Method of Characteristics

Seminar Report submitted by

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ABSTRACT

Partial differential equations governing real life phenomena are complex and behave differently depending on specific situations. Hence, they do not have one particular method of solving and exact solutions are yet to be found. Numerical techniques are usually used to arrive at approximate solutions to these equations. Some analytical techniques have been developed to solve partial differential equations in limited regimes.

Method of characteristics (MOC) is one such technique to solve hyperbolic partial differential equations. This method gives exact solution to the entire flow field but the solution is arrived graphically. MOC can be used to solve all wave phenomena and is particularly useful to solve for flow properties in supersonic flow. The current study looks into the use of MOC for supersonic flow considering a few assumptions and to design nozzles using the same.

MOC can be used to design supersonic nozzles, capture supersonic flow features and flow over any given geometry as long as the flow remains supersonic (hyperbolic). The MOC solution is strongly dependent on the initial points and its effects can be seen in the nozzle design where the sonic line curvature has to be considered to arrive a better design.

Keywords: *Hyperbolic PDEs, Supersonic Flows, Nozzle Design, Flow Features*

1. Introduction and Scope of work

1.1 Review of Partial Differential Equations:

Partial differential equations (PDEs) are the mathematical means to express many real-life phenomena. The solutions of such equations play a major role in solving problems in various fields like biology, physics, engineering and finance. Consider two rods at different temperatures and the temperature variation once they are brought together,

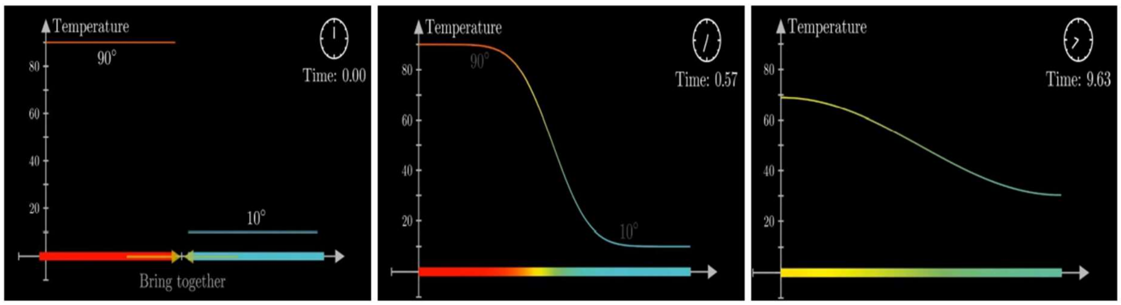


Figure 1: Heat transfer example

The temperature distribution in the previous condition is governed by a PDE called the heat equation which is a diffusion type PDE.

Similarly,

$$\frac{\partial \phi}{\partial t} + c\frac{\partial \phi}{\partial x} = 0 \qquad \text{1D Wave or Advection Equation}$$

$$\frac{\partial \phi}{\partial t} - D\frac{\partial^2 \phi}{\partial x^2} = 0 \qquad \text{Diffusion Equation}$$

$$\frac{\partial^2 \phi}{\partial t^2} - c^2\frac{\partial^2 \phi}{\partial x^2} = 0 \qquad \text{2D Wave Equation}$$

The highest derivates decide the order of the PDEs. If the powers or products of dependent variables or their partial derivates do not exists, the PDE is linear.

$$\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} = xyu \qquad x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = u \qquad (x+y^2)\frac{\partial u}{\partial t} + y\frac{\partial^2 u}{\partial x \partial y} = u$$

If the highest order derivatives of the dependent variables have coefficients which are functions of lower order derivatives, the PDE is quasi-linear. The lower order derivatives can occur in any manner.

$$u\frac{\partial^2 u}{\partial x^2}+\left(\frac{\partial u}{\partial y}\right)^2=u \quad u\frac{\partial u}{\partial y}\frac{\partial^2 u}{\partial x^2}+u^2\left(\frac{\partial u}{\partial y}\right)^2=0$$

If any of the previous conditions are not satisfied, the PDE is said to be non-linear.

1.2 Introduction to the Method of Characteristics:

The method of characteristics (MOC) is used to solve PDEs by converting the PDEs into an ordinary differential equation (ODEs) to make the problem solvable which is same as other approaches like separation of variables or change of coordinates. The basic principle of MOC is finding curves on the surface where the PDE becomes an ODE. The curves are called characteristic curves.

Consider 1D wave equation also known as advection equation for demonstration and simplicity:

$$\frac{\partial u}{\partial t}+c\frac{\partial u}{\partial x}=0$$

We know that

$$\frac{\mathrm{d}u}{\mathrm{d}t}=\frac{\partial u}{\partial t}+\frac{\partial u}{\partial x}\frac{\mathrm{d}x}{\mathrm{d}t}$$

By comparing the above equations

$$\frac{\mathrm{d}x}{\mathrm{d}t}=c \quad \text{and} \quad \frac{\mathrm{d}u}{\mathrm{d}t}=0$$

which means that u is constant along the line

$$\frac{\mathrm{d}x}{\mathrm{d}t}=c$$

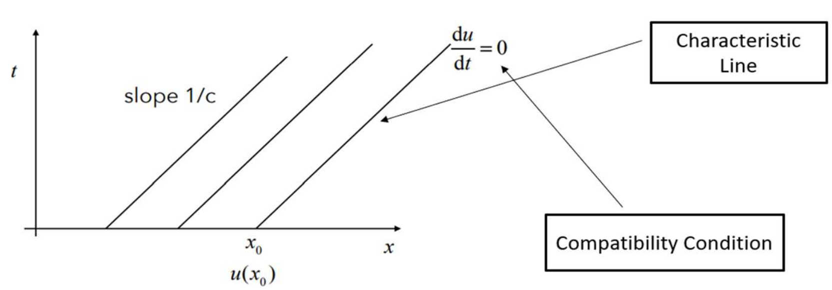


Figure 2: Characteristics Lines

$$\frac{\mathrm{d}x}{\mathrm{d}t} = c \Rightarrow x = ct + x_0 \Rightarrow x_0 = x - ct$$

$$\frac{\mathrm{d}u}{\mathrm{d}t} = 0 \Rightarrow u = u(x_0) = u_0(x - ct)$$

It can be seen that the solution is dependent entirely on the initial function u0. Solutions can be obtained in a similar manner for quasi linear and two-dimensional form of the wave equation as this method is particularly well suited for hyperbolic equations.

The classification of PDEs into hyperbolic, parabolic and elliptic is done based on characteristics.

Consider a generic PDE

$$a\frac{\partial^2 u}{\partial x^2} + b\frac{\partial^2 u}{\partial x \partial y} + c\frac{\partial^2 u}{\partial y^2} = d$$

Taking

$$v = \frac{\partial u}{\partial x} \qquad w = \frac{\partial u}{\partial y}$$

We get

$$a\frac{\partial v}{\partial x} + b\frac{\partial v}{\partial y} + c\frac{\partial w}{\partial y} = d$$

Proceeding with algebraic manipulation for obtaining the characteristics we arrive at a eigen value problem and solving for a non-trivial solution, we obtain a result for the characteristic curves which is used to classify the PDEs

$$\alpha = \frac{1}{2a}\left(b \pm \sqrt{b^2 - 4ac}\right)$$

2. Details of the Topic

Even in the case of 2-D inviscid, irrotational flows, the governing partial differential equations in fluid mechanics are nonlinear. Although linearization has been successfully developed as an approximate method for thin, slender bodies at small angles of attack in subsonic and supersonic flows, the accuracy of solution is reduced when perturbations are no longer small. Transonic flows cannot be linearized even for slender bodies. supersonic flows, where the governing partial differential equation is hyperbolic, there are unique curves in physical space called characteristics that turn the governing nonlinear partial differential equations into ordinary differentials that can be integrated. Therefore, we have exact solutions in supersonic flow along the characteristic curves, once we can identify them in the flowfield. The integration of the governing equations produces the compatibility conditions that are held constant along the characteristics. The method of characteristics is inherently graphical since it relies on developing a network of characteristic curves. Thus, mapping of a supersonic flowfield starts with an initial data line like an initial-value problem and literally propagates downstream along characteristic directions in the flow. Since the analysis does not rely on the linearization of the governing equations, the upper transonic flow regime is also included.

2.1 MOC for Supersonic Flow:

The flow is assumed to be is steady, irrotational and isentropic for the sake of mathematical simplicity. Two-dimensional flows in Cartesian coordinates are described by the velocity field

$$\boldsymbol{V} = u\mathbf{i} + v\mathbf{j}$$

For irrotational flows where the curl of the velocity field vanishes

$$\nabla \times V = 0 \quad \rightarrow \quad \frac{\partial u}{\partial y} = \frac{\partial v}{\partial x}$$

We know that,

$$V = \nabla \Phi$$

$$u \equiv \frac{\partial \Phi}{\partial x}$$

$$v \equiv \frac{\partial \Phi}{\partial y}$$

Applying mass conservation and momentum for steady two-dimensional flow we arrive at the velocity potential equation given by,

$$\left(1-\frac{\Phi_x^2}{a^2}\right)\Phi_{xx}-\frac{2\Phi_x\Phi_y}{a^2}\Phi_{xy}+\left(1-\frac{\Phi_y^2}{a^2}\right)\Phi_{yy}=0$$

and

$$du=\frac{\partial u}{\partial x}dx+\frac{\partial u}{\partial y}dy=\frac{\partial^2\Phi}{\partial x^2}dx+\frac{\partial^2\Phi}{\partial x\partial y}dy$$

$$dv=\frac{\partial v}{\partial x}dx+\frac{\partial v}{\partial y}dy=\frac{\partial^2\Phi}{\partial x\partial y}dx+\frac{\partial^2\Phi}{\partial^2 y}dy$$

Therefore, the system of three equations for the three unknowns, Φ_{xx}, Φ_{xy} and Φ_{yy}

Can be written as,

$$\left(1-\frac{u^2}{a^2}\right)\Phi_{xx}-\frac{2uv}{a^2}\Phi_{xy}+\left(1-\frac{v^2}{a^2}\right)\Phi_{yy}$$

$$dx\Phi_{xx}+dy\Phi_{xy}=du$$

$$dx\Phi_{xy}+dy\Phi_{yy}=dv$$

The solution can be expressed in terms of ratio of determinants using Cramer's rule and Φ_{xy} Must be indeterminate in order to satisfy the decoupling condition.

$$\Phi_{xy}=\frac{\begin{vmatrix}(1-\frac{u^2}{a^2}) & 0 & (1-\frac{v^2}{a^2})\\ dx & du & 0\\ 0 & dv & dy\end{vmatrix}}{\begin{vmatrix}(1-\frac{u^2}{a^2}) & \frac{-2uv}{a^2} & (1-\frac{v^2}{a^2})\\ dx & dy & 0\\ 0 & dx & dy\end{vmatrix}}$$

Setting the denominator to zero, we get

$$\left(1-\frac{u^2}{a^2}\right)(dy)^2-dx\left[\left(\frac{-2uv}{a^2}\right)dy-\left(1-\frac{v^2}{a^2}\right)dx\right]=0$$

$$\left(1-\frac{u^2}{a^2}\right)\left(\frac{dy}{dx}\right)^2+\frac{2uv}{a^2}\left(\frac{dy}{dx}\right)+\left(1-\frac{v^2}{a^2}\right)=0$$

$$\left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)_{\text{characteristic}} = \frac{\frac{uv}{a^2} \pm \sqrt{\left(\frac{uv}{a^2}\right)^2 - \left(1 - \frac{u^2}{a^2}\right)\left(1 - \frac{v^2}{a^2}\right)^2}}{\left(1 - \frac{u^2}{a^2}\right)}$$

$$= \frac{\frac{uv}{a^2} \pm \sqrt{\frac{u^2}{a^2} + \frac{v^2}{a^2} - 1}}{\left(1 - \frac{u^2}{a^2}\right)} = \frac{\frac{uv}{a^2} \pm \sqrt{M^2 - 1}}{\left(1 - \frac{u^2}{a^2}\right)}$$

Expressing the velocity components in terms of velocity magnitude and the angle

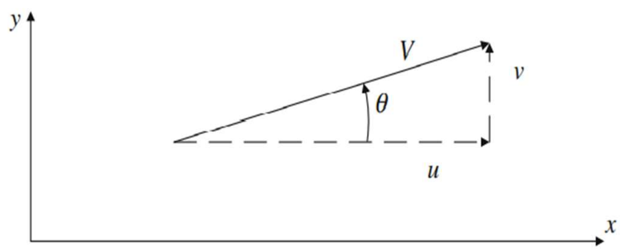


Figure 3: Velocity Components

$$u = V\cos\theta$$

$$v = V\sin\theta$$

$$\left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)_{\text{characteristic}} = \frac{M^2\sin\theta\cos\theta \pm \sqrt{M^2 - 1}}{1 - M^2\cos^2\theta}$$

Expressing Mach number in terms of Mach angle

$$\left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)_{\text{characteristic}} = \frac{\frac{\sin\theta\cos\theta}{\sin^2\mu}\cot\mu}{1 - \frac{\cos^2\theta}{\sin^2\mu}} = \frac{\sin\theta\cos\theta \pm \sin\mu\cos\mu}{\sin^2\mu - \cos^2\theta}$$

This simplifies to

$$\left.\frac{\mathrm{d}y}{\mathrm{d}x}\right|_{\text{characteristic}} = \tan(\theta \mp \mu)$$

Similarly, setting the numerator to zero

$$\begin{vmatrix} \left(1-\frac{u^2}{a^2}\right) & 0 & \left(1-\frac{v^2}{a^2}\right) \\ \mathrm{d}x & \mathrm{d}u & 0 \\ 0 & \mathrm{d}v & \mathrm{d}y \end{vmatrix} = 0$$

Expanding the determinant

$$\left(1-\frac{u^2}{a^2}\right)\mathrm{d}u\mathrm{d}y + \left(1-\frac{v^2}{a^2}\right)\mathrm{d}v\mathrm{d}x$$

$$\frac{\mathrm{d}v}{\mathrm{d}u} = \frac{1-\frac{u^2}{a^2}}{1-\frac{v^2}{a^2}}\left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)_{\text{characteristic}} = \frac{\mathrm{d}\,(V\sin\theta)}{\mathrm{d}\,(V\cos\theta)} = -\frac{1-\frac{u^2}{a^2}}{1-\frac{v^2}{a^2}}\left(\frac{-\frac{uv}{a^2}\pm\sqrt{M^2-1}}{1-\frac{u^2}{a^2}}\right)$$

$$\frac{\mathrm{d}\,(V\sin\theta)}{\mathrm{d}\,(V\cos\theta)} = \frac{M^2\sin\theta\cos\theta\mp\sqrt{M^2-1}}{1-M^2\sin^2\theta}$$

$$\mathrm{d}\theta = \mp\sqrt{M^2-1}\frac{\mathrm{d}V}{V}$$

The right-hand side of the equation is the differential of the Prandtl-Meyer function.

Hence, the compatibility conditions reduce to

$$\mathrm{d}\theta \mp \mathrm{d}\nu = 0$$

$$\theta - \nu = K_+ \quad \text{Constant along } C_+ \text{ characteristic}$$

$$\theta + \nu = K_- \quad \text{Constant along } C_- \text{ characteristic}$$

	Slope		Compatibility condition
C^-	$tan(\theta - \mu)$	K^-	$\theta + \nu(M) = K^-$
C^+	$\tan(\theta + \mu)$	K^+	$\theta - \nu(M) = K^+$

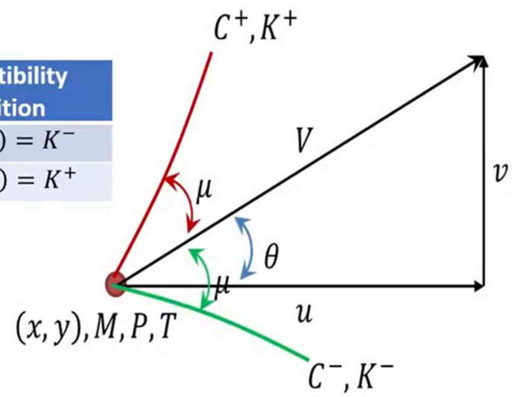


Figure 4: C+ and C- Characteristics

Using the equations for the slopes and the compatibility conditions can be used to solve for properties along both the characteristic curves as long properties at an initial point are known. The solution can be marched forward throughout the flow domain. This is possible due to the behavior of hyperbolic flows.

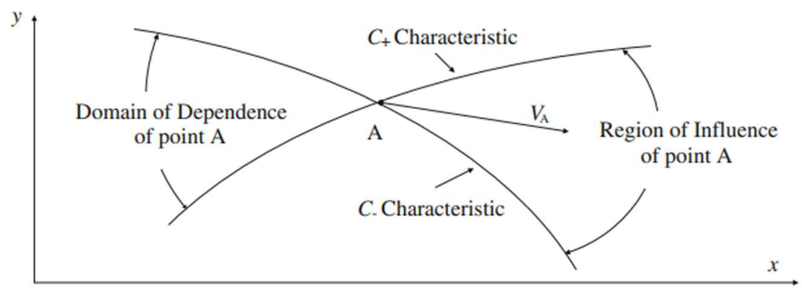


Figure 5: Characteristics for Hyperbolic PDEs

The region of influence is bound within the characteristic lines which makes the solution marching in one direction. In order to solve for properties in various conditions, subroutines have to be established and properties can be solved by developing algorithms for specific conditions using the subroutines.

2.2 Application of MOC:

Using the slope equations and compatibility conditions subroutines can be developed and the subroutines can be used to design geometries to control the flow or to analyze the flow at different conditions. In the current study, MOC is used to design planar CD nozzles and the required subroutines are established.

<u>Interior Point Subroutine:</u>

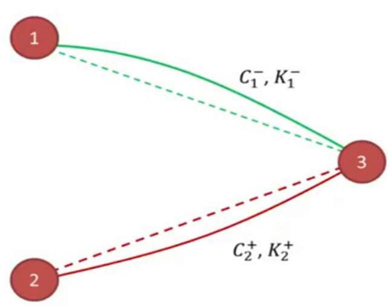


Figure 6: Interior Point

Consider points 1 and 2 in the x − y plane for which we know the flow inclination/direction, θ1 and θ2 as well as the Mach numbers, M1 and M2. From local Mach numbers, we calculate the Mach angles, μ1 and μ2 as well as the Prandtl-Meyer (P-M) angles, v1 and v2. These angles are sufficient to establish the characteristic directions for C+1, C−1, C+2 and C−2 as well as the compatibility constants K+1, K−1, K+2 and K−2 along those characteristics respectively. The intersection of C−1 and C+2 creates point 3, which shares the same compatibility constant with K−1 and K+2 that form two equations with two unknowns.

$$\theta_3 + \nu_3 = \theta_1 + \nu_1 = K_{-1}$$

$$\theta_3 - \nu_3 = \theta_2 - \nu_2 = K_{+2}$$

$$\theta_3 = \frac{K_{-1} + K_{+2}}{2}$$

$$\nu_3 = \frac{K_{-1} - K_{+2}}{2}$$

To graphically construct point 3, from the known points 1 and 2, we graph the C+2 characteristic at the average angle corresponding to points 2 and 3 along a C+ characteristic. The C+2 is graphed at the angle of

$$\frac{\theta_2 + \theta_3}{2} + \frac{\mu_2 + \mu_3}{2}$$

Similarly, the C−1 will be graphed at the average angles corresponding to points 1 and 3 on the C− characteristic,

$$\frac{\theta_1 + \theta_3}{2} - \frac{\mu_1 + \mu_3}{2}$$

since straight lines are drawn for the characteristic network between the adjacent grid points, in general the accuracy will improve if the points are closer to each other.

Symmetry Point Subroutine:

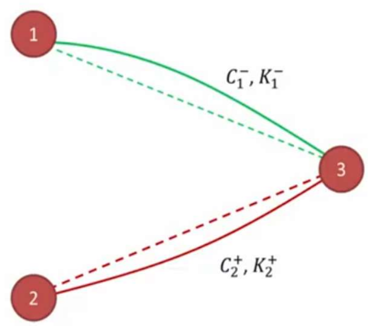


Figure 7: Symmetry Point

If the point 3 is along the axis, the properties at point 3 can be expressed as a function of either one of the previous points as the flow is parallel to the axis. This is particularly useful when solving or designing for symmetric flows. The equations simplify to

$$\theta_3 = 0, \nu_3 = \nu_1 + \theta_1$$

Wall point Subroutine:

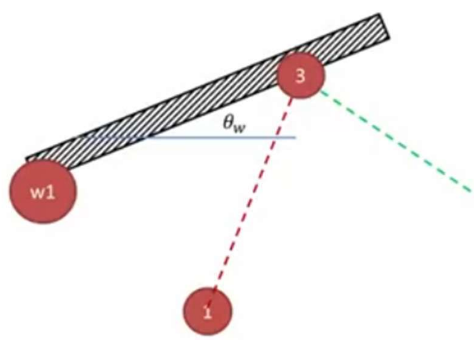


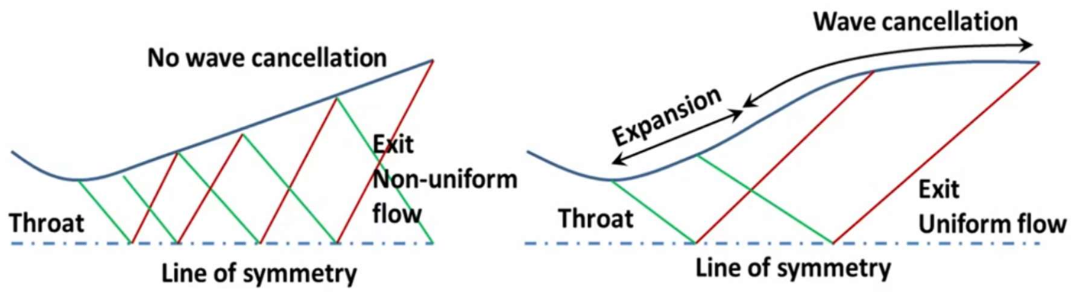
Figure 8: Wall Point

In order to design a CD Nozzle, the flow must be along the wall and the characteristic lines must not reflect after hitting the wall. This condition is satisfied when

$$\theta 3 = \theta w$$

$$\theta w = \theta 1$$

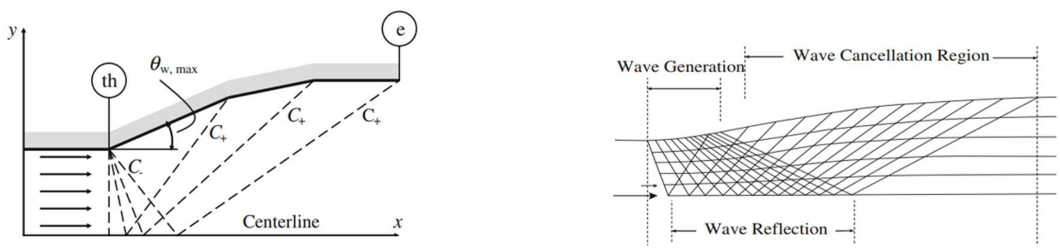
The flow remains tangential to the wall and the phenomenon is call wave cancellation. This condition must be satisfied while designing a nozzle with no shocks and uniform flow at the exit.



Non-Contoured Supersonic Nozzle　　Contoured Supersonic Nozzle

Figure 9: Conical Nozzle vs Contoured Nozzle

Contoured nozzles are necessary to have a uniform flow at the nozzle exit. Using the aforementioned subroutines, the wall of the divergent section of the CD nozzle can be designed. Flow in the divergent section alone can be solved because the flow remains supersonic after the throat and the convergent section can be a smooth function until the throat.



Minimum Length Nozzle　　Gradual Expansion Nozzle

Figure 10: Nozzle Design Approach

There are two possible types of nozzles that can be designed using the current method. One is the minimum length nozzle where expansion occurs from a point at the throat. The other one he’s a gradual expansion nozzle where the expansion after the nozzle throat occurs over a length. The former as the name suggests is used to design nozzles whose length is equal to the minimum required length to develop a flow of the required Mach number. The latter however does not have a length constraint and has an extended throat region. Planar nozzles for both the configurations were designed for an exit Mach number of 3 with throat height of 6cm using MATLAB. CFD analysis was done for both viscous (K-w SST) and inviscid cases for a Stagnation Pressure of 10bar and ambient pressure of 1bar using commercially available solver ANSYS Fluent.

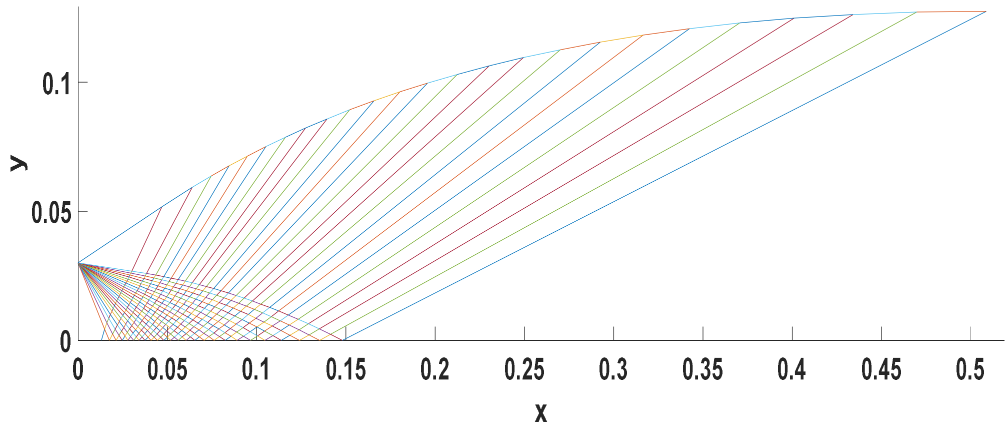


Figure 11: MATLAB Result for Minimum Length Nozzle

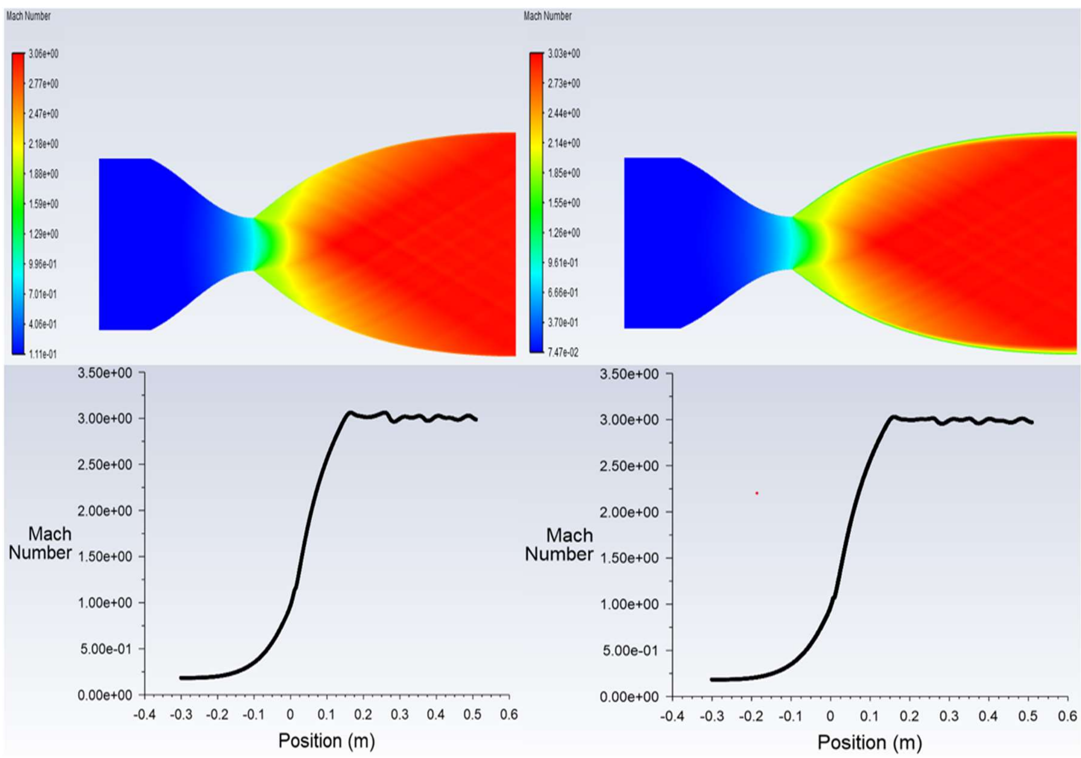


Figure 12: Inviscid (Left) and Viscous Simulation Results for Minimum Length Nozzle

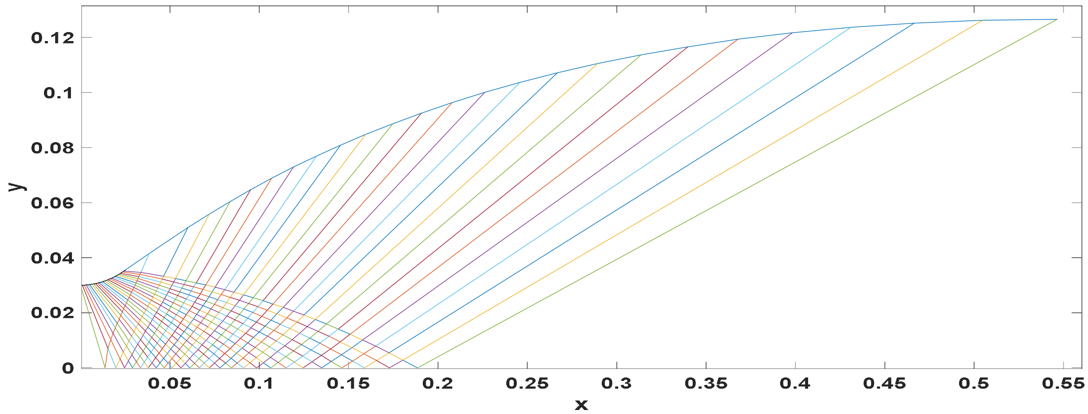


Figure 13: MATLAB Result for Gradual Expansion Nozzle

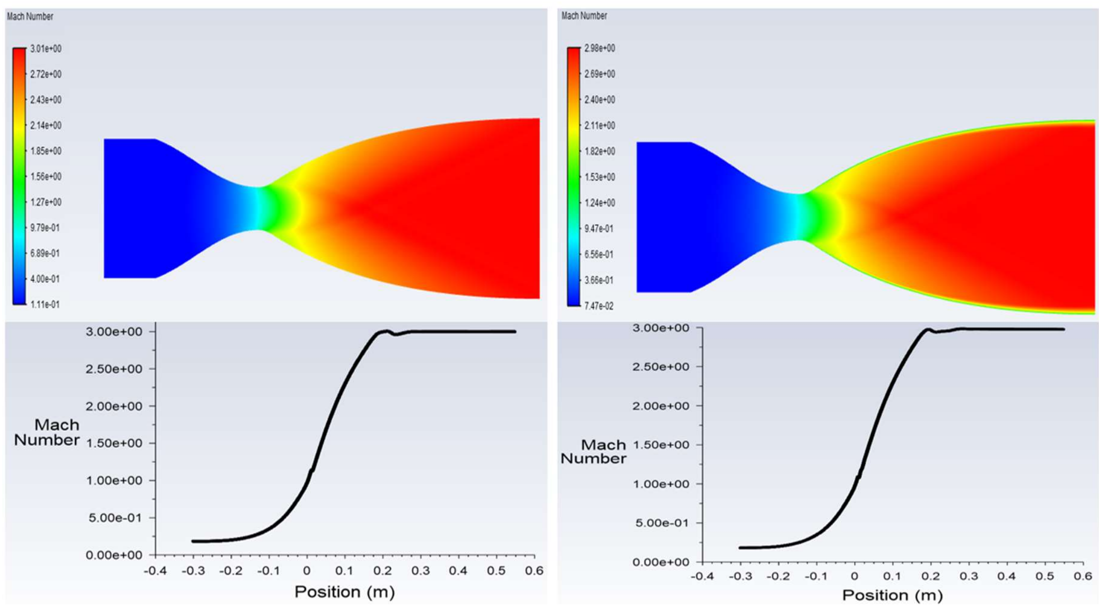


Figure 14: Inviscid (Left) and Viscous Simulation Results for Gradual Expansion Nozzle

For the sake of Simplicity, the radius of curvature of the extended throat region was taken as two times the radius of the throat.

3. Conclusion and Suggestions for Future Work

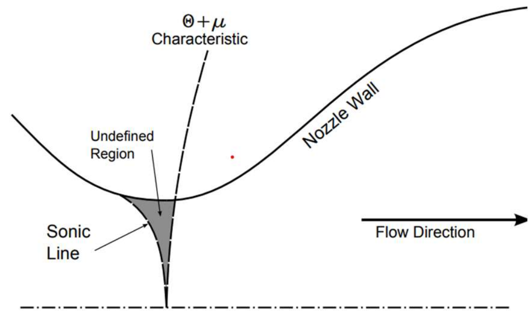


Figure 15: Sonic Line and the Undefined Region

- The present study considers the sonic line to be a straight line which is not the case in reality and the initial data must be evaluated considering the curvature in order to have better flow quality.
- The CFD results both the nozzle types for both viscous and inviscid cases yield the required exit Mach numbers but irregularities can be noticed in the case of minimum length nozzle due the point expansion.
- Gradual expansion nozzle yields much better results in terms of the smoothness of the Mach number variation as expected.
- Similar subroutines can be introduced for pressure boundary condition and solved for supersonic nozzle exhaust flow.

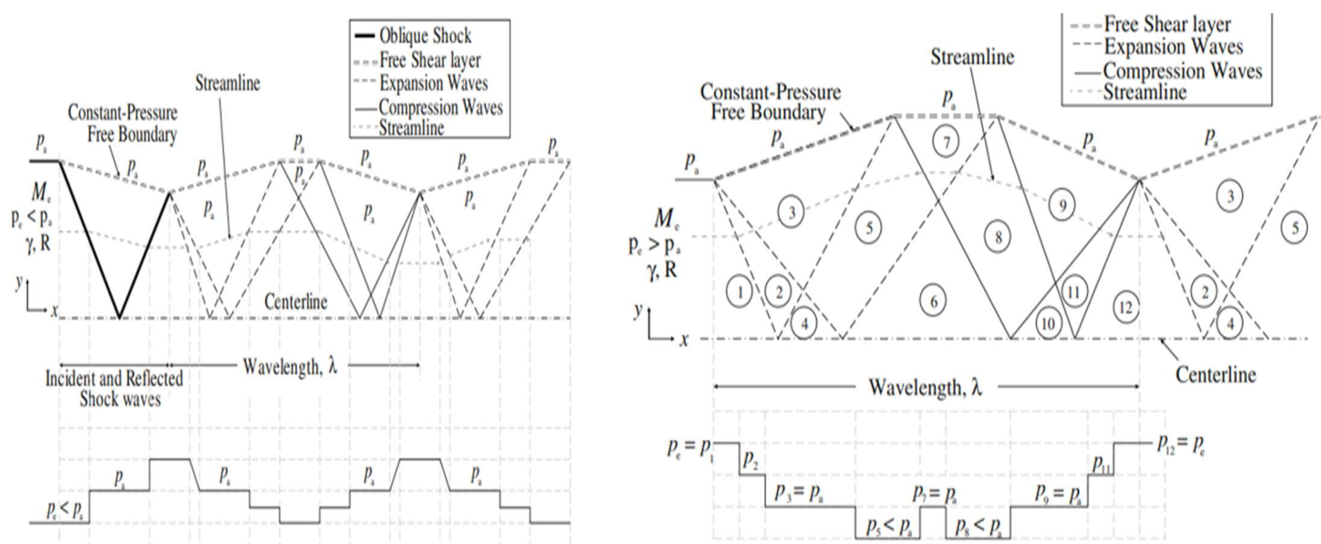


Figure 16: Evaluation of overexpanded and Under expanded Jets using MOC

- MOC can further used to solve for 3D, axisymmetric and rotational flows as mentioned in [1].

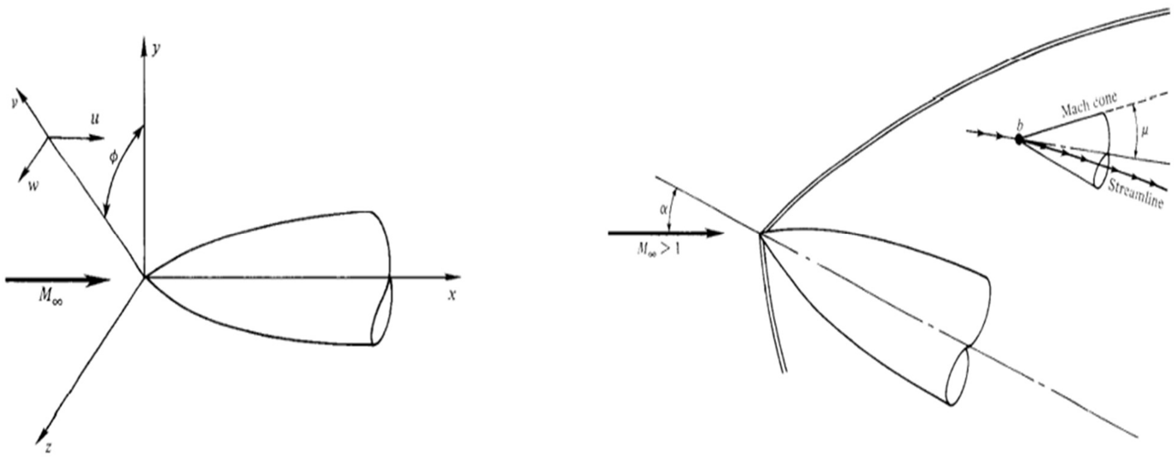


Figure 17: MOC for Axisymmetric and 3D Cases

- MOC can be modified and can applied to other problems like:
  - Shock Tubes
  - Displacement problems in porous media
  - Underwater Explosion Problems
  - Flow in open channels and shallow waters
  - Population balancing problem
  - Ground water solute transport problem
  - Traffic Movement problems
  - Advection and Diffusion Combination problems
  - Nuclear Engineering

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