$\lambda_r = \Omega r/U = (r/R)\lambda$$

The results for the two assumptions are tabulated below:

| $r/R$ | $C_P$ $C_d = 0.00$ | $C_P$ $C_d = 0.02$ |
|---|---|---|
| 0.15 | 0.015 | 0.015 |
| 0.25 | 0.028 | 0.026 |
| 0.35 | 0.040 | 0.037 |
| 0.45 | 0.052 | 0.047 |
| 0.55 | 0.064 | 0.057 |
| 0.65 | 0.076 | 0.066 |
| 0.75 | 0.088 | 0.074 |
| 0.85 | 0.100 | 0.082 |
| 0.95 | 0.112 | 0.090 |
| Total | 0.576 | 0.493 |

When wake rotation is included, the power coefficient is slightly less than the Betz limit when the drag is assumed to be zero. If the drag is assumed to be 0.02, the $C_P$ is only 85.6% of the value if the drag is zero. The contributions to the total $C_P$ from each section indicate that the outer part of the blade produces the majority of the power.

c) The expected power can be calculated from:

$$P = C_P \tfrac{1}{2} \rho A U^3$$

At-22.8°C the density of dry air at standard atmospheric pressure is 1.41 kg/m$^3$. The swept area is 314.2 m$^3$. Assuming that the turbine $C_P$ is 0.493 (including realistic drag), the rotor power would be 188.7 kW. Even with tip losses, losses due to a more easily manufactured non-ideal shape, and mechanical and electrical losses, the turbine would easily meet the design criteria.

**3.9** A two-bladed wind turbine is designed using one of the LS-1 family of airfoils. The 13 m long blades for the turbine have the following specifications (Table B.4).

**Table B.4** LS-1 Airfoil Blade Geometry

| r/R | Section Radius (m) | chord (m) | twist (degrees) |
|---|---|---|---|
| 0.05 | 0.65 | 1.00 | 13.000 |
| 0.15 | 1.95 | 1.00 | 11.000 |
| 0.25 | 3.25 | 1.00 | 9.000 |
| 0.35 | 4.55 | 1.00 | 7.000 |
| 0.45 | 5.85 | 0.87 | 5.000 |

| | | | |
|---|---|---|---|
| 0.55 | 7.15 | 0.72 | 3.400 |
| 0.65 | 8.45 | 0.61 | 2.200 |
| 0.75 | 9.75 | 0.54 | 1.400 |
| 0.85 | 11.05 | 0.47 | 0.700 |
| 0.95 | 12.35 | 0.42 | 0.200 |

Note: $\theta_{p,0}$ = -1.97 degrees (pitch at tip)

Assume that the airfoil's aerodynamics characteristics can be approximated as follows (note, $\alpha$ is in degrees):

For $\alpha$ < 21 degrees:

$$C_l = 0.42625 + 0.11628\,\alpha - 0.00063973\,\alpha^2 - 8.712\times10^{-5}\,\alpha^3 - 4.2576\times10^{-6}\,\alpha^4$$

For $\alpha > 21$: $C_l = 0.95$

$$C_d = 0.011954 + 0.00019972\,\alpha + 0.00010332\,\alpha^2$$

For the *midpoint* of section 6 ($r/R$ = 0.55) find the following for operation at a tip speed ratio of 8: a) angle of attack, $\alpha$; b) angle of relative wind, $\theta$; c) $C_l$ and $C_d$; d) the local contributions to $C_P$. Ignore the effects of tip losses.

**SOLUTION**

a) The angle of attack ($\alpha$) can be determined by equating the lift coefficient from the measured $C_l$ for the blade with the lift coefficient required by strip theory. From experiment:

For $\alpha$ < 21 degrees:

$$C_l = 0.42625 + 0.11628\,\alpha - 0.00063973\,\alpha^2 - 8.712\times10^{-5}\,\alpha^3 - 4.2576\times10^{-6}\,\alpha^4$$

For $\alpha > 21$: $C_l = 0.95$

From strip theory, for each section:

$$C_{l,i} = 4\sin\varphi_i \frac{(\cos\varphi_i - \lambda_{r,i}\sin\varphi_i)}{\sigma_i'(\sin\varphi_i + \lambda_{r,i}\cos\varphi_i)}$$

The pitch, local solidity, and local tip speed ratio for the center of each section are determined from:

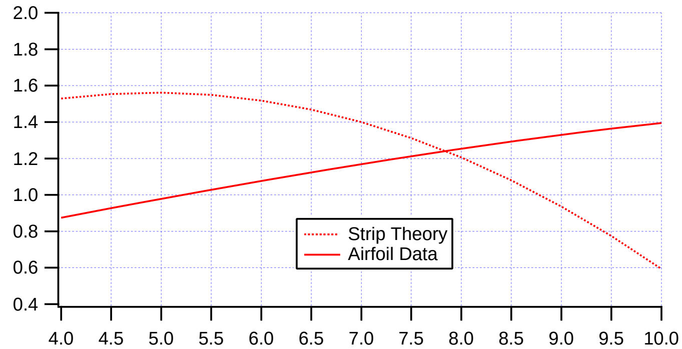
$$\theta_P = \theta_T + \theta_{P,0}$$

where $\theta_{P,0}$ = blade pitch angle at the tip.

$$\sigma' = Bc/2\pi r$$

$$\lambda_r = \lambda(r/R)$$

The simultaneous equations can be solved graphically, iteratively, or numerically. The graphical solution is illustrated below.




A numerical solution for the lift coefficient gives:

| $\alpha$ | Strip Theory | Lift Curve |
|---|---|---|
| 1 | 0.968622 | 0.541799 |
| 1.5 | 1.11033 | 0.598915 |
| 2 | 1.23256 | 0.655486 |
| 2.5 | 1.33541 | 0.711424 |
| 3 | 1.41894 | 0.766635 |
| 3.5 | 1.48322 | 0.821019 |
| 4 | 1.52832 | 0.874469 |
| 4.5 | 1.5543 | 0.926871 |
| 5 | 1.56122 | 0.978106 |
| 5.5 | 1.54914 | 1.02805 |
| 6 | 1.51811 | 1.07656 |
| 6.5 | 1.46819 | 1.12352 |
| 7 | 1.39942 | 1.16876 |
| 7.5 | 1.31184 | 1.21214 |
| 8 | 1.20551 | 1.2535 |
| 8.5 | 1.08044 | 1.29268 |
| 9 | 0.936697 | 1.32951 |
| 9.5 | 0.774296 | 1.3638 |
| 10 | 0.593272 | 1.39538 |
| 10.5 | 0.393653 | 1.42406 |

| | | |
|---|---|---|
| 11 | 0.175462 | 1.44963 |

By either method, the operating angle of attack is 7.84 degrees at a tip speed ratio of 8.

b) From the blade geometry $\varphi = \alpha + \theta_P$ and $\theta_P = \theta_T + \theta_{P,0}$. Thus for section 6 the angle of the relative wind ($\varphi$) is 9.27 degrees.

c) $C_l$ and $C_d$: From the equations for the lift and drag coefficients, $C_l$ = 1.24 and $C_d$ = .020.

d) The local contribution to $C_P$ can be determined from:

$$C_{P,i} = \left(\frac{8\Delta\lambda_r}{\lambda^2}\right)\sin^2\varphi_i\left(\cos\varphi_i - \lambda_{ri}\sin\varphi_i\right)\left(\sin\varphi_i + \lambda_{ri}\cos\varphi_i\right)\left[1 - \left(\frac{C_d}{C_l}\right)\cot\varphi_i\right]\lambda_{ri}^{\ 2}$$

Plugging in the angle of the relative wind and the local tip speed ratio, the local contribution to $C_P$ is 0.057.

**3.10** This problem is based on the blades used for the UMass wind machine WF-1. Refer to Table B.5 for the blade geometry at specific locations along the blade. There is no airfoil below *r/R* = 0.10

In addition, note that for the NACA 4415 airfoil: for $\alpha$ < 12 degrees: $C_l$ = 0.368 + 0.0942$\alpha$, $C_d = 0.00994 + 0.000259\,\alpha + 0.0001055\,\alpha^2$ (note, $\alpha$ is in degrees). Radius: 4.953 m (16.25 ft); No. of blades: 3; Tip speed ratio: 7; Rated wind speed: 11.62 m/s (26 mph). The pitch at the tip is: $\theta_{P,0}$ = -2 degrees

a) Divide the blade into 10 sections (but assume that the hub occupies the innermost 1/10). For the *midpoint* of each section find the following: i) angle of attack, $\alpha$; ii) angle of relative wind, $\varphi$; iii) $C_l$ and $C_d$; iv) the local contributions to $C_P$ and thrust. Include the effects of tip losses.

b) Find the overall power coefficient. How much power would the blades produce at 11.62 m/s (26 mph)? Include drag and tip losses. Assume an air density of 1.23 kg/m$^3$.

**Table B.5** WF-1 Airfoil Geometry

| *r/R* | Radius (ft) | Radius (m) | Chord (ft) | Chord (m) | Twist (degrees) |
|---|---|---|---|---|---|
| 0.10 | 1.63 | 0.495 | 1.35 | 0.411 | 45.0 |
| 0.20 | 3.25 | 0.991 | 1.46 | 0.455 | 25.6 |
| 0.30 | 4.88 | 1.486 | 1.26 | 0.384 | 15.7 |
| 0.40 | 6.50 | 1.981 | 1.02 | 0.311 | 10.4 |
| 0.50 | 8.13 | 2.477 | 0.85 | 0.259 | 7.4 |
| 0.60 | 9.75 | 2.972 | 0.73 | 0.223 | 4.5 |
| 0.70 | 11.38 | 3.467 | 0.63 | 0.186 | 2.7 |
| 0.80 | 13.00 | 3.962 | 0.55 | 0.167 | 1.4 |
| 0.90 | 14.63 | 4.458 | 0.45 | 0.137 | 0.40 |

| | | | | | |
|---|---|---|---|---|---|
| 1.00 | 16.25 | 4.953 | 0.35 | 0.107 | 0.00 |

**SOLUTION**

a) The physical measurements for the blade is given for each 1/10 of the blade radius. The blade radius ($r$), chord ($c$), twist ($\theta_T$), pitch ($\theta_p$), local solidity ($\sigma'$), and local tip speed ratio ($\lambda_r$) must all be determined for the center of each blade section to be used in the calculation. Interpolation from the given data gives:

| Section | Section Radius, $r/R$ | Section radius $r$ (ft) | Section radius $r$ (m) | Section twist $\theta_T$ (deg) | Section Chord $c$ (ft) | Section Chord $c$ (m) |
|---|---|---|---|---|---|---|
| 1 | 0.05 | 0.81 | 0.248 | hub | hub | hub |
| 2 | 0.15 | 2.44 | 0.743 | 35.30 | 1.405 | 0.428 |
| 3 | 0.25 | 4.06 | 1.239 | 20.65 | 1.360 | 0.415 |
| 4 | 0.35 | 5.69 | 1.734 | 13.05 | 1.140 | 0.347 |
| 5 | 0.45 | 7.31 | 2.229 | 8.90 | 0.935 | 0.285 |
| 6 | 0.55 | 8.94 | 2.724 | 5.95 | 0.790 | 0.241 |
| 7 | 0.65 | 10.56 | 3.219 | 3.60 | 0.680 | 0.207 |
| 8 | 0.75 | 12.19 | 3.715 | 2.05 | 0.590 | 0.180 |
| 9 | 0.85 | 13.81 | 4.210 | 0.90 | 0.500 | 0.152 |
| 10 | 0.95 | 15.44 | 4.705 | 0.20 | 0.400 | 0.122 |
| Tip | 1 | 16.25 | 4.953 | 0.00 | 0.350 | 0.107 |

The pitch, local solidity, and local tip speed ratio for the center of each section are determined from:

$$\theta_P = \theta_T + \theta_{P,0}$$

$$\sigma' = Bc/2\pi r$$

$$\lambda_r = \lambda(r/R)$$

The results of the calculations are:

| Section | Section radius r/R | Local solidity $\sigma'$ | Local tip speed ratio, $\lambda_r$ | Section pitch $\theta_p$, deg |
|---|---|---|---|---|
| 1 | 0.05 | hub | hub | hub |
| 2 | 0.15 | 0.275 | 1.05 | 33.30 |
| 3 | 0.25 | 0.160 | 1.75 | 18.65 |
| 4 | 0.35 | 0.096 | 2.45 | 11.05 |
| 5 | 0.45 | 0.061 | 3.15 | 6.90 |
| 6 | 0.55 | 0.042 | 3.85 | 3.95 |
| 7 | 0.65 | 0.031 | 4.55 | 1.60 |
| 8 | 0.75 | 0.023 | 5.25 | 0.05 |
| 9 | 0.85 | 0.017 | 5.95 | -1.10 |
| 10 | 0.95 | 0.012 | 6.65 | -1.80 |
| Tip | 1 | | 7.00 | -2.00 |

The angle of attack ($\alpha$) can be determined by equating lift coefficient from the measured $C_l$ for the blade with the lift coefficient required by strip theory. From experiment:

$$C_l = 0.368 + 0.0942\,\alpha$$

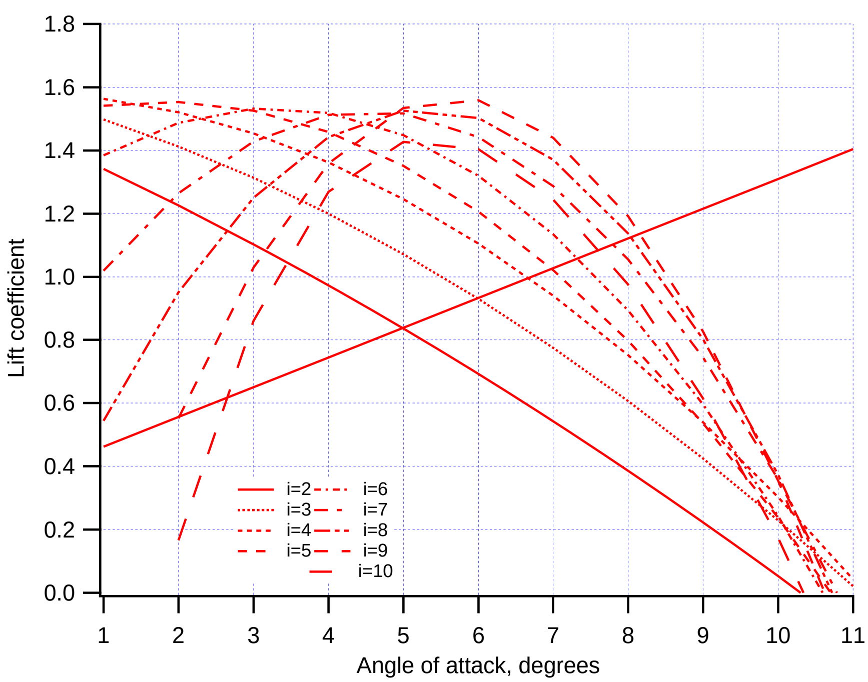
From strip theory:

$$C_{l,i} = 4F_i \sin\varphi_i \frac{(\cos\varphi_i - \lambda_{r,i} \sin\varphi_i)}{\sigma_i'(\sin\varphi_i + \lambda_{r,i} \cos\varphi_i)}$$

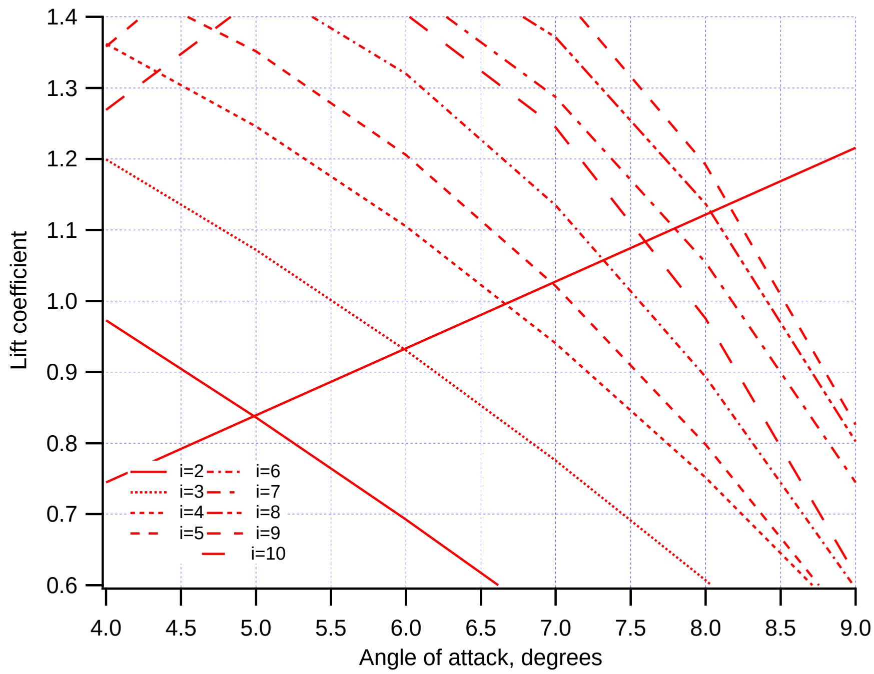
where
$$F_i = (2/\pi)\cos^{-1}\left[\exp\left(-\left\{\frac{(B/2)[1-(r_i/R)]}{(r_i/R)\sin\varphi_i}\right\}\right)\right]$$

where
$$\varphi = \theta_P + \alpha$$

The simultaneous equations can be solved graphically, iteratively, or numerically. The graphical solution is illustrated below. The curves for $i$ =8, 9, and 10 cross the airfoil lift curve in two places. The right-hand intersection points 1) provide a solution in which the angles of attack vary smoothly along the airfoil, and 2) yield axial induction factors that are less than 1/2, a requirement of momentum theory. A calculation shows that left hand intersection points yield solutions with axial induction factors much greater than 1/2.



A detailed look at the graph follows:




Numerical solution gives:

| Section | r/R | $\alpha$, deg | $C_l$ | $C_d$ | F | $\varphi$, deg |
|---|---|---|---|---|---|---|
| 1 | 0.05 | hub | hub | hub | hub | hub |
| 2 | 0.15 | 4.99 | 0.838 | 0.0139 | 1.000 | 38.29 |
| 3 | 0.25 | 5.99 | 0.932 | 0.0153 | 1.000 | 24.64 |
| 4 | 0.35 | 6.67 | 0.997 | 0.0164 | 1.000 | 17.72 |
| 5 | 0.45 | 6.98 | 1.025 | 0.0169 | 1.000 | 13.88 |
| 6 | 0.55 | 7.34 | 1.059 | 0.0175 | 0.999 | 11.29 |
| 7 | 0.65 | 7.81 | 1.104 | 0.0184 | 0.995 | 9.41 |
| 8 | 0.75 | 8.04 | 1.125 | 0.0188 | 0.982 | 8.09 |
| 9 | 0.85 | 8.17 | 1.138 | 0.0191 | 0.926 | 7.07 |
| 10 | 0.95 | 7.63 | 1.087 | 0.0181 | 0.696 | 5.83 |

The local contribution to $C_P$ and thrust can be determined from:

$$C_{P,i} = \left(\frac{8\Delta\lambda_r}{\lambda^2}\right)\sin^2\varphi_i\left(\cos\varphi_i - \lambda_{ri}\sin\varphi_i\right)\left(\sin\varphi_i + \lambda_{ri}\cos\varphi_i\right)\left[1-\left(\frac{C_d}{C_l}\right)\cot\varphi_i\right]\lambda_{ri}^{\ 2}$$

and $$dF_N = B\tfrac{1}{2}\rho U_{rel}^{\,2}\left(C_l\cos\varphi + C_d\sin\varphi\right)c\Delta r$$

where $$U_{rel} = \frac{U(1-a)}{\sin\varphi} = \frac{U}{(\sigma' C_l/4F)\cot\varphi + \sin\varphi}$$

To determine the thrust an air density must be assumed. Assuming an air density of $\rho$ = 1.23 kg/m$^3$ (0.0765 lbm/ft$^3$) the following values can be determined.

| Section | Local Cp | Relative m/s | Relative ft/s | Thrust N | Thrust lbf | Torque Nm | Torque ft-lb |
|---|---|---|---|---|---|---|---|
| 1 | hub | hub | hub | hub | hub | hub | hub |
| 2 | 0.0092 | 16.78 | 55.06 | 73.1 | 16.43 | 41.42 | 30.55 |
| 3 | 0.0211 | 23.34 | 76.56 | 175.3 | 39.4 | 95.27 | 70.27 |
| 4 | 0.0329 | 30.66 | 100.6 | 283.7 | 63.77 | 148.30 | 109.38 |
| 5 | 0.0432 | 38.33 | 125.76 | 380.7 | 85.59 | 194.90 | 143.75 |
| 6 | 0.0536 | 46.16 | 151.44 | 486.4 | 109.35 | 241.77 | 178.32 |
| 7 | 0.0639 | 54.07 | 177.41 | 602.1 | 135.35 | 288.20 | 212.57 |
| 8 | 0.0725 | 62.05 | 203.58 | 703.4 | 158.13 | 326.86 | 241.08 |
| 9 | 0.0769 | 70.06 | 229.85 | 769.9 | 173.07 | 346.88 | 255.85 |
| 10 | 0.0652 | 78.08 | 256.17 | 732.2 | 164.6 | 293.96 | 216.82 |
| Total | 0.4385 | | | 4206.8 | 945.69 | 1977.53 | 1458.6 |

b) The overall power coefficient can be determined by summing the individual contributions from each blade section. The result, from the table above is 0.439. This calculation uses the equation above, which includes drag and tip losses. The total power produced by the blades is:

$$P = C_P P_{wind} = C_P \tfrac{1}{2}\rho\pi R^2 U^3$$

where $U$ is the free stream wind velocity, 11.62 m/s (26 mph). The total power produced by the blades is 32.51 kW.

c) The total thrust and torque at 11.62 m/s (26 mph) can also be determined by summing the individual contributions from each blade section. The total thrust on three blades is 4.21 kN and the total torque from the three blades is 1.98 kNm.

**3.11** A two-bladed wind turbine is designed using one of the LS-1 family of airfoils. The 13 m long blades for the turbine have the specifications listed previously in Table B.4.

Assume that the airfoil's aerodynamics characteristics can be approximated as follows (note, $\alpha$ is in degrees):

For $\alpha$ < 21 degrees:

$$C_l = 0.42625 + 0.11628\,\alpha - 0.00063973\,\alpha^2 - 8.712\times 10^{-5}\,\alpha^3 - 4.2576\times 10^{-6}\,\alpha^4$$

For $\alpha > 21$: $C_l = 0.95$

$$C_d = 0.011954 + 0.00019972\,\alpha + 0.00010332\,\alpha^2$$

For the *midpoint* of the outermost section of the blade ($r/R$ = 0.95) find the following for operation at a tip speed ratio of 8: a) angle of attack, $\alpha$, with and without tip losses; b) angle of relative wind, $\theta$; c) $C_l$ and $C_d$; d) the local contributions to $C_P$; e) the tip loss factor, $F$, and the axial induction factor, $a$, with and without tip losses. How do tip losses affect aerodynamic operation and the local contribution to $C_P$ at this outermost section?

**SOLUTION**

a) The angle of attack ($\alpha$) can be determined by equating the lift coefficient from the measured $C_l$ for the blade with the lift coefficient required by strip theory. From experiment:

For $\alpha < 21$ degrees:

$$C_l = 0.42625 + 0.11628\,\alpha - 0.00063973\,\alpha^2 - 8.712\times 10^{-5}\,\alpha^3 - 4.2576\times 10^{-6}\,\alpha^4$$

For $\alpha > 21$: $C_l = 0.95$

From strip theory, for each section:

$$C_{l,i} = 4F_i \sin\varphi_i \,\frac{\left(\cos\varphi_i - \lambda_{r,i}\sin\varphi_i\right)}{\sigma_i'\left(\sin\varphi_i + \lambda_{r,i}\cos\varphi_i\right)}$$

where

$$F_i = \left(2/\pi\right)\cos^{-1}\left[\exp\left(-\left\{\frac{(B/2)\left[1-(r_i/R)\right]}{(r_i/R)\sin\varphi_i}\right\}\right)\right]$$

and

$$\varphi = \theta_P + \alpha$$

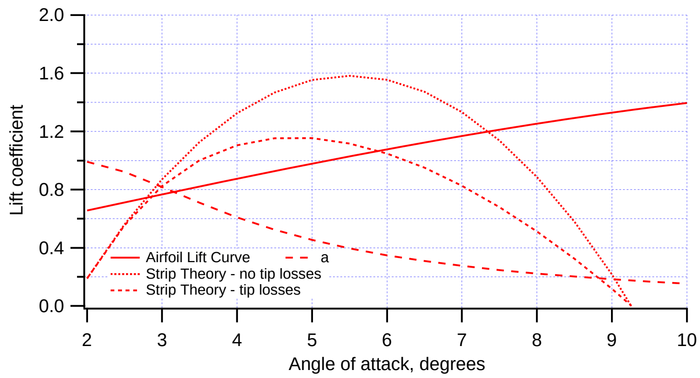
The pitch, local solidity, and local tip speed ratio for the center of each section are determined from:

$$\theta_P = \theta_T + \theta_{P,0}$$

$$\sigma' = Bc/2\pi r$$

$$\lambda_r = \lambda\left(r/R\right)$$

The simultaneous equations can be solved graphically, iteratively, or numerically. The graphical solution is illustrated below. It can be seen that the curve of the angle of attack calculated from strip theory intersects the airfoil lift curve at two points. The axial induction factor as a function of angle of attack is the same for each case and is also shown on the graph below. Strip Theory requires that $a$ <= 0.5. Thus the right intersection points are the correct solutions.




A numerical solution gives:

| Angle of Attack $\alpha$ | Airfoil lift curve | Strip Theory without tip losses | Strip Theory with tip losses | Chord c |
|---|---|---|---|---|
| 2 | 0.655 | 0.189 | 0.189 | 0.991 |
| 2.5 | 0.711 | 0.559 | 0.553 | 0.922 |
| 3 | 0.767 | 0.871 | 0.823 | 0.818 |
| 3.5 | 0.821 | 1.126 | 1.000 | 0.709 |
| 4 | 0.874 | 1.325 | 1.105 | 0.610 |
| 4.5 | 0.927 | 1.467 | 1.152 | 0.525 |
| 5 | 0.978 | 1.553 | 1.154 | 0.454 |
| 5.5 | 1.028 | 1.582 | 1.117 | 0.396 |
| 6 | 1.077 | 1.555 | 1.048 | 0.348 |
| 6.5 | 1.124 | 1.472 | 0.951 | 0.308 |
| 7 | 1.169 | 1.333 | 0.827 | 0.275 |
| 7.5 | 1.212 | 1.137 | 0.680 | 0.247 |
| 8 | 1.254 | 0.886 | 0.512 | 0.223 |
| 8.5 | 1.293 | 0.579 | 0.324 | 0.202 |
| 9 | 1.330 | 0.216 | 0.117 | 0.184 |

By either method, without including tip losses, the operating angle of attack is 7.35 degrees at a tip speed ratio of 8. When tip losses are included in the calculation the angle of attack is determined to be 5.89 degrees.

b) From the blade geometry $\varphi = \alpha + \theta_P$ and $\theta_P = \theta_T + \theta_{P,0}$. $C_l$ and $C_d$ can be calculated from the equations for the lift and drag coefficients using the angle of attack. The local contribution to Cp can be determined from:

$$C_{P,i} = \left(\frac{8\Delta\lambda_r}{\lambda^2}\right)\sin^2\varphi_i\left(\cos\varphi_i - \lambda_{ri}\sin\varphi_i\right)\left(\sin\varphi_i + \lambda_{ri}\cos\varphi_i\right)\left[1-\left(\frac{C_d}{C_l}\right)\cot\varphi_i\right]\lambda_{ri}{}^2$$

where
$$F_i = (2/\pi)\cos^{-1}\left[\exp\left(-\left\{\frac{(B/2)[1-(r_i/R)]}{(r_i/R)\sin\varphi_i}\right\}\right)\right]$$

And the axial induction factor, $a$, can be determined from:

$$a = 1\Big/\left[1+4F\sin^2\varphi\big/(\sigma' C_l\cos\varphi)\right]$$

The results of these calculations are tabulated below:

| | $\alpha$ (degrees) | $\varphi$ (degrees) | $C_l$ | $C_d$ | Local $C_P$ | F | a |
|---|---|---|---|---|---|---|---|
| No Tip Losses | 7.35 | 5.58 | 1.199 | 0.019 | 0.0898 | 1.00 | 0.52 |
| Tip Losses | 5.89 | 4.12 | 1.066 | 0.017 | 0.0548 | 0.68 | 0.45 |

The tip losses result in a reduced angle of attack and resulting reductions in lift coefficient and angle of the relative wind. In the outermost blade section of this example the contribution to the power produced by the rotor is reduced by 39% due to the effect of tip losses.

**3.12** A two-bladed wind turbine is operated at two different tip speed ratios. At 8.94 m/s (20 mph) (tip speed ratio = 9) one of the blade sections has an angle of attack of 7.19 degrees. At 16.09 m/s (36 mph) (tip speed ratio = 5) the same 1.22 m (4 ft) section of the blade is starting to stall, with an angle of attack of 20.96 degrees. Given the following operating conditions and geometric data, determine the relative wind velocities, the lift and drag forces, and the tangential and normal forces developed by the blade section at the two different tip speed ratios. Determine, also, the relative contribution (the fraction of the total) of the lift and drag forces to the tangential and normal forces developed by the blade section at the two different tip speed ratios.

How do the relative velocities and the lift and drag forces compare? How do the tangential and normal forces compare? How do the effects of lift and drag change between the two operating conditions?

Operating Conditions are listed in Table B.6:

**Table B.6** Operating conditions

| $\lambda$ | $\alpha$ | $\varphi$ | a |
|---|---|---|---|
| 5 | 20.96 | 22.21 | .070 |
| 9 | 7.19 | 8.44 | .390 |

This particular blade section is 1.22 m (4 ft) long, has a chord length of 0.811 m (2.66 ft), and has a center radius of 5.49 m (18 ft). The following lift and drag coefficients are valid for $\alpha < 21$ degrees, where $\alpha$ is in degrees:

$$C_l = 0.42625 + 0.11628\,\alpha - 0.00063973\,\alpha^2 - 8.712 \times 10^{-5}\,\alpha^3 - 4.2576 \times 10^{-6}\,\alpha^4$$
$$C_d = 0.011954 + 0.00019972\,\alpha + 0.00010332\,\alpha^2$$

**SOLUTION**

The information given in the problem statement can be used to solve for the relative wind velocity at the blades using:

$$U_{rel} = U(1-a)/\sin\varphi$$

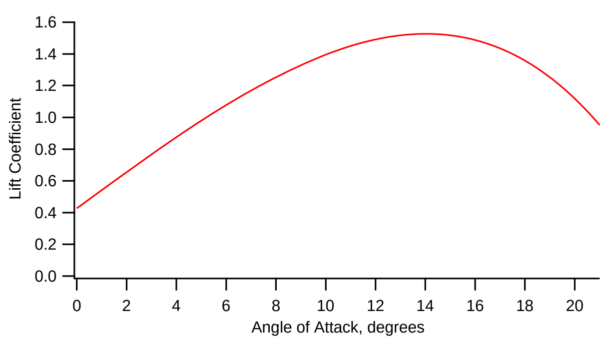
The results are:

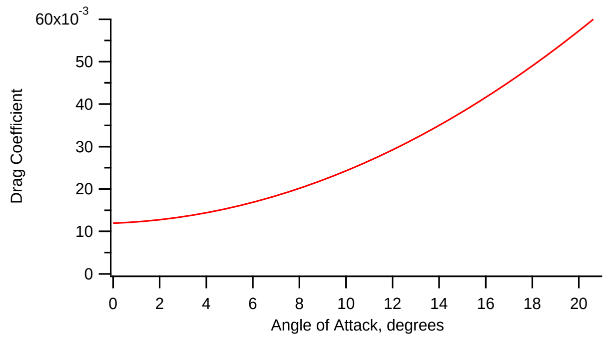
| $\lambda$ | U m/s | $U_{rel}$ m/s |
|---|---|---|
| 5 | 16.09 | 39.61 |
| 9 | 8.94 | 37.13 |

The airfoil characteristics can be used to determine the lift and drag coefficients at the prescribed angles of attack using:

$$C_l = 0.42625 + 0.11628\,\alpha - 0.00063973\,\alpha^2 - 8.712 \times 10^{-5}\,\alpha^3 - 4.2576 \times 10^{-6}\,\alpha^4$$
$$C_d = 0.011954 + 0.00019972\,\alpha + 0.00010332\,\alpha^2$$

The lift and drag curves are shown below.






For lambda=5: $C_l = 0.959$ and $C_d = 0.062$.
For lambda=9: $C_l = 1.186$ and $C_d = 0.019$.

To determine the forces an air density must be assumed. Assuming an air density of $\rho$ = 1.23 kg/m$^3$, the forces can be determined using:

$$dF_l = C_l \, \tfrac{1}{2} \rho U_{rel}{}^2 c \, \mathrm{dr}$$

$$dF_d = C_d \, \tfrac{1}{2} \rho U_{rel}{}^2 c \, \mathrm{dr}$$

$$dF_N = dF_l \cos\varphi + dF_d \sin\varphi$$

$$dF_T = dF_l \sin\varphi - dF_d \cos\varphi$$

The fraction of the normal force caused by lift is:

$$dF_{Nl} = \frac{dF_l \cos\varphi}{dF_N}$$

The fraction of the normal force caused by drag is:

$$dF_{Nd} = \frac{dF_d \sin\varphi}{dF_N}$$

The fraction of the tangential force caused by lift is:

$$dF_{Tl} = \frac{dF_l \sin\varphi}{dF_T}$$

The fraction of the tangential force caused by drag is:

$$dF_{Td} = \frac{-dF_d \cos\varphi}{dF_T}$$

The results of these calculations appear below:

| $\lambda$ | 5 | 9 | units |
|---|---|---|---|
| $U_{rel}$ | 39.61 | 37.13 | m/s |
| Angle of Attack | 20.96 | 7.19 | degrees |
| | | | |
| Lift Force | 915.55 | 994.93 | N |
| Drag Force | 59.19 | 15.94 | N |
| | | | |
| Normal Force | 870.00 | 986.49 | N |
| Contribution from lift | 0.97 | 1.00 | |
| Contribution from drag | 0.03 | 0.00 | |
| | | | |
| Tangential Force | 291.28 | 130.26 | N |
| Contribution from lift | 1.19 | 1.12 | |
| Contribution from drag | -0.19 | -0.12 | |

In English units:

| $\lambda$ | 5 | 9 | |
|---|---|---|---|
| $U_{rel}$ | 129.95 | 121.82 | ft/s |
| Angle of Attack | 20.96 | 7.19 | degrees |
| | | | |
| Lift Force | 204.69 | 222.45 | lbf |
| Drag Force | 13.23 | 3.56 | lbf |
| | | | |
| Normal Force | 194.5 | 220.56 | lbf |
| Contribution from lift | 0.97 | 1.00 | |
| Contribution from drag | 0.03 | 0.00 | |
| | | | |
| Tangential Force | 65.12 | 29.13 | lbf |
| Contribution from lift | 1.19 | 1.12 | |
| Contribution from drag | -0.19 | -0.12 | |

An inspection of the table shows that:

1. The relative velocities and lift forces for the two conditions are very similar.

2. The drag force is greater at the lower tip speed ratio (stalled) condition and drag forces at both conditions are much less than the lift forces.

3. The normal forces are also very similar for the two conditions but drag contributes a slightly greater proportion of the force in the stalled condition.

4. The tangential forces at the blade section are very different for the two operating conditions. Due to the much greater angle of attack at the higher wind speed, the tangential force is much greater than at the lower wind speed even though the lift coefficient is somewhat less and the drag coefficient is greater at the higher wind speed. The drag decreases the torque produced by the rotor and the relative negative contribution of the drag to the tangential force is greater in the stalled condition. Due to the much higher tangential force at the higher wind speed, the absolute contribution of both the lift and the drag forces to the tangential force is greater at the higher wind speed.

**3.13** Assume a VAWT operates with an axial unduction factor of 0.3 and with a tip speed ratio of 12. Equations 3.147 and 3.149 can be used to determine the ratio of $U_{rel}$ to $U$ and the angle of attack. Equations 3.148 and 3.150 can be used estimate those same quantities for high tip speed ratios. What is the maximum percentage deviation of the estimates of these quanties from the exact values as the blade of the VAWT makes one full rotation? If the blades are replaced and now the turbine operates at a tip speed ratio of 20 and the same axial induction factor?

**SOLUTION**

At a tip speed ratio of 12, the error in the approximation of $U_{rel}/U$ is only 0.2% but the error of the approximation of the axial induction factor is up to 5.8%. At tip speed ratios of 20, these errors are reduced to less than 0.1% and less than or equal to 3.5%. The details of the solution can be found in *VAWT_High Tipspeed Approximations_Solution.xls*.

**3.14** Determine the operating axial induction factor of a vertical axis wind turbine with straight blades using Equation 3.156. Guess an "input" axial induction factor and solve the right hand side of the equation. Using the quadratic equation solve the left side for the value of the axial induction factor that is less than 0.5. If the two do not agree, adjust the input value until it does. The resulting axial induction factor is the solution. Assume the following input conditions and that the lift coefficient as a function of the angle of attack is the same as that for a flat plate (Equation 3.51):

| | | |
|---|---|---|
| Number of blades | B | 3 |
| Radius | R | 6 |
| Chord | c | 0.1 |
| Tip speed ratio | lambda | 15 |

**SOLUTION**

The axial induction factor for these conditions is 0.295. The details of the solution are contained in *VAWT_Lambda_a_Solution.xls*.

## B.4 Chapter 4 Problems

**4.1** A wind turbine rotor turning at 60 rpm is brought to a stop by a mechanical brake. The rotor inertia is 13,558 kg $m^2$.

a) What is the kinetic energy in the rotor before it is stopped? How much energy does the brake absorb during the stop?

b) Suppose that all the energy is absorbed in a steel brake disc with a mass of 27 kg. Ignoring losses, how much does the temperature of the steel brake disc rise during the stop? Assume a specific heat for steel of 0.46 kJ/kg C.

**SOLUTION**

a) A rotating body contains kinetic energy given by:

$$E = \frac{1}{2} J \Omega^2$$

Here the angular speed = 6.283 rad/s. $J$ = 13 558 kgm$^2$ so the total energy that must be absorbed by the brake is 267.6 kJ.

b) The brake plate has a mass, $m$, of 27 kg. Steel has a specific heat, $c_p$, of 0.46 kJ/kg-C. Therefore the temperature increase, $\Delta T$ , is:

$$\Delta T = \frac{Q}{mc_p}$$

where $Q$ is the energy absorbed by the brake. Therefore:

$$\Delta T = (267.6 \text{ kJ}) / [\ (0.46)(27)\ ] = 21.5 \text{ C} = 38.8 \text{ F}$$

**4.2** A wind turbine has a rotor with polar mass moment of inertia, $J$, of 4.2 x $10^6$ kg $m^2$. It is operating in steady winds at a power of 1,500 kW and a rotational speed of 20 rpm. Suddenly the connection to the electrical network is lost and the brakes fail to apply. Assuming that there are no changes in the aerodynamic forces, how long does it take for the operating speed to double?

**SOLUTION**

Find the rotational speed in rad/s:

$$\Omega = N * \pi / 30 = 2.09 \text{ rad/s}$$

Find the torque on the rotor

$$Q = P / \Omega = (1{,}500\, kNm/s) / 2.09\, rad/s = 716 \text{ kNm}$$

Find angular acceleration, $\alpha$

$$\alpha = Q / J = 716\, kNm / 4.2E6\, kg\, m^2 = 0.171 \text{ rad/s}^2$$

The time to double speed is:

$$t = \frac{2\Omega_0 - \Omega_0}{\alpha} = \frac{2.09}{0.171} = 12.3 \text{ sec}$$

Note: in a real situation the aerodynamic torque would not remain constant over this speed range, so this simple approach to a solution would not be sufficient.

**4.3** A cantilevered 2 m long main shaft of a wind turbine holds a 1500 kg hub and rotor at its end. At rated power the turbine develops 275 kW and rotates at 60 rpm. The shaft is a 0.15 m in diameter cylindrical steel shaft.

a) How much does the shaft bend down at its end as a result of the load of the rotor and hub?

b) How much does the shaft twist when the turbine is operating at rated power? What is the maximum shear stress in the shaft?

**SOLUTION**

a) The shaft can be treated as a beam with a point load at the end. The area moment of inertia for a shaft is:

$$I = \frac{\pi d^4}{64}$$

Using $d = 0.15$ m, the area moment of inertia comes out to be $I = 2.48\ 10^{-5}$ m$^4$.

The maximum deflection of a beam with a load at the end is:

$$y_{\max} = \frac{WL^3}{3EI}$$

Here $L$ =2 m and the modulus of elasticity for steel is: $E = 210$ GPa.

The load on the shaft is $W = mg$ = (1500 kg)(9.81 m/s$^2$) = 14.715 kN. Using the equation for the maximum beam deflection, the shaft bends:

$$y_{\max} = 0.0075 \text{ m} = 7.5 \text{ mm.}$$

b) At rated load the torque, $Q$, is:

$$Q = \text{(Power/Angular Speed)} = 275 \text{ kW/ (6.283 rad/s)} = 42.77 \text{ kNm.}$$

For a solid shaft the polar moment of inertia, $J$, is given by:

$$J = \frac{\pi d^4}{32} = \frac{\pi r^4}{2}$$

$d =$ 0.15 m so $J = 4.97 * 10^{-5}$ m$^4$

The angular deflection is:

$$\varphi = \frac{QL}{JG}$$

If we estimate $G$ from E and Poisson's ratio ( $\mu$ ):

$$G = \frac{E}{2(1+\mu)}$$

Assuming $\mu$ = 0.3, $E$ = 210 GPa, then $G$ = 80.8 GPa. The length $L$ = 2 m. Therefore the deflection at the end of the shaft is 0.0213 radians = 1.22 degrees.

The shear stress can be determined from

$$\tau = \frac{\varphi G r}{L}$$

Thus the maximum shear stress at the outside of the shaft is 64.5 MPa.

**4.4** A wind turbine on a 24.38 m (80 ft) tower is subject to a thrust load of 26.69 kN (6000 lbf) during operation at 250 kW, the rated power of the turbine. In a 44.7 m/s (100 mph) hurricane the thrust load on the stopped turbine is expected to be 71.62 kN (16,100 lbf).

a) If the tower is a steel tube 1.22 m (4 ft) in outer diameter (O.D.) with a 0.0254 m (1 in) thick wall, how much will the top of the tower move during rated operation and in the hurricane force winds?

b) Suppose the tower were a three-legged lattice tower with the specifications given in Figure B.2, how much will the top of the tower move during rated operation and in the hurricane force winds? Ignore any effect of cross bracing on the tower.

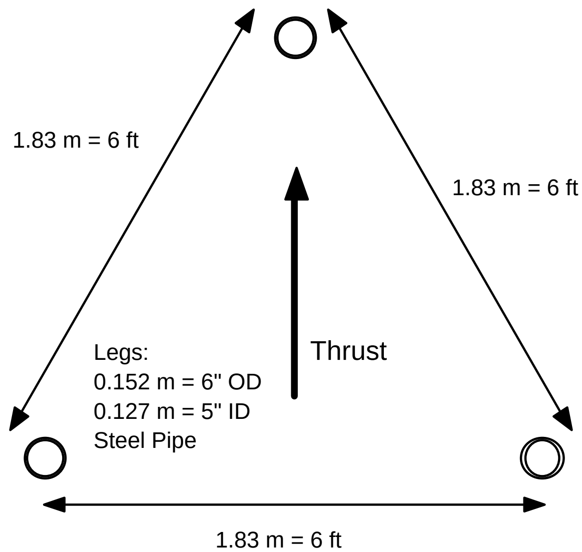



**Figure B.2** Lattice tower cross-section; ID, inner diameter; OD, outer diameter

**SOLUTION**

a) The tower can be treated as a beam with a point load at the top. The area moment of inertia, $I$, for an annular ring is:

$$I = \frac{\pi\left(d_0^4 - d_i^4\right)}{64}$$

Using 1.22 m for the O.D. ($d_0$) and 1.17 m for the I.D. ($d_i$), the area moment of inertia comes out to be $I = 0.0168$ m$^4$. Using 48 in. for the O.D. and 46 in. for the I.D., the area moment of inertia is $I = 40790$ in$^4$ = 1.97 ft$^4$.

The maximum deflection of a beam with a load at the end is:

$$y_{\max} = \frac{WL^3}{3EI}$$

The modulus of elasticity for steel is usually assumed to be $E = 210$ GPa or $E = 30 * 10^6$ psi = $4.32 * 10^9$ lbf/ft$^2$.

The tower length is $L = 24.38$ m = 80 ft. and the load at the top of the tower, $W$, is 26.69 kN = 6000 lbf. Using the equation for the maximum beam deflection, the tower moves:

$$y_{\max} = 0.0365 \text{ m} = 0.120 \text{ ft} = 1.44 \text{ in.}$$

In a hurricane the tower moves:

$$y_{\max} = 0.098 \text{ m} = 0.322 \text{ ft} = 3.86 \text{ in.}$$

b) Once again the tower can be treated as a beam with a point load at the top, but in this case the tower is a composite of 3 pipes. First the centroid of the tower structure, the area of the pipes, and the area moment of inertia of the pipe need to be determined. Then the area moment of inertia of the composite tower needs to be determined using the parallel axis theorem.

The area moment of inertia for the pipe is the same as for the annular ring:

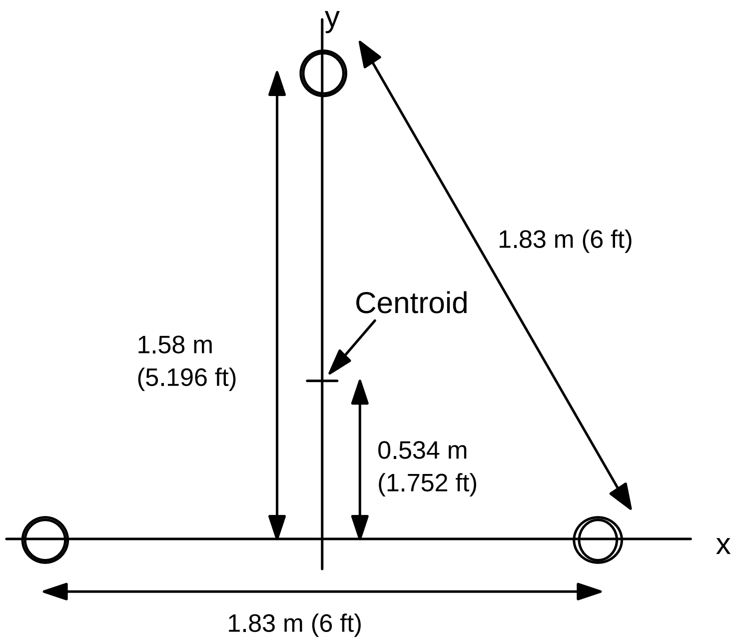
$$I = \frac{\pi\left(d_0^4 - d_i^4\right)}{64}$$

The area moment of inertia is $I = 1.34 * 10^{-5}$ m$^4$ = 32.9 in$^4$ = 0.0016 ft$^4$. The area of the pipe cross section is 0.0055 m$^2$ = 0.060 ft$^4$.

The centroid of the structure can be determined from:

$$\bar{y} = \frac{\sum y_i A_i}{\sum A_i}$$

where $A_i$is the area of each pipe and $y_i$ is the distance from a convenient axis. Using the axes illustrated below, it can be determined that the upper pipe is 1.58 m (5.196 ft) above the two lower ones and that the centroid is located 0.534 m (1.752 ft) above the two lower ones. The centroid is located on the y axis due to symmetry.




The area moment of inertia about an axis through the centroid of the tower can be determined using:

$$I_C = \sum I_i + A_i \bar{y}_i^{\,2}$$

where:
$I_C$ = moment of inertia about the centroid of each pipe
$\bar{y}_i$ = Distance between the centroid of each pipe and the centroid of the tower.
$A_i$ = Cross section area of each pipe.

Thus:

$$I_C = 1.085\ \mathrm{ft}^4 = 0.0092\ \mathrm{m}^4$$

The maximum deflection of a tower, treated as a beam with a load at the end is:

$$y_{\max} = \frac{WL^3}{3EI_C}$$

The modulus of elasticity for steel is usually assumed to be $E$ = 210 GPa or $E = 30 * 10^6$ psi = $4.32*10^9$ lbf/ft$^2$.

Using the equation for the maximum beam deflection, the tower moves:

$$y_{\max} = 0.067 \text{ m} = 0.218 \text{ ft under operation}$$

$$y_{\max} = 0.179 \text{ m} = 0.586 \text{ ft in a hurricane}$$

**4.5** The wind turbine rotor shown in Figure B.3 has a rotor rotation velocity, $\Omega$, of 1 Hz (60 rpm) and is yawing at an angular velocity, $\omega$, of 10 degrees per second. The polar mass moment of inertia of the rotor is 13,558 kg m$^2$ (10,000 slug ft$^2$). The rotor weighs 1459 kg (100 slugs) and is 3.05 m (10 feet) from the center of the bed plate bearing support. Centered over the bed plate bearing support are the bearings holding the main shaft. These bearings are 0.91 m (3 ft) apart. The directions of positive moments and rotation are indicated in the figure.

a) What are the bearing loads when the turbine is not yawing?
b) What are the bearing loads when the turbine is yawing?

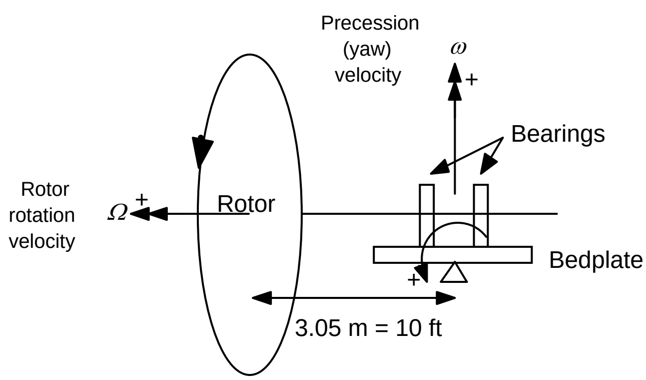



**Figure B.3**

**SOLUTION**

a) The sum of the moments about the tower top and the sum of the forces must both be equal to zero. If $W$ is the weight of the rotor, $F_1$ is the downward force of the left bearing on the shaft and $F_2$ is the upward force of the right bearing on the shaft then the sum of the moments about the tower top requires that:

$$(W)(3.05 \text{ m}) - F_1(0.455 \text{ m}) - F_2(0.455 \text{ m}) = 0$$

The balance of the forces requires:

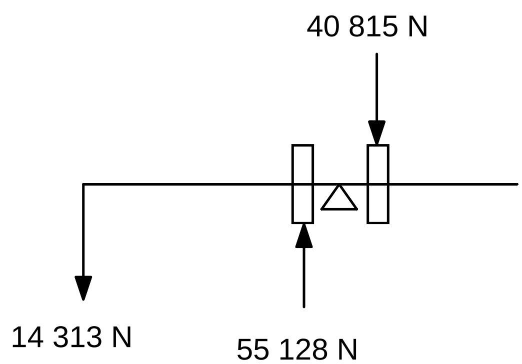
$$W - F_1 + F_2 = 0$$

$W$ = (1459 kg) (9.81 m/s$^2$). Solving these two equations gives:

$$F_1 = 55{,}128 \text{ N} = 12\,393 \text{ lbf}$$

$$F_2 = 40{,}815 \text{ N} = 9175 \text{ lbf}$$

The resulting forces at the bearings are shown below.




b) The moment caused by the yaw rotation is:

$$\vec{M} = \vec{\omega} \times I\vec{\Omega}$$

Thus, the magnitude of $M$ is:

$$M = \Omega \omega I$$

The yaw rate, $\omega$, is 10 degrees/s or 0.174 rad/s. The rotor angular velocity is, $\Omega$, $2\pi$ rad/s and the rotor polar moment of inertia, $I$, is 13,558 kgm$^2$. Thus, the moment caused by the rotation rate is + 14,823 Nm = 10,932 ft-lbf, adding to the static moment caused by the blade weight. The resulting gyroscopic force at the bearings is:

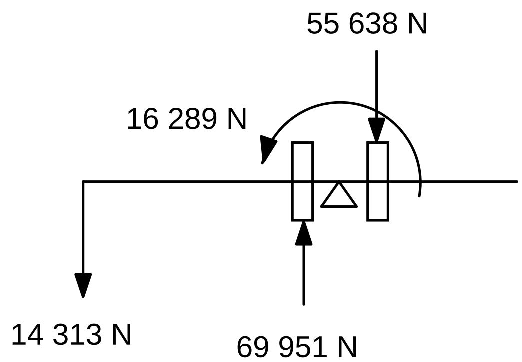
$$F_G = 14{,}823 \text{ Nm} / 0.91 \text{ m} = 16\,289 \text{ N} = 3{,}662 \text{ lbf}$$

Adding the two forces together, the resulting forces are:

$$F_1 = 69{,}951 \text{ N} = 15{,}724 \text{ lbf}$$

$$F_2 = 55{,}638 \text{ N} = 12{,}507 \text{ lbf}$$

These are shown in the figure below.




**4.6** A 3 bladed wind turbine has a rotor diameter of 38.1 m. The blade chord is constant and equal to 1.0 m. Each blade has a non-rotating natural frequency of 1.67 Hz. In operation, rotating at 50 rpm, each blade has a natural frequency of 1.92 Hz. The pitch angle of the blade is 1.8 degrees. (The hub is sufficiently small that it can be ignored.) The turbine is operating in a wind of 8 m/s. The airfoil lift curve slope is equal to 2π. Air density is 1.225 kg/m$^3$. If the blade has a mass of 898 kg, what (a) hinge–spring stiffness, (b) moment of inertia and (c) offset would be used to model the blade dynamics with the hinge spring blade model?

**SOLUTION**

As described in the text, the non-dimensional hinge offset is given by:

$$e = \frac{2(Z-1)}{3+2(Z-1)}$$

where

$$Z = \frac{\omega_R^2 - \omega_{NR}^2}{\Omega^2}$$

The mass moment of inertia of the hinged blade can be approximated by:

$$I_b = M_b \frac{R^2}{3}[1-e]^3$$

and the flapping spring constant is:

$$K_\beta = \omega_{NR}^2 I_b$$

In this case the non-rotating natural frequency is 1.67 Hz (= 10.49 rad/s), the rotating natural frequency is 1.92 Hz (= 12.06 rad/s), and the rotor speed is 50 rpm (= 5.236 rad/s).

Thus $Z$ is 1.2902 and

$e = 0.162$.

(The MiniCodes confirm this)

Then the moment of inertia of the equivalent blade is

$I_b = 63{,}900$ kg m$^2$

(The Minicode say 63,903 kg m$^2$)

The spring constant (stiffness) is:

$K_\beta = 7{,}007$ kNm/rad.

(The MiniCodes say 7,001 kNm/rad)

**4.7** Consider the wind turbine rotor used in Problem 4.6. Using the hinge spring blade model, find the flapping angle during these conditions. How far is the tip of the blade from the plane of rotation?

Hint: In solving this, there are a number of intermediate calculations that need to be made. It may be helpful to confirm your results with the mini-codes. In any case, check the units of all parameters carefully.

**SOLUTION**

This is solved by the following: $\beta_0 = \dfrac{\gamma A}{2K}$

It is first necessary to find $\gamma$, $K$ and $A$.

Find $\gamma$:

$$\gamma = \rho C_{L\alpha} c\, R^4 / I_b = 15.85$$

Find $K$ from:

$$K = 1 + \varepsilon + K_\beta / I_b \Omega^2$$

Use this for $I_B$:

$$I_b = m_B (R^2/3)(1-e)^3 = 63{,}900\, kg\, m^2 \text{ (from Problem 4.6)}$$

$$\varepsilon = \frac{3+e}{2(1-e)} = 0.29 \text{ , using } e = 0.164\text{, from Problem 4.6}$$

Then, using $K_\beta = 7{,}007$ from Problem 4.6:

$$K = 1 + \varepsilon + K_\beta / I_b \Omega^2 = 1 + 0.29 + \frac{7{,}007{,}000}{(63{,}900)(12.47)} = 5.29$$

Find the axisymmetric flow term, $A$, using:

$$A = (\Lambda / 3) - (\theta_p / 4) = 0.0097$$

Assume, $\lambda = \Omega R / U = (5.236)(19.04) / 18 = 12.46$

First find non-dimensional inflow, assuming $a$ = 1/3:

$$\Lambda = U(1-a) / \Omega R = 0.0535$$

(Note that this definition is slightly different than Eggleston and Stoddard's original, which is used in the MiniCodes. The answers are almost the same, however.)

Therefore:

$$A = (\Lambda / 3) - (\theta_p / 4) = \frac{0.535}{3} - \frac{(1.8)(\pi / 180)}{4} = 0.0097$$

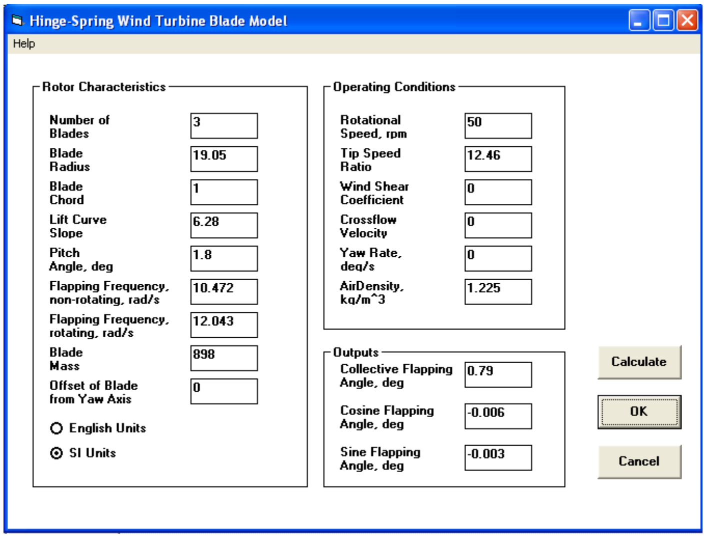
Find flapping angle, $\beta_0$, from $\beta_0 = \frac{\gamma A}{2K}$

$$\beta_0 = \frac{\gamma A}{2K} = \frac{(15.85)(0.0097)}{(2)(5.29)} = 0.0145\, rad = 0.83 \deg$$

(Compare: The MiniCodes say 0.79 deg. The difference is due to method used to calculate non-dimensional inflow, $\Lambda$)

The deflection = sin($\beta$)*radius = 0.277 m

The output from the MiniCodes is shown below.




**4.8** Blade bending moments are being measured on a research wind turbine on a day with mean hub height winds of 9.14 m/s (30 ft/s), but a wind shear results in winds of 12.19 m/s (40 ft/s) at the top of the blade tip path and 6.09 m/s (20 ft/s) at the bottom of the blade tip path. A gust of wind has started the wind turbine yawing at a steady rate of 0.1 radians per second in the $+X'$ direction (see Figure 4.17 in the text). Meanwhile a wind direction change results in a crosswind of +0.61 m/s (2.0 ft/s). The 24.38 m (80 ft) diameter turbine starts with a flap hinge angle, $\beta$, of 0.05 radians and, at the moment that measurements of are being made, the rate of change of the flap hinge angle is 0.01 rad/s. The rotor is 3.05 m/s (10 ft) from the yaw axis on this fixed speed turbine that rotates at a speed of 1 Hz. The very efficient turbine is operating with an axial induction factor of 1/3.

Ignoring tower shadow and transient effects, if these operating conditions were to persist for one revolution of the rotor, what would the perpendicular and tangential wind as a function of azimuth angle be half way out on the blades? What would the magnitude of the contributions to the perpendicular wind be from yaw rate, shear, crosswind, and blade flapping? How would the angle of attack vary at this part of the blade as the azimuth changes? Assume that the blade pitch angle, $\theta_p$, is 0.05 radians (2.86 degrees).

**SOLUTION**

The perpendicular and tangential components of the wind at the blade are:

$$U_P = U\,(1-a) - r\,\dot{\beta} - (V_0\,\beta + q\,r)\sin(\psi) - U(r/R)K_{vs}\cos(\psi)$$

$$U_T = \Omega\,r - (V_0 + q\,d_{yaw})\cos(\psi)$$

where:
$U$ = Free stream wind velocity
$a$ = Axial induction factor

$V_0$ = Cross wind velocity (due to yaw error)
$q$ = Yaw rate (constant)
$K_{sh}$ = Wind shear coefficient (linear)
$d_{yaw}$ = Distance from rotor plane to yaw axis

The component of the perpendicular wind velocity due to the mean wind and axial induction factor is:

$$U_{PU} = U(1-a) = (9.14)\left(\frac{2}{3}\right) = 6.09 \text{ m/s} = 20 \text{ ft/s}$$

The component of the perpendicular wind velocity due to the flapping motion is:

$$U_{P\dot{\beta}} = -r\,\dot{\beta} = -6.1(0.01) = -0.061 \text{ m/s} = -0.2 \text{ ft/s}$$

The component of the perpendicular wind velocity due to the crosswind is:

$$U_{P,crs} = -V_0\,\beta\sin(\psi) = -(0.61)(0.05)\sin(\psi) = -0.0305\sin(\psi)\,\text{m/s} = -0.1\sin(\psi)\,\text{ft/s}$$

The component of the perpendicular wind velocity due to the yaw rate is:

$$U_{P,yaw} = -q\ r\ \sin(\psi) = -(0.1)(6.1)\sin(\psi) = -0.61\sin(\psi)\,\text{m/s} = -2\sin(\psi)\,\text{ft/s}$$

The component of the perpendicular wind velocity due to the wind shear is:

$$U_{P,vs} = -U(r/R)K_{vs}\cos(\psi) = -9.14\left(\frac{1}{2}\right)\left(\frac{1}{3}\right)\cos(\psi) = -1.52\cos(\psi)\,\text{m/s} = -5\cos(\psi)\,\text{ft/s}$$

Taking all of the terms into account, the perpendicular wind velocity is:

$$U_P = 6.04 - 0.64\sin(\psi) - 1.52\cos(\psi)\ \text{m/s} \quad = 19.8 - 2.1\sin(\psi) - 5\cos(\psi)\ \text{ft/s}$$

The component of the tangential wind velocity due to the rotor rotation is:

$$U_T = \Omega r = 2\pi(6.1) = 12.2 \text{ m/s} = 125.7 \text{ ft/s}$$

The component of the tangential wind velocity due to the crosswind is:

$$U_{T,crs} = -V_0\cos(\psi) = -0.61\cos(\psi)\,\text{m/s} = -2\cos(\psi)\,\text{ft/s}$$

The component of the tangential wind velocity due to the yaw rate is:

$$U_{T,yaw} = -q\ d_{yaw}\cos(\psi) = -(0.1)(3.05)\cos(\psi) = -0.31\cos(\psi)\,\text{ft/s} = -1\cos(\psi)\,\text{ft/s}$$

Taking all of the terms into account, the tangential wind velocity is:

$$U_T = 38.3 - 0.915\cos(\psi)\ \text{ m/s} = 125.7 - 3\cos(\psi)\ \text{ ft/s}$$

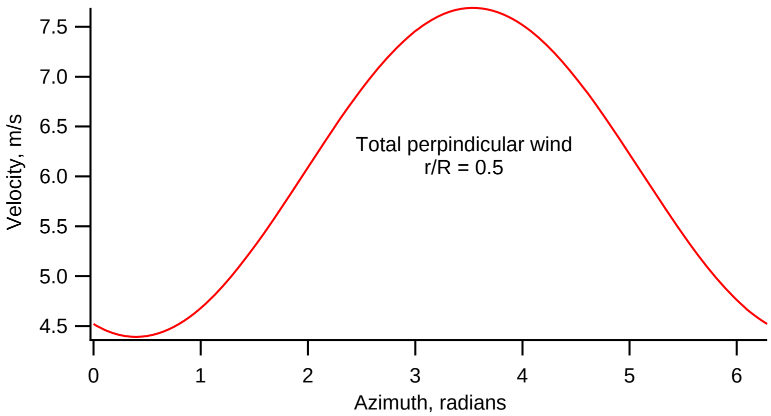
The angle of attack is:

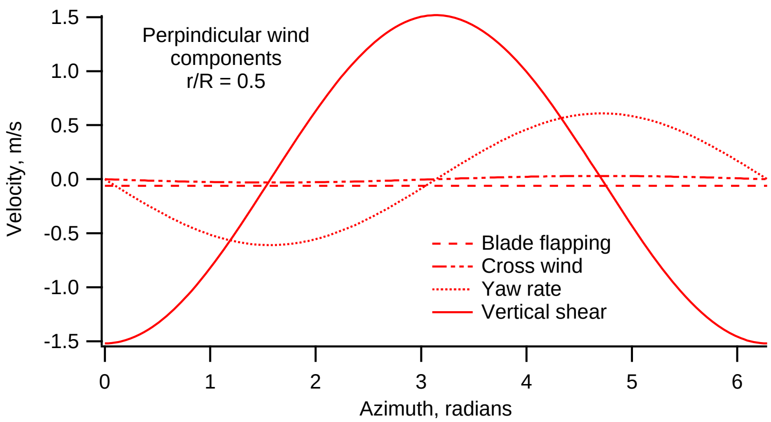
$$\alpha = \phi - \theta_p = \tan^{-1}(U_P/U_T) - \theta_p \approx U_P/U_T - \theta_p$$

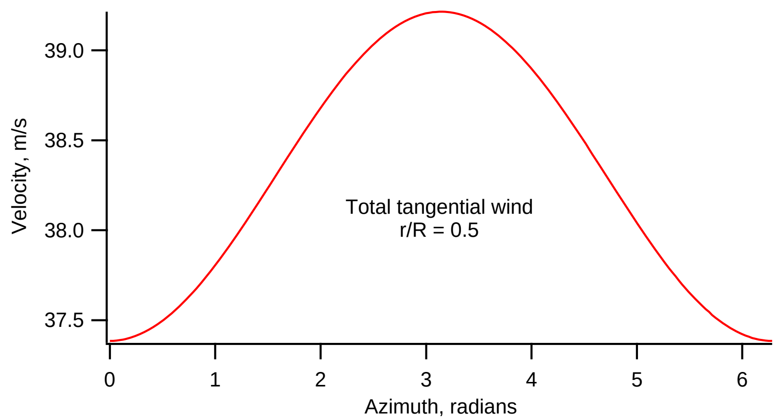
or

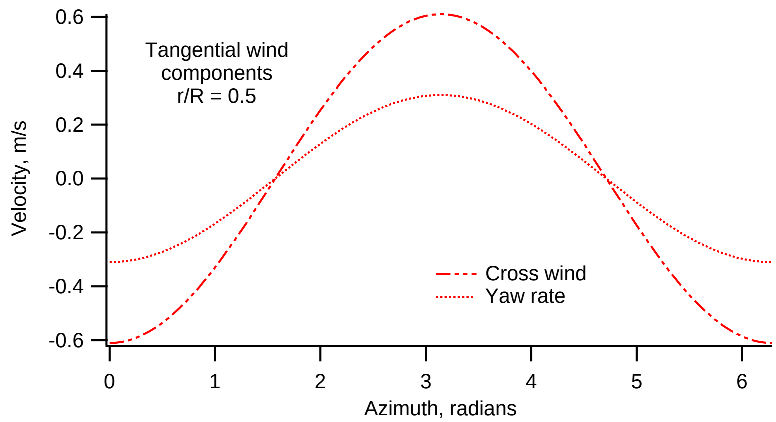
$$\alpha \approx \frac{6.04 - 0.64\sin(\psi) - 1.52\cos(\psi)}{38.3 - 0.915\cos(\psi)} - 0.05$$

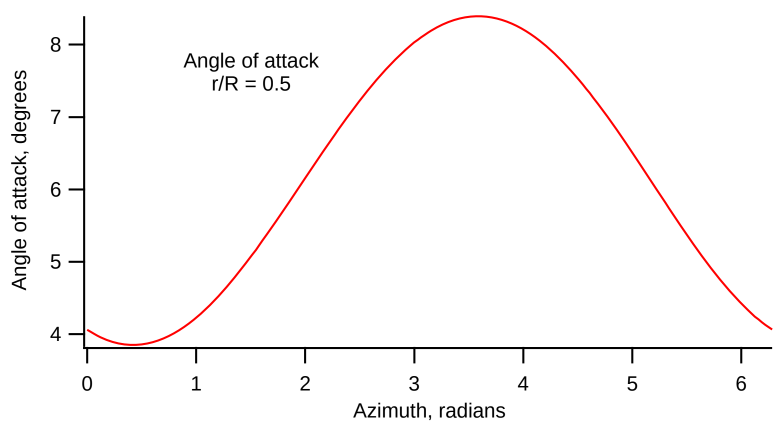
Graphs of these terms appear below.




39.0
38.5
38.0
37.5
Velocity, m/s
Total tangential wind
r/R = 0.5
0
1
2
3
4
5
6
Azimuth, radians


0.6
0.4
0.2
0.0
-0.2
-0.4
-0.6
Velocity, m/s
Tangential wind
components
r/R = 0.5
Cross wind
Yaw rate
0
1
2
3
4
5
6
Azimuth, radians


8
7
6
5
4
Angle of attack, degrees
Angle of attack
r/R = 0.5
0
1
2
3
4
5
6
Azimuth, radians

**4.9** When only blade rotation and the hinge–spring and offset are included in the calculation of the flap angle, the only non-zero term is the steady state flap angle, $\beta_0$ .

a) What is the derivative of the steady state flap angle with respect to wind velocity? Assume that the axial induction factor is also a function of wind velocity.

b) Assuming that the Lock number is positive, what would the effect of a negative value of d*a*/d*U* be on the steady state flap angle?

c) Suppose a = 1/3 and d*a*/d*U* = 0. In this case, what is the expression for the derivative of the steady state flap angle with respect to wind velocity?

d) What is the derivative of the steady state flap angle with respect to blade pitch?

**SOLUTION**

The flap equations of motion, including only rotation and the spring and offset can be expressed as:

$$\begin{bmatrix} K & 0 & 0 \\ 0 & K-1 & \frac{\gamma}{8} \\ 0 & -\frac{\gamma}{8} & K-1 \end{bmatrix} \begin{bmatrix} \beta_0 \\ \beta_{1c} \\ \beta_{1s} \end{bmatrix} = \begin{bmatrix} \frac{\gamma}{2}A \\ 0 \\ 0 \end{bmatrix}$$

where

$K$ = Flapping inertial natural frequency (includes rotation, offset, hinge spring), $K = 1+\varepsilon+K_\beta / I_b\Omega^2$

$A$ = Axisymmetric flow term, $= (\Lambda/3)-(\theta_p/4)$

$\Lambda$ = Non-dimensional inflow, $\Lambda = U(1-a)/\Omega R$

$\gamma$ = Lock number, $\gamma = \rho C_{L\alpha} c R^4 / I_b$

The solution is:

$$\beta_0 = \frac{1}{K}\frac{\gamma}{2}A$$

The coning angle results from a balance between the aerodynamic moments on the one hand and the centrifugal force and hinge spring moments opposing them.

a) The derivative of the steady state flap angle with respect to wind velocity is:

$$\frac{d\beta_0}{dU} = \frac{\gamma}{2K}\left(\frac{dA}{dU}\right) = \frac{\gamma}{2K}\left(\frac{(1-a)}{3\Omega R} - \frac{U}{3\Omega R}\left(\frac{da}{dU}\right)\right)$$

b) If the Lock number is positive and d*a*/d*U* is negative then, from the equation above, the steady state coning angle increases as the wind speed increases.

c) If *a*=1/3 and d*a*/d*U*=0 then the derivative of the steady state flap angle with respect to wind velocity reduces to:

$$\frac{d\beta_0}{dU} = \frac{\gamma}{2K}\left(\frac{(1-a)}{3\Omega R} - \frac{U}{3\Omega R}\left(\frac{da}{dU}\right)\right) = \frac{1}{9}\left(\frac{\gamma}{K\Omega R}\right)$$

d) The derivative of the steady state flap angle with respect to blade pitch is:

$$\frac{d\beta_0}{d\theta_P} = \frac{\gamma}{2K}\left(\frac{dA}{d\theta_P}\right) = \frac{\gamma}{2K}\frac{d}{d\theta_P}\left(\frac{1}{3}\left[\frac{U(1-a)}{\Omega R}\right] - \frac{\theta_P}{4}\right) = -\frac{\gamma}{8K}$$

**4.10** Just as the flap angle can be represented by a the sum of a constant term, a sine term, and a cosine term, the lead-lag angle in the simplified dynamics model can be represented by the sum of a constant term, a sine term, and a cosine term:

$$\zeta \approx \zeta_0 + \zeta_{1c}\cos(\psi) + \zeta_{1s}\sin(\psi)$$

The following matrix equation for the solution of the lead–lag motion can be derived by substituting this solution into the lead–lag equations of motion (note that flap coupling terms have been omitted for simplicity):

$$\begin{bmatrix} 2B & (K_2-1) & 0 \\ 0 & 0 & (K_2-1) \\ K_2 & B & 0 \end{bmatrix}\begin{bmatrix}\zeta_0 \\ \zeta_{1c} \\ \zeta_{1s}\end{bmatrix} = \begin{bmatrix} -\frac{\gamma}{2}\left[K_{vs}\overline{U}A_4 - \frac{\theta_P \Lambda}{2}\left(\overline{V}_0 + \overline{q}\overline{d}\right)\right] \\ -2B - \frac{\gamma}{2}\overline{q}A_4 \\ \frac{\gamma}{2}\Lambda A_2 \end{bmatrix}$$

where:

$K_2$ = inertial natural frequency (includes offset, hinge–spring) $= \varepsilon_2 + \dfrac{K_\zeta}{I_b\Omega^2} = \left(\dfrac{\omega_\zeta}{\Omega}\right)^2$

$A_2$ = axisymmetric flow term (includes tip speed ratio and pitch angle) $= \dfrac{\Lambda}{2} + \dfrac{\theta_p}{3}$

$A_4$ = axisymmetric flow term (includes tip speed ratio and pitch angle) $= \dfrac{2}{3}\Lambda - \dfrac{\theta_p}{4}$

The other terms are all defined in the text.

a) Write the matrix equation for lead–lag motion, including the terms for the steady mean wind and the hinge spring model, but assuming that gravity is zero and that other aerodynamic forcing functions are zero.

b) Solve the equations and find the expression for the steady state lead–lag angle as a function of the Lock number, the non-dimensional wind speed, the lead–lag natural frequency, and the blade pitch.

c) Suppose the pitch were 5.7 degrees, the rotor speed were 50 rpm and the diameter were 9.14 m (30 ft). By what factor would the steady state lead–lag angle increase if the wind speed increased from 7.62 m/s (25 ft/s) to 15.24 m/s (50 ft/s)? Assume that the axial induction factor decreases from 0.30 to 0.25 as the speed increases to 15.24 m/s (50 ft/s).

**SOLUTION**

a) The equation of motion, assuming zero yaw rate, shear, gravity, and crosswind, is:

$$\begin{bmatrix} 0 & (K_2-1) & 0 \\ 0 & 0 & (K_2-1) \\ K_2 & 0 & 0 \end{bmatrix} \begin{bmatrix} \zeta_0 \\ \zeta_{1c} \\ \zeta_{1s} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \frac{\gamma}{2}\Lambda A_2 \end{bmatrix}$$

b) The steady state lead lag angle can be found using Cramer's Rule:

$$\zeta_0 = \frac{\begin{vmatrix} 0 & (K_2-1) & 0 \\ 0 & 0 & (K_2-1) \\ \frac{\gamma}{2}\Lambda A_2 & 0 & 0 \end{vmatrix}}{\begin{vmatrix} 0 & (K_2-1) & 0 \\ 0 & 0 & (K_2-1) \\ K_2 & 0 & 0 \end{vmatrix}}$$

The determinant of the top matrix is:

$$\begin{vmatrix} 0 & (K_2-1) & 0 \\ 0 & 0 & (K_2-1) \\ \frac{\gamma}{2}\Lambda A_2 & 0 & 0 \end{vmatrix} = -(K_2-1)\begin{vmatrix} 0 & (K_2-1) \\ \frac{\gamma}{2}\Lambda A_2 & 0 \end{vmatrix}$$

$$= -(K_2-1)\left[0-(K_2-1)\left(\frac{\gamma}{2}\Lambda A_2\right)\right] = (K_2-1)^2\left(\frac{\gamma}{2}\Lambda A_2\right)$$

The determinant of the bottom matrix is:

$$\begin{vmatrix} 0 & (K_2-1) & 0 \\ 0 & 0 & (K_2-1) \\ K_2 & 0 & 0 \end{vmatrix} = -(K_2-1)\begin{vmatrix} 0 & (K_2-1) \\ K_2 & 0 \end{vmatrix}$$

$$= -(K_2-1)[0-(K_2-1)(K_2)] = K_2(K_2-1)^2$$

Thus:

$$\zeta_0 = \frac{(K_2 - 1)^2\left(\frac{\gamma}{2}\Lambda A_2\right)}{K_2(K_2 - 1)^2} = \frac{\gamma \Lambda A_2}{2K_2}$$

or

$$\zeta_0 = \frac{\gamma\Lambda}{2K_2}\left(\frac{\Lambda}{2} + \frac{\theta_p}{3}\right)$$

c) The non-dimensional inflow can be written as:

$$\Lambda = \frac{U - u_i}{\Omega R} = \frac{U(1-a)}{\Omega R}$$

Thus, the equation can be written as:

$$\zeta_0 = \frac{\gamma}{2K_2}\left(\frac{U(1-a)}{\Omega R}\right)\left(\frac{1}{2}\left(\frac{U(1-a)}{\Omega R}\right) - \frac{\theta_p}{3}\right) = \frac{\gamma(1-a)}{2K_2 \Omega R}\left(\frac{U^2(1-a)}{2\Omega R} - \frac{U\theta_p}{3}\right)$$

The ratio of the angles at 15.24 m/s (50 ft/s), designated with a subscript of *f*, to the lead lag angle at 7.62 m/s (25 ft/s), designated with a subscript of *i*, can be written, after some manipulation, as:

$$\frac{\zeta_{o,f}}{\zeta_{o,i}} = \frac{(1-a_f)\left(3U_f^{\,2}(1-a_f) - U_f(2\Omega R\theta_p)\right)}{(1-a_i)\left(3U_i^{\,2}(1-a_i) - U_i(2\Omega R\theta_p)\right)}$$

The pitch is 5.7 degrees = 0.1 radians.
The radial velocity is 50 rpm = 5.24 rad/s.
The axial induction factor at 7.62 m/s (25 ft/s), $a_i$, is 0.30.
The axial induction factor at 15.24 m/s (50 ft/s), $a_f$, is 0.25.
The radius is 4.57 m = 15 feet.

Substituting, one gets:

$$\frac{\zeta_{o,f}}{\zeta_{o,i}} = \frac{(.75)\left(3\left(15.24^2(.75)\right) - 15.24\left(2(5.24)(4.57)(0.1)\right)\right)}{(.7)\left(3\left(7.62^2(.7)\right) - 7.62\left(2(5.24)(4.57)(0.1)\right)\right)} = 5.64$$

Thus, this two fold increase in wind speed increases the steady state lead lag angle 5.6 times.

**4.11** Using the lead–lag equations described in Problem 4.10:

a) Write the matrix equation for lead–lag motion if all of the terms are ignored except those that have to do with steady winds and vertical wind shear.

b) Solve the equations and find the expression for the lead–lag angle with steady winds and vertical shear. What is the effect of vertical wind shear on each term of the lead–lag angle in the absence of yaw rate, gravity, and crosswind?

**SOLUTION**

a) The equation for the lead lag motion, assuming zero yaw rate, gravity, and crosswind, is:

$$\begin{bmatrix} 0 & (K_2-1) & 0 \\ 0 & 0 & (K_2-1) \\ K_2 & 0 & 0 \end{bmatrix}\begin{bmatrix} \zeta_0 \\ \zeta_{1c} \\ \zeta_{1s} \end{bmatrix} = \begin{bmatrix} -\frac{\gamma}{2}K_{vs}\overline{U}A_4 \\ 0 \\ \frac{\gamma}{2}\Lambda A_2 \end{bmatrix}$$

b)

**Constant Term, $\zeta_0$**

The steady state lead lag angle can be found using Cramer's Rule:

$$\zeta_0 = \frac{\begin{vmatrix} -\frac{\gamma}{2}K_{vs}\overline{U}A_4 & (K_2-1) & 0 \\ 0 & 0 & (K_2-1) \\ \frac{\gamma}{2}\Lambda A_2 & 0 & 0 \end{vmatrix}}{\begin{vmatrix} 0 & (K_2-1) & 0 \\ 0 & 0 & (K_2-1) \\ K_2 & 0 & 0 \end{vmatrix}}$$

The determinant of the top matrix is:

$$\begin{vmatrix} -\frac{\gamma}{2}K_{vs}\overline{U}A_4 & (K_2-1) & 0 \\ 0 & 0 & (K_2-1) \\ \frac{\gamma}{2}\Lambda A_2 & 0 & 0 \end{vmatrix} = -(K_2-1)\begin{vmatrix} -\frac{\gamma}{2}K_{vs}\overline{U}A_4 & (K_2-1) \\ \frac{\gamma}{2}\Lambda A_2 & 0 \end{vmatrix}$$

$$= -(K_2-1)\left[0-(K_2-1)\left(\frac{\gamma}{2}\Lambda A_2\right)\right] = (K_2-1)^2\left(\frac{\gamma}{2}\Lambda A_2\right)$$

The determinant of the bottom matrix is:

$$\begin{vmatrix} 0 & (K_2-1) & 0 \\ 0 & 0 & (K_2-1) \\ K_2 & 0 & 0 \end{vmatrix} = -(K_2-1)\begin{vmatrix} 0 & (K_2-1) \\ K_2 & 0 \end{vmatrix}$$

$$= -(K_2 - 1)[0 - (K_2 - 1)(K_2)] = K_2 (K_2 - 1)^2$$

Thus:

$$\zeta_0 = \frac{(K_2 - 1)^2 \left( \frac{\gamma}{2} \Lambda A_2 \right)}{K_2 (K_2 - 1)^2} = \frac{\gamma \Lambda A_2}{2K_2}$$

or

$$\zeta_0 = \frac{\gamma \Lambda}{2K_2} \left( \frac{2}{3} \Lambda - \frac{\theta_p}{4} \right)$$

This constant blade bending in the lead lag direction is the result of steady wind and shows no effect of wind shear in the absence of yaw rate, gravity, and crosswind.

**Cosine Term,** $\zeta_{1c}$

The coefficient of the cosine term can also be found using Cramer's Rule:

$$\zeta_{1c} = \frac{\begin{vmatrix} 0 & -\frac{\gamma}{2} K_{vs} \overline{U} A_4 & 0 \\ 0 & 0 & (K_2 - 1) \\ K_2 & \frac{\gamma}{2} \Lambda A_2 & 0 \end{vmatrix}}{\begin{vmatrix} 0 & (K_2 - 1) & 0 \\ 0 & 0 & (K_2 - 1) \\ K_2 & 0 & 0 \end{vmatrix}}$$

The determinant of the top matrix is:

$$\begin{vmatrix} 0 & -\frac{\gamma}{2} K_{vs} \overline{U} A_4 & 0 \\ 0 & 0 & (K_2 - 1) \\ K_2 & \frac{\gamma}{2} \Lambda A_2 & 0 \end{vmatrix} = -(K_2 - 1) \begin{vmatrix} 0 & -\frac{\gamma}{2} K_{vs} \overline{U} A_4 \\ K_2 & \frac{\gamma}{2} \Lambda A_2 \end{vmatrix}$$

$$= -(K_2 - 1) \left[ 0 - \left( -\frac{\gamma}{2} K_{vs} \overline{U} A_4 \right) K_2 \right] = -K_{vs} (K_2 - 1) \left( \frac{\gamma}{2} K_2 \overline{U} A_4 \right)$$

The determinant of the bottom matrix is the same as before:

$$\begin{vmatrix} 0 & (K_2-1) & 0 \\ 0 & 0 & (K_2-1) \\ K_2 & 0 & 0 \end{vmatrix} = K_2(K_2-1)^2$$

Thus:

$$\zeta_{1c} = \frac{-K_2(K_2-1)\left(\frac{\gamma}{2}K_{vs}\overline{U}A_4\right)}{K_2(K_2-1)^2} = \frac{-\frac{\gamma}{2}K_{vs}\overline{U}A_4}{(K_2-1)}$$

or

$$\zeta_{1c} = \frac{-\frac{\gamma}{2}K_{vs}\overline{U}}{(K_2-1)}\left(\frac{2}{3}\Lambda - \frac{\theta_p}{4}\right)$$

Thus, the perturbation introduced on top of the steady state term, in the absence of yaw rate, gravity, and crosswind, is negative when the cosine of the azimuth is positive (in the lower half of the rotor disk) where there is less wind. The lead lag bending angle is increased in the upper half of the rotor disk.

**Sine Term, $\zeta_{1s}$**

The coefficient of the sine term can also be found using Cramer's Rule:

$$\zeta_{1s} = \frac{\begin{vmatrix} 0 & (K_2-1) & -\frac{\gamma}{2}K_{vs}\overline{U}A_4 \\ 0 & 0 & 0 \\ K_2 & 0 & \frac{\gamma}{2}\Lambda A_2 \end{vmatrix}}{\begin{vmatrix} 0 & (K_2-1) & 0 \\ 0 & 0 & (K_2-1) \\ K_2 & 0 & 0 \end{vmatrix}}$$

The determinant of the top matrix is:

$$\begin{vmatrix} 0 & (K_2-1) & -\frac{\gamma}{2}K_{vs}\overline{U}A_4 \\ 0 & 0 & 0 \\ K_2 & 0 & \frac{\gamma}{2}\Lambda A_2 \end{vmatrix} = K_2\begin{vmatrix} (K_2-1) & -\frac{\gamma}{2}K_{vs}\overline{U}A_4 \\ 0 & 0 \end{vmatrix} = 0$$

So:

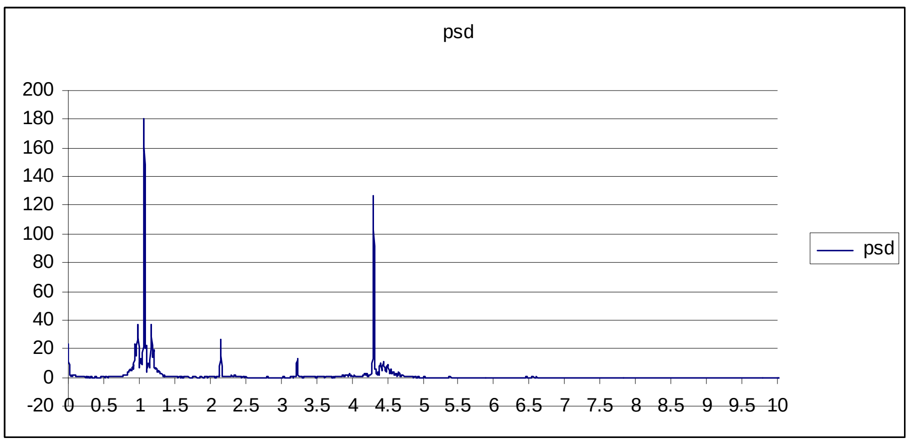
$$\zeta_{1s} = 0$$

Thus, in the absence of yaw rate, gravity, and crosswind, wind shear only affects the cosine term in the lead lag angle equation.

**4.12** Consider the data file, *Flap1.txt*. This data represents 10 minutes of blade bending moment data taken from an operating wind turbine. The turbine has a nominal operating speed of 60 rpm. The data was sampled at 40 Hz. Find the psd of the data. What are the predominant frequencies? What do you suspect that these frequencies might be caused by?

**<u>SOLUTION</u>**

Using the MiniCodes, the psd the range of interest is shown below. The peak frequencies are 1.07 Hz and 4.307 Hz. The spike at 1.07 is presumably due to rotation. The other one may be due to a blade vibration mode.




**4.13** Consider the following turbine. The tower is 50 m high, has diameter of 2 m at the top and 3.36 m at the bottom. The mass of the nacelle and rotor is 20,400 kg. The wall thickness is 0.01384 m. Assume that the density of steel is 7,700 kg/m$^3$ and that its modulus of elasticity is 160 GPa. The problem considers a number of approaches (successively improving) to estimate the natural frequency (1$^{st}$ mode) of the turbine.

a) Assume that the tower is not tapered, but has a constant diameter of 2 m. What is the natural frequency using the simple equation, but ignoring the mass of the rotor nacelle assembly (RNA) on the top of the tower.

b) Repeat (a) using the Euler method. Do this directly or with the MiniCodes.

c) Repeat (a) using the Myklestad method. Use the MiniCodes.

d) Repeat (a), taking into account the weight of the RNA the top, using the simple equation.

e) Repeat (d), using the Myklestad method. This will require some approximations to account for the weight of the RNA.

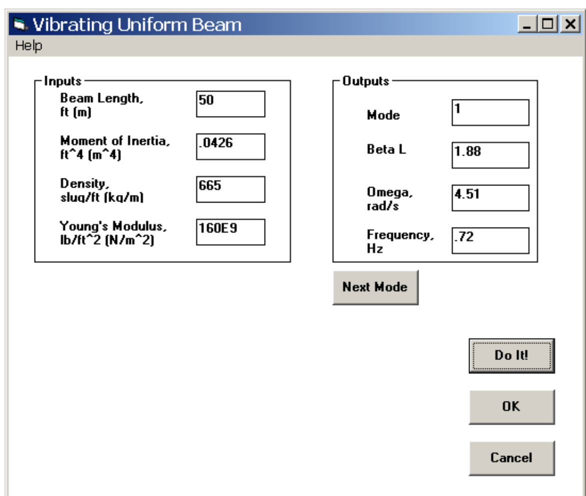
f) Repeat (e), but take into account the actual taper of the tower.

**SOLUTION**

Parts of this problem can be solved with the help of the MiniCodes. An Excel spreadsheet is also useful for solving the simpler parts and for preparing the input files for the MiniCodes. See, for example, *V47.xls*.

a) $$f_0 = \frac{1}{2\pi}\sqrt{\frac{3EI}{(0.23\,m_{tower} + m_{nacelle})L^3}} = \frac{1}{2\pi}\sqrt{\frac{3(160E9)(0.0426)}{((0.23)(33{,}248))50^3}} = 0.736\,Hz$$

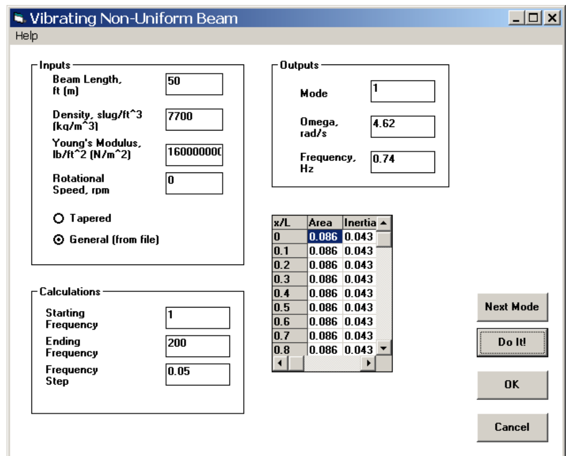
b) See MiniCodes screenshot: $\omega = 4.51\,rad/s$, $f_o = 0.72$ Hz




c) This is for straight tower with no weight, using the Myklestad method. The answer is almost the same. Ten identical sections works well enough:

Input: *tower straight_2.csv*
Use:

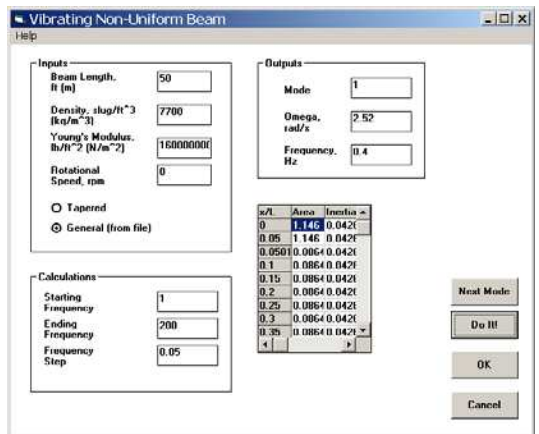
| | | |
|---|---|---|
| 0 | 0.086 | 0.043 |
| 0.1 | 0.086 | 0.043 |
| 0.2 | 0.086 | 0.043 |
| 0.3 | 0.086 | 0.043 |
| 0.4 | 0.086 | 0.043 |
| 0.5 | 0.086 | 0.043 |
| 0.6 | 0.086 | 0.043 |
| 0.7 | 0.086 | 0.043 |
| 0.8 | 0.086 | 0.043 |
| 0.9 | 0.086 | 0.043 |
| 1 | 0.086 | 0.043 |




d) $$f_0 = \frac{1}{2\pi}\sqrt{\frac{3EI}{\left(0.23 m_{tower} + m_{nacelle}\right)L^3}} = \frac{1}{2\pi}\sqrt{\frac{3(160E9)(0.0426)}{\left((0.23)(33{,}248) + 20{,}400\right)50^3}} = 0.384\,Hz$$

e) We used 20 sections. The top one was heavier to include effect of weight. Note that method uses averages, so we needed to introduce an extra section.
Use:

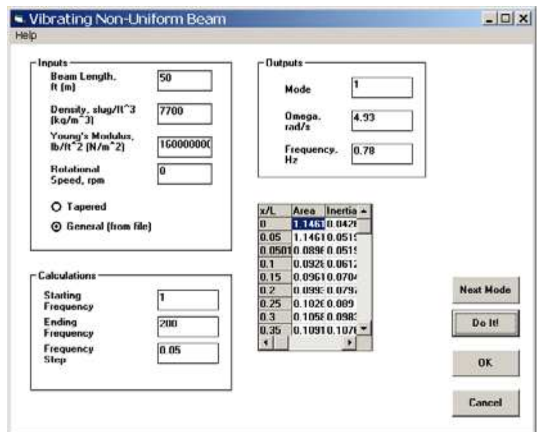
| | | |
|---|---|---|
| 0 | 1.146 | 0.0426 |
| 0.05 | 1.146 | 0.0426 |
| 0.0501 | 0.0864 | 0.0426 |
| 0.1 | 0.0864 | 0.0426 |
| 0.15 | 0.0864 | 0.0426 |
| 0.2 | 0.0864 | 0.0426 |
| 0.25 | 0.0864 | 0.0426 |
| 0.3 | 0.0864 | 0.0426 |
| 0.35 | 0.0864 | 0.0426 |
| 0.4 | 0.0864 | 0.0426 |
| 0.45 | 0.0864 | 0.0426 |
| 0.5 | 0.0864 | 0.0426 |
| 0.55 | 0.0864 | 0.0426 |
| 0.6 | 0.0864 | 0.0426 |
| 0.65 | 0.0864 | 0.0426 |
| 0.7 | 0.0864 | 0.0426 |
| 0.75 | 0.0864 | 0.0426 |
| 0.8 | 0.0864 | 0.0426 |
| 0.85 | 0.0864 | 0.0426 |
| 0.9 | 0.0864 | 0.0426 |
| 0.95 | 0.0864 | 0.0426 |
| 1 | 0.0864 | 0.0426 |




f) This is whole thing, much wider at the base, much more steel. The RNA is only approximated here; could have used average area of top 2 sections to find factor to include effect of weight:

For inputs we used:

| | | |
|---|---|---|
| 0 | 1.1461 | 0.0426 |
| 0.05 | 1.1461 | 0.0519 |
| 0.0501 | 0.0896 | 0.0519 |
| 0.1 | 0.0928 | 0.0612 |
| 0.15 | 0.0961 | 0.0704 |
| 0.2 | 0.0993 | 0.0797 |
| 0.25 | 0.1026 | 0.089 |
| 0.3 | 0.1058 | 0.0983 |
| 0.35 | 0.1091 | 0.1076 |
| 0.4 | 0.1123 | 0.1169 |
| 0.45 | 0.1156 | 0.1262 |
| 0.5 | 0.1188 | 0.1354 |
| 0.55 | 0.122 | 0.1447 |
| 0.6 | 0.1253 | 0.154 |
| 0.65 | 0.1285 | 0.1633 |
| 0.7 | 0.1318 | 0.1726 |
| 0.75 | 0.135 | 0.1819 |
| 0.8 | 0.1383 | 0.1912 |
| 0.85 | 0.1415 | 0.2004 |
| 0.9 | 0.1448 | 0.2097 |
| 0.95 | 0.148 | 0.219 |
| 1 | 0.1512 | 0.2283 |




Summary

| Tower | RNA | Method | Frequency, Hz |
|---|---|---|---|
| Straight | No | Simple | 0.736 |
| Straight | No | Euler | 0.72 |
| Straight | No | Myklestad | 0.74 |
| Straight | Yes | Simple | 0.38 |
| Straight | Yes | Myklestad | 0.40 |
| Tapered | Yes | Myklestad | 0.78 |

**Conclusion:**
Adding weight to top reduces stiffness; making the tower wider at the base increases stiffness.

## B.5 Chapter 5 Problems

**5.1** A 48 V DC wind turbine is hooked up to charge a battery bank consisting of four 12 V batteries in series. The wind is not blowing, but the batteries must supply 2 loads. One is a light bulb, rated at 175 W at 48 V; the other is a heater, rated at 1000 W at 48 V. The loads are in parallel. The batteries may be considered to be constant 12 V voltage sources, with an internal series resistance of 0.05 Ohm each. How much power is actually supplied to the loads? How does that power compare to that which is expected? (Ignore the effect of temperature on resistance of filament and heating elements.)

**SOLUTION**

Find the resistance of each of the loads, based on the nominal ratings.

$$R_{light} = V_{rated}^2 / P_{rated} = 48^2 / 175 = 13.17\,\Omega$$

$$R_{heater} = V_{rated}^2 / P_{heater,rated} = 48^2 / 1000 = 2.304\,\Omega$$

The series resistance in the battery bank is:

$$R_{bank} = 4 \text{ x } 0.05 = 0.2\,\Omega$$

The total resistance is:

$$R = R_{\text{bank}} + 1/(1/R_{heater} + 1/R_{light}) = 2.161\,\Omega$$

The current is:

$$I = V/R = 48/2.161 = 22.21\ A$$

The voltage across the battery bank is:

$$V_{bank} = E - R_{bank} I = 48 - (0.2)(22.21) = 43.56\,V$$

where $E$ = the internal voltage of the bank.
The actual power dissipated in the light is

$$P_{light} = V_{bank}^2 / R_{light} = 43.56^2 / 13.17 = 144\ W$$

The actual power dissipated in the heater is:

$$P_{heater} = V_{bank}^2 / R_{heater} = 43.56^2 / 2.304 = 824\ W$$

The net effect is to reduce the power provided to the loads by nearly 18%

**5.2** An electromagnet is used to hold a wind turbine's aerodynamics brakes in place during operation. The magnet supplies 30 lbs (133.4 N) of force to do this. The magnet is supplied by a 60 V DC source. The coil of the electromagnet draws 0.1 A. The core of the electromagnet is assumed to have a relative permeability of $10^5$. The diameter of the core is 3 in (7.62 cm).

Find the number of turns in the coil and the wire size. Assume that the relation between force, magnetic flux, and area of the core is given by $F = 397\,840\, B^2\ A_c$ :
where: $F$= force (N), $B$ = magnetic flux (Wb/m$^2$), $A_c$ = area of core (m$^2$). Also, assume that the length of each turn is equal to the circumference of the core. The resistivity of copper is $\rho = 1.72 \times 10^{-6}\, \Omega\,\text{cm}$.

**SOLUTION**

Find the area of core from the given diameter

$$A_c = \frac{\pi(0.0762)^2}{4} = 0.00456\, m^2$$

Find magnetic flux, $B$, from the relation given in the problem statement

$$B = \sqrt{\frac{F}{(397{,}840)(A_c)}} = \sqrt{\frac{134.4}{(397{,}840)(.00456)}} = 0.272\, Wb/m^2$$

Find the magnetic field strength, $H$, from Equation 5.31

$$H = \frac{B}{\mu_0 \mu_r} = \frac{0.272}{(10^4)(4x10^{-7})} = 68\, A\, turns$$

The total number of turns is

$$N_t = \frac{H}{I} = \frac{68}{0.1} = 680\ turns$$

Find the resistance

$$R = \frac{V}{I} = \frac{60}{0.1} = 600\, \Omega$$

Find length of each turn:

$$L_t = \pi D = \pi(0.0762) = 23.9\, cm$$

The total length of wire is

$$L = L_t N_t = 162.5\ m$$

The cross sectional area is

$$a_c = \rho \frac{L_t}{R} = (1.72x10^{-6}\,\Omega\, cm)\frac{16252\, cm}{600\,\Omega} = 0.0000466\ cm^2$$

The diameter of the wire is

$$d_w = (2)\sqrt{\left(\frac{a_c}{\pi}\right)} = (2)\sqrt{\left(\frac{0.0000466\, cm^2}{\pi}\right)} = 0.0077\, cm = 0.003"$$

The wire size in this case is very close to 40 AWG. (For a table on wire gauge sizes, see, for example, http://en.wikipedia.org/wiki/American_wire_gauge .)

**5.3** Consider the following phasors: $\hat{\mathbf{X}} = 10 + j14$, $\hat{\mathbf{Y}} = -4 + j5$. Find the following, and express in both rectangular and polar form: $\hat{\mathbf{X}} + \hat{\mathbf{Y}}$, $\hat{\mathbf{X}} - \hat{\mathbf{Y}}$, $\hat{\mathbf{X}}\hat{\mathbf{Y}}$, $\hat{\mathbf{X}} / \hat{\mathbf{Y}}$ .

**SOLUTION**

Using the rules given in the chapter or the wind engineering mini-codes

$$\hat{\mathbf{X}} = 10 + j14 = 17.2\angle 54.5\deg$$

$$\hat{\mathbf{Y}} = -4 + j5 = 6.40\angle 128.7\deg$$

$$\hat{\mathbf{X}} + \hat{\mathbf{Y}} = 6 + \mathrm{j}19 = 19.9\angle 72.5\deg$$

$$\hat{\mathbf{X}} - \hat{\mathbf{Y}} = 14 + \mathrm{j}9 = 16.6\angle 32.7\deg$$

$$\hat{\mathbf{X}} \cdot \hat{\mathbf{Y}} = \text{-}110 - j6 = 110.2\angle 183.1\deg$$

$$\hat{\mathbf{X}} / \hat{\mathbf{Y}} = 0.732 - j2.585 = 2.69\angle -74.2\deg$$

**5.4** Show that the magnitude of the line-to-neutral voltage in a balanced, Y connected three phase system ($V_{LN}$) is equal to the line-to-line voltage ($V_{LL}$) divided by the square root of 3, i.e.

$$V_{LN} = V_{LL} \big/ \sqrt{3}\ .$$

The line feeding a Y-connected three-phase generator has a line-to-line voltage of 480 V. What is the line-to-neutral voltage?

**SOLUTION**

Consider three phases a, b, c with line to line voltages $\hat{V}_{aa}, \hat{V}_{bb}, \hat{V}_{cc}$. The relation between any two of the line to neutral voltages and the line to line voltage is given by the vector sum: $\hat{\mathbf{V}}_{ab} = \hat{\mathbf{V}}_{an} - \hat{\mathbf{V}}_{bn}$. Suppose $\hat{\mathbf{V}}_{an} = \left|\hat{\mathbf{V}}_{an}\right|\angle 0$ and $\hat{\mathbf{V}}_{bn} = \left|\hat{\mathbf{V}}_{an}\right|\angle 120$. Then

$$\left|\frac{\hat{\mathbf{V}}_{ab}}{\hat{\mathbf{V}}_{an}}\right| = \left|\frac{\left|\hat{\mathbf{V}}_{an}\right|\angle 0 - \left|\hat{\mathbf{V}}_{an}\right|\angle 120}{\left|\hat{\mathbf{V}}_{an}\right|\angle 0}\right| = \left|\frac{1\angle 0 - 1\angle 120}{1\angle 0}\right| = \sqrt{3}$$

If the line to line voltage is 480 V, then the line to neutral voltage is $\left|\hat{\mathbf{V}}_{\ln}\right| = \left|\hat{\mathbf{V}}_{ll}\right| / \sqrt{3} = 480/\sqrt{3} = 277\,V$

**5.5** A circuit has a resistance and inductance in series. The applied AC voltage is $240\angle 0^\circ$ V, 60 Hz. The resistance is 8 Ω and the induction is 10 Ω. Find the current in the circuit.

**SOLUTION**

Find the impedance: $Z = R + jX_L$ = 8+10*j*. The current is:

$$I = \frac{V}{Z} = \frac{240\angle 0^\circ}{8+10j} = 15.6\angle -51.3^\circ \text{ A}$$

**5.6** An AC circuit has a resistor, capacitor and inductor in series with a 120 V, 60 Hz voltage source. The resistance of the resistor is 2 Ohms, the inductance of the inductor is 0.01 Henry; the capacitance of the capacitor is 0.0005 Farads. Find the following: reactance of the capacitor and inductor, current, apparent power, real power, reactive power, power factor angle and power factor.

**SOLUTION**

The reactance of the inductor is:

$$X_L = 2\pi f L = (2)(\pi)(60)(0.01) = 3.77\,\Omega$$

The reactance of the capacitor is:

$$X_C = \frac{1}{2\pi f C} = \frac{1}{(2)(\pi)(60)(0.0005)} = 5.30\,\Omega$$

The total impedance is:

$$Z = R + j\,X_L - j\,X_C = 2 + j\,3.77 - j\,5.30 = 2.52\angle -37.5\,\Omega$$

The current is

$$I = \frac{V}{Z} = \frac{120\angle 0\,V}{2.52\angle -37.5\,\Omega} = 47.6\angle 37.5\,A$$

The apparent power is

$$S = I\,V = (47.6\,A)(120\,V) = 5712\,W$$

The power factor angle is

$$\theta = \angle V - \angle I = 0 - 37.5 = -37.5$$

The real power is

$$P = I\,V\cos(\theta) = (47.6)(120)\cos(-37.5) = 4532\ W$$

The reactive power is

$$Q = I\,V\sin(\theta) = (47.6)(120)\sin(-37.5) = -3478\ VAr$$

The power factor is

$pf = \cos(\theta) = 0.793$, leading since the power factor angle < 0.

**5.7** A circuit as a resistor and an inductor in parallel. In addition a second resistor is in series with a capacitor in parallel and the two of them are in parallel with the inductor and first resistor. The resistance of the first resistor is 8 Ω and that of the second resistor is 12 Ω. The impedance of the inductor is $j16$ Ω and that of the capacitor is $-j22$ Ω. The applied voltage is 200 V. Find the apparent power, the power factor and the real power.

**SOLUTION**

First find the impedance

$$Z = \frac{1}{1/8 + 1/\,j16 + 1/(12 - j22)} = \frac{1}{0.125 - j0.0625 + 0.0191 + j0.035}$$

$$Z = \frac{1}{0.1441 - j0.0275} = 6.82\angle 10.8°\,\Omega$$

The current is

$$I = \frac{V}{Z} = \frac{200}{6.82\angle 10.8°} = 29.33\angle = 10.8°\text{ A}$$

The apparent power is:

$$S = |I||V| = (200)(29.33) = 5866\text{ W}$$

The power factor is the cosine of the angle

$$pf = \cos(10.8) = 0.982\text{ , lagging}$$

The real power is

$$P = (S)(pf) = (5866)(0.982) = 5762\text{ W}$$

**5.8** A transformer rated at 120 V/480 V, 10 kVA has an equivalent circuit as shown in Figure B.4. The low voltage side is connected to a heater, rated at 5 kW, 120 V. The high voltage side is connected to a 480 V, 60 Hz, single-phase power line. Find the actual power transferred, the magnitude of the measured voltage across the heater, and the efficiency (power out/power in) of the transformer.

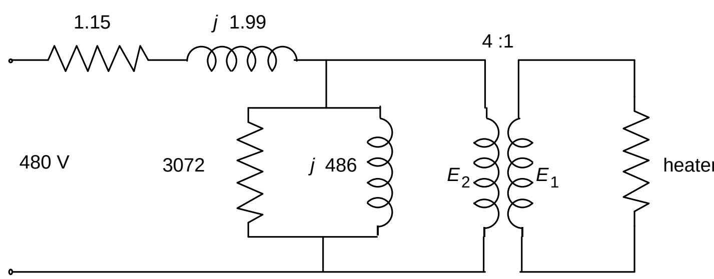



**Figure B.4** Transformer equivalent circuit

<u>**SOLUTION**</u>

Find resistance of the heater:

$$R_{heater} = V^2 / P_{heater} = 120^2 / 5000 = 2.88\,\Omega$$

Refer the heater's resistance to the high voltage side of the transformer:

$$R_{heater}^{'} = R_{heater}\left(\frac{N_1}{N_2}\right)^2 = 2.88\left(\frac{480}{120}\right)^2 = 46.08\,\Omega$$

The total parallel resistance is then:

$$R_p = 1/(1/R_{heater}^{'} + 1/3072) = 1/(1/46.08 + 1/2072) = 45.4\,\Omega$$

The total impedance is:

$$Z = 1.15 + j1.99 + 1/(1/45.4 + 1/j486) = 46.21 + j6.209\,\Omega$$

The current is:

$$I = V/Z = 480/(46.21 + j6.209) = 10.295\angle -7.653\,A$$

The parallel impedance is

$$Z_p = 1/(1/45.4 + 1/j486) = 45.06 + j\ 4.219\,\Omega$$

The voltage across the load (referred to high side)

$$V_L = I\,Z_p = (10.295\angle -7.653\,A)(45.06 + j\ 4.219) = 465.92\angle -2.304 = 465.55 - j18.73V$$

The current in the heater is

$$I_h = V_L / R = (465.55 - j18.73)/46.08 = 10.111\angle -2.304\,A$$

The actual power in the heater is

$$P_{h,actual} = I_h^2 R = (10.111^2)(46.08) = 4710\,W$$

The total power entering the transformer is

$$P_{total} = |V||I|\cos(-7.653) = (480)(10.295)\angle -7.653 = 4898\ W$$

The efficiency of the transformer is:

$$\eta = P_{h,actual} / P_{total} = 4710/4898 = 96.2\%$$

The magnitude of the voltage across the heater, referred to the low side is

$$\left|V_{h,low}\right| = \left|V_h\right| / 4 = 116.4V$$

**5.9** A small wind turbine generator (single phase) produces a 60 Hz voltage at 120 V rms. The output of the generator is connected to a diode bridge full-wave rectifier, which produces a fluctuating DC voltage. What is the average DC voltage? A silicon-controlled rectifier (SCR) is substituted for the diode rectifier. Under one condition the SCRs are turned on at 60 deg after the beginning of each half cycle. What is the average DC voltage in that case?

**SOLUTION**

The maximum voltage is

$$V_{\max} = V_{rms}\sqrt{2} = 169.7V$$

The average DC voltage is:

$$\overline{V}_{DC} = V_{\max}\frac{1}{\pi}\int_0^{\pi}\sin(\theta) = -V_{\max}\frac{1}{\pi}\cos(\theta)\Big|_0^{\pi} = V_{\max}\frac{2}{\pi} = 169.7\frac{2}{\pi} = 108\ V$$

Using the SCR rectifier the average DC voltage is

$$\overline{V}_{DC} = V_{\max}\frac{1}{\pi}\int_{\pi/3}^{\pi}\sin(\theta) = -V_{\max}\frac{1}{\pi}\cos(\theta)\Big|_{\pi/3}^{\pi} = V_{\max}\frac{1.5}{\pi} = 169.7\frac{1.5}{\pi} = 81\ V$$

**5.10** The parameters of an induction machine can be estimated from test data taken under a few specified conditions. The two key tests are the no-load test and blocked-rotor test. Under the no-load test, the rated voltage is applied to the machine and it is allowed to run at no-load (i.e. with nothing connected to the shaft). In the blocked-rotor test the rotor is prevented from turning and a reduced voltage is applied to the terminals of the machine. In both cases the voltage, current and power are measured. . Under the no-load test, the slip is essentially equal to 0, and the loop with the rotor parameters may be ignored. The magnetizing reactance accounts for most of the impedance and can be found from the test data. Under blocked-rotor conditions, slip is equal to 1.0 so the magnetizing reactance can be ignored, and the impedance of the leakage parameters can be found. A third test can be used to estimate the windage and friction losses. In this test the machine being tested is driven by a second machine but it is not connected to the power system. The power of the second machine is measured, and from that value is subtracted the latter's no-load power. The difference is approximately equal to the test machine's windage and friction.

In a simplified version of the analysis, all of the leakage terms may be assumed to be on the same side of the mutual inductance, as shown in Figure B.5. In addition, the resistance in parallel with the magnetizing reactance is assumed to be infinite, the stator and rotor resistances, $R_S$ and $R'_R$, are assumed equal to each other and the leakage inductances are also assumed to be equal to each other.

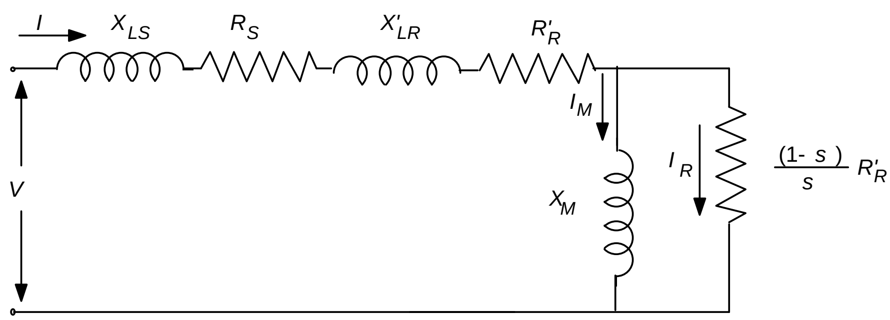



**Figure B.5** Simplified induction machine equivalent circuit

The following test data are available for a wye-connected, three-phase induction machine:

No-load test: $V_0 = 480$ V, $I_0 = 46$ A, $P_0 = 3500$ W

Blocked rotor test: $V_B = 70$ V, $I_B = 109$ A, $P_B = 5600$ W

Windage and friction losses equal 3000 W.

Find the parameters for the induction generator model.

**SOLUTION**

The voltage for each phase is

$$V_{1,0} = \frac{480}{\sqrt{3}} = 277\,V$$

The power per phase is:

$$P_{1,0} = \frac{P_0 - P_{windage}}{3} = \frac{3500 - 3000}{3} = 166\,W$$

The power factor is

$$\phi = \cos^{-1}\left(\frac{P_{1,0}}{(I_0)(V_{1,0})}\right) = 89.3\,\text{deg}$$

The inductive reactance is the imaginary part of the impedance

$$\hat{Z} = \frac{\hat{V}}{\hat{I}} = \frac{277\angle 0}{46\angle 89.3} = 0.074 - j\,6.02\,\Omega$$

To a good approximation, then: $X_m = 6.02\,\Omega$

The sum of the two leakage resistances can be found from:

$$2\,r_l = \frac{P_{1\phi,B}}{I_B^2} = \frac{5600/3}{109^2} = 0.157\ \Omega$$

The power factor under blocked rotor conditions is:

$$\phi = \cos^{-1}\left(\frac{P_{1\phi,B}}{\left(V_{1\phi,B}\right)\left(I_B\right)}\right) = \cos^{-1}\left(\frac{5600/3}{\left(70/\sqrt{3}\right)(109)}\right) = 64.9\,\text{deg}$$

The sum of the resistances and sum of inductances may be found by phasor division of the voltage by the current.

$$\hat{Z}_l = \frac{\hat{V}_{1\phi,B}}{\hat{I}_B} = \frac{\left(70/\sqrt{3}\right)\angle 0}{109\angle 64.9} = 0.157 - j\,0.336$$

Therefore:

$$r_{ls} = r_{lr} = 0.157/2 = 0.0785\,\Omega$$

$$x_{ls} = x_{lr} = 0.336/2 = 0.168\,\Omega$$

**5.11** A four pole induction generator is rated at 300 kVA and 480 V. It has the following parameters: $X_{LS} = X_{LR} = 0.15\ \mathbf{\Omega}$, $R_S = 0.014\ \mathbf{\Omega}$, $R_R = 0.0136\ \mathbf{\Omega}$, $X_M = 5\ \mathbf{\Omega}$. How much power does it produce at a slip of -0.025? How fast is it turning at that time? Also, find the torque, power factor and efficiency. (Ignore mechanical losses.)

Suppose the generator is used in a wind turbine, and the torque due to the wind is increased to a value of 2100 Nm. What happens?

**SOLUTION**

The rotor speed of a 4 pole generator at slip of -0.025 is:

$$N = N_{sync} - s\,N_{sync} = 1800 - (-0.025)1800 = 1845\,rpm$$

Voltage in each phase is $V_{\ln} = V_{ll} / \sqrt{3} = 277.13\angle 0\,V$

Find the total impedance:

$$Z = r_{ls} + j\,x_{ls} + \cfrac{1}{\cfrac{1}{jx_m} + \cfrac{1}{r_{lr}/s + j\,x_{lr}}} =$$

$$0.014 + j\,0.145 + \cfrac{1}{\cfrac{1}{j\,5} + \cfrac{1}{0.0136/(-0.025) + j\,0.15}} = -0.49311 + j\,0.3492\,\Omega$$

The current in each phase is:

$$I_p = \frac{V_{\ln}}{Z} = \frac{277.13\angle 0}{-0.49311 + j\,0.3492} = 458.64\angle 215.3\,\text{deg}$$

The voltage drop across the rotor is:

$$V_r = V_{\ln} - I_p(r_{ls} + j\,x_{ls}) = 277.13 - 458.64\angle 215.3(0.014 + j\,0.15) = 242.89\angle 13.86\,V$$

The current through the rotor is:

$$I_r = V_r / Z_r = \cfrac{242.89\angle 13.86}{\cfrac{1}{0.0136/(-0.025) + j\,0.15}} = 442.83\angle 209.27\,A$$

The power in (that which is converted to electricity) in all 3 phases is:

$$P_{in} = 3\,I_r^2\,\frac{r_{lr}\,(1-s)}{s} = (3)(442.83^2)\frac{0.0136(1-(-0.025))}{-0.025} = 328.0\,kW$$

The power lost in the rotor resistance is:

$$P_{loss,rotor} = 3\,I_r^2 r_{lr} = (3)(442.83^2)(0.0136) = 8.00\,kW$$

The power loss in the stator resistance is:

$$P_{loss,stator} = 3\,I^2 r_{ls} = (3)(458.6^2)(0.014) = 8.83\,kW$$

The power out of the terminals is:

$$P_{out} = P_{in} - P_{loss,rotor} - P_{loss,stator} = 328.3 - 8.00 - 8.83 = 311.2\,kW$$

The power factor is:

$$\cos^{-1}(\theta) = \frac{P_{out}}{3V\,I_p} = \frac{311{,}200}{(3)(277.13)(458.6)} = 0.816$$

The efficiency (not including windage and friction) is:

$$\eta = \frac{P_{out}}{P_{in}} = \frac{311{,}200}{328{,}000} = 0.949$$

The speed is:

$$\Omega = (1.025)(1800\,rpm)(1/60\,s/\min)(2\pi\,rad/rev) = 193.2\,rad/s$$

Torque is then:

$$Q = P/\Omega = 328000/193.8 = 1698\,Nm$$

The above calculations can be repeated for increasing values of slip to find the largest value of torque the generator can supply. It is convenient to use the Wind Engineering Mini-Codes for this. It can be seen that a torque of 2100 Nm would just exceed the maximum possible torque (breakaway torque.) The rotor would thus begin to accelerate without limit.

**5.12** A resistive load of 500 kW at 480 V is supplied by an eight-pole synchronous generator, connected to a diesel engine, and a wind turbine with the same induction generator as that in Problem 5.11. The synchronous machine is rated to produce 1000 kVA. Its synchronous reactance is 0.4 Ohms. A voltage regulator maintains a constant voltage of 480 V across the load and a governor on the diesel engine maintains a fixed speed. The speed is such that the grid frequency is 60 Hz. The power system is three-phase, and the electrical machines are both Y-connected. The power from the wind is such that the induction generator is operating at a slip of -0.01. At this slip, the power 153.5 kW and the power factor is 0.888.

a) Find the synchronous generator's speed, power factor, and power angle.

b) Confirm that the power and power factor are as stated.

**SOLUTION**

a) Since the grid frequency is fixed at 60 Hz, and the synchronous generator has 8 poles, the speed is 900 rpm.

According to the problem statement, the electrical output power of the induction generator is 153.5 kW and the power factor is 0.888.

The apparent power is therefore:

$$S = P/\cos(\theta) = 153.5/.888 = 173.2\,kVA$$

The power factor angle of the induction machine is

$$\theta = \cos^{-1}(0.888) = 27.38 \text{ deg}$$

The synchronous machine must supply the additional required real power and the reactive power. Since the load is 500 kW, the synchronous machine supplies 500-153.5 = 346.5 kW.

The reactive power required is:

$$Q = S\sin(\theta) = 173.2\ \sin(27.38) = 79.7\,kVA \text{ , lagging}$$

The apparent power from the synchronous machine is:

$$S_{SM} = \sqrt{Q_{SM}^2 + P_{SM}^2} = \sqrt{79.7^2 + 346.5^2} = 355.5\,kVA$$

The power factor of the synchronous machine is:

$$pf_{SM} = \frac{P_{SM}}{S_{SM}} = \frac{346.5}{355.5} = .975$$

The power factor angle for the synchronous machine is:

$$\theta = \cos^{-1}(pf_{SM}) = \cos^{-1}(0.957) = 12.8 \text{ deg , lagging}$$

The per phase current is:

$$I_P = \frac{S_{SM}}{V_P} = \frac{1}{3}\frac{355{,}500\,W}{\left(480/\sqrt{3}\right)V} = 427.6\angle -12.8\,A$$

The internal voltage, $E$, is:

$$E = V + I_p Z_{SM} = 277.13 + (427.6\angle -12.8)(j\,0.4) = 356.4\angle 27.9\,\text{deg}$$

Since the power angle, $\delta$, is, by definition, the angle between the field voltage and the terminal voltage and since the voltage is assumed to have a reference angle of zero, the power angle is immediately 27.9 deg.

b) Using a slip of –0.01 and solving the circuit for the induction generator as in problem 5.11, or using the MiniCodes, one can verify that the power and power factor are as stated for this induction generator.

**5.13** A diesel engine driven synchronous generator and a wind turbine with a wound rotor induction generator (WRIG) are used to supply a resistive load as shown in Figure B.6. The power system is three-phase, and the electrical machines are both Y-connected.

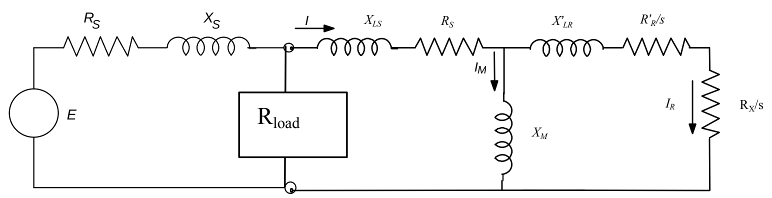



**Figure B.6** System with synchronous generator and WRIG

The WRIG has four poles and is rated at 350 kVA and 480 V. It has the following parameters: $X_{LS} = X_{LR} = 0.15\ \Omega$, $R_S = 0.014\ \Omega$, $R_R^{'} = 0.0136\ \Omega$, $X_M = 5\ \Omega$. The external resistor in the rotor circuit may be varied.

The resistive load consumes 800 kW at 480 V.

The synchronous generator is the same one as in Problem 5.12. It has eight-poles and is rated to produce 1000 kVA. Its synchronous reactance is 0.4 Ohms. A voltage regulator maintains a constant voltage of 480 V across the load and a governor on the diesel engine maintains a fixed speed. The speed is such that the grid frequency is 60 Hz.

All mechanical losses may be ignored

The external resistance has been bypassed. The wind conditions are such that the WRIG is operating at a slip of -0.02

a) How fast (rpm) is the generator turning at this time?

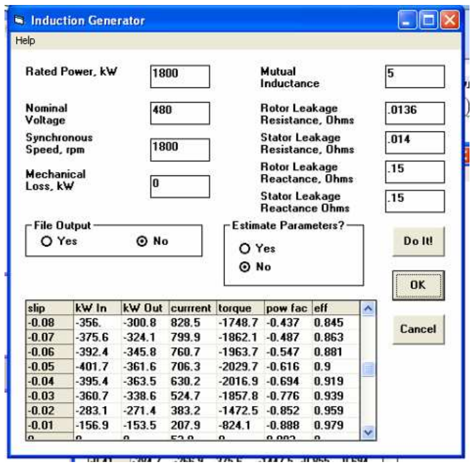
b) How much mechanical power is coming into the generator rotor from the wind turbine's rotor?

c) What is the power factor of the WRIG?

d) What is (i) the power from the synchronous generator; (ii) what is its power factor, (iii) what is the power angle; (iv) what is the generator speed (rpm) and what is the internal voltage, *E*?

**SOLUTION**

a) The speed is 1.02 x 1800 = 1836 rpm




b) The input power is 283.1 kW (see screen shot)

c) The power factor is 0.852, lagging (see screen shot)

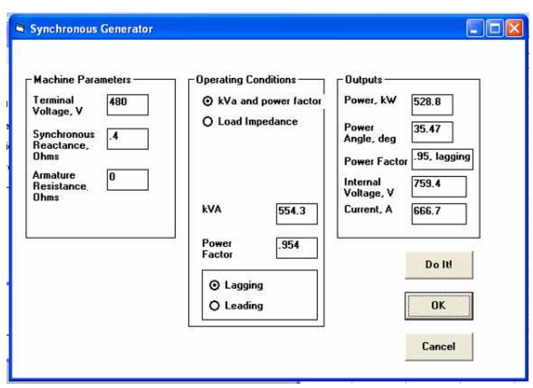
d) What is (i) the power from the synchronous generator; (ii) what is its power factor, (iii) what is the power angle; (iv) what is its speed (rpm) and what is the internal voltage, *E*?
The power from synchronous generator is remaining power required by load, = 800 - 271.4 = 528.6 kW, reactive power supplied = that required by the IG, = kVA sin(power factor angle); kVa = $P_{out}$/pf = 271.4/0.852 = 318.5; Reactive power = 318.5 sin(31.57) = 166.7 kVAr; the required kVa is:

$$kVA_{req} = \sqrt{528.6^2 + 166.7^2} = 554.3$$

The power factor angle is: $\theta_{Pf} = \cos^{-1}(528.6/554.3) = 17.5\,\mathrm{deg}$

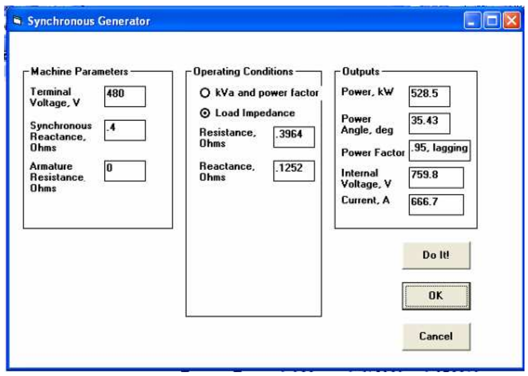
The power factor is: pf = kW/$kVa_{apparent}$ =528.6/554.3 = 0.954

With the MiniCodes:



Another way to do this is via impedances. One can show that the impedance of the induction generator at a slip of -0.02 is -0.615982 + 0.378812j. The resistance of the load to give 800 kW at 480 V is found from R=800,000/480 = 0.288 Ω. The overall impedance is then:

$$Z = \frac{1}{\dfrac{1}{Z_{load}} + \dfrac{1}{Z_{IG}}} = \frac{1}{\dfrac{1}{0.288} + \dfrac{1}{-0.615982 + 0.378812}} = 0.3964 + .1252\,j$$




The answer is almost the same!

**5.14** This problem involves the same generator and load as in Problem 5.13, except now the external resistance of the WRIG is included and it may be varied as required. The wind speed has increased and the control system has responded such that the slip is now -0.42 but the stator currents are unchanged.

a. What is the value (Ω) of the external resistance?
b. What is the speed (rpm) of the generator?
c. How much mechanical power from the turbine's rotor is now coming into the generator's rotor?
d. How much electrical power is coming out of the stator windings?
e. How much power is dissipated in the external resistance?

**SOLUTION**

a) Find *k*:

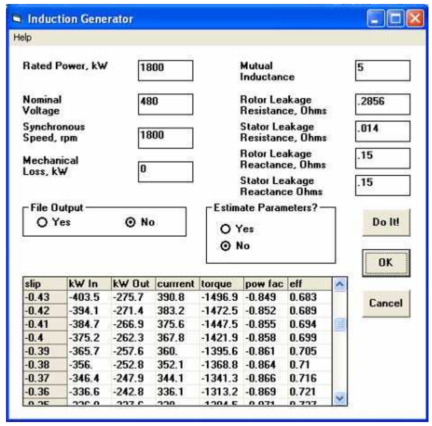
$$\frac{R_R^{'} + R_X}{s} = k \text{ so } k = \frac{R_R^{'}}{s} = \frac{0.0136}{-.02} = -0.68$$

Thus:

$$R_X = ks - R_R^{'} = (-0.68)(-0.42) - 0.0136 = 0.272\ \Omega$$

b) Speed = 1.42 x 1800 = 2556 rpm

c) The power in = 394.1 kW (see screen shot for slip = -0.42)




d) Power out of stator = 271.4 kW (same as total power above)

e) To find the stator loss and rotor loss, use values from slip = -0.02
Total loss = 283.1 – 271.4 = 11.7 kW
Stator loss = (3)($I^2R$)=(3)($383.2^2$)(0.014)/(1000 W/kW)=6.17 kW
Rotor loss = Total loss- Stator loss= 11.7 – 6.17 = 5.53 kW
Total loss with $R_x$ is 394.1-271.4 = 122.7 kW
Loss in external resistance = 122.7 – 11.7 = 111 kW

**5.15** This problem involves the same situation as in Problem 5.14. Instead of an external resistor in the WRIG's rotor circuit, however, an ideal inverter has been inserted. The inverter feeds the same power as was previously dissipated in the rotor resistance into the local network.

a. How much total useful power is produced by the WRIG?
b. What is the power and power factor of the synchronous generator?

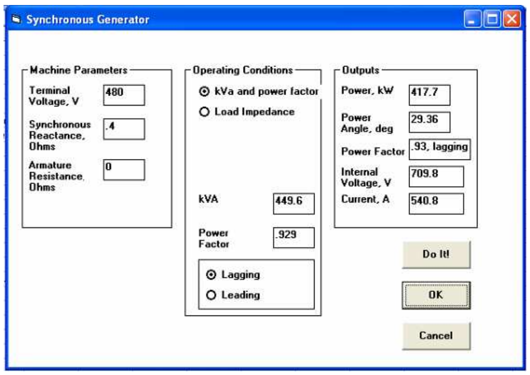
**<u>SOLUTION</u>**

a) See solution to Problem 5.14. The useful power = stator power + rotor power = 271.4 + 111 = 382.4 kW

b) The synchronous generator real power is 800-382.4=417.6 kW. The reactive power requirement is the same as before, so:

$$kVA_{req} = \sqrt{417.6^2 + 166.7^2} = 449.6$$

The power factor is given by: pf = kW/kVA= =417.6/449.6 = 0.929
See also the screen shot from the MiniCodes:




**5.16** A six-pulse inverter has a staircase voltage, two cycles of which are shown in Figure B.7. The staircase rises from 88.85 V to 177.7 V, etc. The frequency is 60 Hz.

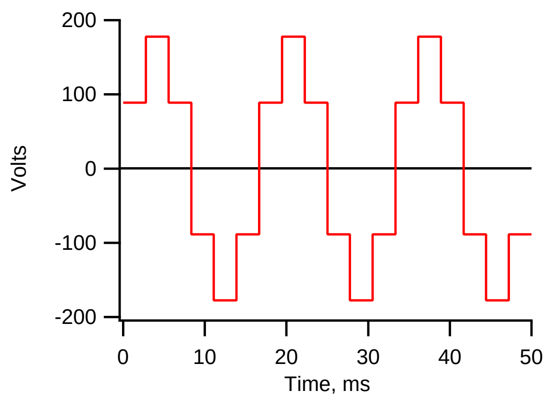



**Figure B.7** Inverter staircase voltage

a) Show that the voltage can be expressed as the following Fourier series:

$$V_{inv}(t) = A\left[\sin(\omega t) + \frac{1}{5}\sin(5\omega t) + \frac{1}{7}\sin(7\omega t) + \frac{1}{11}\sin(11\ \omega t) + ...\right]$$

where $A = (3/\pi)(177.7)$ and $\omega = 2\pi f$

b) A series parallel resonance filter is connected across the terminals. The reactance of each component is the same, so that:

$$f(n) = \left| n^2 \Big/ \left\{ \left(n^2 - 1\right)^2 - n^2 \right\} \right| .$$

Show how the filter decreases the harmonics. Find the reduction to the filter of the fundamental and at least the first three non-zero harmonics. Illustrate the filtered and unfiltered voltage waveform.

Hint: in general, a Fourier series for a pulse train (a square wave) can be expressed as:

$$V(t) = \frac{4}{\pi} H\left[\sin(\omega t) + \frac{1}{3}\sin(3\ \omega t) + \frac{1}{5}\sin(5\omega\ t) + \frac{1}{7}\sin(7\omega t) + \frac{1}{9}\sin(9\omega t) + ...\right]$$

where $\omega = 2\pi f$ and $H$ = the height of the square wave above zero.

**SOLUTION**

a) The voltage can be considered to be the difference of two pulse trains, one with a fundamental at 60 Hz, the other with a fundamental at 3 x 60 Hz = 180 Hz and a magnitude one third of the first one. Call them $V_{p,1}, V_{p,2}, V_{inv}$.

Assuming a magnitude $A = \frac{4}{\pi} H$ and an angular frequency $\omega = 2\pi f$ the first pulse train is

$$V_{p,1}(t) = A\left[\sin(\omega t) + \frac{1}{3}\sin(3\omega t) + \frac{1}{5}\sin(5\omega t) + \frac{1}{7}\sin(7\omega t) + \frac{1}{9}\sin(9\omega t) + ....\right]$$

The second pulse train is

$$V_{p,2}(t) = \frac{1}{3} A\left[\sin(3\omega t) + \frac{1}{3}\sin(9\omega t) + \frac{1}{5}\sin(15\omega t) + \frac{1}{7}\sin(21\omega t) + \frac{1}{9}\sin(27\omega t) + ....\right]$$

The difference is

$$\begin{aligned} &V_{inv}(t) = V_{p,1}(t) - V_{p,2}(t) \\ &= A\left[\sin(\omega t) + \frac{1}{3}\sin(3\omega t) + \frac{1}{5}\sin(5\omega t) + \frac{1}{7}\sin(7\omega t) + \frac{1}{9}\sin(9\omega t) + ....\right] \\ &- \frac{1}{3} A\left[\sin(3\omega t) + \frac{1}{3}\sin(9\omega t) + \frac{1}{5}\sin(15\omega t) + \frac{1}{7}\sin(21\omega t) + \frac{1}{9}\sin(27\omega t) + ....\right] \\ &= A\left[\sin(\omega t) + \left(\frac{1}{3} - \frac{1}{3}\right)\sin(3\omega t) + \frac{1}{5}\sin(5\omega t) + \frac{1}{7}\sin(7\omega t) + \left(\frac{1}{9} - \frac{1}{9}\right)\sin(9\omega t) + ....\right] \end{aligned}$$

Simplifying yields:

$$V_{inv}(t) = A\left[\sin(\omega t) + \frac{1}{5}\sin(5\omega t) + \frac{1}{7}\sin(7\omega t) + \frac{1}{11}\sin(11\,\omega t) + ....\right]$$

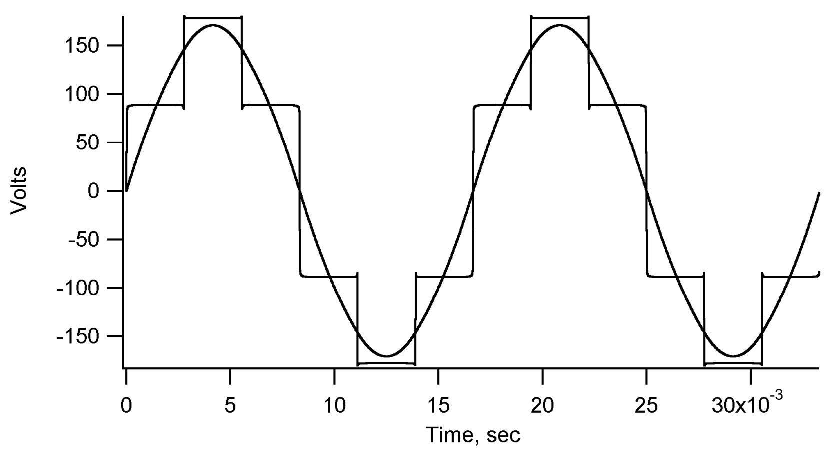
b) As stated:

$$f(n) = \left| n^2 / \left\{ \left(n^2 - 1\right)^2 - n^2 \right\} \right|$$

Therefore, for the non-zero terms:

$$f(1) = 1, f(5) = \frac{25}{576 - 25} = 0.045, f(7) = \frac{49}{2304 - 49} = 0.021, f(11) = \frac{121}{14400 - 121} = 0.00847,....$$

A comparison of the filtered and non-filtered voltages is shown below.




## B.6 Chapter 6 Problems

**6.1** A wind turbine has a hollow steel shaft, 5 m long. The outer radius is 1 m and the inner radius is 0.8 m. Find the area moment of inertia, $I$, and the polar mass moment of inertia, $J$. The moments are to be taken about the long axis of the shaft. Assume that the density of steel, $\rho_s$, is 7800 kg/m$^3$.

**SOLUTION**

$$I = \frac{\pi\left(1^4 - 0.8^4\right)}{2} = 0.9274 \text{ m}^4$$

Find area density:

$$\rho_{area} = (5\,m)(7800\,kg/m^3) = 39{,}000 \text{ kg/m}^2$$

$$J = \rho_{area} I = (39{,}000\,kg/m^2)(0.9274\,m^4) = 36{,}170 \text{ kg m}^2$$

**6.2** The low speed shaft of a wind turbine is 10 m long and has a diameter of 0.5 m. It is made of steel with a shear modulus of elasticity of G = 80 GPa. It is rotating at 12.1 rpm and the turbine is generating 5 MW. (a) Find the applied rotor torque, assuming an overall drive train efficiency of 90%. Find (b) the angle of deflection, (c) the energy stored in the shaft and (d) the maximum stress in the shaft. Hint: The rotational stiffness of a shaft, $k_\theta$, is given by $k_\theta = \dfrac{JG}{l}$

**<u>SOLUTION</u>**

(a) The applied rotor torque is $Q = \dfrac{P}{\Omega\eta} = \dfrac{5{,}000{,}000}{2\pi\left(\dfrac{12.1}{60}\right)(0.9)} = 4.384\,MNm/rad$

(b) The polar moment of inertia is $J = \dfrac{\pi(D)^4}{32} = \dfrac{\pi(0.5)^4}{32} = 0.00614 \text{ m}^4$

The rotational stiffness is $k_\theta = \dfrac{JG}{l} = \dfrac{(0.00614)(80\,x10^9)}{10} = 49.12$ MNm/rad

The angular deflection is $\theta = Q/k_\theta = 4{,}384{,}000/(49.12\,x10^6) = 0.08925$ rad = 5.11°

(c) The strain energy in the shaft is:

$$U = \frac{1}{2}Q\theta = (1/2)(4.384\,MNm/rad)(0.08925\,rad) = 195.6\,kJ$$

(d) The maximum stress is $\sigma_{\max} = Qc/J = (4{,}384{,}000)(0.5/2)/0.00614 = 178.5$ MPa

**6.3** For a helical coil spring, the spring constant, $k$, can be expressed in terms of the spring's dimensions and its material properties:

$$k = \frac{Gd^4}{64R^3 N_c}$$

where: $G$ = modulus of elasticity in shear, N/m$^2$ (lb/in$^2$); $d$ = diameter of wire, m (in); $R$ = radius of coil, m (in), and $N_c$ = number of coils. The average value of $G$ for spring steel is

$11.5x10^6$ lb/in$^2$ ($7.93x10^{10}$ N/m$^2$). The coil radius is based on the distance from the longitudinal axis of the spring to the center of the wire making up the coil.

A spring is needed to return the aerodynamic brakes on a wind turbine blade to their closed position after deployment. The spring should exert a force of 147 lb. (653.9 N) when extended 6 inches (0.1524 m) (brake deployed) and a force of 25 lb. (111.2 N) when the brake is closed. Space limitations dictate that the diameter of the spring be equal 1.5 inches (0.0381 m). A spring of wire gage 7 (0.177 inches or 4.5 mm diameter) is available. How many coils would be required for this spring to have the desired force? How much would the spring coil weigh? Assume a density for steel of 498 lb/ft$^3$ (76 815 N/m$^3$).

**SOLUTION**

The spring constant should be:

$$k = \frac{147-25}{6} = 20.3\ lb/in$$
$$= \frac{653.9-111.2}{0.1524} = 3561\, N/m$$

The radius of the coil is:

$$R = D_{coil}/2 - d_{wire}/2 = 1.5/2 - 0.177/2 = 0.6615\, in$$
$$= 0.0381/2 - 0.0045/2 = 0.0168\, m$$

The number of coils should be

$$N_c = \frac{G\, d^4}{64\, R^3 k} = \frac{(11.5x10^6\ lb/in^2)(0.177\, in)^4}{(64)(0.6625\, in)^3(20.3\, lb/in)} = 29.9\, coils \approx 30\, coils$$
$$= \frac{(7.93x10^{10}\ N/m^2)(0.0045)^4}{(64)(0.0168\, m)^3(3561\, N/m)} \approx 30\, coils$$

Using a weight density for steel, $\rho_s$, of 489 lb/ft$^3$ (76,815 N/m$^3$) the weight of the coil is:

$$W_c = \pi(d_{wire}/2)^2(2)(\pi)(R_{coil})(N_c)(\rho) = \frac{1}{2}\pi^2 R_{coil} N_c \rho$$
$$= \frac{1}{2}\pi^2(0.177\, in)^2(0.6615\, in)(30)(489\, lb/ft^3)/(12\, in/ft)^3 = 0.87\, lb$$
$$= \frac{1}{2}\pi^2(0.0045\, m)^2(0.0168\, m)(30)(76{,}815\, N/m^3) = 3.87\, N$$

**6.4** The basic rating load for ball bearings, $L_R$, is that load for which a bearing should perform adequately for at least one million revolutions. A general equation relates the basic rating load for ball bearings [with balls up to 1 inch (0.083 m diameter)] to a few bearing parameters:

$$L_R = f_c\left[n_R \cos(\alpha)\right]^{0.7} Z^{2/3} D^{1.8}$$

where $f_c$ = constant, between 3,500 and 4,500 (260,200 to 334,600 for ball diameter in m), depending on $D\cos(\alpha)/d_m$; $D$ = ball diameter, inches (m); $\alpha$ = angle of contact of the balls (often 0 deg); $d_m$ = pitch diameter of ball races [= (bore + outside diameter)/2]; $n_R$ = number of rows of balls; $Z$ = number of balls per row.

If two bearings are to operate for a different number of hours, the rating loads must be adjusted according to the relation:

$$L_2 / L_1 = \left(N_1 / N_2\right)^{1/3}$$

A bearing used on the output shaft of a wind turbine's gearbox has the following characteristics: OD = 4.9213 inches (0.125 m), ID (bore) =2.7559 inches (0.07 m). The bearing has 13 balls, each 11/16 inches (0.01746 m) in diameter. Find the basic rating load of the bearing. Suppose that the gearbox is intended to run for 20 years, at 4000 hrs/yr. How does the bearing load rating change? Assume that $f_c$ = 4500. The angle of contact of the balls is equal to zero. The shaft is turning at 1800 rpm.

**SOLUTION**

The equation for basic load rating yields:

$$C = f_c\left(n_R \cos(\alpha)\right)^{0.7} Z^{2/3} D^{1.8} = (4500)(1)^{0.7}(13)^{2/3}(11/16)^{1.8} = 12700\ lb$$
$$= (334{,}600\ )(1)^{0.7}(13)^{2/3}(0.01746)^{1.8} = 56{,}400\ N$$

If the gearbox is intended to operate for 20 years, 4000 hrs/year, the bearing will experience (20 yr)(4000 hrs/yr)(1800 rpm)(60 rev/hr)= 8.64x10$^9$ revolutions. The relative loading is

$$L_2 / L_1 = \left(N_1 / N_2\right)^{1/3} = \left(10^6 / 8.64x10^9\right)^{1/3} = 0.0847$$

Thus the load rating for the bearing should be

$$L_2 = (12{,}700)(0.0487) = 625\ lb$$
$$= (56{,}400)(0.0487) = 2750\ N$$

**6.5** A wind turbine is to be mounted on a three-legged truss tower 30 m high. The turbine has a mass of 5000 kg. The rotor diameter is 25 m. The tower is uniformly tapered and has a mass of 4000 kg. The tower legs are bolted in an equilateral pattern, 4 m apart, to a concrete foundation. Each leg is held in place by six ¾ inch bolts with coarse threads. The wind is blowing in line with one of the tower legs and perpendicular to a line through the other two legs (which are downwind of the single leg). The wind speed is 15 m/s. Find the torque in the bolts such that the upwind tower leg is just beginning to lift off the foundation. Assume that the rotor is operating at the maximum theoretical power coefficient. Hint: the torque in a bolt of this type can be approximately related to the load in the bolt by the following equation:

$$Q_B = 0.195\,\mathrm{d}\,W$$

where $Q_B$ = bolt torque, Nm; $d$ = nominal bolt diameter, m; $W$ = load in bolt, N.

**SOLUTION**

The thrust load at the conditions given is:

$$T = \frac{8}{9}\frac{1}{2}\rho\pi R^2 V^2 = \frac{8}{9}\frac{1}{2}(1.22)(\pi)(12.5)^2(15)^2 = 59{,}900\,N$$

The tower and turbine weight on each leg is:

$$F_T = (m_{tower} + m_{turbine})(g)/3 = (5{,}000 + 4{,}000)(9.81)/3 = 29{,}400\,N$$

Taking moments about the two down wind legs:

$$(F_T + N_B F_B)L_{legs} - (T)(H) = 0$$

This implies that:

$$F_B = \frac{(T)(H)/L_{legs} - F_T}{N_B} = \frac{(59{,}900\,N)(30\,m)/\left(4\left(\sqrt{3}/2\right)\right) - 29{,}400}{6} = 81{,}600\,N/bolt$$

The torque is then:

$Q_B$=(0.195)(.75in)(.0254 m/in)(81,600 N)=303 Nm = 223 ft lb

**6.6** An inventor has proposed a multi-blade wind turbine. The downwind rotor is to have a diameter of 40 ft (12.2 m). Each blade is 17.6 ft (5.36 m) long, with a thickness of 3 inches (7.62 cm) and width of 8 inches (20.3 cm). Assume that the blades have a rectangular shape. The blades are to be made of wood, which can be assumed to have a modulus of elasticity of 2.0 x$10^6$ lb/in$^2$ (1.38x$10^{10}$ Pa) and a weight density of 40 lb/ft$^3$ (6280 N/m$^3$). The rotor turns at 120 rpm. Ignoring other potential problems, are any vibration problems to be expected? (Ignore rotational stiffening.) Explain.

**SOLUTION**

Each blade will be excited as it passes behind the tower. The speed of ration is 120 rpm, which corresponds to 12.57 rad/s. The blades are clearly flexible. Use the Euler method (Chapter 4) or the Engineering Mini-Codes that may be found on the accompanying web site for a cantilevered beam to see if the blade natural frequency is close to that of the excitation. Finding the appropriate terms and converting to consistent units:

$$I = \frac{b\,h^3}{12} = \frac{(8/12)(3/12)^3}{12} = .000868\,ft^4$$
$$= \frac{(.203)(.0762)^3}{12} = 7.48x10^{-6}\,m^4$$

$$\rho' = \rho\,A_c = (40\,lb/ft^3)(8/12\,ft)(3/12\,ft)/(32.2\,lb/slug) = 0.2071\,slug/ft$$
$$= \left(6280\,N/m^3\right)(0.203\,m)(0.0762\,m)/\left(9.81\,m/s^2\right) = 9.9\,kg/m$$

The first mode natural frequency is

$$\omega_1 = \frac{(\beta L)_1^2}{L^2}\sqrt{\frac{E\,I}{\rho'}} = \frac{3.525}{17.6^2}\sqrt{\frac{\left(0.000868\,ft^4\right)\left(2.0x10^6\,lb/in^2\right)\left(144\,in^2/ft^2\right)}{0.02071\,slug/ft}} = 12.5\,rad/s$$
$$= \frac{3.525}{5.36^2}\sqrt{\frac{\left(7.48x10^{-6}\,m^4\right)\left(1.38x10^{10}\,N/m^2\right)}{9.9\,kg/m}} = 12.5\,rad/s$$

The blade natural frequency is very close to the excitation which the blades would experience. This might be expected to be a problem.

**6.7** A gearbox is to be selected for providing speedup to a generator on a new turbine. The turbine is to have a rotor diameter of 40 ft (12.2 m) and to be rated at 75 kW. The generator is to be connected to the electrical grid, and has a synchronous speed of 1800 rpm. A parallel shaft gearbox is being considered. The gearbox has three shafts (an input, intermediate, and output shaft) and four gears. The gears are to have a circular pitch of 0.7854 inches (1.995 cm). The characteristics of the gears are summarized in Table B.7. Find the diameter, rotational speed and peripheral velocity of each gear. Find the tangential forces between gears 1 and 2 and between gears 3 and 4.

**Table B.7** Gear teeth

| Gear | Shaft | Teeth |
|---|---|---|
| 1 | 1 | 96 |
| 2 | 2 | 24 |
| 3 | 2 | 84 |
| 4 | 3 | 28 |

**SOLUTION**

The diameter of each gear is found from the circular pitch:

$$d = N\, p / \pi$$

where :
$p$ = Circular pitch
$N$ = Number of teeth

The relative rotational speeds, $n$, of each gear is given by the inverse of the ratio of the diameters:

$$\frac{n_1}{n_2} = \frac{d_2}{d_1}$$

The peripheral speed is found from.

$$V = (RPM)(\pi)(1/60\, s/\min)(d)$$

The results are summarized in the table below.

| Gear | Shaft | Teeth | d (in) | d (m) | n (RPM) | V (ft/s) | V (m/s) |
|---|---|---|---|---|---|---|---|
| 1 | 1 | 96 | 24 | 0.61 | 150 | 15.7 | 4.79 |
| 2 | 2 | 24 | 6 | 0.152 | 600 | 15.7 | 4.79 |
| 3 | 2 | 84 | 21 | 0.533 | 600 | 55.0 | 16.8 |
| 4 | 3 | 28 | 7 | 0.178 | 1800 | 55.0 | 16.8 |

The tangential force is found from the power divided by the peripheral speed of the gear:

$$F_t = \frac{P}{V}$$

Converting to consistent units, the tangential force between Gears 3 and 4 is:

$$F_{t,3-4} = \frac{P}{v_4} = \frac{(75\,kW)(737.3\left(ft\,lb/s\right)/kW)}{55\;ft/s} = 1006\;lb$$
$$= \frac{\left(75\,kW\right)\left(1000\;W/kW\right)}{16.8\,m/s} = 4464\,N$$

The force between Gears 1 and 2 is

$$F_{t,1-2} = \frac{P}{v_1} = \frac{(75\,kW)(737.3\left(ft\,lb/s\right)/kW)}{15.7\;ft/s} = 3520\,lb$$
$$= \frac{\left(75\,kW\right)\left(1000\,W/kW\right)}{4.79\,m/s} = 15{,}700\,N$$

**6.8** Plate clutches are often used as brakes on wind turbines. According to "uniform wear theory", the load carrying capability of each surface of a plate clutch can be described by the following equation for torque. The torque, $Q$, is simply the normal force times the coefficient of friction, $\mu$ , times the average radius, $r_{av}$, of the friction surface.

$$Q = \mu\frac{r_o + r_i}{2}F_n = \mu\, r_{av}F_n$$

where: $r_o$ = outer diameter; $r_i$ = inner diameter; $\mu$ = coefficient of friction; $F_n$ = normal force.

A clutch type brake is intended to stop a wind turbine at a wind speed 50 mph (22.4 m/s). The brake is to be installed on the generator shaft of the turbine. The turbine has the following characteristics: rotor speed = 60 rpm, generator power at 50 mph (22.4 m/s) = 350 kW, generator synchronous speed = 1800 rpm. The brake will consists of four surfaces, each with an outer diameter of 20 inches (0.508 m) and an inner diameter of 18 inches (0.457 m). The coefficient of friction is 0.3. The brake is spring applied. It is released when air at 10 lb/in$^2$ (68.95 kPa) is supplied to a piston which counteracts the springs. The effective diameter of the piston is 18 inches (0.457 m). What is the maximum torque that the brake can apply? Should that be enough to stop the turbine if the generator is disconnected from the grid? Ignore inefficiencies.

**SOLUTION**

The force that the springs apply must be equal to the force in the piston which can release the brake. This is given by:

$$F_n = p_{air}\pi R^2_{piston} = (10\,lb/in^2)(\pi)(9\,in)^2 = 2545\,lb$$
$$= \left(68950\,Pa\right)\left(\pi\right)\left(0.457\,m/2\right)^2 = 11{,}310\,N$$

The maximum torque is:

$$Q_B = N_s \mu \frac{r_o + r_i}{2} F_n = (4)(0.3)\frac{20/2\,in + 18/2\,in}{2}(2545\,lb)/(12\,in/ft) = 2418\,ft\,lb$$

$$(4)(0.3)\left(\frac{0.508/2\,m + 0.457/2\,m}{2}\right)(11{,}310\,N) = 3274\,Nm$$

where $N_s$ = number of surfaces.

The rotor torque at the speed of the generator is given by:

$$Q_{Rotor} = \frac{P_{Rotor}}{\Omega_{rotor}} = \frac{(350\,kW)(44236\,(ft\,lb/\min)/kW)}{(1800\,rpm)(2\,\pi\,rad/rev)} = 1369\,ft\,lb$$

$$= \frac{(350\,kW)(1000\,W/kW)}{(1800\,rpm)(2\,\pi\,rad/rev)(1/60\,s/\min)} = 1857\,Nm$$

Based on the above calculations, the brake should be able to stop the turbine.

**6.9** A cantilevered steel pipe tower, 80 ft (24.38 m) tall is being considered for a new wind turbine. The weight of the turbine is 12 000 lb (53.4 kN). The pipe under consideration has a constant outer diameter of 3.5 ft (1.067 m) and a wall thickness of 3/4" (1.905 cm). The turbine is to have three blades. The nominal rotor is 45 rpm. Would the tower be considered stiff, soft or soft–soft? Suppose that a tapered tower were being considered instead. How would the natural frequency be analyzed, using the methods discussed in this book? Assume that the density of steel is 489 lb/ft$^3$ (76.8 kN/m$^3$) and its elasticity is 30 x 10$^6$ lb/in$^2$ (2.069 x 10$^{11}$ Pa).

## SOLUTION

Use the approximate equation suggested in this chapter to find the first mode natural frequency:

$$f_0 = \frac{1}{2\pi}\sqrt{\frac{3EI}{(0.23m_{Tower} + m_{Turbine})L^3}}$$

The moment of inertia, $I$, is found from:

$$I = \frac{\pi\left(D_0^4 - D_i^4\right)}{64} = \frac{\pi\left(3.5^4 - 3.375^4\right)}{64} = 0.997\,ft^4$$

$$= \frac{\pi\left(1.067^4 - 1.0289^4\right)}{64} = 0.00856\,m^4$$

The cross sectional area of the tower is:

$$A_c = \frac{\pi\left(D_0^2 - D_i^2\right)}{4} = \frac{\pi\left(3.5^2 - 3.375^2\right)}{4} = 0.6750\,ft^2$$
$$= \frac{\pi\left(1.0668^2 - 1.0289^2\right)}{4} = 0.0624\,m^2$$

The density of steel is 489 lb/ft$^3$ (76.8 kN/m$^3$). The weight of the tower is then:

$$W_{tower} = H_{tower} A_c \rho_{steel} = (80\,ft)\left(0.6750\,ft^2\right)\left(489\,lb/ft^3\right) = 26{,}400\,lb$$
$$= (24.38\,m)\left(0.0624\,m^2\right)\left(76.8\,kN/m^3\right) = 117.4\,kN$$

The natural frequency is:

$$f_0 = \frac{1}{2\pi}\sqrt{\frac{3\,E\,I}{\left(0.23\,m_{tower} + m_{turbine}\right)H^3}}$$
$$= \frac{1}{2\pi}\sqrt{\frac{(3)\left(30x10^6\,lb/in^2\right)\left(144\,in^2/ft^2\right)\left(0.997\,ft^4\right)}{\left((0.23)(26{,}400\,lb) + (12{,}000\,lb)\right)(80\,ft)^3(1/32.2\,lb/slug)}} = 1.07\,Hz$$
$$\frac{1}{2\pi}\sqrt{\frac{(3)\left(2.069x10^{11}\,N/m^2\right)\left(0.00856\,m^4\right)}{\left((0.23)(117.4\,kN) + (53.4\,kN)\right)(24.38\,m)^3(1000\,N/kN)\left(1/9.81\,m/s^2\right)}} = 1.07\,Hz$$

The rotor frequency is:

$$f_R = \frac{N_{\text{rotor}}}{60} = \frac{45}{60} = 0.75\,Hz$$

The blade passing frequency is 3 times the rotor frequency, or 2.25 Hz. Because the tower frequency is between the two it may be considered a soft tower.

For a tapered tower, the Myklestad method, developed in Chapter 4, could be used.

**6.10** A wind turbine has a rotor with a polar mass moment of inertia, $J$, of 55,000 kg m$^2$. The equivalent stiffness of the drive train, $k_\theta$, is 45,000 kNm/rad. The turbine is operating in high winds, such that the angular deflection, $\theta$, in the drive train is 0.05 radians. Suddenly a brake is applied at the far end of the drive train, bringing it to a stop. What is the natural frequency of vibration (in Hz) of the rotor/drive train system as it "rings" against the brake? Write the equation for the angular deflection as a function of time. Ignore any damping.

**SOLUTION**

The natural frequency is found from:

$$\omega = \sqrt{k_\theta / J} = \sqrt{(45{,}000{,}000)(55{,}000)/45} = 28.6 \text{ rad/s}$$

Converting to Hz, the natural frequency is $f = \omega / 2\pi = 28.6/2\pi = 4.55\ Hz$
The equation for angular deflection is:

$$\theta = \theta_0 \cos(\omega t) = (0.05)\cos(28.6\,t)$$

**6.11** An operating turbine is having a problem. The emergency brake is applied, bringing the rotor to a stop in one second. The turbine has a rotor diameter of 80 m. The tower height is also 80 m. The mass of the rotor is 35,000 kg. The turbine is running at 20 rpm before being stopped.

Estimate the overturning moment at the base of the tower due to the application of the brake. Use the impulse-momentum theorem, and assume that all the mass is concentrated at a distance equal to one half of the radius from the center of the rotor.

**SOLUTION**

Start from Newtons' 2nd Law:

$$\sum F = ma = m\frac{dV}{dt}$$

Where $F$ = force, $m$ = mass, $a$ = acceleration, and $V$ = velocity.

Newton's 2nd Law may be written in impulse momentum form as:

$$\int \sum F dt = m(V_2 - V_1)$$

In the case of rotation, the above equation is:

$$\int \sum F dt = mr(\omega_2 - \omega_1)$$

Where $\omega$ = angular speed of rotor. Assuming an average force of $\overline{F}$ over the one second stopping period, and that the final speed ($\omega_2$) is zero, the relation is:

$$\overline{F} = mr\omega_1 / \Delta t$$

Assuming that the mass is concentrated at ½ the radius, the average force would be:

$\overline{F} = mr\omega_1 / \Delta t = (35000)(20)(20)(2\pi / 60) = 1{,}467\,kN$

Assuming the force to act at hub height, the corresponding overturning moment would be:

$M_{OT} = \overline{F}h = (1{,}467\,kNm)(80\,m) = 117{,}300\,kNm$

## B.7 Chapter 7 Problems

**7.1** A turbine is being designed to operate in winds up to 15 m/s (at a hub height of 40 m). The turbine is to have a rotor diameter of $D$ = 45 m. Consider an extreme operating gust which it might be expected to experience at least once every 50 years. What is wind speed at the peak of the gust? For how long would the wind speed exceed that of the nominal maximum operating wind speed? Graphically illustrate the hub height wind speed during the gust. Assume a higher turbulence site.

The IEC 50-year gust for turbines with hub heights greater than 30 m is defined by:

$$U(t) = \begin{cases} U - 0.37\,U_{gust50}\sin(3\pi t / T)[1 - \cos(2\pi t / T)], & \text{for } 0 \le t \le T \\ U, & \text{for } t < 0 \text{ and } t > T \end{cases}$$

where $T$ = 14 s and

$$U_{gust50} = 6.4\left(\frac{\sigma_x}{1 + 0.1(D/21)}\right).$$

**<u>SOLUTION</u>**

From Equation 7.1, the standard deviation of the wind is found from:

$$\sigma_x = I_{ref}(0.75U_{hub} + 5.6)$$

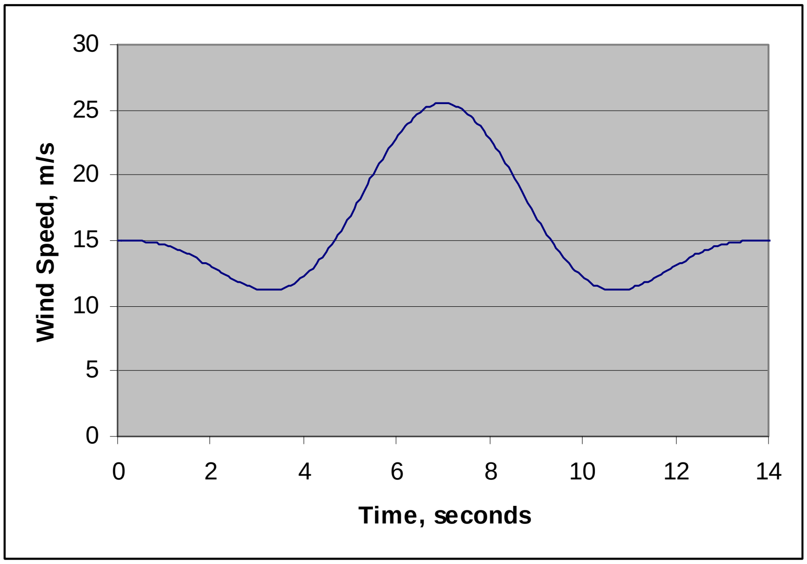
At a higher turbulence site, $I_{15} = 0.16$ and $\sigma_x = 2.7$

The peak value of the gust is then

$$U_{gust50} = 6.4\left(\frac{2.7}{1 + 0.1(\frac{45}{21})}\right) = 14.2\,m/s$$

The time series can be plotted as shown below and is determined by using the design gust equation:

$$U(t)=\begin{cases} U-0.37U_{gust50}\sin(3\pi t/T)(1-\cos(2\pi t/T)) & for\ 0\le t\le T \\ U & for\ t<0\ and\ t>T \end{cases}$$




From the analysis, the peak wind speed is 25.52 m/s.

The wind speed drops from nominal before it begins to rise to its peak. The point at which the wind speed begins to rise is found from:

$$15 = 15 - 0.37\,U_{gust50}\sin(3\pi t/T)(1-\cos(2\pi t/T)$$

In other words, the wind speed begins to rise when $t$ is such that

$$\sin(3\pi t/T)(1-\cos(2\pi t/T) = 0$$

This occurs when $t$ = $T$/3 = 4.67 sec. Since the gust is symmetric over the 14 second interval, the length of time that the wind speed exceeds the nominal is 14-(2)(4.67) = 4.67 sec.

**7.2** The maximum hourly average wind speeds (m/s) recorded at Logan Airport over a sixteen year period are shown in the Table B.8. Using this data, find the 50 yr and 100 yr expected maximum hourly wind speeds.

**Table B.8** Annual maximum wind speeds

| Year | Maximum |
|---|---|
| 1989 | 17.0 |
| 1990 | 16.5 |
| 1991 | 20.6 |
| 1992 | 22.6 |
| 1993 | 24.2 |
| 1994 | 21.1 |
| 1995 | 19.0 |
| 1996 | 20.6 |
| 1997 | 18.5 |
| 1998 | 17.0 |
| 1999 | 17.5 |
| 2000 | 17.5 |
| 2001 | 17.0 |
| 2002 | 16.0 |
| 2003 | 17.0 |
| 2004 | 16.5 |

**SOLUTION**

Use the Gumbel distribution (Equation 2.71):

$$F(x)=\exp\left(-\exp\left(\frac{-(x-\mu)}{\beta}\right)\right)$$

First find the average and the standard deviation of the data.

$\overline{U}_{\max} = 18.6\, m/s \qquad \sigma_{\overline{U}_{\max}} = 2.5\, m/s$

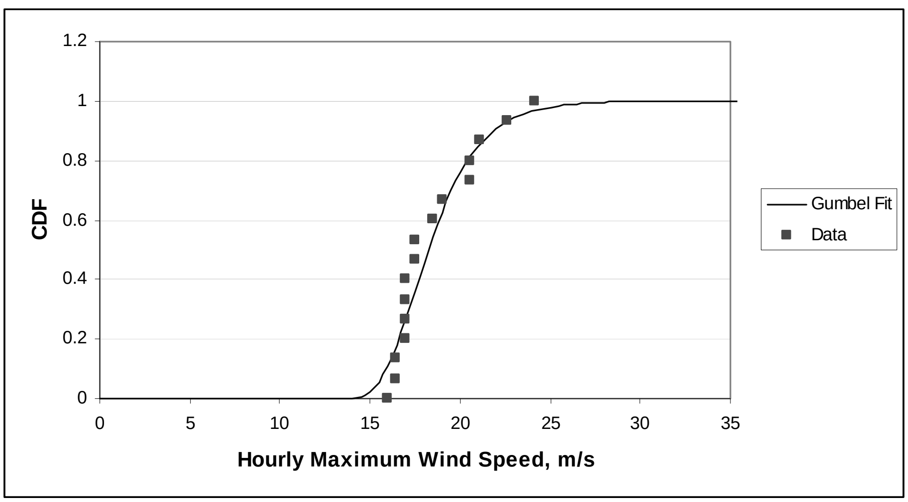
The Gumbel parameters are: $\beta = \dfrac{\sigma\sqrt{6}}{\pi} \quad \mu = \overline{U}_{\max} - 0.577\beta$

Therefore: $\beta = 1.916 \qquad \mu = 17.541$

We need to find the wind speed, $U_{max}$, which is exceeded once in 50 years or once in 100 yrs. That is, the wind such that 1-F($U_{max}$) = 1/50 = 0.02 and 1-F($U_{max}$) = 0.01. In other words, F($U_{max}$) = 0.98 or F($U_{max}$) = 0.99. One can make a table in Excel (see *Gumbel Dist Solution.xls)*:

| $U_{max}$ | $F(U_{max})$ |
|---|---|
| 24 | 0.966 |
| 24.5 | 0.974 |
| **25** | **0.980** |
| 25.5 | 0.984 |
| 26 | 0.988 |
| **26.5** | **0.991** |
| 27 | 0.993 |
| 27.5 | 0.994 |
| 28 | 0.996 |
| 28.5 | 0.997 |
| 29 | 0.997 |
| 29.5 | 0.998 |
| 30 | 0.998 |

Thus the 50 year maximum wind is 25 m/s, and the 100 year wind is 26.5 m/s.
The figure below illustrates the data and the model.




**7.3** Show that maximum stresses due to flapping in a wind turbine blade are independent of size for turbines of similar design. For simplicity, assume a rectangular blade and an ideal rotor.

**SOLUTION**

Maximum stresses are found from the bending moments, the moment of inertia and the maximum thickness of the blade. Without loss of generality, the blades may be considered to be rectangular, with thickness $h$ and width $b$. In the flapwise direction, the bending moment at a given wind speed $U$ is:

$$M_{A,F} = \frac{1}{B}\left(\frac{2R}{3}\right)C_T \frac{1}{2}\rho\pi R^2 U^2$$

where
$B =$ Number of blades
$C_T$ = Rotor thrust coefficient

The ratio of the stresses, $\sigma_{AF} = M\,c/I$, due to the aerodynamic flapping moment in blades of different lengths is:

$$\frac{\sigma_{AF,1}}{\sigma_{AF,2}} = \left(\frac{\frac{1}{B}C_T\left(\frac{2R_1}{3}\right)\frac{1}{2}\rho\pi R_1^2 U^2}{b_1 h_1^2/6}\right) \Bigg/ \left(\frac{\frac{1}{B}C_T\left(\frac{2R_2}{3}\right)\frac{1}{2}\rho\pi R_2^2 U^2}{b_2 h_2^2/6}\right)$$

$$= \left(\frac{R_1^3}{b_1 h_1^2}\right) \Bigg/ \left(\frac{R_2^3}{b_2 h_2^2}\right) = 1$$

Note that here $c/I = (h/2)/(bh^3/12) = 1/(bh^2/6)$

**7.4** Derive the relationship for blade bending stiffness (*EI*) as a function of rotor radius (assuming that the tip speed ratio remains constant, the number of blades, airfoil, and blade material remain the same at different rotor radii and that geometric similarity is maintained to the extent possible).

**SOLUTION**

Stiffness is moment per unit deflection (*EI*). If the same materials are used *E*, a material property, stays the same. For simplicity, consider the blade root to be approximated by a rectangular cross-section of width $c$ (corresponding to the chord) and thickness $t$. The moment of inertia about the flapping axis is $I = c\,t^3/12$. If the radius is doubled, then the moment of inertia goes up by a factor 16. In fact for any constant cross section, if all dimensions are scaled using $R$, the area moment of inertia increases by a factor of $R^4$. Thus the blade flap bending stiffness, *EI*, scales proportional to $R^4$.

**7.5** A wind turbine is being designed to supply power to a load which is not connected to a conventional electrical grid. This could be a water pumping turbine, for example. This problem concerns the estimate of the power curve for this wind turbine–load combination.

The power coefficient vs. tip speed ratio of many wind turbine rotors can be described by the following simple third-order polynomial:

$$C_p = \left(\frac{3\,C_{p,\max}}{\lambda_{\max}^2}\right)\lambda^2 - \left(\frac{2\,C_{p,\max}}{\lambda_{\max}^3}\right)\lambda^3$$

where: $C_{p,\max}$ = maximum power coefficient and $\lambda_{\max}$ = the tip speed ratio corresponding to the maximum power coefficient.

Using the assumptions that a) the power coefficient = zero at a tip speed ratio of zero [i.e. $C_p(\lambda = 0) = 0)$ ], b) the slope of the power coefficient curve is zero at $\lambda = 0$ and c) the slope of the power coefficient curve is also zero at tip speed ratio $\lambda_{\max}$, derive the above relation.

The load the wind turbine is supplying is assumed to vary as the square of its rotational speed, $N_L$, which is related to the rotor speed by the gear ratio, g, so that $N_L = g\,N_R$. Using the rotor speed as reference, the load power is:

$$P_L = g^2\,k\,N_R^2$$

where $k$ is a constant.

A closed-form expression can be derived for the power from the turbine to the load, as a function of rotor size, air density, wind speed, etc. Find that expression. Ignore the effect of inefficiencies in the turbine or the load.

For the following turbine and load, find the power curve between 5 mph (2.24 m/s) and 30 mph (13.4 m/s): $C_{p,\ max}$ = 0.4, $\lambda_{\max} = 7$, $R$ = 5 ft (1.524 m), gear ratio = 2:1 speed up, rated load power = 3 kW, rated load speed = 1800 rpm.

**SOLUTION**

The general form for the polynomial is

$$C_p = c_1 + c_2\lambda + c_3\lambda^2 + c_4\lambda^3$$

Since $C_p = 0$ when $\lambda = 0$ then $c_1 = 0$

The derivative of the polynomial is:

$$\frac{dC_p}{d\lambda} = c_2 + 2\,c_3\lambda + 3\,c_4\lambda^2$$

By the same logic as before, since $dC_p / d\lambda = 0$ when $\lambda = 0$ then $c_2 = 0$ and:

Then $$C_p = c_3\lambda^2 + c_4\lambda^3$$

and $$\frac{dC_p}{d\lambda} = 2c_3\lambda + 3c_4\lambda^2$$

Because $C_p\left(\lambda_{\max}\right) = C_{p,\max}$ then $C_{p,\max} = c_3\lambda_{\max}^2 + c_4\lambda_{\max}^3$ and because the slope at $\lambda_{\max}$ is zero: $0 = 2c_3\lambda_{\max} + 3c_4\lambda_{\max}^2$ .

With two equations in two unknowns, the remaining constants may be found directly:

$$c_4 = -\frac{2C_{p,\max}}{\lambda_{\max}^3} \text{ and } c_3 = -\frac{3C_{p,\max}}{\lambda_{\max}^2}$$

Thus,

$$C_p = \left(\frac{3C_{P,\max}}{\lambda_{\max}^2}\right)\lambda^2 - \left(\frac{2C_{P,\max}}{\lambda_{\max}^3}\right)\lambda^3$$

Rotor power is given by

$$P_R = C_p \frac{1}{2}\rho\pi R^2 V^3$$

Using the expression for $C_p$, and expressing the tip speed ratio in terms of the rotor speed and wind velocity the rotor power is:

$$P_R = \left[\left(\frac{3Cp,\max}{\lambda_{\max}^2}\right)\left(\frac{N_R(2\pi/60)R}{V}\right)^2 - \left(\frac{2Cp,\max}{\lambda_{\max}^3}\right)\left(\frac{N_R(2\pi/60)R}{V}\right)^3\right]\frac{1}{2}\rho\pi R^2 V^3$$

Ignoring efficiciencies, the rotor power must equal the load power:

$$g^2 k N_R^2 = \left[\left(\frac{3Cp,\max}{\lambda_{\max}^2}\right)\left(\frac{N_R(2\pi/60)R}{V}\right)^2 - \left(\frac{2Cp,\max}{\lambda_{\max}^3}\right)\left(\frac{N_R(2\pi/60)R}{V}\right)^3\right]\frac{1}{2}\rho\pi R^2 V^3$$

To simplify, it is convenient to define some constants:

$$a_1 = \left(\frac{Cp,\max}{\lambda_{\max}^2}\right)\frac{1}{600}\rho\pi^3 R^4$$

$$a_2 = -\left(\frac{Cp,\max}{\lambda_{\max}^3}\right)\rho\,\pi\left(\frac{2\,\pi}{60}\right)^3 R^5$$

The equation equating rotor and load power is then:

$$g^2\,k\,N_R^2 = \left[a_1 V\,N_R^2 + a_2 N_R^3\right]$$

Solving this equation yields:

$$N_R = \frac{k\,g^2 - a_1 V}{a_2}$$

Substituting the above equation for speed into the load power equation yields the expression for the power as a function of wind speed:

$$P = g^2\,k\left[\frac{g^2\,k - a_1 V}{a_2}\right]^2$$

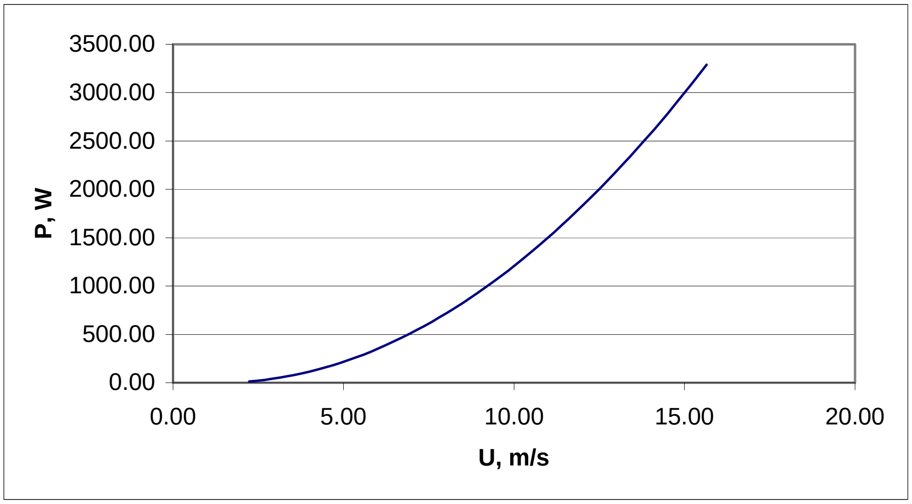
For the specific example turbine and load, the equations above yield:

$$\begin{aligned}
a_1 &= \left(\frac{Cp,\max}{\lambda_{\max}^2}\right)\frac{1}{600}\rho\,\pi^3\,R^4 \\
&= \frac{0.4}{7^2}\left(\frac{1}{600}\right)(0.002378\,slug/ft^3)(\pi^3)(5\,ft)^4 = 0.00062698\ lb\,s^2 \\
&= \frac{0.4}{7^2}\left(\frac{1}{600}\right)(1.225\,kg/m^3)(\pi^3)(1.524\,m)^4 = 0.002788\,N\,s^2
\end{aligned}$$

$$\begin{aligned}
a_2 &= -\left(\frac{Cp,\max}{\lambda_{\max}^3}\right)\rho\,\pi\left(\frac{2\,\pi}{60}\right)^3 R^5 \\
&= -\left(\frac{0.4}{7^3}\right)\left(0.002378\,slug/ft^3\right)\pi\left(\frac{2\,\pi}{60}\right)^3(5\,ft)^5 = -0.00003127\,lb\,s^2\,ft \\
&= -\left(\frac{0.4}{7^3}\right)\left(1.225\,kg/m^3\right)\pi\left(\frac{2\,\pi}{60}\right)^3(1.524\,m)^5 = -0.00004237\ N\,s^2 m
\end{aligned}$$

$$k = 3000/1800^2 = 0.000926\ W/rpm^2$$
$$= (3000W/1.3558W/(ft\,lb/s))(1800^2)\,ft\,lb/s\,rpm^2$$

The resulting power curve is shown below:




**7.6** Assume that current 50 m blade designs have masses of 12,000 kg. Assume also that at the design aerodynamic load, the blade tip deflects in the flap-wise direction 15 m from the blade root axis and the resulting root bending moment is 16 MNm. A new blade testing laboratory is to be designed that will handle blades up to 85 m long. In order to design the new laboratory, the expected mass, tip deflection and root bending moment of 85 m blades that will be designed in the future need to be estimated. Use the scaling relationships derived in the text or derive additional scaling relationships to estimate these quantities. Assume that the tip speed ratio remains constant, the number of blades, airfoil, and blade material remain the same at different rotor radii and that geometric similarity is maintained to the extent possible. Assume that the blade has a uniform cross section and a uniform distributed aerodynamic load of $q$. Note that in that case, the tip deflection, $v$, with an be determined from

$$v = \frac{qR^4}{EI}$$

<u>**SOLUTION**</u>

Mass and Aerodynamic Bending Moment: From previously derived relationships, the weight (or mass) and the aerodynamic root bending moment both scale proportional to $R^3$. So the expected blade mass is:

$$(12{,}000 \text{ kg})(85/50)^3 = 59{,}000 \text{ kg}.$$

The expected aerodynamic bending load at the root is:

$$(16 \text{ MNm})(85/50)^3 = 78.6 \text{ MNm}$$

Deflections: Since:

$$v = \frac{qR^4}{EI}$$

The scaling of $q$, the distributed load along the blade (in N/m) and $EI$ need to be determined.

If the same materials are used $E$, a material property, stays the same. For simplicity, consider the blade root to be approximated by a rectangular cross-section of width $c$ (corresponding to the chord) and thickness $t$. The moment of inertia about the flapping axis is $I = c\,t^3/12$. If the radius is doubled, then the moment of inertia goes up by a factor 16. In fact for any constant cross section, if all dimensions are scaled using $R$, the area moment of inertia increases by a factor of $R^4$. Thus the blade flap bending stiffness, $EI$, scales proportional to $R^4$.

The uniform bending load is due to thrust, which, from the text, scales as $R^2$.

Thus, the deflection scales as:

$$\frac{R^2 R^4}{R^4} = R^2$$

The expected deflection of an 85 m blade would be:

$$(15\text{m})(85/50)^2 = 43.35 \text{ m}$$

**7.7** The data file *ESITeeterAngleDeg50Hz.xls* contains 10 minutes of 50 Hz teeter angle data from a two-bladed 250 kW wind turbine with a down-wind rotor.

a. Determine the average, maximum, minimum and standard deviation of the data.
b. Graph the frequency distribution of the data (a histogram) in bins that are 0.5 degrees wide. If the hub includes dampers to damp teeter motion greater than 3.5 degrees, how many times in this 10-minute data set did the teetering hub contact the teeter dampers?

c. Using the MiniCodes, generate and graph the power spectral density for the data using a segment length of 8192 data points. What is the dominant frequency of oscillation of the data?

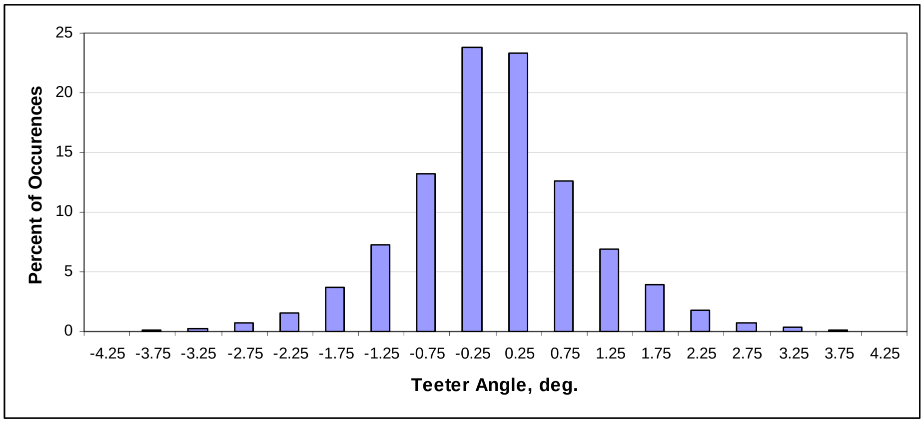
**SOLUTION**

a. The statistics of the data (in degrees) are (see *ESITeeterAngleDeg50HzSolution.xls*):

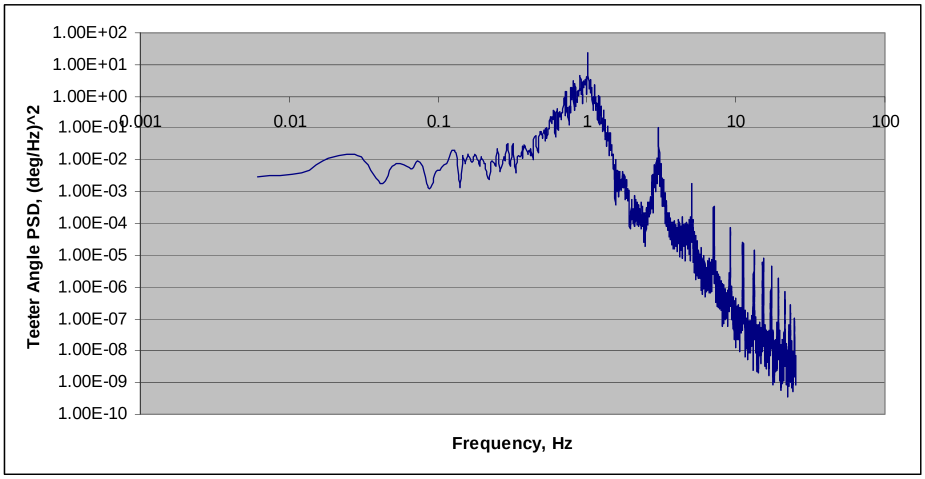
| | |
|---|---|
| Average | -0.00259 |
| Maximum | 4.108 |
| Minimum | -4.021 |
| Standard Deviation | 0.987 |

b. The teetered hub contacts the dampers a total of 58 times in this ten minute data set. The frequency distribution and its graph are provided below.

| BinCenter | Count | Percentage | Damper Contacts |
|---|---|---|---|
| -4.75 | 0 | 0.0 | 0 |
| -4.25 | 2 | 0.0 | 2 |
| -3.75 | 28 | 0.1 | 28 |
| -3.25 | 84 | 0.3 | |
| -2.75 | 226 | 0.8 | |
| -2.25 | 472 | 1.6 | |
| -1.75 | 1097 | 3.7 | |
| -1.25 | 2173 | 7.2 | |
| -0.75 | 3955 | 13.2 | |
| -0.25 | 7127 | 23.8 | |
| 0.25 | 6985 | 23.3 | |
| 0.75 | 3774 | 12.6 | |
| 1.25 | 2058 | 6.9 | |
| 1.75 | 1163 | 3.9 | |
| 2.25 | 529 | 1.8 | |
| 2.75 | 197 | 0.7 | |
| 3.25 | 103 | 0.3 | |
| 3.75 | 24 | 0.1 | 24 |
| 4.25 | 4 | 0.0 | 4 |
| | 30001 | 100 | 58 |




c. The dominant oscillation frequency is 1 Hz. The PSD is provided below.




**7.8** The file *12311215Data.xls* includes data from full scale testing of a 250 kW two-bladed constant-speed wind turbine with an induction generator. From kW and RPM data in *12311215Data.xls* determine the rated slip (in %) of the generator:

(RPM at rated kW- Rated RPM)/(Rated RPM).

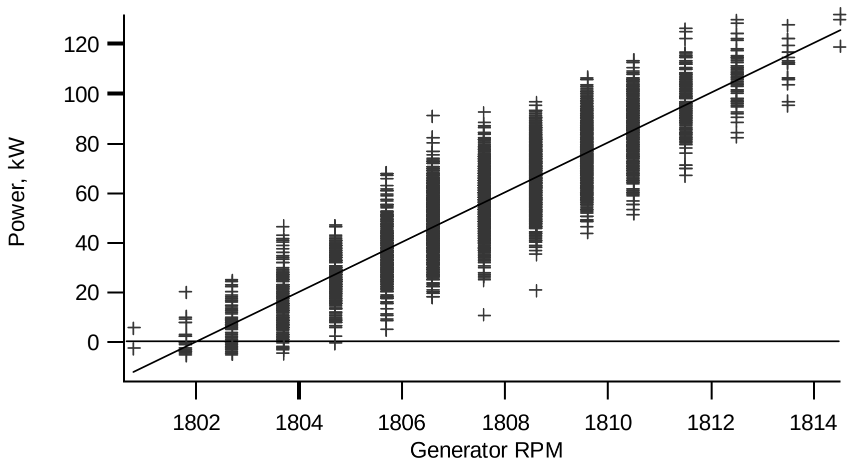
Rated RPM = 1800, Rated kW = 250 kW

## SOLUTION

From a linear fit to the data, the generator speed at rated power, 250 kW is 1826.92 rpm.

From the formula in the problem solution the slip is 1.5% (see *12311215DataSolution.xls*)

From the slope of a linear fit to the data it can be seen that there is a slight offset in the RPM transducer circuit.




The actual no-load generator speed is 1802 rpm. Using this as the rated slip, the slip is 1.38% at 250 kW.

## B.8 Chapter 8 Problems

**8.1** Wind power has been used for centuries for productive purposes. One early yaw control system for windmills, invented by Meikle about 1750, used a fan tail and gear system to turn the rotor into the wind. The windmill turret supported the rotor that faced into the wind. The fan tail rotor, oriented at right angles to the power producing rotor, was used to turn the whole turret and rotor (see Figure B.8). The gear ratio between the fan tail rotor shaft rotation and the turret rotation was about 3000:1. Explain the operation of the yaw orientation system, including the feedback path that oriented the rotor into the wind.

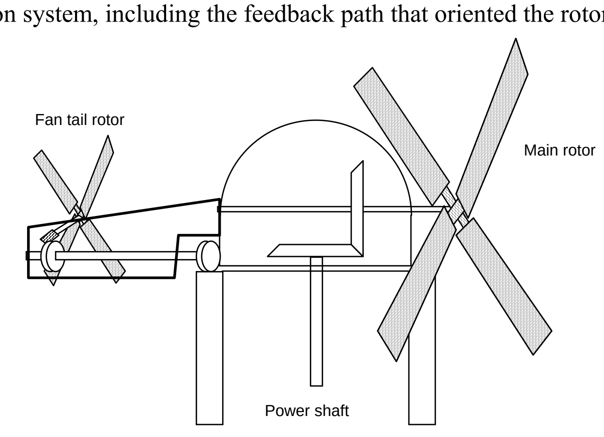



**Figure B.8** Meikle yaw control

**SOLUTION**

The torque from the rotor drove the mill and resulted in a net torque on the power shaft. That torque would drive the turret, including the fan tail rotor out of the wind. As the yaw error increased, the aerodynamic torque on the fan tail rotor would produce a counter torque, to stabilize the yaw position. The system would operate with a yaw error that depended on the design and wind speed. The yaw error sensor (the fan tail) produced a torque that increased as the yaw error increased from the desired set point (no yaw error). The 3000:1 gear reduction provided the force amplification to turn the turret.

**8.2** A supervisory control system, as part of its tasks, is to monitor gearbox operation and the need for gearbox maintenance or repairs. What information should be collected by the supervisory controller and what information should be reported to the system operators?

**SOLUTION**

Numerous variations on input information and reporting might be acceptable, but important input sensor information should include operating temperatures (bearing or lubricant temperatures), gearbox vibration, lubricant level, and measures of lubricant circulation system operation (flow, pressure, or power, etc.). Operators should be appraised of total operating hours on the gearbox and lubrication systems, peak vibration and temperature levels, low lubricant levels, and hours since the last regular maintenance work. Additional information might include files of sensor data collected at frequent intervals, long term temperature trends, and correlations between other operating measures and gearbox operation (gearbox temperatures as a function of power, etc.).

**8.3** A variable speed wind turbine control system uses blade pitch and speed variations to provide constant power to the grid. What are the tradeoffs between fluctuations in rotor speed and the time response of the pitch control system?

**SOLUTION**

In a variable speed wind turbine, the converter is set to provide constant power or torque as wind speed fluctuations cause rotor speed changes. The blade pitch is used to smooth out the rotor speed fluctuations. A slower time response of the pitch control system would allow the rotor speed to increase or decrease more than with a faster pitch control system. This has the advantage of possibly reducing pitch system loading and wear, but may result in undesirable excursions in rotor speed. Faster pitch system response increases pitch motor power requirements and loads in the pitch system and blades.

**8.4** An accelerometer on an experimental wind turbine is used to measure tower vibration. Measurements indicate that the sensor is very sensitive to temperature variations. To complete a series of tests in cold weather the test engineers rig up a quick electrical heating element and controller to keep the sensor's temperature constant. The system includes 1) a small electronic chip that provides a millivolt output that varies with temperature, 2) an electronic circuit, 3) a transistor, 4) a resistance heater, and 5) a housing to encase the chip, the heater, and the accelerometer. The electronic circuit provides a voltage output related to the difference between the voltages at the two inputs. A small current (milliamps) flows into the transistor that is a function of the circuit output voltage. The transistor uses that current to provide up to two amps of current to the heater. The heater maintains the temperature in the enclosure at 70 F.

a) What is the process that is being influenced by the control system?
b) What elements play the roles of the sensor(s), the controller, the power amplifier, and the actuator?
c) What if any are the disturbances in the system?

**SOLUTION**

a) The process that is being influenced by the control system is the energy balance between the accelerometer and components inside the enclosure and the environment on the outside of the enclosure. The introduction of the heating element and the controller adds a heat source inside the enclosure, increasing the temperature inside the enclosure and increasing the heat transfer from the enclosure.

b) The chip is the sensor that responds to temperature changes and the electronic circuit is the controller that responds to differences between the enclosure temperature and the desired enclosure temperature. The transistor is the power amplifier. It takes the small current from the circuit and controls a much larger current that is capable of providing heat. The actuator is the resistance heating element.

c) The heat transfer from the outside of the enclosure is a disturbance to the system. This is mainly a function of ambient temperature, but also of wind speed and possibly the presence of snow, ice, rain, or dirt.

**8.5** An accelerometer on an experimental wind turbine is used to measure tower vibration. Measurements indicate that the sensor is very sensitive to temperature variations. To complete a series of tests in cold weather the test engineers rig up a quick electrical heating element and controller to keep the sensor's temperature constant. The closed loop transfer function of the system is:

$$\frac{T}{T_{ref}} = \frac{0.1}{s^2 + 0.5s + 0.1}$$

Plot the step response of the system to a step increase of the reference temperature of one degree F.

Hint: The step response is formed by multiplying the transfer function by $T_{ref} = 1/s$. The solution is determined by taking the inverse Laplace transform of the resulting equation:

$$T = \frac{0.1}{s^2 + 0.5s + 0.1}\left(\frac{1}{s}\right)$$

First it must be converted into a sum of terms, using the method of partial fraction expansion:

$$T = \frac{0.1}{s^2 + 0.5s + 0.1}\left(\frac{1}{s}\right) = \left(\frac{A}{s}\right) + \frac{Bs + C}{s^2 + 0.5s + 0.1}$$

Then the inverse Laplace transform of each of the terms can be determined. The fraction with the second order denominator may need to be broken into two terms to find the inverse Laplace transform.

**SOLUTION**

The step response of the system is the response for:

$$T_{ref} = \frac{1}{s}$$

or

$$T(s) = \left(\frac{0.1}{s^2 + 0.5s + 0.1}\right)\left(\frac{1}{s}\right)$$

The inverse Laplace transform of this transfer function can be found using a partial fraction expansion:

$$T(s) = \left(\frac{0.1}{s^2 + 0.5s + 0.1}\right)\left(\frac{1}{s}\right) = \frac{A}{s} + \frac{Bs + C}{s^2 + 0.5s + 0.1}$$

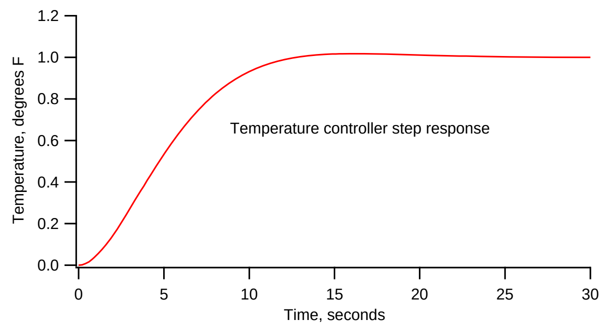
Multiplying through and collecting terms, one finds that $A = 1$, $B = -1$, and $C = -0.5$. A few manipulations result in:

$$T(s) = \frac{1}{s} - \frac{s + 0.5}{s^2 + 0.5s + 0.1} = \frac{1}{s} - \frac{s + 0.25}{(s + 0.25)^2 + 0.194^2} + 1.29\frac{0.194}{(s + 0.25)^2 + 0.194^2}$$

The inverse Laplace transfer of this function is:

$$T(t) = 1 - e^{(-t/4)}\cos(0.194t) + 1.29e^{(-t/4)}\sin(0.194t)$$

The graph looks like:




**8.6** The transfer function for a pitch control system is:

$$\frac{\Theta_m(s)}{\Theta_{m,ref}(s)} = \left(\frac{K}{s^3 + s^2 + s + K}\right)$$

The closed-loop system response to a step command to pitch the blades 1 degree is:

$$\Theta_m(s) = \left(\frac{K}{s^3 + s^2 + s + K}\right)\left(\frac{1}{s}\right) = \frac{K}{(s+a)\left(s^2 + bs + c\right)\ s}$$

a) Calculate and plot the closed-loop time domain system step response for K = 0.5, the initial choice of the designer. Comment on the closed-loop system response, including damping, overshoot, and response time.

b) For no good reason, the control system designer decides to increase the gain of the system, $K$. Calculate the closed-loop time domain system step response for $K$ = 3, the new choice of the designer.

c) What differences are evident between the time responses, each with a different gain?

Hint: perform a partial fraction expansion of the general form of the closed-loop response:

$$\frac{d}{(s+a)\left(s^2 + bs + c\right)s}$$

and find the inverse Laplace transform of each term, using the $a$, $b$, $c$, $d$ variables. This symbolic form will be handy, as it will be used twice in the solution.

For parts a) and b), find the real root, $a$, of the denominator by graphing $s^3 + s^2 + s + d$ , using the appropriate $K$. The values of $s$ at the zero crossing is $-a$. The second order root is found using long division:

$$\frac{s^3 + s^2 + s + d}{(s+a)} = s^2 + (1-a)s + \left(1 - a + a^2\right)s$$

Thus, the roots of the denominator are $s$, $(s + a)$ and

$$s^2 + bs + c = s^2 + (1-a)s + (1 - a + a^2)$$

Insert the solutions for $a$, $b$, $c$, $d$ into the general solution and plot the result.

**SOLUTION**

a) For $K = 0.5$

$$\Theta_m(s) = \frac{0.5}{(s^3 + s^2 + s + 0.5)s}$$

From a graph of $s^3 + s^2 + s + 0.5$ , one root of the denominator is ($s$ +.6478). Long division provides the other imaginary root, so the whole transfer function is:

$$\Theta(s) = \frac{0.5}{(s + 0.6478)(s^2 + 0.3522s + 0.7718)s}$$

which is of the form:

$$\Theta(s) = \frac{d}{(s + a)(s^2 + bs + c)s}$$

Using a partial fraction expansion:

$$\frac{d}{(s+a)(s^2 + bs + c)s} = \frac{A}{s} + \frac{B}{(s+a)} + \frac{Cs + D}{(s^2 + bs + c)}$$

or

$$d = A(s+a)(s^2 + bs + c) + B(s^2 + bs + c)s + (Cs + D)(s + a)s$$

so

$$d = A(s^3 + (a+b)s^2 + (c+ab)s + (ac)) + B(s^3 + bs^2 + cs) + C(s^3 + as^2) + D(s^2 + as)$$

and, therefore, equating coefficients on the left and right hand sides:

$$\begin{aligned} s^3 &: \quad 0 = (A + B + C) \\ s^2 &: \quad 0 = (A(a+b) + Bb + Ca + D) \\ s^1 &: \quad 0 = (A(c+ab) + Bc + Da) \\ s^0 &: \quad d = Aac \end{aligned}$$

So

$$A = \frac{d}{ac}$$

$$C = \frac{d(b-a)}{c(c-ab+a^2)}$$

$$B = -\left(\frac{d}{ac} + C\right)$$

$$D = -\left(\frac{A(c+ab)+Bc}{a}\right)$$

Also, the transfer function:

$$\Theta_m(s) = \frac{A}{s} + \frac{B}{(s+a)} + \frac{Cs+D}{(s^2+bs+c)}$$

can be broken down into functions for which the inverse Laplace transform can be determined:

$$\Theta_m(s) = \frac{A}{s} + \frac{B}{(s+a)} + C\frac{s+b/2}{\left((s+b/2)^2+c-b^2/4\right)} + \frac{(D-C\,b/2)}{\sqrt{c-b^2/4}}\frac{\sqrt{c-b^2/4}}{\left((s+b/2)^2+c-b^2/4\right)}$$

Using those inverse Laplace Transforms:

$$\theta(t) = A + Be^{-at} + Ce^{-bt/2}\cos\left(t\sqrt{c-b^2/4}\right) + \frac{(D-C\,b/2)}{\sqrt{c-b^2/4}}e^{-bt/2}\sin\left(t\sqrt{c-b^2/4}\right)$$

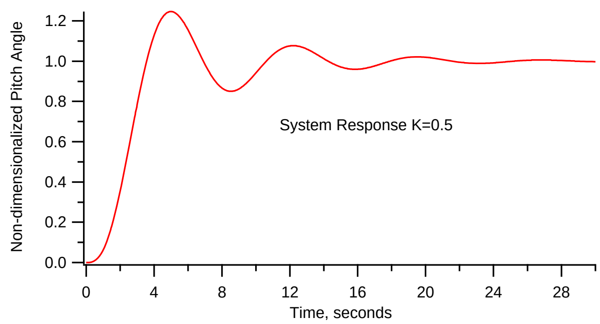
For $K = 0.5$

$$a = 0.6478$$
$$b = 0.3522$$
$$c = 0.7718$$
$$d = 0.5$$

And

$$A = 1$$
$$C = -.199$$
$$B = -.801$$
$$D = -.589$$

A graph of the resulting response is:




The step response has some overshoot and is not highly damped, but is reasonably close to the commanded position after 4 seconds. The response settles at the desired position after about 18 seconds.

b) For $K = 3$

$$\Theta_m(s) = \frac{3}{\left(s^3 + s^2 + s + 3\right)s}$$

From a graph of $s^3 + s^2 + s + 3$, one root of the denominator is ($s$ +1.57474). Long division provides the other imaginary root, so the whole transfer function is:

$$\Theta_m(s) = \frac{3}{(s + 1.57474)\left(s^2 - 0.57474s + 1.90506\right)s}$$

Using the same partial fraction expansion and transformation developed in part a), the system time response is:

$$\theta(t) = A + Be^{-at} + Ce^{-bt/2}\cos\left(t\sqrt{c - b^2/4}\right) + \frac{(D - C\,b/2)}{\sqrt{c - b^2/4}}e^{-bt/2}\sin\left(t\sqrt{c - b^2/4}\right)$$

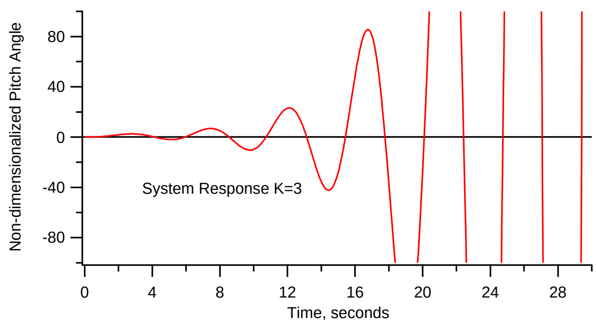
For K=3

$$a = 1.57474$$
$$b = -0.57474$$
$$c = 1.90506$$
$$d = 3$$

And

$$A = 1$$
$$C = -.640$$
$$B = -.360$$
$$D = -.200$$

A graph of the resulting response shows unstable behavior:




c) The response with $K = 3$ is an unstable response whereas the response with $K = 0.5$ is a stable one.

**8.7** A wind turbine manufacturer wants to design a yaw drive control system. To minimize wear on the drive gears the yaw is to be locked with a yaw brake until the 10 minute time-averaged yaw error is more than some specified amount (the "yaw error limit"). At that point the yaw drive would move the turbine to face the previously determined 10 minute time-averaged wind direction.

a) What consequences does the choice of yaw error limit and averaging time have on machine operation?

b) What approach would you take to determining the quantitative tradeoffs between yaw error limit and other factors?

**SOLUTION**

a) Decreased averaging time and decreased yaw error limits would increase the actions of the yaw drive. A larger yaw error limit or a longer averaging time would reduce the operating hours of the yaw drive but increased time at higher yaw errors. High yaw error operation decreases energy capture and may increase machine loads and, consequently, component fatigue. High yaw error operation might also increase electrical power fluctuations, depending on the generator system, and tower torsional fatigue, depending on the yaw system design. Thus, tradeoffs exist between the operating hours of the yaw drive and energy capture and machine life.

b) Two approaches can be used to analyze the consequences of different yaw error limits: time series simulations and statistical approaches.

The first issue to recognize is that the time between yaw orientation corrections may be very site dependent. In turbulent conditions or as weather fronts pass, more yaw corrections may be required because the wind direction is changing rapidly or frequently. In flat sites with constant prevailing winds, few corrections may be required. On the other hand, at sites with diurnal sea and land breezes the turbine may have to rotate at least once a day. This rotation might happen in few or in many steps.

Thus, an analysis might start with a representative time series of wind direction data. From this one could develop 1) a file of running averages, based on the running averaging time to be investigated for use in the controller, and 2) a histogram of averaged yaw errors that would occur between yaw motions. These histograms could be used with turbine models to determine loads, electrical power, and energy capture for different yaw errors and to develop overall estimates of component fatigue, yaw system operating hours, and energy capture.

**8.8** A pitch control system is being designed for a wind turbine. The response of the pitch control system to a unit step command has been determined to be:

$$\theta_p = \theta_{p,ref}\left[1 - 1.67e^{-1.6t}\cos(1.2t - 0.93)\right]$$

where $\theta_p$ is the pitch angle change and $\theta_{p,ref}$ is the magnitude of the commanded pitch angle change. In order to shut down the turbine in high winds the pitch must change by 16 degrees. This motion is opposed by three torques, those due to friction, the pitching moment and inertial. The total friction torque, $Q_{friction}$, is assumed to be a constant 30 Nm, and the total pitching moment, $Q_{pitching}$, in high winds is assumed to be a constant 1000 Nm. Finally, the inertial torque is a function of the inertial moments: $Q_{inertia} = J\ddot{\theta}_p$. The total moment of inertia, $J$, of the blades is 100 kg m$^2$. The total power needed to pitch the blades is:

$$P_{total} = \dot{\theta}_p\left(J\ddot{\theta}_p + Q_{friction} + Q_{pitching}\right)$$

What is the peak power required to pitch the blades 16 degrees (0.279 radians)?

**<u>SOLUTION</u>**

Note: The problem statement in the text mixes two sets of nomenclature. In this solution and problem description, the correct nomenclature has been used.

The power to overcome inertia is:

$$P_{inertia} = \dot{\theta}_p Q_{inertia} = \dot{\theta}_p J\ddot{\theta}_p$$

The power to overcome friction is:

$$P_{friction} = \dot{\theta}_p Q_{friction}$$

The power to overcome the pitching moment is:

$$P_{pitching} = \dot{\theta}_p Q_{pitching}.$$

Thus, the total power required to pitch the blades is

$$P_{total} = \dot{\theta}_p\left(J\ddot{\theta}_p + Q_{friction} + Q_{pitching}\right)$$

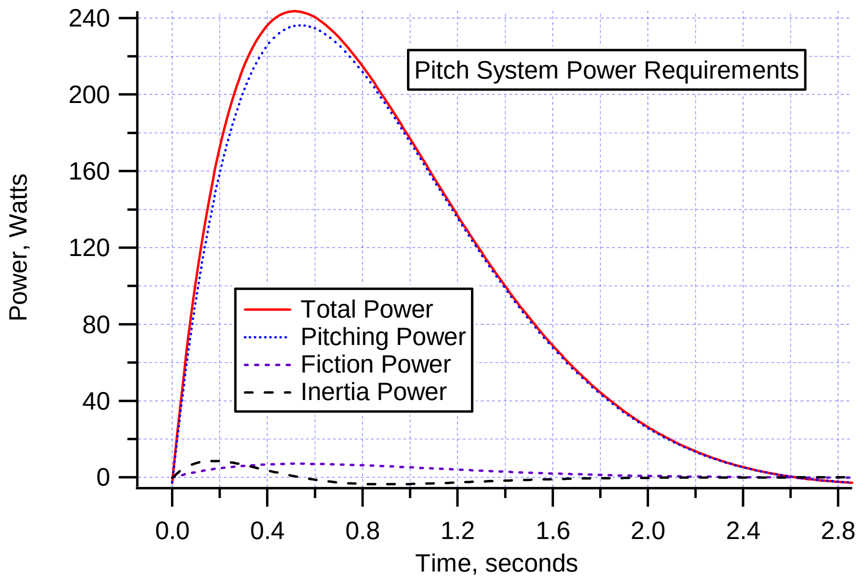
The pitching velocity, under the specified conditions, is:

$$\dot{\theta}_p = \theta_{p,ref}\left(2.672e^{-1.6t}\cos(1.2t-.93)+2.004e^{-1.6t}\sin(1.2t-.93)\right)$$

After differentiating and collecting terms, the pitching acceleration, under the specified conditions, is:

$$\ddot{\theta}_p = \theta_{p,ref}\left(-1.8704e^{-1.6t}\cos(1.2t-.93)-6.4128e^{-1.6t}\sin(1.2t-.93)\right)$$

The total contributions to the required power can be determined using the expressions above. The total power and the contributions to the power are illustrated in the following graph.




The maximum power required is about 243.6 W. Most of the required power, in this case, is needed to overcome the pitching moment.

**8.9** The transfer function for the shaft torque, $Q_S(s)$, of a 250 kW grid-connected, fixed-pitch wind turbine with an induction generator as a function of the aerodynamic torque, $\alpha W(s)$, is:

$$\frac{Q_S(s)}{\alpha W(s)} = \frac{49(s+96.92)}{(s+97.154)(s^2+1.109s+48.88)}$$

a) This transfer function can be characterized as being the product of a first-order system, a second-order system and a term of the form $K(s+a)$. The transfer function for second-order systems can be expressed as:

$$G(s) = \frac{\omega_n{}^2}{s^2 + 2\zeta\omega_n s + \omega_n{}^2}$$

where $\omega_n$ is the system natural frequency and $\zeta$ is the damping ratio. What is the natural frequency and damping ratio of the second-order system component?

b) The transfer function for first-order systems can be expressed as:

$$G(s) = \frac{1/\tau}{s + 1/\tau}$$

where $\tau$ is the time constant of the first-order system. What is the time constant of the first-order system component?

**<u>SOLUTION</u>**

a) The transfer function for second order systems can be expressed as:

$$G(s) = \frac{\omega_n{}^2}{s^2 + 2\zeta\omega_n s + \omega_n{}^2}$$

where $\omega_n$ is the system natural frequency and $\zeta$ is the damping ratio.

The transfer function can be written in the form:

$$\frac{Q_S(s)}{\alpha W(s)} = \frac{1.002}{97.154}(s + 96.92)\frac{97.154}{(s + 97.154)}\left(\frac{6.99^2}{\left(s^2 + 2(0.0793)6.99s + 6.99^2\right)}\right)$$

In this form it can be seen that the natural frequency of the system is 6.99 rad/s and the damping ratio is 0.0793.

b) From the equation above, the time constant for the first order component is 0.0103 seconds.

**8.10** A proportional–integral–derivative (PID) controller can be designed using the electrical circuit pictured in Figure B.9. The differential equation for the controller is:

$$g(t) = -\left[K_P e(t) + K_D \dot{e}(t) + K_I \int e(t)\mathrm{d}t\right] = -\left[\left(\frac{R_2}{R_1} + \frac{C_1}{C_2}\right)e(t) + (R_2 C_1)\dot{e}(t) + \frac{1}{R_1 C_2}\int e(t)\mathrm{d}t\right]$$

where $g(t)$ is the controller output, and $R$ and $C$ are the resistances and capacitances of the respective circuit elements, $e$(t) is the error signal that is input to the controller and the controller constants are $K_P$, $K_I$, and $K_D$.

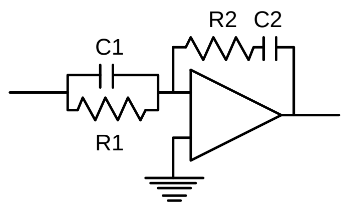



**Figure B.9 PID Controller**

If $R_1$ is 10000 Ohms, what values of $R_2$, $C_1$ and $C_2$ would be required for $K_P$, = 10, $K_I$ = 100 and $K_D$ = 0.01. Note resistances are in Ohms and capacitances are in Farads.

**SOLUTION**

Note, the value for $K_D$ was mistakenly listed as 100 in the problem statement in the text. The correct problem statement is included here.

From

$$g(t) = -\left[K_P e(t) + K_D \dot{e}(t) + K_I \int e(t)\mathrm{d}t\right] = -\left[\left(\frac{R_2}{R_1} + \frac{C_1}{C_2}\right)e(t) + \left(R_2 C_1\right)\dot{e}(t) + \frac{1}{R_1 C_2}\int e(t)\mathrm{d}t\right]$$

$$K_I = \frac{1}{R_1 C_2} = 100$$

so $C_2 = 10^{-6}$ Farads

Combining yields:

$$K_D = R_2 C_1$$

And

$$\left(\frac{K_D}{R_1 C_1} + \frac{C_1}{C_2}\right) = K_P$$

Solving for $C_1$, one obtains:

$$C_1^{\,2} - C_1\left(C_2 K_P\right) + \frac{K_D C_2}{R_1} = 0$$

Substituting the values for $R_1$, $C_2$, $K_D$ and $K_P$, one can solve for $C_1$ and $R_2$:

$$R_2 = 100{,}000 \text{ Ohms}$$
$$C_1 = 10^{-7} \text{ Farads}$$

**8.11** If it is required that a digital control system determine the system behavior at frequencies of at least 10 Hz, a) what is the minimum sampling frequency that can be used and b) what cut-off frequency should be used to filter the input data?

**SOLUTION**

Sampling at 20 Hz is required to correctly represent dynamics at 10 Hz, the Nyquist frequency. Filtering frequencies above 10 Hz is required to make sure that higher frequency components do not affect the signal.

**8.12** The transfer function of a blade pitch controller meant to control the mean rotor power above rated on a constant speed wind turbine is approximated as:

$$\frac{\Theta}{\Theta_{ref}} = \frac{0.5}{s+0.5}$$

Using partial fraction expansions, calculate the step response of the system. Thus, assume that:

$$\Theta_{ref} = \frac{1}{s}$$

What is the step response (the time for the blade pitch to get from 10% to 90% of the desired position)?

**SOLUTION**

From the problem statement:

$$\Theta = \frac{1}{s}\frac{0.5}{s+0.5}$$

Using partial fraction expansions:

$$0.5 = \frac{A}{s} + \frac{B}{s+0.5}$$

Multiplying by the denominator of the transfer function:

$$0.5 = A(s+0.5) + Bs$$

Or

$$0.5 = 0.5A + (A + B)s$$

Thus, equating coefficients of the terms:

$$A + B = 0$$
$$A = 1$$

So:

$$B = -1$$

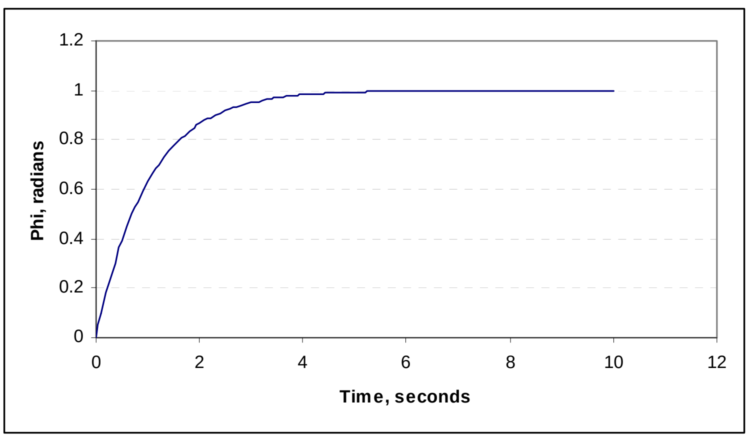
Then:

$$\Theta = \frac{1}{s} - \frac{1}{s+1}$$

The inverse Fourier transform of this is:

$$\theta = 1 - 1e^{-t}$$

A graph of the system response is shown in the following figure.




This can be solved for $t$:

$$t = -\ln(1-\theta)$$

For θ = 0.1, $t$ = 0.105 s; for θ = 0.9, $t$ = 2.303 s. Thus, the step response is 2.2 seconds.

**8.13** An engineer is studying the results of changing the blade pitch control loop gain in the controller that maintains the turbine rotor speed and generator power. The results of her simulations are provided in *WPControlNeg1.csv*, *WPControlNeg10.csv* and *WPControlNeg100.csv*. Note that the time steps for the case with a gain of -100 are less than in the other data sets. Determine the average and standard deviation of the blade pitch angle (degrees), the generator power (kW) and the rotor speed (rpm). Also determine the average and standard deviation of the blade pitch rate (deg/sec). What difference do more negative gains make? What is happening when the gain is -100?

**SOLUTION**

The analysis is included in *WPControlSolution.xls*. The results are:

| Generator Power, kW | | | |
|---|---|---|---|
| | Gain | | |
| | -1 | -10 | -100 |
| Average | 1393.5 | 1419.2 | 1385.9 |
| Std. Dev. | 90.8 | 16.0 | 52.6 |

| Blade Pitch, deg. | | | |
|---|---|---|---|
| | Gain | | |
| | -1 | -10 | -100 |
| Average | 19.4 | 19.3 | 19.5 |
| Std. Dev. | 3.1 | 3.4 | 7.4 |

| Rotor Speed, rpm | | | |
|---|---|---|---|
| | Gain | | |
| | -1 | -10 | -100 |
| Average | 20.5 | 20.5 | 20.3 |
| Std. Dev. | 0.7 | 0.1 | 0.3 |

| Blade Pitch Rate, deg/sec | | | |
|---|---|---|---|
| | Gain | | |
| | -1 | -10 | -100 |
| Average | 0.00 | 0.00 | 0.04 |
| Std. Dev. | 0.92 | 3.76 | 80.08 |

More negative gains (at least up to -10) improve the control in the sense that the rotor speed and generator power variations decrease. On the other hand, the blades much be pitched more actively and the blade pitch variability and pitch rate variability increase. At a gain of -100, the pitch rates are on the order of 80 degrees per second and the controller is loosing control of the rotor speed (the rotor speed and generator power variations increase).

**8.14** If the rotor speed controller on a variable speed turbine can accurately maintain the rotor tip speed ratio at its optimum value, then the rotor $C_P$ during constant tip speed operation will always be at its maximum. In the real world, the control system cannot achieve this level of control. Suppose that the range of tip speeds over which the turbine operates during constant tip speed operation is described by a normal probability density distribution described by:

$$P(\lambda) = \frac{b}{\sqrt{2\pi}} e^{-\left[\frac{(b(\lambda - \lambda_0))^2}{2}\right]}$$

In this equation, $b$ is a parameter that describes the width of the distribution of tip speed ratios. Suppose also that the $C_P(\lambda)$ within a reasonable distance from the optimum tip speed ratio can be described as:

$$C_P(\lambda) = -a(\lambda - \lambda_0)^2 + C_{P,\max}$$

In this equation, $a$ is a parameter that determines the curvature at the top of the $C_P - \lambda$ curve. The average $C_P$ that is described by this type of operation is then:

$$\int C_P(\lambda) P(\lambda) d\lambda$$

Use Excel to numerically integrate the equation above to determine the average $C_P$ and the percentage of optimum $C_P$ when $b$ is either 1 or 2 and when $a$ is either 0.01 or 0.02. When $a$ is fixed at 0.02, how much is the $C_P$ improved when the distribution of the tip speeds is narrower? When $b$ is fixed at 1, how much is the $C_P$ improved when the top of the $C_P - \lambda$ curve has less curvature? Assume that $C_{P,\max} = 0.47$.

**SOLUTION**

See *TrackingOptimumTipSpeed Solution.xls* for the analysis. The results are:

| | | | | |
|---|---|---|---|---|
| a | 0.01 | 0.02 | 0.01 | 0.02 |
| b | 1 | 1 | 2 | 2 |
| Cp | 0.46 | 0.449 | 0.467 | 0.464 |
| Percent of Cp,max | 97.8 | 95.6 | 99.4 | 98.8 |

When $a$ is 0.02 the $C_P(\lambda)$ is less flat near the optimum tip speed ratio and the $C_P$ is 3.4% greater when the distribution of the tip speeds is narrower (the tip speed is more tightly controlled). When $b$ is 1 (a wider range of tip speeds), the $C_P$ improves by 2.1% when the top of the $C_P - \lambda$ curve has less curvature.

**8.15** a) Derive Eq. 8.14 in the text. Hint, Power=Torque x Speed.
b) For a 5 MW turbine, with $R$ = 63 m, $\lambda_{opt}$ = 8, and $C_{P,max}$ = 0.50, and $\rho$ = 1.225 kg/m$^3$, calculate the proportionality constant:

$$k = \frac{\rho \pi R^5 C_{p,\max}}{2(\lambda_{opt})^3}$$

**SOLUTION**

a) The maximum rotor power is:

$$P_{\max} = \frac{1}{2}\rho A U^3 C_{p,\max}$$

From the definition of tip speed ratios:

$$U = \frac{\Omega R}{\lambda}$$

Substituting this into the initial equation, for conditions at $C_{P,max}$ one gets

$$P_{\max} = \frac{1}{2}\rho A\left(\frac{\Omega R}{\lambda_{opt}}\right)^3 C_{p,\max}$$

And using the definition of area, $A = \pi R^2$, and the definition of power, $P = \Omega\ Q$, one gets:

$$Q_{opt}\Omega = \frac{1}{2}\rho\pi R^2\left(\frac{\Omega R}{\lambda_{opt}}\right)^3 C_{p,\max}$$

Collecting terms and designating the optimum torque as the reference torque for the controller one gets:

$$Q_{ref} = \frac{\rho\pi R^5 C_{p,\max}}{2(\lambda_{opt})^3}\Omega^2$$

b) For the stated conditions:

$$k = \frac{\rho\pi R^5 C_{p,\max}}{2(\lambda_{opt})^3} = \frac{1.225\pi 63^5(0.50)}{2(8)^3} = 1864912\ \mathrm{Nms}^2$$

## B.9 Chapter 9 Problems

**9.1** A wind farm is being considered for a ridge top site. Name 10 or more issues that might be considered in evaluating this site.

**SOLUTION**

Considerations might include: mean wind speed, variations of wind speed over the site, turbulence intensity, topographic effects on directional characteristics of the wind, available area for turbines, the presence of nearby wind farms, ease of access, competing land use issues, slope of the terrain, endangered species, environmentally sensitive areas (e.g. vernal pools), visibility, proximity to houses, grid strength, need for a grid extension and cost of any grid extension, legal issues, visual pollution concerns, extreme weather, the possibility of danger to hikers, size of turbine that can be transported to the site, cost of road building, site preparation, and turbine transport, availability of cranes, access for maintenance personnel, interference with microwave transmissions, erosion and run-off from road and site work, and the availability of accurate enough wind data for energy capture projections.

**9.2** Four identical wind turbines that are lined up in a row 12 rotor diameters apart are experiencing wind parallel to the row of wind turbines. Use Katić's wake model to determine the speed of the wind approaching each of the wind turbines. Assume that $k$ = 0.10 and that the thrust coefficient is 0.7.

**SOLUTION**

The most upwind turbine (which will be called #1) experiences the free stream wind speed, $U_0$.

At 12 rotor diameters, the next turbine (#2) experiences a velocity deficit of:

$$1-\frac{U_X}{U_0}=\frac{\left(1-\sqrt{1-C_T}\right)}{\left(1+2k\dfrac{X}{D}\right)^2}=\frac{\left(1-\sqrt{1-0.7}\right)}{\left(1+2(0.1)12\right)^2}=0.039$$

Thus #2 experiences a wind speed of 0.961 $U_0$.

Turbine #3 experiences consequences from both turbines #1 and #2. The wind speed at turbine #3 due to the wake of turbine #2, which sees the wake of turbine #1 is $0.961^2$ = 0.924 $U_0$. Similarly, the wind speed at turbine #4 is $0.961^3$ = 0.888 $U_0$.

The equation for interacting wakes is not needed here as that assumes that spreading wakes of turbines not directly upstream start to merge a distance downstream.

**9.3** A wind farm developer has identified a site with unique winds. They blow all of the time from one direction and at one speed, 15 m/s. The site has enough room for two rows of turbines perpendicular to the prevailing winds. She is wrestling with which size of turbines and how many to put in each row. All turbines being considered have the same hub height. She has developed two options.

*Option 1* – This is an array of 24 1.5 MW turbines in two rows. The turbines just fit onto the site:

- Front row – 12 turbines, each with a rotor diameter of 60 m and a rated power output of 1.5 MW in 15 m/s wind speeds.
- Second row 500 m behind the first one (only 8.33 rotor diameters) – 12 turbines, each with a rotor diameter of 60 m and a rated power output of 1.5 MW in 15 m/s wind speeds.

*Option 2* – This option includes slightly smaller turbines in the first row. This allows a few more turbines in the first row and reduces the velocity deficit at the second row because of the smaller rotor diameters in the first row. In addition, because of the reduced concerns of abutters, these smaller turbines can be moved 100 m closer to the front edge of the site, further reducing the wake effect. The second row is the same as in Option 1:

- Front row – 15 turbines, each with a rotor diameter of 50 m and a power output of 1.0 MW in 15 m/s wind speeds.
- Second row 600 m behind the first one –12 turbines, each with a rotor diameter of 60 m and a power output of 1.5 MW in 15 m/s wind speeds.

To evaluate her options, the developer uses Katić's wake model to determine the wind speed behind the first row of turbines. She assumes that $k$ = 0.10 and that the thrust coefficient is 8/9. She also assumes that in winds lower than the rated wind speed of 15 m/s, the power output of the turbines can be approximated by:

$$P = P_{rated}\left(\frac{U_X}{15}\right)^3$$

a) For both options, what are the power production for each row and the total power production?
b) If the turbines cost $800/installed kW, how much will each option cost to install (assume that no loans are needed)?
c) If the developer's annual operation and maintenance (O&M) costs are 7% of the installed cost plus $10/MWh for maintenance, what are the total annual O&M costs?
d) If the developer's income from the sale of energy is $30/MWh, what is her net income for a year (Sales income - annual O&M costs)?
e) If net income is the deciding factor, which option should she pursue?

**SOLUTION**

a)-d) The results of the calculations for a) through d) are presented in the table below.
e) Based on this analysis, option 1 provides 1.9% more income per year. If net income is the deciding factor, then option 1 should be pursued.

**Option 1**

| | # turbines | $U/U_{free\ stream}$ | wind m/s | MW/turbine | Total MW |
|---|---|---|---|---|---|
| Row 1 | 12 | 1 | 15 | 1.5 | 18 |
| Row 2 | 12 | 0.906 | 13.59 | 1.116 | 13.392 |
| | | | Total Power Production | | 31.392 |

| Installed MW | Installed Cost | O+M | Income | Net Income |
|---|---|---|---|---|
| 36 | $28,800,000 | $4,765,939 | $8,249,818 | $3,483,878 |

**Option 2**

| | # turbines | $U/U_{free\ stream}$ | wind m/s | MW/turbine | Total MW |
|---|---|---|---|---|---|
| Row 1 | 15 | 1 | 15 | 1 | 15 |
| Row 2 | 12 | 0.942 | 14.13 | 1.255 | 15.06 |
| | | | Total Power Production | | 30.06 |

| Installed MW | Installed Cost | O+M | Income | Net Income |
|---|---|---|---|---|
| 33 | $26,400,000 | $4,481,256 | $7,899,768 | $3,418,512 |

**9.4** A wind farm consists of two turbines, located 100 m from each other. They are operating in winds with a mean speed of 10 m/s and an integral time scale of 10 s.

a) What does the graph of the Von Karman spectrum look like (using log–log coordinates)?

b) What is the function for the wind farm filter for these two turbines?

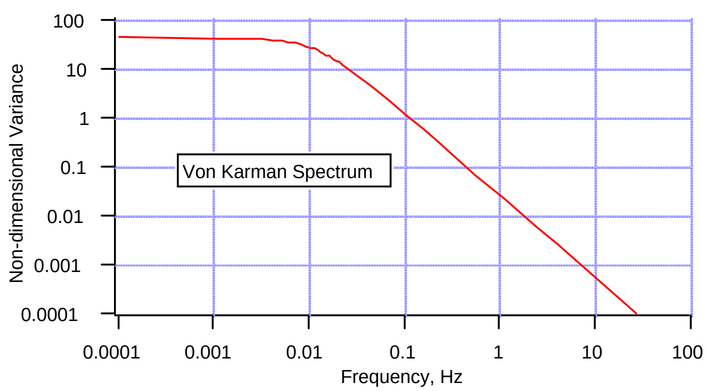
c) Suppose that the spectrum of the local wind can be expressed by:

$$S_1(f) = \sigma_U^2\, 40 e^{-40 f}$$

Assume also that each wind turbine sees the same mean wind speed, $U$, and that the total average wind turbine power, $P_N$, can be expressed as $P_N = NkU$ where $k$ is equal to 10 and $N$ is the number of wind turbines. What is the standard deviation of the wind power from the two wind turbines? How does this compare to the standard deviation of the power from two wind turbines that experience exactly the same wind?

**SOLUTION**

a) The graph of the Von Karman spectrum is presented below:




b) For two wind turbines, the wind farm filter is expressed as

$$\frac{1}{N^2}\sum_{i=1}^{N}\sum_{j=1}^{N}\gamma_{ij}(f) = \frac{1}{2^2}\left(\gamma_{11}(f) + \gamma_{12}(f) + \gamma_{21}(f) + \gamma_{22}(f)\right)$$

where the coherence function is defined as:

$$\gamma_{ij}(f) = e^{-25\frac{x_{ij}}{V}f}$$

For $x_{ij} = 100$

$$\gamma_{12}(f) = \gamma_{21}(f) = e^{-250f}$$

and

$$\gamma_{11}(f) = \gamma_{22}(f) = 1$$

Then the wind farm filter is:

$$\frac{1}{N^2}\sum_{i=1}^{N}\sum_{j=1}^{N}\gamma_{ij}(f) = \frac{1}{2}\left(1 + e^{-250f}\right)$$

c) The variance of the fluctuating power from one wind turbine is given by

$$\sigma_{P,1}^2 = k^2\int_0^\infty S_1(f)df$$

or, for $k = 10$:

$$\sigma_{P,1}^2 = 4000\sigma_V^2\int_0^\infty e^{-40f}\,df = 4000\sigma_V^2\left(\frac{-1}{40}e^{-40f}\right)_0^\infty = 100\sigma_V^2$$

so

$$\sigma_{P,1} = 10\sigma_V$$

The variance of the total fluctuating power is given by

$$\sigma_{P,N}^2 = N^2k^2\int_0^\infty S_N(f)df = k^2\int_0^\infty\left\{S_1(f)\sum_i^N\sum_j^N\gamma_{ij}(f)\right\}df$$

From the previous section:

$$\frac{1}{N^2}\sum_{i=1}^{N}\sum_{j=1}^{N}\gamma_{ij}(f) = \frac{1}{2}\left(1 + e^{-250f}\right)$$

or

$$\sum_{i=1}^{N}\sum_{j=1}^{N}\gamma_{ij}(f)=\frac{N^2}{2}\left(1+e^{-250f}\right)$$

and

$$S_1(f)=\sigma_V^2 40e^{-40f}$$

so

$$\sigma_{P,N}^2=20\sigma_V^2N^2k^2\int_0^\infty\left\{e^{-40f}\left(1+e^{-250f}\right)\right\}df=8000\sigma_V^2\int_0^\infty\left\{e^{-40f}\left(1+e^{-250f}\right)\right\}df$$

Then

$$\sigma_{P,N}^2=8000\sigma_V^2\int_0^\infty\left(e^{-40f}+e^{-290f}\right)df$$

$$\sigma_{P,N}^2=8000\sigma_V^2\left(\tfrac{-1}{40}e^{-40f}+\tfrac{-1}{290}e^{-290f}\right)_0^\infty$$

$$\sigma_{P,N}^2=8000\sigma_V^2\left(\tfrac{1}{40}+\tfrac{1}{290}\right)=227.59\sigma_V^2$$

so

$$\sigma_{P,N}=15.09\sigma_V=.755\left(2\sigma_{P,1}\right)$$

The standard deviation from two turbines experiencing the exact same wind would be $20\sigma_V$. Thus, the standard deviation of the total power from the two wind turbines is 75.5% of what would be expected with two correlated wind turbines.

**9.5** Consider a wind turbine generator connected to a grid system (see Figure B.10) with a line-neutral system voltage, $V_S$. The voltage at the wind turbine, $V_G$, is not necessarily the same as $V_S$. The distribution system resistance is $R$, and the distribution system reactance is $X$.

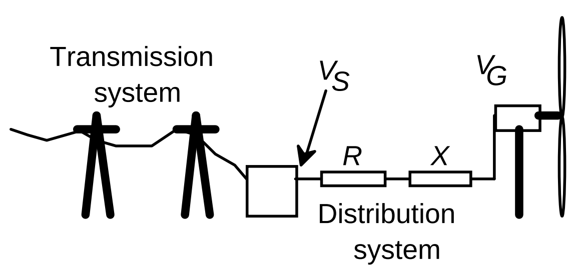



**Figure B.10**

The real generated power is $P$ and the reactive power consumed by the wind turbine generator is $Q$. Suppose $R = 8.8\ \Omega$, $X = 4.66\ \Omega$, $V_s = 11$ kV, $P = 313$ kW, $Q = 162$ kVAr and the power factor at the generator is pf = 0.89. Determine the voltage difference between the grid and the generator, the voltage drop as a percent of the grid voltage, the fault level at the generator and the installed generator capacity as a percent of the fault level.

**SOLUTION**

The voltage at the generator can be determined from:

$$V_G{}^4 + V_G{}^2\left[2(QX - PR) - V_S{}^2\right] + (QX - PR)^2 + (PX - QR)^2 = 0$$

Inserting the known values, one finds:

$$V_G = 11{,}178\ \text{V or } 178\ \text{V}$$

The only realistic answer is 11,178 V at the wind turbine, so:

$$\Delta V = 178\ \text{V}$$

In lightly loaded distribution circuits, the voltage change can be approximated as:

$$\Delta V = V_G - V_S = \frac{PR - QX}{V_S} = 181.8\ V$$

This is 1.65% if the nominal line voltage level. The electrical losses in the distribution system are:

$$W = \frac{(P^2 + Q^2)R}{V_S{}^2} = 9.03\ kW$$

The fault current, $I_F$, is:

$$I_F = \frac{V_S}{(R^2 + X^2)^{1/2}} = 1104\ A$$

and the fault level, *M*, is:

$$M = I_F V_S = 12.1\ MVA$$

The turbine rating is 2.6% of the fault level.

**9.6** A wind turbine is connected to an 11 kV distribution line. The magnitude of the harmonic voltages, $c_n$, at the point of common connection of the wind turbine and other grid users are listed in Table B.9.

**Table B.9**

| $n$ | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 | 13 | 14 | 15 |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| $c_n$, Volts | 11000 | 120 | 0 | 85 | 15 | 0 | 99 | 151 | 0 | 12 | 216 | 0 | 236 | 80 | 0 |

What is the harmonic distortion of each harmonic and the total harmonic distortion? Are these within the allowable IEEE 519 limits?

**SOLUTION**

The harmonic distortion caused by the specific harmonics is:

$$HD_n = \frac{\sqrt{\frac{1}{T}\int_0^T v_n{}^2 dt}}{\sqrt{\frac{1}{T}\int_0^T v_F{}^2 dt}}$$

where

$$\frac{1}{T}\int_0^T v_F{}^2 dt = \frac{1}{T}\int_0^T \left( v_F \sin\left(\frac{\pi t}{L}\right)\right)^2 dt = \frac{v_F{}^2}{T}\int_0^T \tfrac{1}{2}\left(1-\cos\left(\frac{2\pi t}{L}\right)\right) dt$$

$$= \frac{v_F{}^2}{2T}\left(t - \frac{l}{2\pi}\sin\left(\frac{2\pi t}{T}\right)\right)_0^T = \frac{v_F{}^2}{2}$$

Similarly

$$\frac{1}{T}\int_0^T v_n{}^2 dt = \frac{1}{T}\int_0^T \left( v_n \sin\left(\frac{n\pi t}{L} + \phi_n\right)\right)^2 dt = \frac{v_n{}^2}{2}$$

Thus

$$HD_n = \frac{\sqrt{\frac{1}{T}\int_0^T v_n{}^2 dt}}{\sqrt{\frac{1}{T}\int_0^T v_F{}^2 dt}} = \frac{v_n}{v_F}$$

Results for the measured voltages are presented below.

| $n$ | $v_n$ | $HD_n$% |
|---|---|---|
| 1 | 11000 | |
| 2 | 120 | 1.09 |
| 3 | 0 | 0.00 |
| 4 | 85 | 0.77 |
| 5 | 15 | 0.14 |
| 6 | 0 | 0.00 |
| 7 | 99 | 0.90 |
| 8 | 151 | 1.37 |

| | | |
|---|---|---|
| 9 | 0 | 0.00 |
| 10 | 12 | 0.11 |
| 11 | 216 | 1.96 |
| 12 | 0 | 0.00 |
| 13 | 236 | 2.15 |
| 14 | 80 | 0.73 |
| 15 | 0 | 0.00 |

The THD is:

$$THD = \sqrt{\sum_{n=2}^{\infty} (HD_n)^2}$$

The THD is 3.67% and the highest harmonic distortion of any individual harmonic is 2.15%. Thus, these harmonics are within the allowable range of IEEE 519.

**9.7** The data file '*Site1 v Site2 For MCP.csv*' includes five years of wind speed and direction data from Site 1 (the reference site) and Site 2 (the candidate or project site). Data that was determined to be incorrect (due to sensor icing or other problems, for example) is indicated by -999. Use this data to determine the appropriate Variance MCP relationship between the two sites using three different years of concurrent data: 1999, 2001 and 2002. What are the slopes and offsets determined from these three different concurrent data sets? What is the data recovery of the data sets used for the analysis expressed as a percentage of a year worth of data? What might be the cause of the difference between the results for different years?

**SOLUTION**

The calculations have been done in *Site1 v Site2 For MCP_Years 1999, 2001, 2002 Solution.xls*. The results are shown here:

| Year of Concurrent Data | Slope | Offset | Data Recovery |
|---|---|---|---|
| 1999 | 0.998 | 0.214 | 0.958 |
| 2001 | 0.912 | 0.347 | 0.914 |
| 2002 | 0.915 | 0.492 | 0.948 |

There is a significant variability between the slopes, which range from 1.00 to 0.91. There is some variability between the offsets that were determined. All of the data recovery percentages were over 91%. The causes of the variability is unknown but could be due to sensor changes, changes in the ground cover upwind of either site over the four year period or variability in the relationship between the winds at the two locations.

**9.8** A study determines that the maximum chord size in meters of blades can be expressed as:

$$c = 0.0005L^2 + 0.024L + 1.4$$

where $L$ is the blade length and the mass of blades in kilograms can be approximated as

$$m = 9.043L^2 - 340L + 6300$$

The clearance under the bridges on the only road to a proposed project are 5.5 m and the trailer carrying the blades to the project must maintain a clearance between the blade and the road of 1 m. Additionally, the overpasses on the road are only rated for loads (tractor trailer plus cargo) of 36,000 kg. Note that the empty tractor trailer has a mass is 16,000 kg.

a) If the underpass clearances are determined by the maximum blade chord what is the largest blade that can be transported to the project that will fit under the road underpasses and over the bridges?

b) Final access to the project is along a narrow dirt road. The minimum radius of curvature of the road, $r$, that will accommodate a blade of length $L$ when the road width is $w$ is:

$$r = \frac{\left(\frac{L}{2}\right)^2 - w^2}{2w}$$

What is the minimum radius of curvature that will accommodate the blade chosen in part a if $w = 8$ m?

**SOLUTION**

a. The available clearance is 4.5 m between the trailer and the top of the underpasses. The available blade weight is only 20,000 kg due to the weight of the tractor and trailer. The 4.5 m maximum chord length limits the blades to 58.5 m. The mass limit of 20,000 kg limits the blades to about 62 m long. Thus, the underpass limit is the more restrictive and the project must use wind turbines with blades no longer than 58.5 m long.

b. Plugging the values into the formula, the minimum radius of curvature is 68.3 m.

**9.9** A developer decides to install a 45 turbine offshore wind farm. The wind map of the proposed project is shown in the Figure B-11. The rows are 2 km apart in the north-south direction and 1 km apart in the east-west direction. The closet turbine to land is 4 km to the east of Eastern Point.

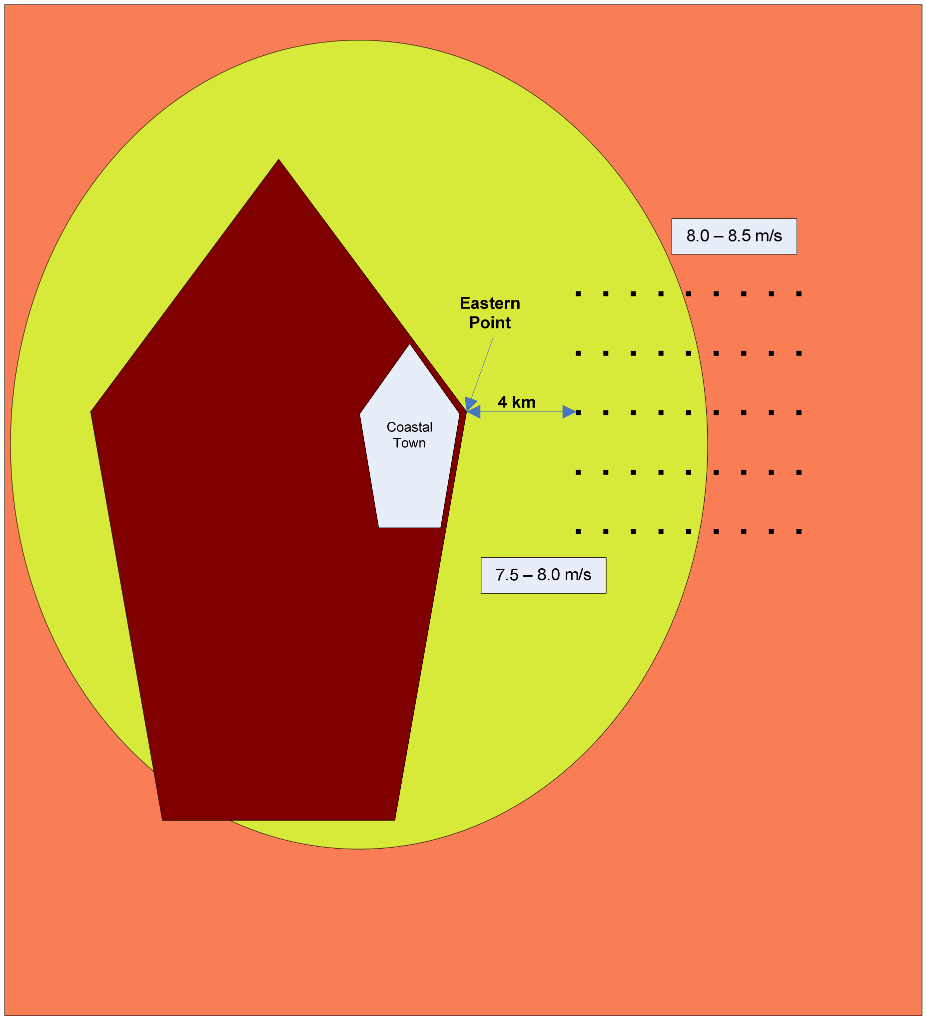



**Figure B.11 Map of proposed project.**

The wind map shows the expected range of mean hub height wind speeds at the site in each shaded area. The Weibull shape factors found at this site are 2.4. As the permitting for the project moves forward, the developer realizes that he will need to eliminate 4 turbines in the higher wind speed region due to concerns at archeologically valuable shipwrecks nearby. He also decides to eliminate all turbines within 5.7 km of Eastern Point to avoid conflicts with lobstermen who are concerned about environmental damage to lobster beds. What is the expected range of annual energy generation if the turbines that will be used have a rated power of 4.0 MW above 13 m/s and has a cubic power curve from 4 m/s to rated and a cut-out wind speed of 24 m/s?

**SOLUTION**

The final number of wind turbines in the lower wind speed area is 14 and in the high wind area it is 17. The average capacity factors are shown in the table below. The analysis can be found in *OffshoreWindFarmSolution.xls*.

| | |
|---|---|
| Minimum expected CF | 0.277 |
| Maximum expected CF | 0.315 |

**9.10** A proposed wind farm will have 18 wind turbines in a row. Each will be 8 rotor diameters apart. The wind speeds in each 30 degree direction bin are characterized in the Table B.10. The turbine that will be used in this project generates rated power of 2.0 MW above 11 m/s and has a cubic power curve from 4 m/s to rated and a cut-out wind speed of 24 m/s. The array losses are assumed to be 15% when the wind is from the north (0 degrees) and 0% when the wind is from other directions. Additionally, it is assumed that other losses (downtime, electrical losses, weather conditions, etc.) will be 20%. Under these assumptions, what will the total generation of the wind farm be? After consulting the manufacturer, the developer learned that the wind turbine manufacturer would not honor the turbine warranty if the wind turbines were operated when the wind was directly along the row of the turbines (in direction sectors bins of 0 and 180 degrees) due to excessive wake turbulence. Given this new information, how much can the developer expect to generate at this wind farm and preserve the warranty on his turbines?

**Table B.10** Wind speed and direction data

| Direction Bin Center | Mean Wind Speed, m/s | Weibull Shape Factor | Percent of Time |
|---|---|---|---|
| 0 | 7.9 | 2.0 | 8.0 |
| 30 | 7.0 | 2.1 | 6.0 |
| 60 | 6.7 | 2.0 | 4.0 |
| 90 | 5.6 | 2.1 | 10.0 |
| 120 | 6.9 | 2.3 | 8.0 |
| 150 | 7.2 | 2.1 | 7.0 |
| 180 | 8.8 | 2.2 | 7.0 |
| 210 | 9.2 | 2.1 | 10.0 |
| 240 | 9.0 | 2.0 | 8.0 |
| 270 | 8.3 | 2.1 | 14.0 |
| 300 | 8.4 | 2.1 | 12.0 |
| 330 | 7.6 | 2.2 | 6.0 |

**SOLUTION**

The answers are presented in the table below and the details of the analysis are in *DirSectorMgtSolution.xls*.

| | |
|---|---|
| Project CF including array losses | 0.430 |
| Net Project CF including all losses | 0.344 |
| | |
| Project CF w/ Dir. Sector Management including array losses | 0.368 |
| Net Project CF w/ Dir. Sector Management, incl. all losses | 0.294 |

**9.11** *HourlyLoadAndPowerOneYearGW.csv* contains average hourly electrical load (GW) in a utility control area and the wind power that is expected to be generated by future planned wind farms.

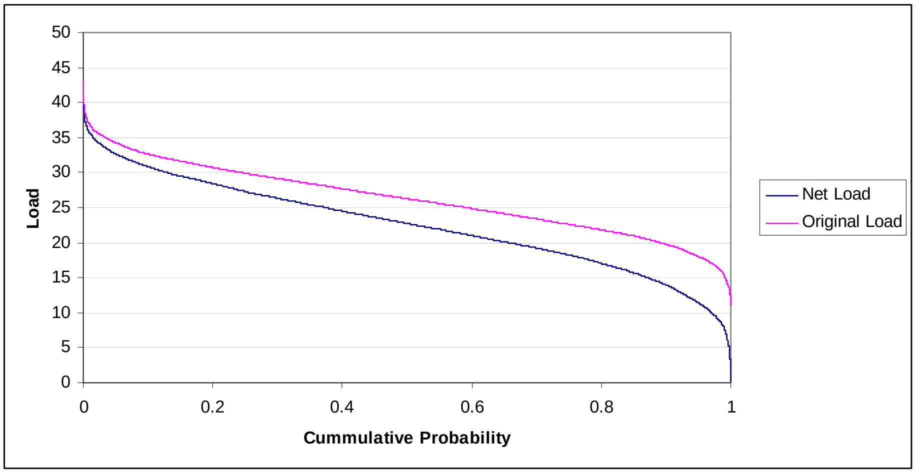
a. Determine the total GWh, mean, maximum, minimum and standard deviation of the gird load and the generated wind power. Graph the load duration curve of the utility with the existing load (before the installation of the wind plants).
b. Determine the total GWh, mean, maximum, minimum and standard deviation of the net load once the wind power plants are installed. Graph the load duration curve of the net load of the utility after the installation of the wind plants.
c. How much does the mean electrical load decrease with the installation of the wind power plants? How much does the standard deviation of the load seen by the grid operators (the net load) increase with the inclusion of wind? How much does the peak net load change?

**SOLUTION**

Answers to a) and b): The statistics of the data are:

| | **Original Load** | **Wind Power** | **Net Load** |
|---|---|---|---|
| Average GW | 26.26 | 3.67 | 22.58 |
| Total GWh | 229,995 | 32,183 | 197,812 |
| Standard Deviation, GW | 4.96 | 4.16 | 6.43 |
| Max GW | 43.06 | 12.00 | 43.06 |
| Min GW | 11.03 | 0.00 | 0.24 |

The cumulative distributions are (see *LoadGWSolution.xls*):




c. The mean electrical net load decreased from 26.26 GW to 22.58 GW but the standard deviation increased from 4.96 GW to 6.43 GW. Meanwhile the peak net electrical load remains the same.

## B.10 Chapter 10 Problems

**10.1** An isolated power system serving the community of Cantgettherefromhere uses a 100 kW diesel generator. The community plans to add 60 kW of wind power. The hourly average load and power from the wind turbine over a 24 hour period is detailed in Table B.11. The diesel generator has a no-load fuel usage of 3 liter per hour and an additional incremental fuel use of 1/4 liter per kilowatt-hour.

For the day in question, using the hybrid system design rules, determine:

a) The maximum renewable energy that can possibly be used in an ideal system.

b) The maximum renewables contribution without storage, and the maximum with storage.

c) The maximum fuel savings that can be achieved.

d) The minimum diesel fuel use that can be achieved with intelligent use of storage and controls.

**Table B.11** Diesel system load data

| Hour | Load (kW) | Wind (kW) | Hour | Load (kW) | Wind (kW) |
|---|---|---|---|---|---|
| 0 | 25 | 30 | 12 | 85 | 45 |
| 1 | 20 | 30 | 13 | 95 | 45 |
| 2 | 15 | 40 | 14 | 95 | 50 |
| 3 | 14 | 30 | 15 | 90 | 55 |
| 4 | 16 | 20 | 16 | 80 | 60 |
| 5 | 20 | 10 | 17 | 72 | 60 |
| 6 | 30 | 5 | 18 | 60 | 48 |
| 7 | 40 | 5 | 19 | 74 | 50 |
| 8 | 50 | 15 | 20 | 76 | 55 |
| 9 | 70 | 20 | 21 | 60 | 60 |
| 10 | 80 | 25 | 22 | 46 | 60 |
| 11 | 90 | 40 | 23 | 35 | 55 |

## SOLUTION

The hybrid system design rules are:

Rule 1: The maximum renewable energy that can be used is limited by the load.
Rule 2: The use of renewable energy will be further limited by temporal mismatch between the load and the renewables.
Rule 3: The maximum possible benefit with improved controls or operating strategies is a system approaching the fuel use of the ideal diesel generator - fuel use proportional to the diesel-served load.
Rule 4: The maximum fuel savings arising from the use of renewables in an optimised system is never greater than the fuel savings of an ideal generator supplying the proportional reduction in load resulting from use of renewables.

a) Based on rule number 1, the maximum renewables contribution with any system cannot be larger than the load. Thus the maximum renewables contribution over the day would be 1338 kWh, the total daily energy need.

b) Based on the rules 1 and 2, the maximum renewables contribution in any hour without storage is the same as the load, if it is smaller than the available wind power, or the available wind power. An examination of the data shows that the maximum renewables contribution without storage would be 819 kWh. The maximum with storage would be the total daily load if it were less than the total available wind power, or the total available wind power. The total daily load is 1338 kWh; the total available wind power over the day is 913 kWh. Thus the maximum renewables contribution with storage is 913 kWh.

c) The total fuel use of the original system is 1338(1/4)+ 24(3) = 406.5 liters/day. The maximum fuel savings occurs with optimum use of storage, controls and renewables. Under these conditions, the diesel load with would be 1338 – 913 = 425 kWh. The fuel use of an optimised system would be ¼ l/kWh or 106.25 l/day, a savings of 300.25 l/day.

d) Based on rule 3, the minimum diesel fuel use that can be achieved with intelligent use of storage and controls (and without renewables) is that part of the fuel use that is not proportional to the load: 24 hrs of 3 l/hr = 72 l/day.

**10.2** Based on the analysis in Problem 10.1 and other input, the community of Cantgettherefromhere has upgraded its power system. The hybrid power system includes a 100 kW diesel generator, 100 kW of installed load, 60 kW of wind power, a 100 kW dump load, and energy storage with 100 kWh capacity. The hourly average load and power from the wind over a 24 hour period have remained the same and are listed in Table B.11.

Assume that the mean hourly data accurately describe the load and power from the wind and that fluctuations about the mean load are handled by the energy storage. Determine the hourly energy flows in the system for the following system control and operating approaches.

a) Diesel-only system – In this baseline system, the diesel provides all of the power to the load. The diesel provides power down to 0 kW with no provision to limit the ensure a minimum diesel load to avoid diesel engine wear.

b) Minimum diesel – In this system minimizing diesel power takes priority, except that the diesel may not be shut off and must run at a minimum of 30 kW to ensure long diesel engine life. Thus, the system operates by the rules:

1. The minimum diesel power is 30 kW
2. Only the minimum diesel power that is needed above 30 kW is used (no power above 30 kW is used to fill up the storage)
3. Storage can only be used between 20 and 95 kWh of capacity, to maximize storage efficiency and battery life
4. Storage capacity starts at 50 kWh
5. The sum of the energy into the power sources in the system (diesel, wind, battery) must equal the sum of the energy into the power sinks in the system (the load, dump load, and battery).
6. If there is excess energy in the system it is first stored, if possible, and dumped only if necessary.

c) Diesel shut off – In this system a rotary inverter between the batteries and the grid provides reactive power and the diesel can be shut off. It shuts off whenever possible, but when running, needs to be at a minimum load of 30 kW. When running it is also used to fill up the storage to use the fuel most efficiently. The battery level is maintained between 20 and 90 kWh. This insures that there is adequate capacity to handle fluctuating loads when there is no diesel in the system. Thus, the system operates by the rules:

1. The diesel can be shut off
2. The minimum diesel power is 30 kW when the diesel is running
3. When the diesel is running, the storage is also filled up if possible to improve diesel fuel efficiency
4. Storage can only be used between 20 and 90 kWh of capacity, to maximize storage efficiency and battery life
5. Storage capacity starts at 50 kWh

6. The sum of the energy into the power sources in the system (diesel, wind, battery) must equal the sum of the energy into the power sinks in the system (the load, dump load, and battery)
7. If there is excess energy in the system it is first stored, if possible, and dumped only if necessary

Specifically, for each operating approach, determine how much energy is supplied by the diesel over the 24 hour period and the reduction in diesel power compared to the diesel only case.

**SOLUTION**

Diesel only system

The baseline system with just the diesel would result in the diesel providing 1338 kWh of power over the 24 hour period.

Results for the other two systems are detailed in the charts below.

## Minimum Diesel System

**Minimum Diesel**

Run diesel as little as poss., use wind, storage
Diesel - 30-100 kW, no diesel shut-off
Storage - 100kWh capacity, 20-95 kWh range
Dump - 100 kW

| Hour | Load | Wind | Diesel | Dump | Storage | Storage Level | Energy Balance |
|---|---|---|---|---|---|---|---|
| | kW | kW | kW | kW | kW | kWh | kW |
| | L | W | D | X | B | S | L-W-D+X+B |
| | | | Initial Storage Level: | | | 50 | |
| 0 | 25 | 30 | 30 | 0 | 35 | 85 | 0 |
| 1 | 20 | 30 | 30 | 30 | 10 | 95 | 0 |
| 2 | 15 | 40 | 30 | 55 | 0 | 95 | 0 |
| 3 | 14 | 30 | 30 | 46 | 0 | 95 | 0 |
| 4 | 16 | 20 | 30 | 34 | 0 | 95 | 0 |
| 5 | 20 | 10 | 30 | 20 | 0 | 95 | 0 |
| 6 | 30 | 5 | 30 | 5 | 0 | 95 | 0 |
| 7 | 40 | 5 | 30 | 0 | -5 | 90 | 0 |
| 8 | 50 | 15 | 30 | 0 | -5 | 85 | 0 |
| 9 | 70 | 20 | 30 | 0 | -20 | 65 | 0 |
| 10 | 80 | 25 | 30 | 0 | -25 | 40 | 0 |
| 11 | 90 | 40 | 30 | 0 | -20 | 20 | 0 |
| 12 | 85 | 45 | 40 | 0 | 0 | 20 | 0 |
| 13 | 95 | 45 | 50 | 0 | 0 | 20 | 0 |
| 14 | 95 | 50 | 45 | 0 | 0 | 20 | 0 |
| 15 | 90 | 55 | 35 | 0 | 0 | 20 | 0 |
| 16 | 80 | 60 | 30 | 0 | 10 | 30 | 0 |
| 17 | 72 | 60 | 30 | 0 | 18 | 48 | 0 |
| 18 | 60 | 48 | 30 | 0 | 18 | 66 | 0 |
| 19 | 74 | 50 | 30 | 0 | 6 | 72 | 0 |
| 20 | 76 | 55 | 30 | 0 | 9 | 81 | 0 |
| 21 | 60 | 60 | 30 | 21 | 9 | 90 | 0 |
| 22 | 46 | 60 | 30 | 44 | 0 | 90 | 0 |
| 23 | 35 | 55 | 30 | 50 | 0 | 90 | 0 |
| | | | | | | | |
| Totals | 1338 | 913 | 770 | 305 | kWh | | |
| Diesel kWh saved | | | 568 | | | | |

## Diesel Shutoff System

**Diesel Shut Off**

When diesel runs >=30
Run diesel as little as poss., use wind, storage
If not enough wind, storage, run diesel at max eff.
Dump as little as poss.

Diesel - 0-100 kW, diesel shut-off allowed
Storage - 100kWh capacity, 20-90 kWh range

| Hour | Load | Wind | Diesel | Dump | Storage | Storage Level | Energy Balance |
|---|---|---|---|---|---|---|---|
| | kW | kW | kW | kW | kW | kWh | kW |
| | L | W | D | X | B | S | L-W-D+X+B |
| | | | Initial Storage Level: | | | 50 | |
| 0 | 25 | 30 | 0 | 0 | 5 | 5 | 0 |
| 1 | 20 | 30 | 0 | 0 | 10 | 15 | 0 |
| 2 | 15 | 40 | 0 | 0 | 25 | 40 | 0 |
| 3 | 14 | 30 | 0 | 16 | 0 | 40 | 0 |
| 4 | 16 | 20 | 0 | 4 | 0 | 40 | 0 |
| 5 | 20 | 10 | 0 | 0 | -10 | 30 | 0 |
| 6 | 30 | 5 | 0 | 0 | -25 | 5 | 0 |
| 7 | 40 | 5 | 0 | 0 | -35 | -30 | 0 |
| 8 | 50 | 15 | 100 | 0 | 65 | 35 | 0 |
| 9 | 70 | 20 | 60 | 0 | 10 | 45 | 0 |
| 10 | 80 | 25 | 0 | 0 | -55 | -10 | 0 |
| 11 | 90 | 40 | 100 | 0 | 50 | 40 | 0 |
| 12 | 85 | 45 | 0 | 0 | -40 | 0 | 0 |
| 13 | 95 | 45 | 90 | 0 | 40 | 40 | 0 |
| 14 | 95 | 50 | 0 | 0 | -45 | -5 | 0 |
| 15 | 90 | 55 | 80 | 0 | 45 | 40 | 0 |
| 16 | 80 | 60 | 0 | 0 | -20 | 20 | 0 |
| 17 | 72 | 60 | 0 | 0 | -12 | 8 | 0 |
| 18 | 60 | 48 | 0 | 0 | -12 | -4 | 0 |
| 19 | 74 | 50 | 0 | 0 | -24 | -28 | 0 |
| 20 | 76 | 55 | 89 | 0 | 68 | 40 | 0 |
| 21 | 60 | 60 | 0 | 0 | 0 | 40 | 0 |
| 22 | 46 | 60 | 0 | 14 | 0 | 40 | 0 |
| 23 | 35 | 55 | 0 | 20 | 0 | 40 | 0 |
| | | | | | | | |
| Totals | 1338 | 913 | 519 | 54 | kWh | | |
| Diesel kWh saved | | | 819 | | | | |

**10.3** This problem concerns a hypothetical island wind/diesel system. Wind and load data for the island for one year are in the files *Wind_TI.txt* and *Load_TI.txt*. Assume that the wind turbines available are (1) and AOC 15-50, (2) one with twice the output or (3) one with 4 times the output. The power curves for these turbines are, respectively, *AOC_pc.csv*, *AOC_pc_x2.csv* and *AOC_pc_x4.csv*.

The diesel to be considered is one rated at 100 kW, with no load fuel consumption of 2 fuel units per hour and full load fuel consumption of 10 units per hour.

Using the MiniCodes, your job is to find: (1) the average wind power, (2) the useful wind power, (3) the average dumped power, and (4) the average fuel use for the three different turbine options, and for storage of 0 kWh, 100 kWh, and 1000 kWh.

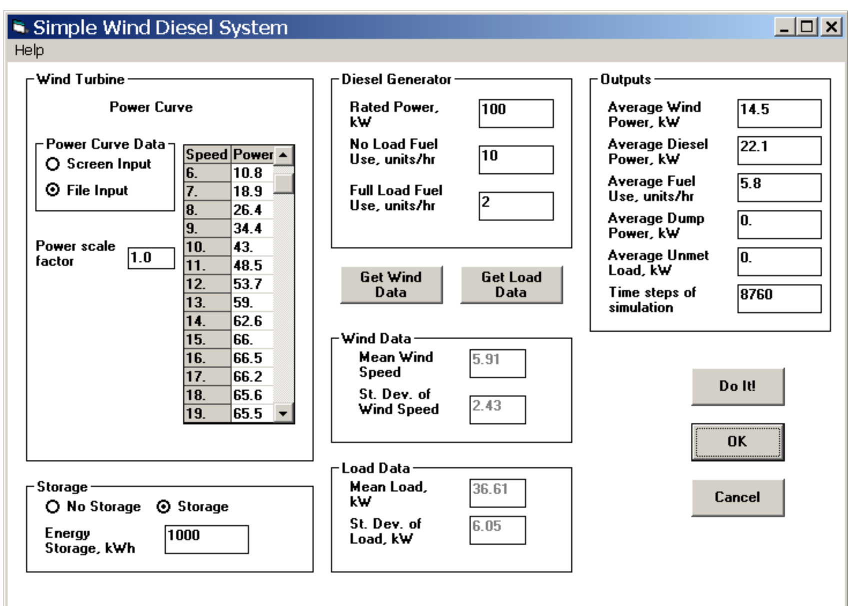
What do the results suggest about the usefulness of storage?

## SOLUTION

From the MiniCodes: (Powers in kW, Fuel usage in "units per hour")

| | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|
| Storage | 0 | 0 | 0 | 100 | 100 | 100 | 100 | 100 | 100 |
| Turbines | 1 | 2 | 4 | 1 | 2 | 4 | 1 | 2 | 4 |
| P average | 14.5 | 29 | 58.1 | 14.5 | 29 | 58.1 | 14.5 | 29 | 58.1 |
| Useful | 12.8 | 17.8 | 21.3 | 13.5 | 19.4 | 23.1 | 14.5 | 24.5 | 30.5 |
| Dumped | 1.7 | 11.2 | 36.8 | 1 | 9.6 | 35 | 0 | 4.5 | 27.6 |
| Fuel use | 6.7 | 5.2 | 4 | 6.2 | 4.3 | 3.1 | 5.8 | 3 | 1.3 |

Sample screen, for 1 turbine, 1000 kWh storage:



The results indicate that storage can have a significant effect on fuel usage when the wind generation is high compared to the load, but not much when generation is low.

**10.4** The 10 minute average wind speed at an offshore buoy is measured to be 8.5 m/s at an elevation of 10 m. The buoy is 10 km from land in the direction of the oncoming wind.

a) If the surface roughness length is assumed to be 0.0002 m, what is the mean wind speed at 80 m?

b) If the Charnock constant, $A_C$, is assumed to be to be 0.018 and $C_{D,10}$ is 0.0015, what is the mean wind speed at 80 m, assuming that the friction velocity is given by $U_* = U_{10}\sqrt{C_{D,10}}$ ?

c) Assume that the water depth is 20 m, the wave height is 3 m, and the peak period of the waves is 9 sec. What is the mean wind speed at 80 m in this case?

**SOLUTION**

a) From Equation 2.34

$$\frac{U(z)}{U(z_r)} = \frac{\ln(z/z_0)}{\ln(z_r/z_0)}$$

$$\frac{U(80)}{8.5} = \frac{\ln(80/0.0002)}{\ln(10/0.0002)} = 1.192$$

So the wind speed at 80 meters is estimated to be 10.1 m/s.

b) From the given equation and Equation 10.2:

$$U_* = U_{10}\sqrt{C_{D,10}} = 8.5\sqrt{0.0015} = 0.329$$

$$z_0 = A_C\frac{{U_*}^2}{g} = 0.018\frac{0.329^2}{9.81} = 0.00020$$

So the wind speed at 80 meters is estimated to be the same, 10.1 m/s.

c) Use Equation 10.3:

$$z_0 = H_s(1200)(H_s / {L_P}^{4.5})$$

Where it may be assumed that the given wave height of 3 m is equal to the significant wave height, $H_s$. The water depth, $d$ = 20 m, is moderately shallow, so Equation 10.6 may be used to estimate the wave length, $L_P$, from the peak period, $T_P$ = 9 sec.

$$L_P = \left(\frac{gT_p^2}{2\pi}\right)\sqrt{\tanh\left(\frac{4\pi^2 d}{T_p^2 g}\right)}$$

The wavelength is then:

$$L_P = \left(\frac{(9.81)(9^2)}{2\pi}\right)\sqrt{\tanh\left(\frac{4\pi^2(20)}{(9^2)(9.81)}\right)} = 110.2\,m$$

The roughness length is:

$$z_0 = H_s(1200)\left(H_s / L_P{}^{4.5}\right) = (3)(1200)(3/110.2)^{4.5} = 0.003266\ m$$

The wind speed is:

$$U(80) = (8.5)\frac{\ln(80/0.003266)}{\ln(10/0.003266)} = 10.2\,m/s$$

Note: the wave length in part c) could also be determined from Equation 10.5. In that case, $L_P = 105.2$, $z_0 = 0.0004$ m and $U = 10.25$ m/s.

**10.5** Derive the equation for total inertial force on a monopile (see Equation 10.15). Note: the equation for the acceleration as a function of depth is: $\dot{U}_W(z) = \frac{g\pi H}{L}\frac{\cosh(k(z+d))}{\cosh kd}\cos(\theta)$ Assume that the depth is much greater than the wave height.

**<u>SOLUTION</u>**

From the second term in Equation 10.14

$$\hat{F}_I = C_m \rho A \dot{U}_w = C_m \rho \frac{\pi D^2}{4}\dot{U}_w$$

Therefore:

$$\hat{F}_I = C_m \rho \frac{\pi D^2}{4}\frac{g\pi H}{L}\frac{\cosh(k(z+d))}{\cosh(kd)}$$

The total inertial force is found by integrating from the mudline to surface:

$$F_I = \int_{-d}^{0} C_m \rho \frac{\pi D^2}{4}\frac{g\pi H}{L}\frac{\cosh(k(z+d))}{\cosh(kd)}dz$$

Let $u = z + d$, then d$z$ =d$u$, so

$$F_I = \int_0^d C_m \rho \frac{\pi D^2}{4} \frac{g\pi H}{L} \frac{\cosh(ku)}{\cosh(kd)} du = C_m \rho \frac{\pi D^2}{4} \frac{g\pi H}{L} \frac{1}{\cosh(kd)} \int_0^d \cosh(ku) du$$

$$F_I = C_m \rho \frac{\pi D^2}{4} \frac{g\pi H}{L} \frac{1}{\cosh(kd)} \left. \frac{\sinh(ku)}{k} \right|_0^d$$

$$F_I = C_m \rho \frac{\pi D^2}{4} \frac{g\pi H}{kL} \frac{\sinh(kd)}{\cosh(kd)} = C_m \rho \frac{\pi D^2}{4} \frac{g\pi H}{(2\pi / L)L} \tanh(kd)$$

$$F_I = \rho_w g \frac{C_m \pi D^2}{4} \hat{\zeta} \tanh(kd) \text{q.e.d.}$$

(Recall that the wave height is 2 x the amplitude.)

**10.6** A wind turbine has rotor diameter of 90 m. It is installed on monopile with a diameter of 4 m in diameter in water 15 m deep. The wind speed is 12 m/s , the wave height is 2 m and the wave length is 100 m. Estimate the force due to the wind and the maximum force due to waves. Use the Airy model for the waves and assume that the rotor thrust coefficient is the one corresponding to the Betz power coefficient. Assume that inertia coefficient is 2.0 and the drag coefficient 1.5. (Assume that the maximum forces due to drag and inertia occur at the same time.)

**SOLUTION**

The rotor force, $F_R$, is:

$$F_R = 0.5 C_T \rho \pi \frac{D^2}{4} U^2 = 0.5 \left( \frac{8}{9} \right) (1.225) \pi \frac{90^2}{4} 12^2 = 2545\, kN$$

The maximum inertial wave force with wavelength $L$ = 100 m is:

$$F_I = \rho_w g \frac{C_m \pi D^2}{4} \hat{\zeta} \tanh(kd) = \frac{(1000)(9.81)(2)(\pi 4^2)}{4} (1^2) \tanh\left( \frac{2\pi}{100} (10) \right) = 181.6\, kN$$

The maximum wave force due to drag is:

$$F_D = \rho_w g \frac{C_d D}{2} \hat{\zeta}^2 \left[ \frac{1}{2} + \frac{kd}{\sinh(2kd)} \right]$$
$$= \frac{(1000)(9.81)(1.5)(4)}{4} (1^2) \left[ \frac{1}{2} + \frac{(2\pi / 100) 15}{\sinh((2)((2\pi / 100) 15))} \right] = 17.0\, kN$$

**10.7** A small community has 2000 residents and each of them consumes an average of 100 liters of water per day. They have access to an aquifer 100 m below ground level and are considering acquiring a wind electric water pump to bring the water to the surface. What would be the rated power of wind turbine (kW) that would pump, on the average, an amount of water equal to what the community uses? Assume that the capacity factor of the wind turbine is 0.25 and that the efficiency of the water pump is 80%.

**SOLUTION**

Assuming 2000 people, consuming 100 liters/day person (0.1 $m^3$/day person), the total community water requirement is 200 $m^3$/day.
Assuming a water density of 1000 kg/$m^3$, the average mass flow rate is:

$$\dot{m} = \left(200\, m^3/day\right)\left(1000\, kg/m^3\right)/\left[\left(24\, hr/day\right)\left(3600\, \sec/hr\right)\right] = 2.315\, kg/s$$

The power requirement is

$$P_{water} = \dot{m}gh/\eta = \left(2.315\, kg/s\right)\left(9.81\, m/s^2\right)\left(100\, m\right)/0.8 = 2839\, W$$

The rated power of the turbine would be:

$$P_R = P_{water}/cap.fac. = 2.839\, kW/0.25 = 11.35\, kW$$

**10.8** An island community has 2000 residents and each of them consumes an average of 100 liters of water per day. The community is considering acquiring a wind powered desalination plant to produce pure water from sea water (density = 1020 kg/$m^3$). The sea water is 20° C and has salinity of 38 g/kg. What size wind turbine would produce, on the average, an amount of fresh water equal to what the community uses? Assume that the capacity factor of the wind turbine is 0.25, and that the desalination plant uses 5 times the minimum theoretical amount of power.

**SOLUTION**

Since 20° C =293° K, the osmotic pressure is:

$$p_\pi = iS\rho_w R_u T/M$$
$$= \left(1.8\right)\left(38\, g/kg\right)\left(1020\, kg/m^3\right)\left(8.31447\, m^3 Pa/mol\, K\right)\left(293\, K\right)/58.5\, g/kmol = 2{,}909\, kPa$$

The minimum amount of power required is

$$P_{\min} = p_\pi/3{,}600 = 2{,}909\, kPa/3{,}600\, m^3\, kPa/kWh = 0.807\, kWh/m^3$$

(Note that 1 kPa = 1 kNm/$m^3$ = 1 kJ/$m^3$ =1 kWs/$m^3$ = (1/3600) kWh/$m^3$.)

Assuming 4 times theoretical, the power consumption is 4.04 kWh/m$^3$.

Assuming 2000 people, consuming 0.1 m$^3$/day person, the total community water requirement is 200 m$^3$/day. The average power requirement is then 807 kWh/day or 33.6 kW.

The wind turbine rated power would be $P_R = P_{av}/cap\ fac$ = 33.6/.25 = 134.5 kW. It may be noted that the required turbine power required to desalinate water is considerably greater than that required to pump the equivalent amount of water in Problem 10.7.

**10.9** A lead acid battery has a nominal capacity of 200 Ahrs. The constant $c$ is equal to 0.5 and the rate constant $k$ is equal to 2.0. Find apparent capacity of the battery (charge removed) and corresponding current when it is discharged in (i) 1 hour and (ii) 10 hours.

**SOLUTION**

Use Equation 10.30:

$$q_{\max}(I) = \frac{q_{\max} kct}{1 - e^{-kt} + c\left(kt - 1 + e^{-kt}\right)}$$

Making the appropriate substitutions, one has:

| **Discharge time, hrs** | **Capacity, Ah** |
|---|---|
| 1 | 139.6 |
| 10 | 190.4 |

**10.10** A bank of 12 lead acid batteries in series is being used to supply a resistive load with nominal power consumption of 1200 W at 120 V. The voltage constants for each battery (in discharging) are $E_0$ = 12 V, $A$ = -0.05, $C$ = -0.5, $D$ = 1. The internal resistance is $R_{int}$ = 0.1 Ω. At a certain point the batteries are 70% discharged. Find the terminal voltage of the batteries and the actual power consumption at this point.

**SOLUTION**

When the batteries are 70% discharged, X = 0.7
The internal voltage is:

$$E = E_0 + AX + CX/(D - X) = 12 + (-0.05)(0.7) + (-0.5)(0.7)/(1 - 0.7) = 10.8V$$

The resistance of the electrical load is found from:

$$R_{load} = \frac{V^2_{Load,nom}}{P_{nom}} = \frac{120^2}{1200} = 12\ \Omega$$

Find the current such that the voltage is equal for batteries and load.

The battery voltage is

$$V_{bat} = N_{bat}\left(E - IR_{\text{int}}\right)$$

The load voltage is

$$V_{load} = IR_{load}$$

The two equations may be solved:

$$I = N_{bat}E/\left(R_{load} + N_{bat}R_{\text{int}}\right)$$

Substituting the values:

$$I = (12)(10.8)/(12 + (12)(0.1)) = 9.818\text{ A}$$

The power is then

$$P = I^2 R_{load} = \left(9.818^2\right)(12) = 1157 \text{ W}$$

Note that the actual power delivered is less than the nominal value.

**10.11** A natural gas fired turbine operating on an ideal Brayton cycle is used with a wind powered compressed air storage facility. The cycle has the following characteristics: the temperature at the input to the compressor is 27° C and at the input to the turbine the temperature is 727° C. The maximum pressure is 2 MPa and the minimum pressure is 100 kPa. (a) Find the heat input from the natural gas if the net output of the gas turbine is 1 MW and the compressed air from the wind turbine is not being used. (b) Now assume that the gas turbine's compressor is not being used, but the compressed air is coming from the compressed air storage. The compressed air storage is being continually recharged and the pressure remains constant at 2 MPa. All other things being the same, what would be the net power output of gas turbine now? Assume that the constant pressure heat capacity of air is 1.005 kJ/kg K and that the ratio of $c_p$ to $c_v$ (i.e. k) is equal to 1.4.

**<u>SOLUTION</u>**

Assume that this is an ideal Brayton cycle (as stated in the corrected problem statement). Find the temperatures $T_2$ and $T_4$, using the temperature/pressure relation for isentropic compression or expansion, such as is given by Equation 10.36:

$$T_2 = T_1\left(\frac{P_2}{P_1}\right)^{(k-1)/k} = 300\left(\frac{2000}{100}\right)^{0.4/1.4} = 706\,K$$

$$T_4 = T_3 \left( \frac{P_4}{P_3} \right)^{(k-1)/k} = 1000 \left( \frac{100}{2000} \right)^{0.4/1.4} = 424.9\,K$$

Using Equation 10.37, the compressor specific work is:

$$w_{comp,in} = c_p (T_2 - T_1) = (1.005\,kJ/kg\,K)(706\,K - 300\,K) = 408.1\,kJ/kg$$

Similarly, the turbine specific work is:

$$w_{turb,out} = c_p (T_3 - T_4) = (1.005\,kJ/kg\,K)(1000\,K - 424.9\,K) = 578\,kJ/kg$$

The net specific work is

$$w_{net} = w_{turb,out} - w_{comp,in} = 578\,kJ/kg - 408.1\,kJ/kg = 169.9\,kJ/kg$$

The specific heat input is:

$$q_{in} = c_p (T_3 - T_2) = (1.005\,kJ/kg\,K)(1000\,K - 706\,K) = 295.4\,kJ/kg$$

The efficiency is:

$$\eta = w_{net} / q_{in} = 169.9\,kJ/kg / 295.4\,kJ/kg = 0.575$$

The total required heat is:

$$Q_{in} = W_{out} / \eta = 1000/0.575 = 1738.8 \text{ kJ/s (kW)}$$

(b) If the air came from the storage, there would be no compressor work, so the net specific work would be equal to the turbine work (578 kJ/kg). The total net output would be proportionally higher:

$$W_{out,CAES} = \frac{w_{turb,out}}{w_{net}} (W_{out}) = \left( \frac{578}{169.9} \right) (1000) = 3402 \text{ kW}$$

**10.12** An island community has 2000 residents, each of whom uses, on the average, 8 kWh/day. The electricity is presently produced by a diesel generator, but they are considering acquiring a wind turbine. They are would like to incorporate pumped water storage in their plans. An open spot on top of a hill in could be dug out to form a circular reservoir. The reservoir could be 30 m in diameter and 5 m deep. The hill is 70 m above the town. A reservoir of similar size could be next at the same elevation as the town. How many hours could the reservoir supply the average electrical load of the community?

**SOLUTION**

The solution involves applying Equation 10.41. The inputs and results are summarized in the table below.

| | | |
|---|---|---|
| Enegy per capita | 8 | kWh/person day |
| Population | 2000 | people |
| Av power required | 666.67 | kW |
| | | |
| Reservoir diameter | 30 | m |
| Depth | 5 | m |
| Volume | 3,534 | m^3 |
| | | |
| Head | 70 | m |
| Water density | 1000 | kg/m^3 |
| | | |
| Energy | 2,426,998 | kJ |
| Energy | 674 | kWh |
| | | |
| Hrs | 1.011 | |

**10.13** The famous dirigible Hindenburg had a volume of approximately 199,000 m$^3$. It was filled with hydrogen to give it buoyancy. Assume that the pressure inside was 100 kPa and that the temperature was 15° C. How much hydrogen did the Hindenburg contain? How much energy (kWh) would it take to produce that hydrogen by electrolysis of water, assuming an electrolyzer efficiency of 65%? How long would it take to produce that hydrogen using 1.5 MW turbine, operating at full power?

**SOLUTION**

Use the ideal gas law to find the mass of hydrogen. First convert the temperature, 15° C to 288° Kelvin.

$$m = \frac{PVM}{R_u T} = \frac{(100\,kPa)(199{,}000\,m^3)(2.016\,kg/kmol)}{(8.31447\,m^3 kPa/kmol\,K)(288\,K)} = 16{,}754\,kg$$

The energy, $E$, required to produce that amount of hydrogen would be:

$$E = (16{,}754\,kg)(39.4\,kWh/kg)/0.65 = 1{,}016{,}000\,kWh$$

The time, $t$, that the turbine would need to run at full output would be:

$$t = 1500\,kW/1{,}016{,}000\,kWh = 677\,hr$$

**10.14** A hydrogen powered bus has four roof top mounted storage cylinders with a total capacity of 1.28 $m^3$. When full the pressure is 45 MPa. How many times could the cylinders be refilled with the hydrogen that would fill the Hindenburg, as found in Problem 10.13? Assume that the cylinders are completely emptied before being refilled and assume that the ideal gas law is applicable.

**SOLUTION**

The volume, $V_{45\ Mpa}$, required to hold the hydrogen at 45 MPa, using the ideal gas law, would be given by the inverse ratio of the pressures times the volume of the Hindenburg:

$$\mathrm{V}_{45\,\mathrm{MPa}} = \frac{100\,kPa}{45{,}000\,kPa}\left(199{,}000\,m^3\right) = 422.2\,m^3$$

The number of times, $N$, that the top mounted storage cylinders could be filled is then:

$$N = \frac{422.2\,m^3}{1.28\,m^3} = 345.5$$

## B.11 Chapter 11 Problems

**11.1** The first unit of a new wind turbine costs \$100,000. The system is estimated to follow an $s$ = 0.83 learning curve. What is the cost of the 100th unit?

**SOLUTION**

This problem solution is based on the application of the learning curve equation (11.1). This is expressed as:

$$\frac{C(V)}{C(V_0)} = \left(\frac{V}{V_0}\right)^b$$

The progress ratio, $s$, is given by $s = 2^b$. For this example, $s$ = 0.83, and thus $b$ = -0.2688

The cost for the 100th unit, $C(V)$ is then:

$$C(V) = C(V_0)\left(\frac{V}{V_0}\right)^b = \$100{,}000\left(\frac{100}{1}\right)^{-0.2688} = \$29{,}000$$

**11.2** Estimate the progress ratio ($s$) of modern utility-scale wind turbines. Use the time period of about 1980 to the present.

**SOLUTION**

The solution to this problem will require some literature research. A good reference on the subject is given by L. Neij ('Cost Dynamics of Wind Power', Energy, 24, 1999, p. 375-389). In this paper, the author estimates a value of $s$ between 0.92 and 0.98, depending on various turbine size and manufacturer assumptions.

**11.3** A small 50 kW wind turbine with an initial cost of \$50,000 is installed in Nebraska. The fixed cost ratio is 15% and the annual operation and maintenance ($C_{O\&M}$) is 2% of the initial cost. This system produces 65,000 kWh/yr (AkWh). Determine the cost of energy (COE) for this system.

**SOLUTION**

The simple equation for the cost of energy (*COE*) is:

$$\mathrm{COE} = [(C_c \times \mathrm{FCR}) + C_{O\&M}]/E_a$$

Thus, for this example: *COE* = ((\$50,000 x 0.15) + ((0.02) x \$50,000) / (65,000) = \$0.131/kwh

**11.4** A small wind machine, with a diameter of 6 m, is rated at 4 kW in an 11 m/s wind speed. The installed cost is \$10,000. Assuming Weibull parameters at the site to be $c = 8$ m/s and $k = 2.2$, it is claimed that the capacity factor is 0.38.

The interest rate is 11% and the period of the loan is 15 years. Find the cost per unit area, per kW, and the levelized cost per kWh. You do not have to include factors such as inflation, tax credits, or operation and maintenance costs.

**SOLUTION**

a) The cost per unit area is: Rated power cost/rotor area = \$354/m$^2$
b) The cost per kW is: \$10,000/ 4 kW = \$2500/kW
c) The levelized cost of energy, $COE_L$, is given from Equation 11.21:

$$COE_L = \frac{(NPV_C)(CRF)}{\text{Annual energy production}}$$

Use Equation 11.12 for the capital recovery factor:

$$CRF = \begin{cases} r/[1-(1+r)^{-N}], \text{ if } r \neq 0 \\ 1/N, \text{ if } r = 0 \end{cases}$$

The capital recovery factor ($CFR$) for use in the $COE$ equation is based on the discount rate, $r$, and 20 year projected life, $L$, and is found to be 0.1175. The capital recovery factor for the loan payments, $CRF_P$, is found by using the loan interest rate, $b$, and 15 year loan period, $L$, is found in a similar manner and is equal to 0.1391. The annual payment is:

$$\text{Annual payment} = (CRF_P)(Cost) = \$1391.$$

The net present value of costs, $NPV_C$, is given by Equation 11.18:

$$NPV_C = P_d + P_a Y\left(\frac{1}{1+r}, N\right) + C_c f_{OM} Y\left(\frac{1+i}{1+r}, L\right)$$

The $Y()$ term is evaluated according to Equation 11.19:

$$Y(k, \ell) = \sum_{j=1}^{\ell} k^j = \begin{cases} \dfrac{k - k^{\ell+1}}{1-k}, & \text{if } k \neq 1 \\ \ell, & \text{if } k = 1 \end{cases}$$

In this case, a number of terms are equal to zero. Thus:

$$NPV_C = P_a Y\left(\frac{1}{1+r}, N\right) = \frac{\dfrac{1}{1.10} - \left(\dfrac{1}{1.10}\right)^{16}}{1 - \dfrac{1}{1.10}} = (\$1391)(7.61) = \$10{,}585$$

Assuming the capacity factor of 0.38 is correct (one could check this out given the Weibull parameters of the problem), the annual energy production is given by:

$$E_a = (0.38)(4)(8760) \text{ kWh/yr} = 13{,}315 \text{ kWh/yr}$$

Thus, the levelized cost of energy is given by:

$$COE_L = (\$10{,}585)(0.1175)/(13{,}315) = \$0.093/\text{kWh}$$

**11.5** The estimated cost and savings from the purchase and operation of a wind machine over 20 years are given in Table B.12. a) Compare the total costs with the total savings. b) Find the present values of the cost and savings. Use an annual discount rate of 10%. Would this be a good investment? c) Determine the pay back period for the wind machine. d) What is the breakeven cost of the machine? e) Estimate the cost of energy produced by the wind machine if it is expected to produce 9,000 kWh annually.

**Table B.12** Annual wind turbine costs

| Year | Costs | Savings | Year | Costs | Savings |
|---|---|---|---|---|---|
| 0 | $9000 | | 11 | 227 | 669 |

| | | | | | |
|---|---|---|---|---|---|
| 1 | 127 | 2324 | 12 | 241 | 719 |
| 2 | 134 | 1673 | 13 | 255 | 773 |
| 3 | 142 | 375 | 14 | 271 | 831 |
| 4 | 151 | 403 | 15 | 287 | 894 |
| 5 | 160 | 433 | 16 | 304 | 961 |
| 6 | 168 | 466 | 17 | 322 | 1033 |
| 7 | 180 | 501 | 18 | 342 | 1110 |
| 8 | 191 | 539 | 19 | 362 | 1194 |
| 9 | 202 | 579 | 20 | 251 | 1283 |
| 10 | 214 | 622 | | | |

**SOLUTION**

A spreadsheet tabulation of the costs and savings for this system is given below. From this spreadsheet, the following answers can be determined:

(a) The total costs are \$13,531 and the total savings are \$17,382. However, this does not tell us if this is a good investment. That must be determined from the net present value.

(b) Using a discount rate of 10%, the following are determined:

Total PVcosts = \$10,638.00

Total PVsavings = \$7,967.60

NPV = \$10,638.00 - \$7,697.60 = - \$2940.40

Therefore, this may not be a good investment since the NPV is negative.

(c) The cumulative savings first exceeds the costs in the 16th year, so this is one measure of payback.

(d) The breakeven cost is equal to the total present value of the savings. Therefore, the breakeven costs = \$7,697.60.

(e) Annual energy production = 9000 kWh. Use Equation 11.21:

$$COE_L = \frac{(NPV_C)(CRF)}{\text{Annual energy production}}$$

The net present value of the costs, found above, are \$10,638. The *CRF* in this case is found from Equation 11.12 using $L$= 20 and $r$ = 0.1:

$$CRF = r/[1-(1+r)^{-L}] = 0.1/[1-(1.1)^{-20}] = 0.11746$$

Therefore:

$$COE_L = \frac{(\$10,638)(0.11746)}{9000\,kWh} = \$0.14/kWh$$

SPEADSHEET RESULTS

| Year | Costs | Annual Savings | Cumulative Costs | Cumulative Savings | Discount Factor | Present Value Costs | Present Value Savings |
|---|---|---|---|---|---|---|---|
| 0 | 9000 | 0 | 9000 | 0 | 1 | 9000 | 0 |
| 1 | 127 | 2324 | 9127 | 2324 | 0.909 | 115.45 | 2112.73 |
| 2 | 134 | 1673 | 9261 | 3997 | 0.826 | 110.74 | 1382.64 |
| 3 | 142 | 375 | 9403 | 4372 | 0.751 | 106.69 | 281.74 |
| 4 | 151 | 403 | 9554 | 4775 | 0.683 | 103.14 | 275.25 |
| 5 | 160 | 433 | 9714 | 5208 | 0.621 | 99.35 | 268.86 |
| 6 | 168 | 466 | 9882 | 5674 | 0.564 | 94.83 | 263.04 |
| 7 | 180 | 501 | 10062 | 6175 | 0.513 | 92.37 | 257.09 |
| 8 | 191 | 539 | 10253 | 6714 | 0.467 | 89.1 | 251.45 |
| 9 | 202 | 579 | 10455 | 7293 | 0.424 | 85.67 | 245.55 |
| 10 | 214 | 622 | 10669 | 7915 | 0.386 | 82.51 | 239.81 |
| 11 | 227 | 669 | 10896 | 8534 | 0.35 | 79.56 | 234.48 |
| 12 | 241 | 719 | 11137 | 9303 | 0.319 | 76.79 | 229.1 |
| 13 | 255 | 773 | 11392 | 10076 | 0.29 | 73.86 | 223.91 |
| 14 | 271 | 831 | 11663 | 10907 | 0.263 | 71.36 | 218.83 |
| 15 | 287 | 894 | 11950 | 11801 | 0.239 | 68.71 | 214.02 |
| 16 | 304 | 961 | 12254 | 12762 | 0.218 | 66.16 | 209.14 |
| 17 | 322 | 1033 | 12576 | 13795 | 0.198 | 63.71 | 204.37 |
| 18 | 342 | 1110 | 12918 | 14905 | 0.18 | 61.51 | 199.64 |
| 19 | 362 | 1194 | 13280 | 16099 | 0.164 | 59.19 | 195.23 |
| 20 | 251 | 1283 | 13531 | 17382 | 0.149 | 37.34 | 190.71 |

**11.6** Consider the application of a small 1 kW wind turbine with a capital cost of $2500. The installation and setup cost raises its total installed cost to $4500. Assume that the $2500 capital cost is to be paid for with a 15 year, 7% loan. Also assume that O&M costs will be $200 per year. Estimate the (simplified) cost of energy) over the 15 year period if the capacity factor (CF) is 0.30.

**SOLUTION**

The answer to this problem is based on the use of Equation 11.7 The operating cost of this wind turbine is:

Operating cost = Annual payments on loan + O&M costs

For a 15 year loan at 7% interest, the annual payment is given by:

Annual Payment = Loan x Capital recovery factor (CRF)

CRF is calculated via Equation 11.12 and equals 0.1098

The annual payment is \$494.10 and the total operating cost is \$694.10

The average energy delivered per year is: CF x Rated Power x 8760 (kWh/yr)

= 0.30 x 1 kW x 8760 hr/yr = 2628 kWh/yr

Thus, COE = \$694.10/ 2628 = \$0.264/ kWh

**11.7** A 2.4 MW wind turbine has an installed cost of \$2.4 x $10^6$. Assume that the annual operating and maintenance costs are 2% of the initial installed cost. The wind regime at this site results in a capacity factor (CF) of 0.35. It is also assumed that the lifetime of this turbine is 25 years and that the electrical output can be sold for \$0.05/ kWh. For a 5% discount rate calculate the following economic parameters for this system:

a) Simple payback period
b) The net present value of the savings ($NPV_S$)
c) The benefit-cost ratio (B/C)
d) The internal rate of return (IRR)
e) The levelized cost of energy ($COE_L$)

**SOLUTION**

a) The simple payback period is calculated via the use of Equation 11.3 The annual energy production is equal to 0.35 x 8760 x 2400 = 7.35 x $10^6$ kWh/yr.
The average annual return, AAR, is equal to 7.35 x $10^6$ x 0.05 = \$3.68 x $10^5$/ yr.

Thus, the simple payback period is \$2.4 x $10^6$ / \$3.68 x $10^5$ = 6.52 yr.

b) The net present value (NPV) of the savings is equal to the net present cost of the system subtracted from the net present value of the income from the wind turbine.

The $NPV_{income}$ is found from Equations 11.14 and 11.19, using discount rate, $d$ = 0.05, and lifetime, $L$ = 25, as:

$$NPV_{income} = (AAR)Y\left[\frac{1}{1+d}, L\right] = \$3.68x10^5\left(\frac{\left(\frac{1}{1+d}\right) - \left(\frac{1}{1+d}\right)^{L+1}}{1-\left(\frac{1}{1+d}\right)}\right)$$

$$= \left(3.68x10^5\right)(14.09) = \$5.185x10^6$$

The net present value of the cost of the system equals the net present value of the capital cost (\$2.4 x $10^6$) plus the net present value of the O&M costs. The NPV of the O&M costs are calculated via Equations 10.18 and 10.19 as:

$NPV_{O\&M}$ = (\$2.4 x $10^6$) x 0.02 x 14.09 =\$6.76 x $10^5$

Thus, $NPV_{cost}$ = \$2.4 x $10^6$ + \$6.76 x $10^5$ = \$3.076 x $10^6$

And, $NPV_{savings}$ = \$5.185 x $10^6$ - \$3.076 x $10^6$ = \$2.11 x $10^6$

c) The Benefit-Cost Ratio (BCR) is found from Equation 11.23

BCR = \$5.185 x $10^6$ / \$3.076 x $10^6$ = 1.69

d) The internal rate of return (IRR) is defined by Equation 11.22

By trial and error, the answer works out to be about 14%

e) The levelized cost of energy is given by Equation 11.20.

$COE_L$ = \$3.076 x $10^6$ x .07095 / 7.35 x $10^6$ = \$0.0296 /kWh

**11.8** A proposed wind farm installation is based on the use of thirty 2 MW wind turbines. The installed capital cost is estimated at \$1000/kW and the annual operation and maintenance costs are estimated to be 3% of the original capital cost. Financing for the project is to come from two sources: 1) A 20 year loan at 7% (75%), and 2) an equity investment that has a return of 15% (25%). The average capacity factor of the wind farm is estimated to be 0.35. Determine the estimated cost of energy (\$/kWh) from this proposed wind farm.

**SOLUTION**

The solution to this problem will be based on the use of Equation 11.20 thus we need to calculate all the annual costs plus the annual electrical production.

For the loan (\$45 x $10^6$) the annual payment is found using Equation 11.12 and is equal to \$4.25 x$10^6$/yr.

The return on the equity (\$15 x $10^6$) is equal to 15 x $10^6$ x 0.15 = \$2.25 x $10^6$/yr

Annual O&M costs are equal to 0.03 x \$60 x $10^6$ = \$1.8 x $10^6$

Thus, the total annual costs are \$(4.25 + 2.25 + 1.8) x $10^6$ = \$8.3 x $10^6$.

The annual energy production is equal to: 30 x 2 x 1000 x 8760 x 0.35 kWh/yr =1.84 x $10^8$ kWh/yr, so the cost of energy is calculated as \$8.3 x $10^6$/ 1.84 x $10^8$ = \$0.045/ kWh

**11.9** This problem is intended to assess the relative economic merits of installing a wind–diesel system rather than a diesel-only system at a hypothetical location. Assume that the existing diesel is due for replacement and that the annual system demand will remain fixed during the lifetime of the project.

Your problem is to calculate the levelized cost of energy for the diesel only and for the wind–diesel system. The following system and economic parameters hold:

| | |
|---|---|
| *System* | |
| Average system load | 96 kW (841,000 kWh/yr) |
| Diesel rated power | 200 kW |
| Mean diesel usage- no wind turbine generator | 42.4 l/hr (371,000l/yr) |
| Wind turbine rated power | 100 kW |
| Average annual wind speed | 7.54 m/s |
| Annual wind speed variability | 0.47 |
| Average wind turbine power | 30.9 kW |
| Useful wind turbine power (based on 25% lower limit on diesel loading) | 21.1 kW |
| Mean diesel usage with WTG | 34.8 l/hr (305,000 l/yr) |
| *Economics* | |
| Wind turbine cost | \$1500/kW installed |
| Diesel cost | \$3500/kW installed |
| System life | 20 yr |
| Initial payment | \$30,000 |
| Loan term | 10 yr |
| Interest | 10% |
| General inflation | 5% |
| Fuel inflation | 6% |
| Discount rate | 9% |
| Diesel fuel cost | \$0.50/l |
| Wind turbine O&M cost | 2% of capital cost/yr |
| Diesel O&M cost | 5% of capital cost/yr |

**SOLUTION:**

This problem is discussed in Chapter 8 of Wind Diesel Systems by Hunter and Elliot (1994). In order to determine the levelized cost of energy, the net present value (or here, net present costs) of installing the different energy supply systems must be calculated.

The net present value equation for a wind/diesel system can be found by adding fuel costs to Equation 11.18:

$$NPV_C = P_d + P_a Y\left(\frac{1}{1+r}, N\right) + C_c f_{OM} Y\left(\frac{1+i}{1+r}, L\right) + (FL)(FC) Y\left(\frac{1+e}{1+r}, L\right)$$

where:

$FL$ = Fuel consumption, units/yr

$FC$ = Fuel cost, \$/unit

*e* = energy inflation rate

The following Series Present Worth and Capital Recovery Factors may be calculated first to facilitate solving the problem:

Series Present worth factor for interest payments, *SWWFP*:

$$Y\left[\frac{1}{1+r}, N\right] = 6.418$$

Series Present worth factor for fuel costs, *SPWFF*

$$Y\left[\frac{1+e}{1+r}, L\right] = 15.144$$

Series Present worth factor for O&M costs, *SWWFO*

$$Y\left[\frac{1+i}{1+r}, L\right] = 13.822$$

Capital Recovery Factor for interest payments, *CRFP*

$$1/Y\left[\frac{1}{1+b}, N\right] = 0.1628$$

Capital Recovery Factor for system income, *CRFI*

$$1/Y\left[\frac{1}{1+r}, L\right] = 0.1095$$

Now, the Net Present Values can be determined using the form of Equation 11.18 as given above. The following values result:

| *Parameter* | *Equation* | *Diesel Only* | *WTG and Diesel* |
|---|---|---|---|
| Capital Cost | $C_c$ | \$70,000 | \$220,000 |
| Initial Payment on System | $P_d$ | \$30,000 | \$30,000 |
| Loan | $C_c$ - $P_d$ | \$40,000 | \$190,000 |
| Annual Payment ($A_p$) | ($C_c$ - $P_d$) CRFP | \$6510 | \$30,922 |
| NPV of Annual Payments | $A_p$ x SPWFP | \$41,779 | \$198,448 |
| Fuel Costs per year (year 1) | FL x FC | \$185,500 | \$152,500 |
| NPV of fuel consumed over system lifetime | FL x FC x SPWFF | \$2,803,647 | \$2,593,176 |
| O&M costs (year 1) OMC | $\Sigma C_c$ x OM | \$3500 | \$6500 |
| NPV of O&M costs over system lifetime | OMC x SPWFO | \$48,377 | \$89,843 |
| Total NPV of system costs, TNPV | \$2,923,803 | \$2,623,176 | |
| Levelized cost of energy, | (TNPV)(CRFI)/SL | \$0.38/kWh | \$0.34/kWh |

## B.12 Chapter 12 Problems

**12.1** Despite its positive environmental attributes, not everyone is in favor of wind power development. Your assignment here is to perform an internet search and summarize the view points of three different organizations that actively oppose wind projects in one way or another. Make sure that you list their objections. You should also comment on their technical validity.

**SOLUTION**

As a start one could look at the web sites of the following organizations:

**1) European Platform Against Windfarms: http://www.epaw.org/**

**2) Wind Turbine Syndrome: http://www.windturbinesyndrome.com/**

**3) Virginia Wind: http://www.vawind.org/**

**4) National Wind Watch: http://www.wind-watch.org/**

**5) Country Guardian: http://www.countryguardian.net/**

**12.2** Assume that you are planning the development of a small wind farm (say 10 turbines of 500 kW) in a rural area. Describe how you would reduce the visual impact of this project. Also prepare a list of the tools that could be used for assessing the impact of this project on the surrounding environment.

**SOLUTION**

This is a literature review problem. Good sources here are references dealing with visual impact assessment, especially ones that give some type of visual simulation. A search of the internet should yield descriptions of available computer software programs that accomplish this.

**12.3** Researchers from Spain (see Hurtado et al. (2004) *Spanish method of visual impact evaluation in wind farms.* Renewable and Sustainable Energy Reviews, 8, pp. 483-491.) have developed a method that gives a numerical value for a factor that represents a coefficient of the affected population.

Review this paper and give an evaluation of the use of this methodology

**SOLUTION**

Once one finds and reads this paper, the solution is left to the originality and thoughts of the participating student.

**12.4** Estimate the sound power level for a three-bladed, upwind wind turbine with the following specifications: rated power = 500 kW, rotor diameter = 40 m, rotor rotational speed = 40 rpm, wind speed = 12 m/s.

**SOLUTION**

This solution is based on solution of three different sound power level prediction equations in the text:

$$L_{WA} = 10\,(\log_{10} P_{WT}) + 50 \tag{12.9}$$

$$L_{WA} = 22\,(\log_{10} D) + 72 \tag{12.10}$$

$$L_{WA} = 50(\log_{10} V_{Tip}) + 10\,(\log_{10} D) - 4 \tag{12.11}$$

From the 1st equation:

$$L_{WA} = 10\,(\log_{10}(500{,}000) + 50 = 56.99 + 50 = 106.99\,\text{dB}$$

From the 2nd equation:

$$L_{WA} = 22\,(\log_{10}(40) + 72 = 35.25 + 72 = 107.25\,\text{dB}$$

Using the 3rd equation:

$$V_{tip} = (2\pi)(20)\,(40/60) = 83.78\text{ m/s}$$

$$L_{WA} = 50\ \log_{10}(83.78) + 10\log_{10}(40) - 4 = 96.16 + 16 - 4 = 108.18\,\text{dB}$$

**12.5** Suppose the sound pressure level at a distance of 100 m from a single wind turbine is 60 dB. What are the sound pressure levels at a distance of 300, 500, and 1000 m?

**SOLUTION**

The solution is based on the use of Equation 12.14 from the text:

$$L_p = L_w - 10\log_{10}(2\pi R^2) - \alpha R$$

The MiniCodes give a Sound Power Level = 108.48 dB

Sound Pressure = 49.5 at 300 m

Sound Pressure = 44 at 500 m

Sound Pressure = 35.5 at 1000 m

**12.6** Assuming the same turbine as in the previous problem, what are the sound pressure levels at a distance of 300, 500, and 1000 m for 2, 5, and 10 turbines?

**SOLUTION**

Use Equation 12.13

$$L_{total} = 10 \log_{10} \sum_{i=1}^{N} 10^{L_i/10}$$

2 turbines
  Sound Pressure = 52.5 at 300 m
  Sound Pressure = 47 at 500 m
  Sound Pressure = 38.5 at 1000 m

5 turbines
  Sound Pressure = 56.4 at 300 m
  Sound Pressure = 51 at 500 m
  Sound Pressure = 42.5 at 1000 m

10 turbines
  Sound Pressure = 59.5 at 300 m
  Sound Pressure = 54 at 500 m
  Sound Pressure = 45.5 at 1000 m

**12.7** Two wind turbines emitting 105 dB(A) at the source are located 200 m and 240 m away from a location of interest. Calculate the sound pressure level (in dB(A)) at the point of interest from the combined acoustic effect of the two turbines. Assume a sound absorption coefficient of 0.005 dB(A)/m.

**SOLUTION**

For the first turbine, the sound pressure level is calculated from Equation 12.14 as

$(L_{p)1} = 105 - 10 \log_{10}(2 \text{ x } \pi \text{ x } 200^2) - 0.005\text{x}200 = 50$ dB(A)

Similarly for the second turbine

$(L_{p)2} = 105 - 10 \log_{10}(2 \text{ x } \pi \text{ x } 240^2) - 0.005\text{x}200 = 48$ dB(A)

The sound pressure level from the two turbines is then calculated from Equation 12.12 as

$L_{total} = 10 \log_{10} (10^{\,L1/10} + 10^{\,L2/10}) = 52.12$ dB(A)

**12.8** A wind turbine has a measured sound power level of 102 dB at 8 m/s. Plot the sound pressure level as a function of distance using both hemispherical and spherical noise propagation models. Do this for sound absorption coefficients of 0.0025, 0.005, and 0.010 (dB(A) $m^{-1}$).

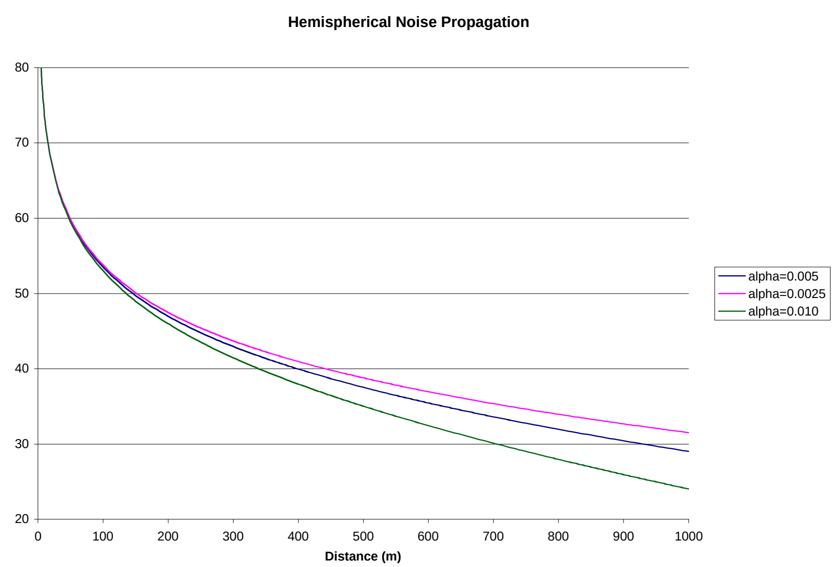
**SOLUTION**

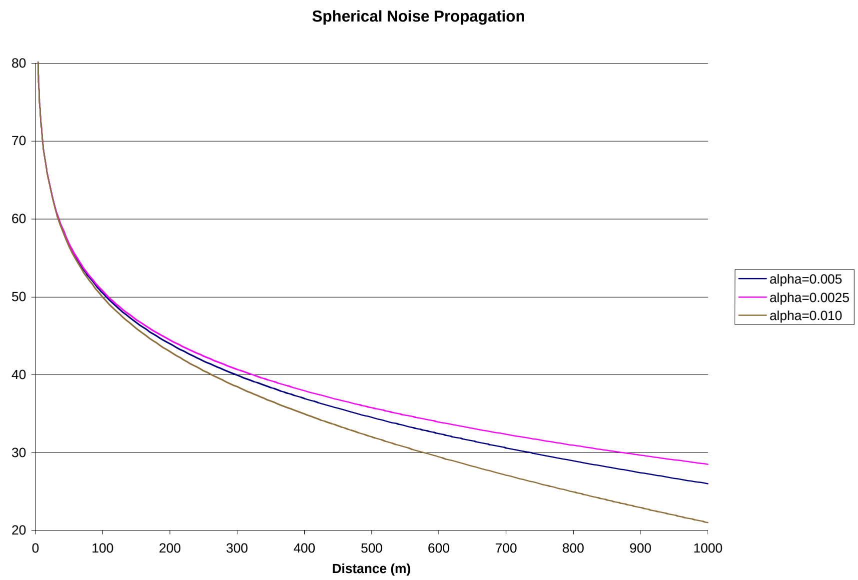
The sound level prediction equation for hemispherical noise propagation is given in the Text as Equation 12.14. A little research is needed to find the sound prediction equation for a spherical noise prediction model (e.g., see http://resource.npl.co.uk/acoustics/techguides/wtnm/).

It is similar to Equation 12.14 and is:

$L_p = L_w - 10 \log10 (4\pi R^2) - \alpha R$

The graphical results are shown below for the three values of sound absorption coefficient.






**12.9** For electromagnetic interference problems once the (bistatic) radar cross-section ($\sigma_b$) is known, the polar coordinate $r$ can be calculated for a required interference ratio ($C/I$). For a TV transmitter operating in an urban environment, $\sigma_b$ was experimentally determined to be 24 and 46.5 dBm$^2$ in the backscatter and forward scatter regions. Determine the value of $r$ in both the backscatter and forward scatter regions for required values of $C/I$ equal to 39, 33, 27, and 20 dB.

**SOLUTION**

This solution is based on experimental and analytical work summarized in the paper of Van Kats and Van Rees (1989). The solution is based on the rewriting of Equation 12.19 in the following manner:

$$20\log(r) = C/I - 10\log(4\pi) + 10\log(\sigma_b) - A_2 + A_1 - A_r - \Delta G$$

For a worst case approximation, $\Delta G = 0$ (no discrimination of the receiving antenna), and when $A_1 = A_2 = A_r = 0$, $r$ can be calculated from:

$$\begin{aligned} 20\log(r) &= C/I - 10\log(4\pi) + 10\log(\sigma_b) \\ &= C/I - 11 + 10\log(\sigma_b) \end{aligned}$$

For the backscattering region, $\sigma_b = 24\,\text{dB}$, thus:

$$20\log(r) = C/I + 13$$

This gives the following results:

| C/I | r (m) |
|---|---|
| 39 | 398 |
| 33 | 200 |
| 27 | 100 |
| 20 | 45 |

For the forward scattering region, $\sigma_b = 46.5\,\mathrm{dB}$, thus:

$$20\log(r) = C/I + 35$$

This gives the following results:

| C/I | r (m) |
|---|---|
| 39 | 5308 |
| 33 | 2660 |
| 27 | 1333 |
| 20 | 595 |

**Lists of Files Used in Problems or Solutions**

A summary table of the data input files is provided below.

| Chapter | Files for Problems | Problem # |
|---|---|---|
| **2** | | |
| | *MtTomData.xls* | 2.11 |
| | *MtTomWindUM.txt* | 2.14 |
| | *MtTom7Hzms.txt* | 2.15 |
| **4** | | |
| | *Flap1.txt* | 4.12 |
| **7** | | |
| | *ESITeeterAngleDeg50Hz.xls* | 7.7 |
| | *12311215Data.xls* | 7.8 |
| **8** | | |
| | *WPControlNeg1.csv* | 8.13 |
| | *WPControlNeg10.csv* | 8.13 |
| | *WPControlNeg100.csv* | 8.13 |
| **9** | | |
| | *Site1 v Site2 For MCP.csv* | 9.7 |
| | *HourlyLoadAndPowerOneYearGW.csv* | 9.11 |
| **10** | | |
| | *AOC_pc.csv* | 10.13 |
| | *AOC_pc_x2.csv* | 10.13 |
| | *AOC_pc_x4.csv* | 10.13 |
| | *Load_TI.txt* | 10.13 |
| | *Wind_TI.txt* | 10.13 |

A summary table of the solution files is provided below.

| Chapter | Solution Files for Problems | Problem # |
|---|---|---|
| 3 | | |
| | *VAWT_High Tipspeed Approximations_Solution.xls* | 3.13 |
| | *VAWT_Lambda_a_Solution.xls* | 3.13 |
| 4 | | |
| | *tower straight_2.csv* | 4.13 |
| | *V47.xls* | 4.13 |
| 7 | | |
| | *Gumbel Dist.xls* | 7.2 |
| 8 | | |
| | *WPControl.xls* | 8.13 |
| | *TrackingOptimumTipSpeed.xls* | 8.14 |
| 9 | | |
| | *Site1 v Site2 For MCP_Years 1999, 2001, 2002 Solution.xls* | 9.7 |
| | *OffshoreWindFarmSolution.xls* | 9.9 |
| | *DirSectorMgtSolution.xls* | 9.10 |
| | *LoadGWSolution.xls* | 9.11 |