HCAI5TML01 – Mathematics of Learning.

Week – 2: Lecture – 02

A Refresher on the Mathematics Behind Machine Learning.

Least Squares and Fundamental Theory of Linear Algebra.

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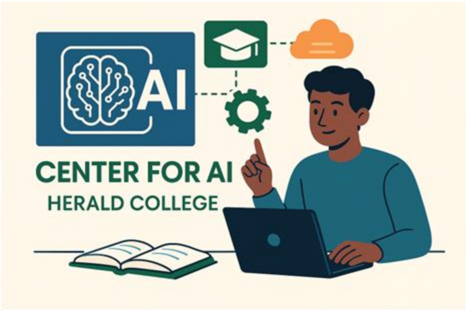

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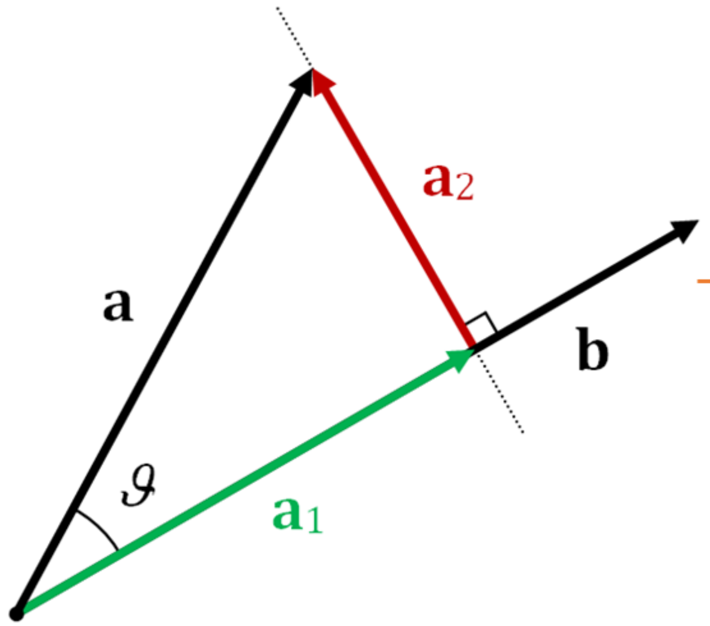
1. Some More Definitions.

1.1 Introduction: Vector projection.

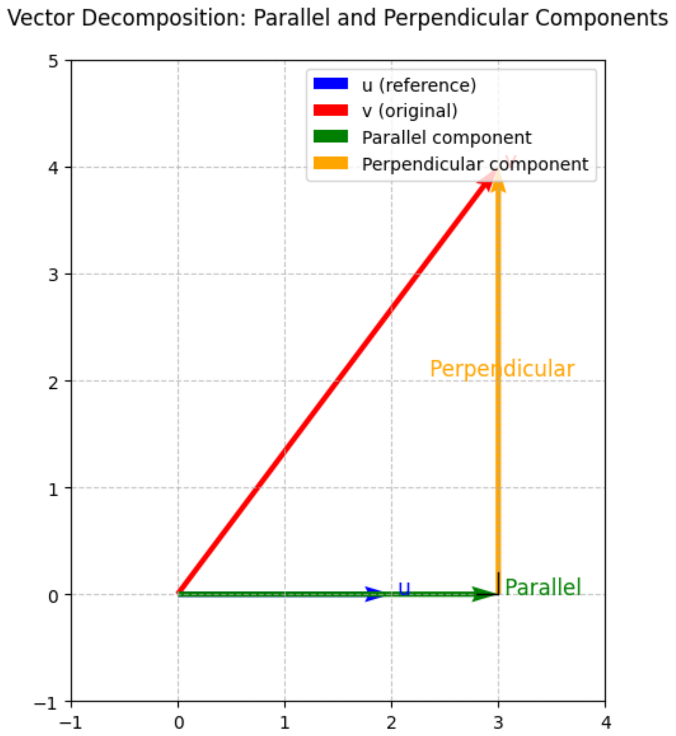
- Vector projection is a fundamental concept in linear algebra that decomposes
  - a vector into components **parallel and perpendicular** to another vector or subspace.



- In this example we **project vector a onto vector b,**
- Then we can indeed **decompose a** into:
  - A **parallel component aligned with b**
  - A **perpendicular component orthogonal to b.**
- Mathematically, this is expressed as:
  - $\mathbf{a} = \mathrm{proj}_{\mathbf{b}}\mathbf{a} + \mathrm{perp}_{\mathbf{b}}\mathbf{a}$
    - $\mathrm{proj}_{\mathbf{b}}\mathbf{a}$ → **the projection a onto b.**
    - $\mathrm{perp}_{\mathbf{b}}\mathbf{a}$ → **is the rejection of a from b (perpendicular to b)**

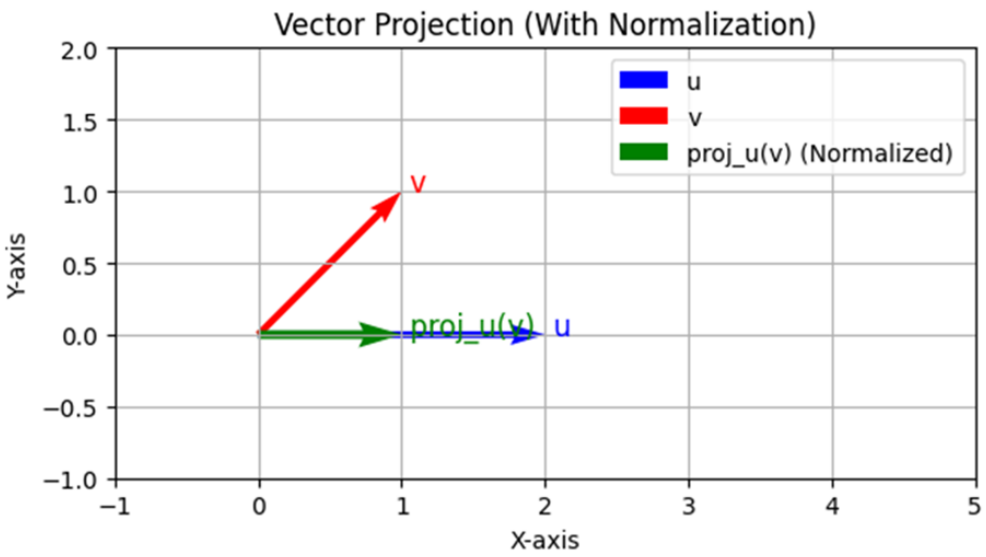
1.2 Projection on to vector.

- **Definition:**
  - Given two vector **u and v,** the projection of **v onto u (denoted $\mathbf{proj_u v}$)**
    - is a **component of v that lies in the direction of u.**
    - The $\mathbf{proj_u v}$ is commuted as:
      - $\mathbf{proj_u v} = \left(\frac{\mathbf{u \cdot v}}{\mathbf{u \cdot u}}\right)\mathbf{u} = \left(\frac{\mathbf{u \cdot v}}{\|\mathbf{u}\|^2}\right)\mathbf{u}$
  - **Key components:**
    - **Dot product** $(\mathbf{u} \cdot \mathbf{v})$:
      - Measures how **much v aligns with u** (the “overlap”).
      - If u and v are orthogonal, $\mathbf{u} \cdot \mathbf{v} = \mathbf{0}$ **(no projection).**
    - **Normalization** $(\mathbf{u} \cdot \mathbf{u} == \|\mathbf{u}\|^2)$:
      - This is the scaled magnitude of u.
      - Purpose: scales the projection to the unit length of u,
        - ensuring the result is **proportional to u’s direction** without **distorting its length**.
      - This ensures the projection depends only on the *direction* of **u**, not its magnitude.



1.2.1 Example: with and without Normalizations.

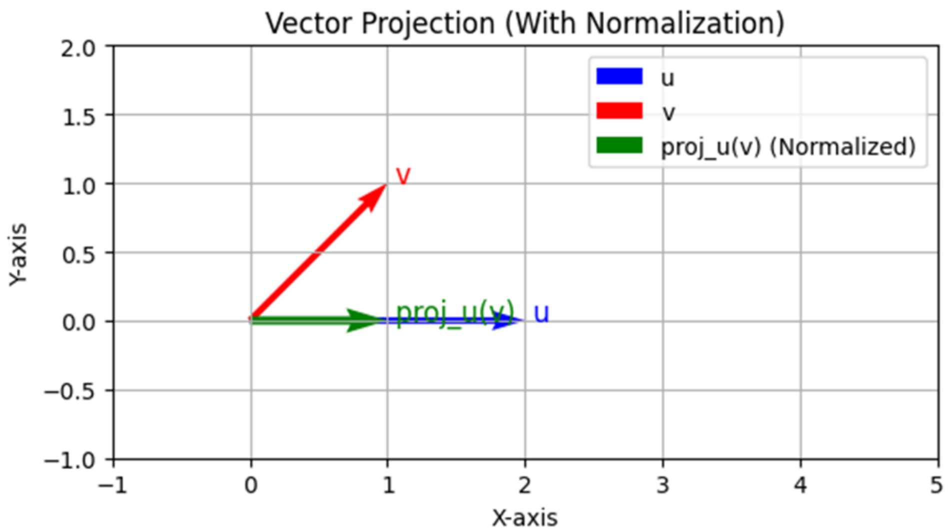
- Let's use the same vectors u and v but omit normalization to see how the projection becomes distorted.
  - Given vectors: $\mathbf{u} = \begin{matrix}2\\0\end{matrix}; \mathbf{v} = \begin{matrix}1\\1\end{matrix}$
  1. **Correct Projection with Normalization:**
     - $\mathbf{u} \cdot \mathbf{v} = [2 \quad 0] \cdot \begin{bmatrix}1\\1\end{bmatrix} = 2 \times 1 + 0 \times 1 = 2$
     - $\mathbf{u} \cdot \mathbf{u} = 2^2 + 0^2 = 4$
     - Projection:
       - $\mathbf{proj_u v} = \left(\frac{\mathbf{u \cdot v}}{\mathbf{u \cdot u}}\right)\mathbf{u} = \left(\frac{2}{4}\right)\begin{bmatrix}2\\0\end{bmatrix} = \begin{bmatrix}1\\0\end{bmatrix}$

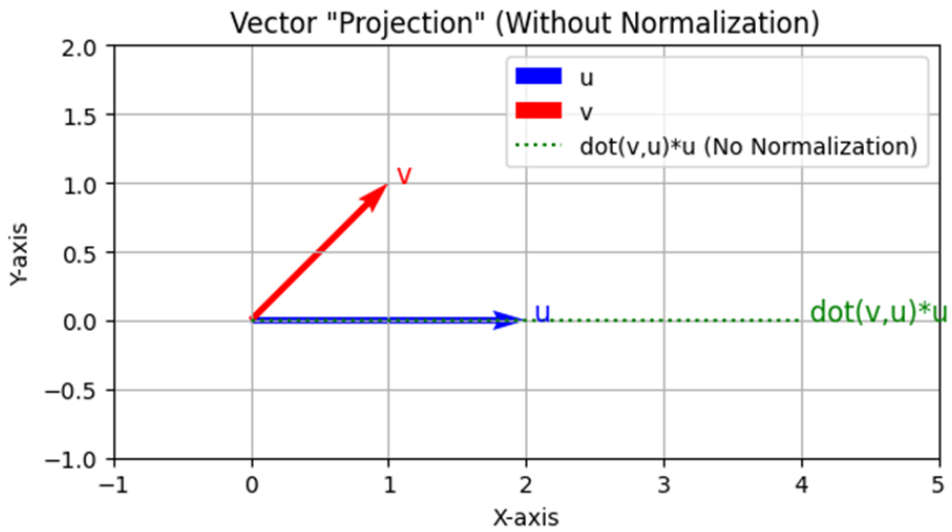


- The result is a vector in the direction of u with length scaled to match the "shadow" of v onto u.

1.2.1 Example: with and without Normalizations.

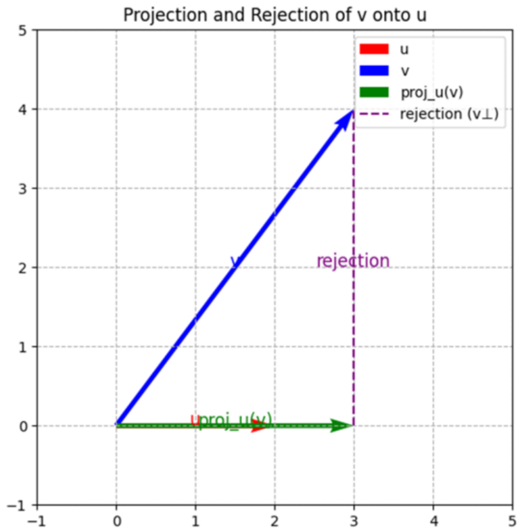
2. **Distorted Projection (without Normalization):**
   - If we **omit normalization**, the projection becomes:
     - **Distored Projection** $= (\mathbf{u} \cdot \mathbf{v})(\mathbf{u}) = 2\begin{bmatrix} \mathbf{2} \\ \mathbf{0} \end{bmatrix} = \begin{bmatrix} \mathbf{4} \\ \mathbf{0} \end{bmatrix}$
       - Without dividing by $\mathbf{u} \cdot \mathbf{u} == \|\mathbf{u}\|^2$, the result, **over scales by the length of u.**
     - Length of **distorted projection**: 4 (2× longer than u itself!).





1.2.2 Orthogonal Projection (Rejection):

- When you **project v onto u**, you're extracting the **part of v that aligns with u** (the **projection**).
  - The **remaining part**—the **rejection**—is literally
    - **"rejected"** from u because **it has no component** in the **direction of u**.
- The perpendicular (rejection) component of v from u is:
  - $\mathbf{rej_u v = v - proj_u v}$
- **Key Property:**
  - The original vector v is the sum of its projection and rejection:
    - $\mathbf{v = proj_u v + rej_u v}$
  - The rejection component is orthogonal to u.
  - To verify orthogonality, take the dot product with u:
    - $\mathbf{rej_u v \cdot u = (v - proj_u v) \cdot u = v \cdot u - \left(\frac{u \cdot v}{u \cdot u}\right)(u \cdot u) = (v \cdot u) - (u \cdot v) = 0}$ {**Dot Product Commutativity**}
  - This confirms $\mathbf{rej_u v}$ is orthogonal to u.



1.3 Projection onto Subspace.

- **Idea:**
  - Instead of projecting **onto a single vector**, we can project onto a **subspace** (e.g., a plane or hyperplane).
- **Using Matrix Projection {only true for $\mathbf{b} \in \mathbf{C}(\mathbf{A})$} :**
  - Given a matrix $\mathbf{A} \in \mathbb{R}^{\mathbf{m}\times\mathbf{n}}$ whose columns form a basis for a subspace (full column rank matrix),
    - the projection of **b onto C(A) (column space of A)** is:
      - $\mathbf{proj_{C(A)}b = A\left(A^T A\right)^{-1} A^T b.}$
- **Special Case: Least Square Solution,**
  - If $\mathbf{b} \notin \mathbf{C}(\mathbf{A})$, this projection gives the best approximation or least square solution to $\mathbf{Ax = b.}$

1.3.1 Derivation: $\mathrm{proj}_{C(A)}b = A\left(A^T A\right)^{-1} A^T b$.

- Derive the projection of a **vector** $\mathbf{b} \in \mathbb{R}^{\mathbf{m}}$ onto the column space $\mathbf{C(A)} \exists\ \mathbf{A} \in \mathbb{R}^{\mathbf{m\times n}}$:
  - $\mathbf{proj_{C(A)} b = A\left(A^T A\right)^{-1} A^T b}$
- **Step 1 Definition of Projection onto a subspace:**
  - We want to find a vector $\mathbf{p} \in \mathbf{C(A)}$ (the projection of $\mathbf{b}$) such that the residual $\mathbf{b - p}$ is **orthogonal** to $\mathbf{C(A)}$.
  - Mathematically:
    - $\mathbf{p = A\,\hat{x}\ \left(since\ p\ is\ in\ C(A)\right)}$
  - And
    - $\mathbf{A^T(b - A\,\hat{x}) = 0\ (orthogonality\ condition)}$.

1.3.1 Derivation: $\text{proj}_{C(A)}b = A\left(A^T A\right)^{-1} A^T b.$

- **Step 2: Solve for $\hat{\mathbf{x}}$:**
  - From the orthogonality condition:
    - $\mathbf{A^T(b - A\,\hat{x}) = 0}$
    - $\mathbf{A^T b - A^T A\,\hat{x} = 0}$
    - $\boxed{\mathbf{A^T A\,\hat{x} = A^T b}}$
  - This is called **Normal Equation.**
- **Step 3: Solve the Normal Equation:**
  - Assuming A has full column rank (columns are linearly independent), **$\mathbf{A^T A}$ is invertible**, and:
    - $\mathbf{\hat{x} = \left(A^T A\right)^{-1} A^T b}$

1.3.1 Derivation: $\text{proj}_{C(A)}b = A\left(A^T A\right)^{-1} A^T b.$

- **Optional:**
  - **Step 4: Projection Matrix:**
    - The *projection p of a vector b onto the column space C(A)* is:
      - $\mathbf{p} = \overbrace{\mathbf{A}\left(\mathbf{A}^T\mathbf{A}\right)^{-1}\mathbf{A}^T}\,\mathbf{b} = \mathbf{P}\mathbf{b}.$
    - Here:
      - $\mathbf{P} = \mathbf{A}\left(\mathbf{A}^T\mathbf{A}\right)^{-1}\mathbf{A}^T$ is the **projection matrix**, when **P multiplies b**, it returns the **projected vector p**.
  - **Step 4: Verification (Orthogonality Check):**
    - The residual $b-p$ must be orthogonal to $C(A)$:
      - $\mathbf{A}^T(\mathbf{b}-\mathbf{p}) = \mathbf{A}^T\mathbf{b} - \mathbf{A}^T\mathbf{A}\left(\mathbf{A}^T\mathbf{A}\right)^{-1}\mathbf{A}^T\mathbf{b} = \mathbf{A}^T\mathbf{b} - \mathbf{I}\mathbf{A}^T\mathbf{b} = \mathbf{A}^T\mathbf{b} - \mathbf{A}^T\mathbf{b} = \mathbf{0}.$

2. Solving Linear Regression.

{ With Normal and Least Squares.}

2.1 With Normal Equation.

- The **Normal Equation** is a fundamental result in linear regression
  - that provides a **closed-form solution** for finding the optimal parameters θ
    - that minimize the **sum of squared errors** (SSE) in linear regression.

Problem Setup: For Linear Regression Problem

Given:
A design matrix

$$\mathbf{X} \in \mathbb{R}^{\mathbf{m} \times \mathbf{n}}$$

(with m examples and n features, including a bias term if applicable).
A target vector $\mathbf{y} \in \mathbb{R}^{\mathbf{m}}$.
A parameter vector $\theta \in \mathbb{R}^{\mathbf{n}}$ (the weights we want to estimate).

Model:
The linear regression model is given by:

$$\mathbf{y} = \mathbf{X}\theta + \varepsilon$$

where ε is the error term.

2.1.1 Objective.

Objective: Minimize Sum of Squared Errors (SSE)

Objective:
Minimize the sum of squared errors (SSE):

$$\mathbf{J}(\theta) = \|\mathbf{y} - \mathbf{X}\theta\|^{\mathbf{2}} = (\mathbf{y} - \mathbf{X}\theta)^{\top}(\mathbf{y} - \mathbf{X}\theta)$$

2.1.2 Derivation of Normal Equation.

Goal: Find Optimal θ Minimizing the Cost Function

Goal: Find the optimal θ that minimizes the cost function $\mathbf{J}(\theta)$.
Start with the cost function:

$$\mathbf{J}(\theta) = \|\mathbf{y} - \mathbf{X}\theta\|^2 = (\mathbf{y} - \mathbf{X}\theta)^\top (\mathbf{y} - \mathbf{X}\theta)$$

Expand the expression:

$$\mathbf{J}(\theta) = \mathbf{y}^\top \mathbf{y} - \mathbf{y}^\top \mathbf{X}\theta - \theta^\top \mathbf{X}^\top \mathbf{y} + \theta^\top \mathbf{X}^\top \mathbf{X}\theta$$

Note: $\mathbf{y}^\top \mathbf{X}\theta$ is a scalar, so

$$\mathbf{y}^\top \mathbf{X}\theta = \theta^\top \mathbf{X}^\top \mathbf{y}$$

Simplify:

$$\mathbf{J}(\theta) = \mathbf{y}^\top \mathbf{y} - 2\theta^\top \mathbf{X}^\top \mathbf{y} + \theta^\top \mathbf{X}^\top \mathbf{X}\theta$$

Next, take the gradient with respect to θ.

2.1.2 Derivation of Normal Equation.

Gradient Calculation of the Least Squares Cost Function

1. Cost Function Definition

The least squares cost function is:

$$\mathbf{J}(\theta) = \mathbf{y}^\top \mathbf{y} - \mathbf{2}\theta^\top \mathbf{X}^\top \mathbf{y} + \theta^\top \mathbf{X}^\top \mathbf{X}\theta$$

2. Term-by-Term Gradient Calculation

We compute the gradient $\nabla_\theta J(\theta)$ by differentiating each term separately.

- **Term 1: $\mathbf{y}^\top \mathbf{y}$**
  This term does not depend on θ.
  Its gradient is zero:
  $$\nabla_\theta(\mathbf{y}^\top \mathbf{y}) = \mathbf{0}$$
- **Term 2:**
  $$-\mathbf{2}\theta^\top \mathbf{X}^\top \mathbf{y}$$
  This is a linear term in θ.

  Using the identity
  $$\nabla_\theta(\theta^\top \mathbf{a}) = \mathbf{a}$$
  where $a = X^\top y$, we get
  $$\nabla_\theta(-\mathbf{2}\theta^\top \mathbf{X}^\top \mathbf{y}) = -\mathbf{2}\mathbf{X}^\top \mathbf{y}$$

2.1.2 Derivation of Normal Equation.

Gradient Calculation (Continued)

- **Term 3:**

$$\theta^\top \mathbf{X}^\top \mathbf{X}\theta$$

This is a quadratic term in θ.

Using the identity

$$\nabla_\theta(\theta^\top \mathbf{A}\theta) = (\mathbf{A} + \mathbf{A}^\top)\theta$$

and noting that $X^\top X$ is symmetric (i.e., $A^\top = A$), we have:

$$\nabla_\theta(\theta^\top \mathbf{X}^\top \mathbf{X}\theta) = \mathbf{2X}^\top \mathbf{X}\theta$$

2.1.2 Derivation of Normal Equation.

Combined Gradient of the Cost Function

Adding the gradients of all three terms, we get:

$$\nabla_\theta \mathbf{J}(\theta) = \mathbf{0} - \mathbf{2X}^\top \mathbf{y} + \mathbf{2X}^\top \mathbf{X}\theta$$

Simplifying:

$$\nabla_\theta \mathbf{J}(\theta) = -\mathbf{2X}^\top \mathbf{y} + \mathbf{2X}^\top \mathbf{X}\theta$$

2.1.2 Derivation of Normal Equation.

Setting Gradient to Zero and Deriving Normal Equation

To find the optimal θ, set the gradient to zero:

$$-2\mathbf{X}^\top \mathbf{y} + 2\mathbf{X}^\top \mathbf{X}\theta = \mathbf{0}$$

Divide both sides by 2:

$$-\mathbf{X}^\top \mathbf{y} + \mathbf{X}^\top \mathbf{X}\theta = \mathbf{0}$$

Rearranged, this gives the **Normal Equation**:

$$\mathbf{X}^\top \mathbf{X}\theta = \mathbf{X}^\top \mathbf{y}$$

- **Why this works?**
  - The **cost function** $\mathbf{J}(\boldsymbol{\theta})$ **is convex (bowl-shaped)**, so **the gradient zero-point gives the global minimum**.
  - The solution $\boldsymbol{\theta} = \left(\mathbf{X}^{\mathbf{T}}\mathbf{X}\right)^{-1}\mathbf{X}^{\mathbf{T}}\mathbf{y}$ is the **least squares estimator**.

2.1.3 Intuition Behind Normal Equation.

Geometric Intuition Behind the Normal Equation

Geometric Insight:
The normal equation arises from the fact that the optimal θ minimizes the distance between the observed vector y and the predicted vector $X\theta$, which lies in the column space of X.
This means the residual vector $r = y - X\theta$ must be **orthogonal to the column space** of X.
Mathematically, this orthogonality condition is expressed as:

$$X^\top (y - X\theta) = 0$$

This condition ensures that $X\theta$ is the orthogonal projection of y onto the column space of X, and the residual lies in the orthogonal complement (i.e., the left null space of X).

2.1.4 Interpretation of the Solution.

- The solution **θ minimizes** the **least-squares error**.
  - $\mathbf{X^TX}$ must be invertible (i.e., **X must have full column rank**).
  - If not, **regularization** (**like Ridge Regression**) can be used.
- **Advantage:** Direct solution (**no iterative optimization needed**).
- **Disadvantage:** Computationally expensive for large $\mathbf{n}$ $\left(\textbf{since } (\mathbf{X^TX})^{-1} \textbf{ is } O(\mathbf{n^3})\right)$.

2.2 What is the Least Square Solution?

- The least squares method solves the overdetermined system
- $X\theta \approx y$; $X \in \mathbb{R}^{m \times n}$ & $m > n$ by minimizing the sum of squared residuals:
- $J(\theta) = \|y - X\theta\|^2$
- The minimizer θ^* is called the least square solution and is:
- $\theta^* = (X^T X)^{-1} X^T y$
- Key Notes:
- **Uniqueness**: If X has full column rank (rank(X) = n), $X^T X$ is invertible, and θ^* is unique.
- **Degenerate case:** If rank(X) < n
  - infinitely many solutions exist (use pseudoinverses or regularization).

2.2.1 Why call it Least Squares?

1. **Minimizes Squared Error**:
   - $\boldsymbol{\theta}^*$ minimizes $\|\mathbf{y} = \mathbf{X}\boldsymbol{\theta}\|^2$ the sum of squared vertical distances (residuals) between data points and the model predictions.
2. **Geometric Interpretation**:
   - Projects $\mathbf{y}$ onto the **column space of X**, giving the **closest point** $\mathbf{X}\boldsymbol{\theta}^*$ in the subspace.
3. **Statistical Justification**:
   - Under Gaussian noise assumptions, $\boldsymbol{\theta}^*$ **is the maximum likelihood estimator**.

2.2.1 Existence and Uniqueness of Least Squares Solution

Conditions for Existence and Uniqueness of Least Squares Solution

Least Squares Solution:

$$\hat{\theta}_{\text{LS}} = (X^\top X)^{-1} X^\top y$$

Condition for Existence and Uniqueness:

- $X^\top X$ must be **invertible**.
- This requires that X has **full column rank**, i.e., $\text{rank}(X) = n$.
- Geometrically: the columns of X must be **linearly independent**.
- If $X^\top X$ is not invertible (i.e., singular or ill-conditioned), the solution:

$$\hat{\theta}_{\text{LS}} = (X^\top X)^{-1} X^\top y$$

  does not exist in the usual sense.

In such cases, we need a remedy — this is where *regularization* comes in.

2.2.2 Regularized Least Squares

Regularization: Ridge Regression (L2 Regularization)

Motivation: When $X^\top X$ is not invertible or when we want to control model complexity, we add a penalty on the size of the coefficients.

Modified Objective:

$$J_{\text{ridge}}(\theta) = \|y - X\theta\|^2 + \lambda\|\theta\|^2$$

where $\lambda > 0$ is the regularization parameter.

Ridge Solution:

$$\hat{\theta}_{\text{ridge}} = (X^\top X + \lambda I)^{-1} X^\top y$$

Why This Helps:

- $X^\top X + \lambda I$ is always invertible when $\lambda > 0$, even if $X^\top X$ is not.
- Helps prevent overfitting by penalizing large weights.
- Especially useful in high-dimensional or multicollinear data settings.

2.2.3 When Solution exist but is Not Unique.

What Does a Unique Solution Mean?

A **unique solution** is one where there is exactly *one* parameter vector θ that minimizes the residual error.

- No other distinct vector $\theta' \neq \theta$ achieves the same minimal error.
- Ensures stability and interpretability of the model.
- Occurs when the design matrix X has full column rank and $X^\top X$ is invertible.
- Geometrically, the projection of y onto the column space of X corresponds to exactly one $\hat{\theta}$.

2.2.3 When Solution exist but is Not Unique.

When Least Squares Solution Exists but Is Not Unique

Scenario:
A least squares solution always exists, but it may not be unique when:

$$\text{rank}(X) < n \quad \text{(i.e., } X \text{ does not have full column rank)}$$

Implications:

- $X^\top X$ is not invertible (singular matrix).
- There are infinitely many solutions θ that minimize $\|y - X\theta\|^2$.
- The system is underdetermined: multiple θ produce the same projection $X\theta$.

Selecting a Unique Solution:
To obtain a unique solution, we can use:

$$\hat{\theta} = X^+ y \quad \text{(minimum norm solution via Moore–Penrose pseudoinverse)}$$

or apply regularization:

$$\hat{\theta}_{\text{ridge}} = (X^\top X + \lambda I)^{-1} X^\top y \quad \text{(ridge regression with } \lambda > 0\text{)}$$

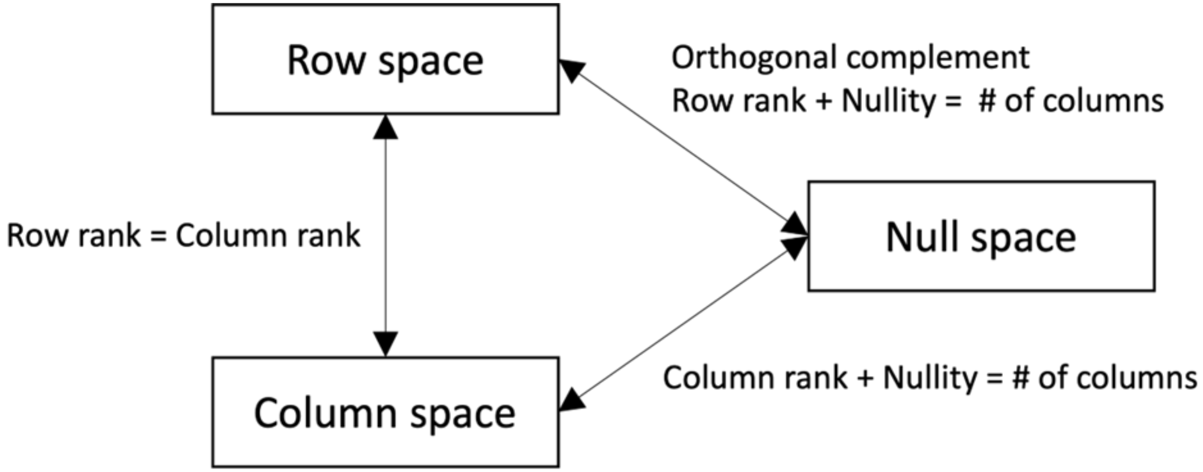
Both methods produce a unique $\hat{\theta}$ that minimizes the residual error.

3. Fundamental Theorem of Linear Algebra.

{ Combining all the fundamental subspaces of Matrix.}

3.1 What is FTLA?

- The **Fundamental Theorem of Linear Algebra (FTLA)** summarizes the relationships between the four fundamental subspaces associated with a matrix.



3.2 Domain vs Range Space.

- In Linear Algebra, if $A \in \mathbb{R}^{m \times n}$, we think of A as a linear transformation:
- $A: \mathbb{R}^n \to \mathbb{R}^m$
- So:
  - Domain → $\mathbb{R}^n$: the space where the input vectors x live.
  - Range/Co-domain → $\mathbb{R}^m$: the space where the output vectors Ax live.
- Fundamental Subspaces in Each space:
- Domain Space $\mathbb{R}^n$
- This is the space of input vectors x, and it contains:
- Row space: $C(A^T) \subseteq \mathbb{R}^n$
- Null space: $\mathcal{N}(A) \subseteq \mathbb{R}^n$
- Together:
  - $\mathbb{R}^{\mathbf{n}} = \mathbf{C}\left(\mathbf{A}^{\mathbf{T}}\right) \oplus \mathbf{N}(\mathbf{A})$

3.2.1 Fundamental Subspaces in Each space:

- **Domain Space $\mathbb{R}^{\mathbf{n}}$:**
  - This is the space of input vectors x, and it contains:
    - Row space: $\mathbf{C}\left(\mathbf{A}^{\mathbf{T}}\right) \subseteq \mathbb{R}^{\mathbf{n}}$
    - Null space: $\boldsymbol{\mathcal{N}}(\mathbf{A}) \subseteq \mathbb{R}^{\mathbf{n}}$
  - Together:
    - $\mathbb{R}^{\mathbf{n}} = \mathbf{C}\left(\mathbf{A}^{\mathbf{T}}\right) \oplus \boldsymbol{\mathcal{N}}(\mathbf{A})$
- **Range/Co-domain Space $\mathbb{R}^{\mathbf{m}}$:**
  - This is the space of **output vectors y = Ax** and it contains:
    - Column Space: $\mathbf{C}(\mathbf{A}) \subseteq \mathbb{R}^{\mathbf{m}}$
    - Left Null Space: $\boldsymbol{\mathcal{N}}\left(\mathbf{A}^{\mathbf{T}}\right) \subseteq \mathbb{R}^{\mathbf{n}}$
  - Together:
    - $\mathbb{R}^{\mathbf{n}} = \mathbf{C}(A) \oplus \boldsymbol{\mathcal{N}}\left(\mathbf{A}^{\mathbf{T}}\right)$

3.3 The Three Standard Statements of the FTLA

- **Statement 1:**
  - The Column Space and Left Null Space Are Orthogonal Complements in $\mathbb{R}^m$:
    - $\mathbf{C(A) \perp \mathcal{N}\left(A^T\right) \text{ and } \mathbb{R}^n = C(A) \oplus \mathcal{N}\left(A^T\right)}$
- **Statement 2:**
  - The Row Space and Null Space Are Orthogonal Complements in $\mathbb{R}^n$:
    - $\mathbf{C\left(A^T\right) \perp \mathcal{N}(A) \text{ and } \mathbb{R}^n = C(A) \oplus \mathcal{N}(A)}$
- **Statement 3:**
  - The Dimensions of the Four Subspaces Are Related by Rank:
    - $\mathbf{dim\big(C(A)\big) = rank(A)}$
    - $\mathbf{dim\left(C\left(A^T\right)\right) = rank(A)}$
    - $\mathbf{dim\big(\mathcal{N}(A)\big) = n - rank(A)}$
    - $\mathbf{dim\left(\mathcal{N}\left(A^T\right)\right) = m - rank(A)}$

Putting them Together.

Range Space and FTAL

Fundamental Theorem of Linear Algebra – Orthogonality and Decomposition (Range Space)

1. **Orthogonality:** $C(A) \perp N(A^\top)$
   - The column space $C(A) \subseteq \mathbb{R}^m$ is orthogonal to the left null space $N(A^\top) \subseteq \mathbb{R}^m$.
   - For $y_1 = Ax \in C(A)$ and $y_2 \in N(A^\top)$:

$$y_1^\top y_2 = x^\top A^\top y_2 = 0 \Rightarrow y_1 \perp y_2$$

2. **Direct Sum:** $\mathbb{R}^m = C(A) \oplus N(A^\top)$
   - Every vector $y \in \mathbb{R}^m$ can be uniquely decomposed as:

$$y = y_c + y_n, \quad \text{with } y_c \in C(A),\ y_n \in N(A^\top)$$

   - Since:

$$\dim(C(A)) + \dim(N(A^\top)) = \text{rank}(A) + (m - \text{rank}(A)) = m$$

   and $C(A) \cap N(A^\top) = \{0\}$

Column Space and FTAL

Fundamental Theorem of Linear Algebra – Orthogonality and Decomposition (Domain Space)

3. Orthogonality: $C(A^\top) \perp N(A)$

- The row space $C(A^\top) \subseteq \mathbb{R}^n$ is orthogonal to the null space $N(A) \subseteq \mathbb{R}^n$.
- For $x_1 \in C(A^\top)$ and $x_2 \in N(A)$, $Ax_2 = 0 \Rightarrow x_1^\top x_2 = 0$

4. Direct Sum: $\mathbb{R}^n = C(A^\top) \oplus N(A)$

- Every vector $x \in \mathbb{R}^n$ can be uniquely written as:

$$x = x_r + x_n, \quad \text{with } x_r \in C(A^\top),\ x_n \in N(A)$$

- Since:

$$\dim(C(A^\top)) + \dim(N(A)) = \operatorname{rank}(A) + (n - \operatorname{rank}(A)) = n$$

and $C(A^\top) \cap N(A) = \{0\}$

Thank You.