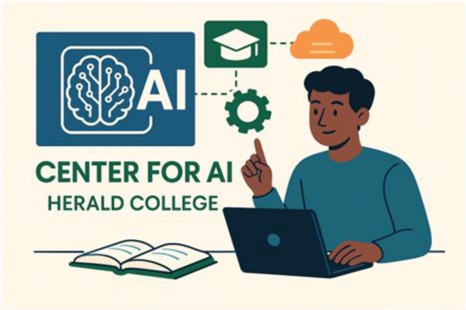
HCAI5TML01 – Mathematics of Learning.

Lecture – 01

A Refresher on the Mathematics Behind Machine Learning.

Vector, Matrix and System of Linear Equations.

Siman Giri



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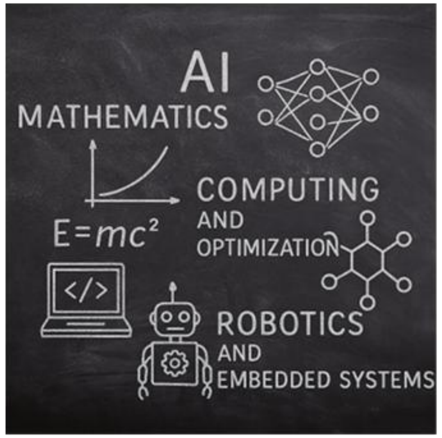
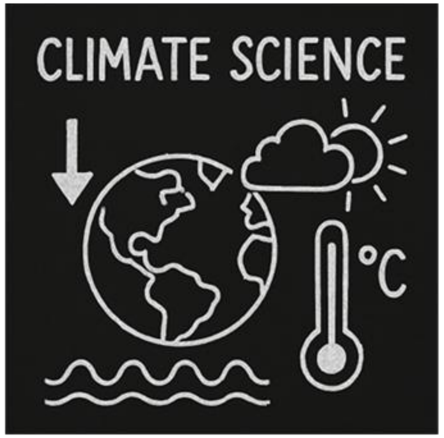
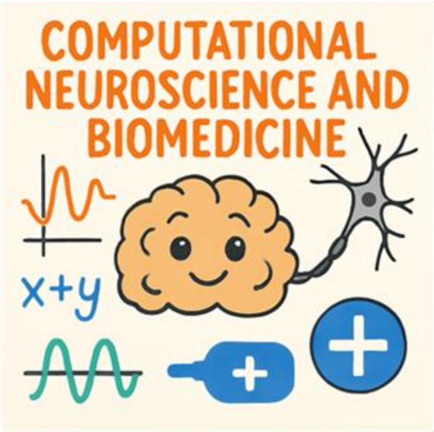
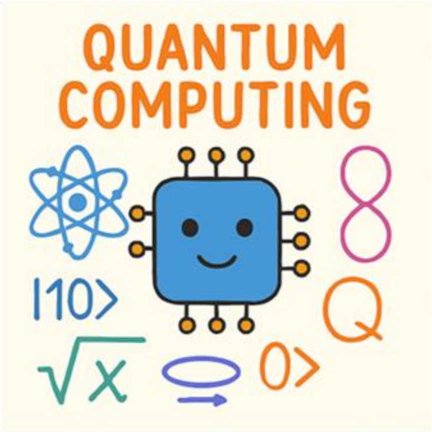








Already Working

Area of Interest …

Coming Soon

A. Why do we need Linear Algebra for ML/DL?

{Why to study Vector and Matrices?}

A.1 What is Linear Algebra?

- **Linear Algebra** is the branch of **mathematics** concerning **linear equations** such as:
  - $\mathbf{a_1 x_1 + \ldots\ldots + a_n x_n = b}$;
  - **linear maps** such as:
    - $(\mathbf{x_1}, \ldots\ldots, \mathbf{x_n}) \mapsto \mathbf{a_1 x_1} + \cdots + \mathbf{a_n x_n}$;
  - and **their representations** in **vector spaces and through matrices.** – **Wikipedia.**
- **Linear algebra** is a branch of mathematics that deals with **vectors, vector spaces** (also **known as linear spaces**),
  - and **linear transformations** between these spaces.
    - It involves operations on matrices and vectors, solving **systems of linear equations**, and understanding geometric concepts like **lines, planes, and subspaces**. – **"chatgpt."**

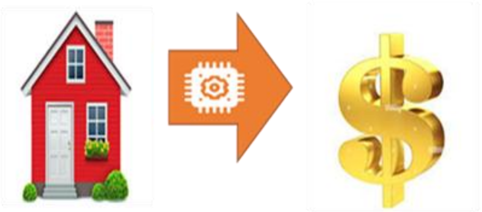
Fig: What is Linear Algebra?

Image: somewhere from web compiled by siman

A.2 Why Linear Algebra for Machine Learning?

- **Representation of Data:**
  - In machine learning, data is typically **represented** as **vectors** and **matrices**. For example, a dataset might be **stored as a matrix** where each row is a data point (vector), and each column is a feature.

Task: House Price Prediction.

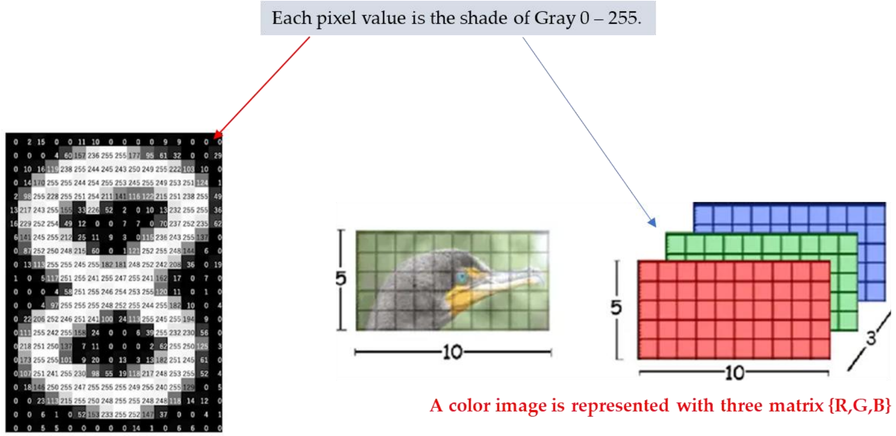


Data: Features/Descriptor of House

Area	Rooms	Price
1080	8	1,00,000.00
1200	10	1,50,000.00

How would you represent this, for computer?
Matrix.

$$\begin{bmatrix} 1080 & 8 \\ 1200 & 10 \end{bmatrix} \begin{bmatrix} 1,00,000 \\ 1,50,000 \end{bmatrix}$$



A color image is represented with three matrix {R,G,B}

A gray scale image is represented with single matrix {R,G,B}

A.2.1 Why Linear Algebra for Machine Learning?

- **Efficient Computing:**
  - Matrix operations allow for efficient computations on large datasets. Libraries like **NumPy**, **TensorFlow**, and **PyTorch** leverage **linear algebra** for operations on large matrices and tensors {**Vectorizations**}, which makes **machine learning models** **faster** and more **scalable**.

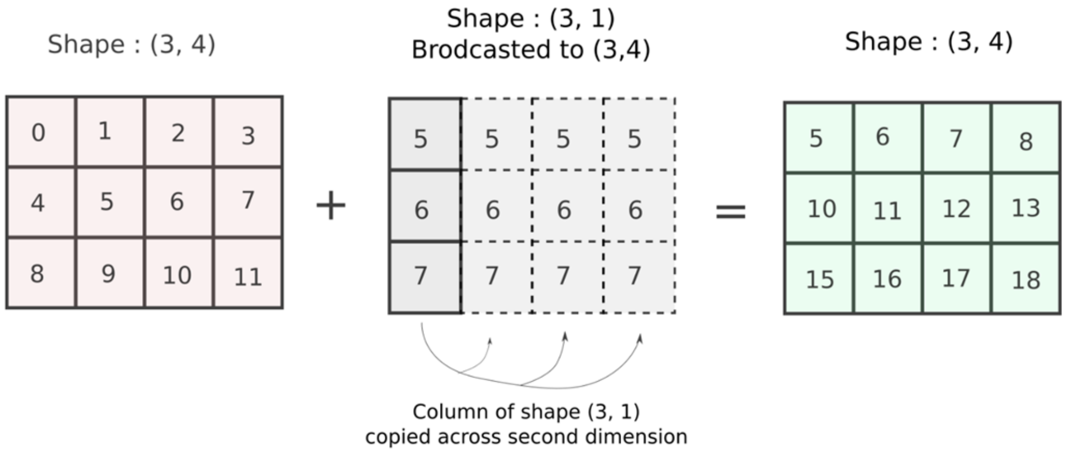


Fig: Idea of Vectorizatons.

A.2.2 Why Linear Algebra for Machine Learning?

- Understanding {Machine or Deep Learning} Algorithms:
  - Training **machine or deep learning models** often involves **solving systems of linear equations**.
    - Linear algebra provides the **necessary tools** to solve these systems efficiently.
  - Many machine learning algorithms are based on linear algebra concepts.
  - For instance:
    - **Linear Regression** involves finding a line (or hyperplane) that best fits the data.
    - **Neural Networks** use matrix multiplication for forward and backward propagation.

B. Summary : Linear Algebra for Machine Learning.

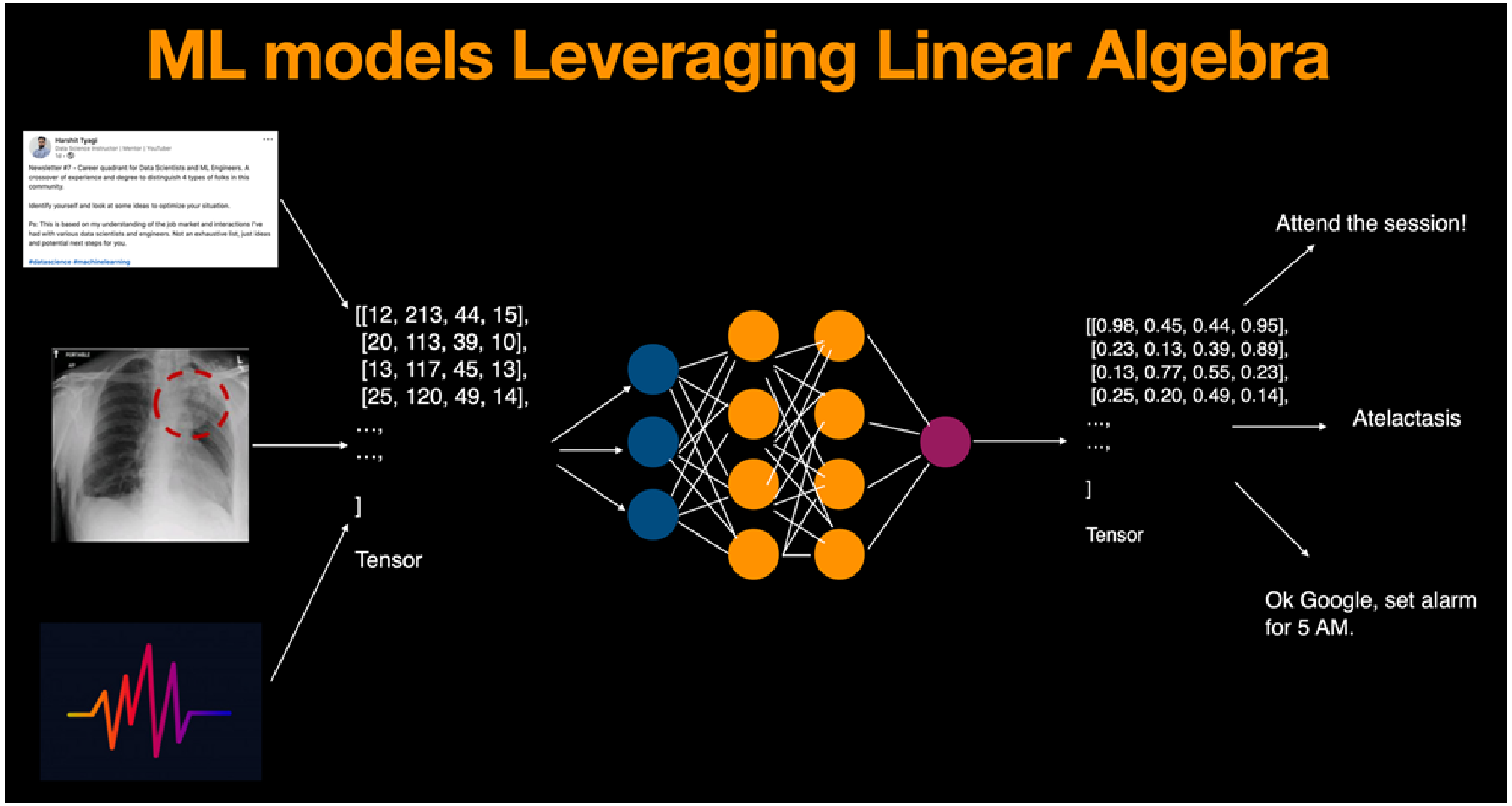


Image By Harshit Tyagi and freeCodeCamp

Understanding Vector and Matrices.

{Basic Concepts, Definition and Notations.}

1.1 What are Vectors?

Interpretation – 1: Point in Space.

- E.g., in **2D{dimension}**
  - we can visualize **the data points** with respect to a **coordinate origin**

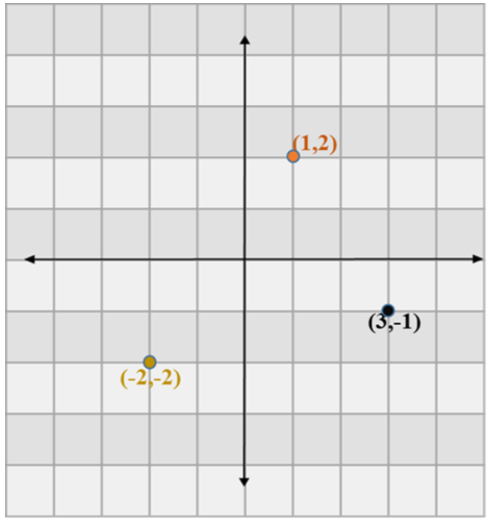


Fig: Vector as a point

Interpretation – 2: Direction in Space.

- E.g., the vector $\vec{\mathbf{v}} = [3, 2]^T$ has a **direction** of **3 steps to the right** and **2 steps up**
- The **notation** $\vec{\mathbf{v}}$ is sometimes used to indicate that the **vectors have a direction**
- All **vectors in the figure** have the **same direction**

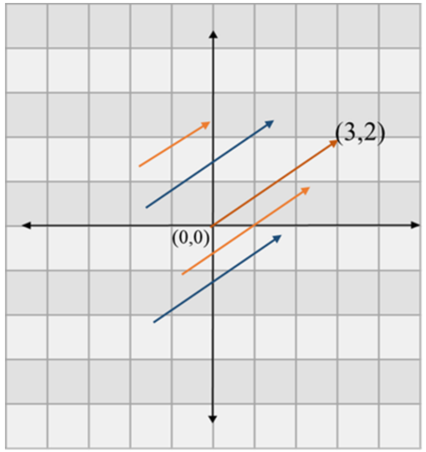


Fig: Vector as Direction

1.2 Vector formal Definition.

- In Linear Algebra and Applied Mathematics, we define vector with in n-dimensional vector space.
- **Vector Space**:
  - If **n** is a positive integer, then an **ordered n-tuple** is a sequence of **n real numbers** $[\mathbf{n_1, n_2, \dots, n_n}]$
  - The set of all **ordered n-tuples** is called **n – space or n – dimensional vector space** and is denoted by $\mathbb{R}^{\mathbf{n}}$.
- **Vectors in** $\mathbb{R}^n$**:**
  - Let $\mathbb{R}^{\mathbf{n}} = \{(\mathbf{x_1, \dots, x_n}) : \mathbf{x_j} \in \mathbb{R} \textbf{ for } \mathbf{j = 1, \dots, n}\}$. Then,
    - $\vec{\mathbf{x}} = [\mathbf{x_1, \dots, x_n}]$ is called **a vector in vector space** $\mathbb{R}^n$.
  - The number $\mathbf{x_j} \rightarrow \mathbf{x_1, \dots x_n}$ are called the **components** of $\vec{\mathbf{x}} \in \mathbb{R}^n$.
- Examples:

$$\mathbf{a} = [\mathbf{a_1, a_2}] \in \mathbb{R}^2$$

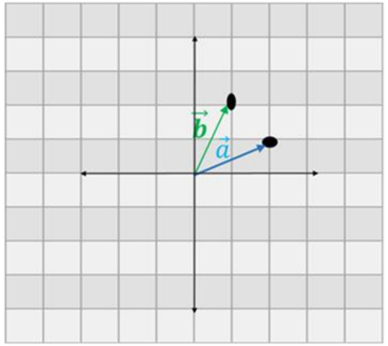


Fig: 2 dimensional vector space

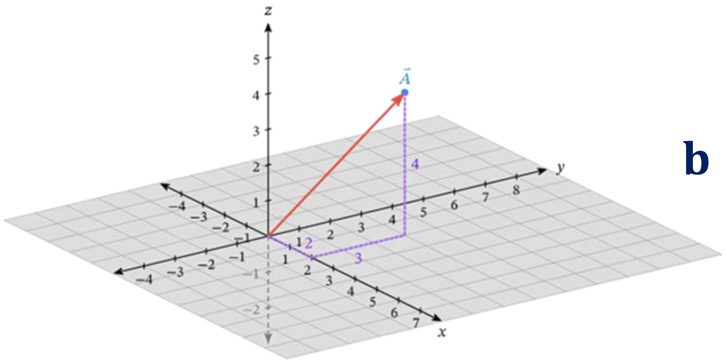


Fig: 3 dimensional vector space

$$\mathbf{b} = [\mathbf{b_1, b_2, b_3}] \in \mathbb{R}^3$$

1.3 Vector in Vector – Space.

- **Vector Space:**
  - A set **V** of ***n*-dimensional vectors** (with a corresponding set of scalars) such that the set of vectors is:
    - **"closed" under vector addition.**
    - **"closed" under scalar multiplication.**
    - **Origins are defined and fixed {0 vector must exist}**
  - In other words:
    - **For addition of two vectors:**
      - takes **two vectors** $\mathbf{u}, \mathbf{v} \in \mathbb{R}^2$, and it produces the **third vector** $\mathbf{u} + \mathbf{v} \in \mathbb{R}^2$.
      - (addition of vectors – gives another vector in the same set)
    - **For scalar Multiplication:**
      - Takes **a scalar c** $\in$ **F** and a vector $\mathbf{v} \in \mathbb{R}^n$ produces a **new vector cv** $\in \mathbb{R}^n$.
      - (multiplying a vector by a scalar – gives another vector in the same set)

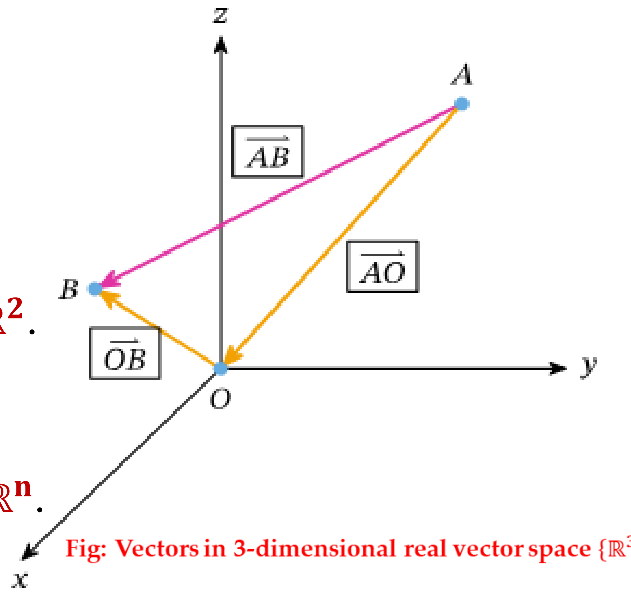


Fig: Vectors in 3-dimensional real vector space $\{\mathbb{R}^3\}$

1.4.1 Axioms of Vector – Space.

- If **V** is a set of vectors satisfying the above definition of a vector space, then it satisfies the following axioms:
  - **Existence of an Additive Identity**: any vector space **V** **must have a zero vector**.
  - **Existence of Negative Vector**: for any vector **v** in **V** its **–ve** must also be in **V**.
  - Has **Athematic /Algebraic Properties** – We can perform valid mathematical operations.

{details in course note}

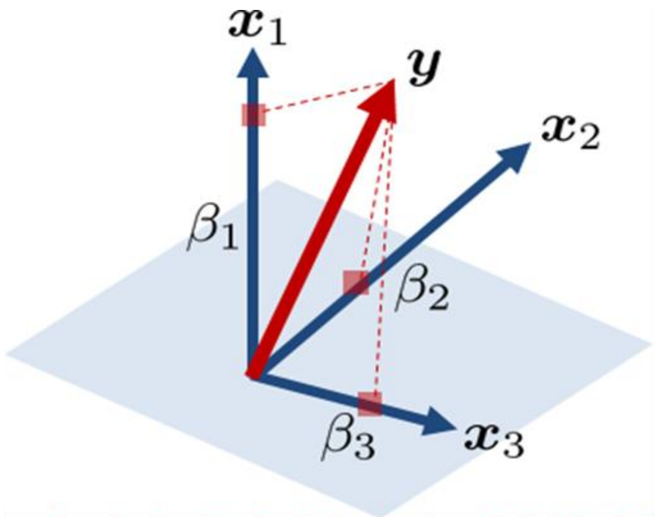


Image from Stanley Chan Book: Introduction to Probability for Data Science.

1.5 Matrices: Introduction.

- In general: A **matrix** is a **rectangular array** of numbers. The **numbers in the array** are called **the entries** in the **matrix**.
    - Array of numbers are an "*ordered collection of vectors*".
    - Like vectors matrices are also fundamentals in machine learning/AI, as matrices are the way computer *interact with data* in practice.
- A **matrix** is represented with *italicized* upper-case letter like "**A**".
    - For two dimensions: we say the matrix **A** has:
        - **m rows** and **n columns**.
        - Each entry/element of **A** is defined as $\mathbf{a_{ij}}$.
    - Thus, a **matrix** $\mathbf{A^{m \times n}}$ is define as:

$$\boxed{A_{m\times n} := \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}, a_{ij} \in \mathbb{R}}$$

- Overview of notation for discussing matrices:
- Given a set $\mathbf{C} \in \mathbb{R}$, we let $\mathbf{C_{m\times n}}$ denote the set of all matrices of **m rows** and **n columns** consisting of items from set **C**.
    - For matrix: $\mathbf{A} \in \mathbf{C_{m\times n}}$: we let $\mathbf{a_{ij}}$ denote the item at the $\mathbf{i^{th}}$ **row** and $\mathbf{j^{th}}$ **column** of **A**.
    - For matrix $\mathbf{A} \in \mathbf{C_{m\times n}}$: we let $\mathbf{a_{i*}}$ denote the $\mathbf{i^{th}}$ **row vector of A**.
    - For matrix $\mathbf{A} \in \mathbf{C_{m\times n}}$: we let a_{*i} denote the $\mathbf{j^{th}}$ **column vector** of **A**.

1.6 Special Matrices.

- Rectangular Matrix:
  - Matrices are said to be rectangular when the number of rows is ≠ to the number of columns, i.e. $\mathbf{A}^{\mathbf{m}\times\mathbf{n}}$ with $\mathbf{m} \neq \mathbf{n}$. For instance:

$$\boxed{A_{2\times3} := \begin{bmatrix} 1 & 2 & 3 \\ 5 & 5 & 4 \end{bmatrix}}$$

- Square Matrix:
  - Matrices are said to be square when the number of rows = the number of columns, i.e. $\mathbf{A}^{\mathbf{m}\times\mathbf{n}}$. For instance:

$$\boxed{A_{2\times2} := \begin{bmatrix} 1 & 3 \\ 2 & 5 \end{bmatrix}}$$

- Diagonal Matrix:
  - Square matrices are said to be diagonal when each of its non-diagonal elements is zero, i.e. for
    - $\mathbf{D} = (\mathbf{d_{ij}})$,we have $\forall \mathbf{i}, \mathbf{j} \in \mathbf{n}\ \mathbf{i} \neq \mathbf{j} \Rightarrow \mathbf{d_{ij}} = \mathbf{0}$.
  - For instance:

$$\boxed{A_{3\times3} := \begin{bmatrix} 9 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 5 \end{bmatrix}}$$

- Upper triangular matrix:
  - Square matrices are said to be upper triangular when the elements below the main diagonal are zero i.e. For $\mathbf{D} = (\mathbf{d_{ij}})$, we have $\mathbf{d_{ij}} = \mathbf{0}$, for $\mathbf{i} > \mathbf{j}$. For instance:

$$\boxed{A_{3\times3} := \begin{bmatrix} 9 & 8 & 4 \\ 0 & 1 & 3 \\ 0 & 0 & 5 \end{bmatrix}}$$

- Lower triangular matrix:
  - Square matrices are said to be lower triangular when the elements above the main diagonal are zero . i.e. $\mathbf{D} = (\mathbf{d_{ij}})$, we have $\mathbf{d_{ij}} = \mathbf{0}$, for $\mathbf{i} < \mathbf{j}$. For instance:

$$\boxed{A_{3\times3} := \begin{bmatrix} 9 & 0 & 0 \\ 8 & 1 & 0 \\ 4 & 2 & 5 \end{bmatrix}}$$

- Identity Matrix:
  - A diagonal matrix is said to be the identity when the elements along its main diagonal are equal to one. For instance:

$$\boxed{A_{3\times3} := \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}}$$

1.6.1 Special Matrices.

- **Symmetric Matrix:**
  - Square matrices are said to be symmetric its equal to its transpose, i.e. $\mathbf{A} = \mathbf{A}^{\mathbf{T}}$. For instance:

$$A_{3\times3} := \begin{bmatrix} 1 & 2 & 3 \\ 2 & 1 & 6 \\ 3 & 6 & 1 \end{bmatrix}$$

- **Scalar Matrix**:
  - Diagonal matrices are said to be scalar when all the elements along its main diagonal are equal, i.e. $\mathbf{D} = \boldsymbol{\alpha}\mathbf{I}$. For instance:

$$A_{3\times3} := \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix}$$

- **Null or Zero Matrix:**
  - Matrices are said to be null or zero matrices when all its elements equal to zero, which is denoted as $\mathbf{0}_{\mathbf{m}\times\mathbf{n}}$. For instance:

$$A_{3\times3} := \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

- **Equal Matrix:**
  - Two matrix are said to be equal if
    - $\mathbf{A}(\mathbf{a}_{\mathbf{ij}}) = \mathbf{B}(\mathbf{b}_{\mathbf{ij}})$.
  - For instance:

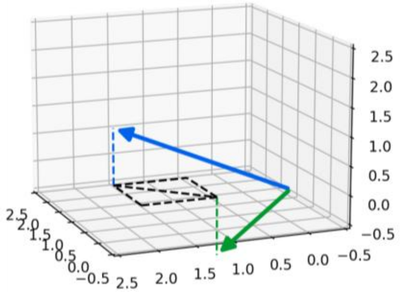
$$B_{2\times2} := \begin{bmatrix} 1 & 3 \\ 2 & 5 \end{bmatrix}$$

$$A_{2\times2} := \begin{bmatrix} 1 & 3 \\ 2 & 5 \end{bmatrix}$$

1.7 Interpretation of a Matrix: Collection of Vectors.

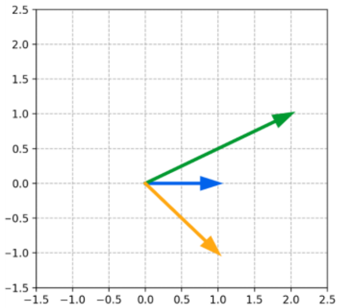
- A matrix can be thought of as a set of vectors.
- For example, for the following matrix:
  - $\mathbf{A} \coloneqq \begin{bmatrix} \mathbf{1} & \mathbf{2} & \mathbf{1} \\ \mathbf{0} & \mathbf{1} & -\mathbf{1} \end{bmatrix}$ can be thought of as
    - a **two** **three-dimensional** row vectors i.e.
    - $\boldsymbol{a}_{1*} \coloneqq [\mathbf{1} \quad \mathbf{2} \quad \mathbf{1}]$ and $\boldsymbol{a}_{2*} \coloneqq [\mathbf{0} \quad \mathbf{1} \quad -\mathbf{1}]$

$$\begin{bmatrix} 1 & 2 & 1 \\ 0 & 1 & -1 \end{bmatrix}$$



    - Or as a **three** **two-dimensional** column vectors:
    - $\mathbf{a}_{*1} \coloneqq \begin{bmatrix} \mathbf{1} \\ \mathbf{0} \end{bmatrix}$; $\mathbf{a}_{*2} \coloneqq \begin{bmatrix} \mathbf{2} \\ \mathbf{1} \end{bmatrix}$ and $\mathbf{a}_{*3} \coloneqq \begin{bmatrix} \mathbf{1} \\ -\mathbf{1} \end{bmatrix}$

$$\begin{bmatrix} 1 & 2 & 1 \\ 0 & 1 & -1 \end{bmatrix}$$



1.7.1 Interpretation of Matrix: As a table of data.

- The simplest interpretation of matrix is as a two – dimensional array of values.
- For example:
  - A numerical dataset represented as a matrix.

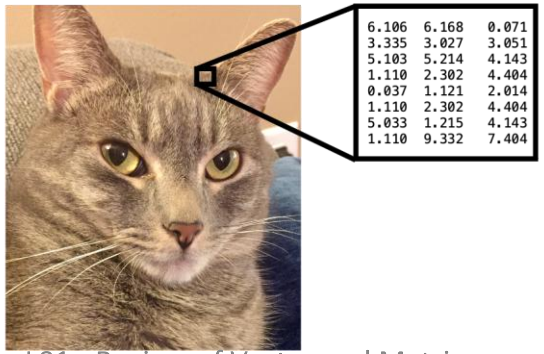
Spreadsheet

	A	B	C	D
1	sepal_length	sepal_width	petal_length	petal_width
2	5.1	3.5	1.4	0.2
3	4.9	3	1.4	0.2
4	7	3.2	4.7	1.4
5	6.5	2.8	4.6	1.5
6	5.8	2.7	5.1	1.9
7	7.1	3	5.9	2.1

Matrix

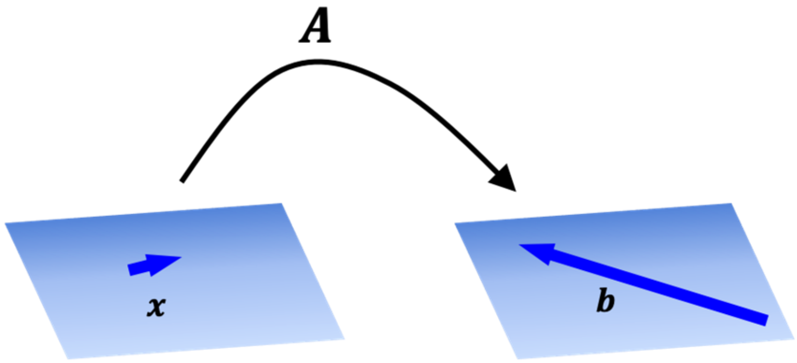
$$\begin{bmatrix} 5.1 & 3.5 & 1.4 & 0.2 \\ 4.9 & 3.0 & 1.4 & 0.2 \\ 7.0 & 3.2 & 4.7 & 1.4 \\ 6.5 & 2.8 & 4.6 & 1.5 \\ 5.8 & 2.7 & 5.1 & 1.9 \\ 7.1 & 3.0 & 5.9 & 2.1 \end{bmatrix}$$

  - The **pixels** of an image can be represented as a matrix.
  - Let's say we have an image of **m × n pixels**.
    - Let **X** be a **matrix** representing this image where $x_{i,j}$ represents the **intensity of the pixel** at row **i** and **j**.



1.7.2 Interpretation of Matrix: As a Function.

- A matrix can also be viewed as a function **that maps**
  - **vectors in one vector space** to **vectors in another vector space.**
- These special kind of **matrix – defined function** are also called
  - **Linear Transformation** and written as:
    - $\mathbf{T(x) \coloneqq Ax}$
- A very simple visualization of such function is **matrix – vector multiplication**.

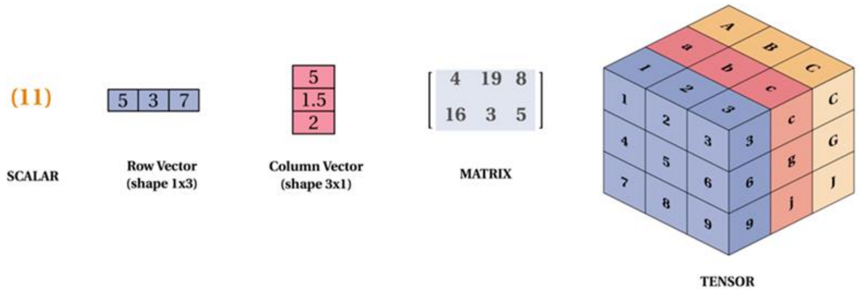


$$A.X \coloneqq \begin{bmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{m1} & \cdots & a_{mn} \end{bmatrix} \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} = x_1 \begin{bmatrix} a_{11} \\ \vdots \\ a_{m1} \end{bmatrix} + x_2 \begin{bmatrix} a_{12} \\ \vdots \\ a_{m2} \end{bmatrix} + x_n \begin{bmatrix} a_{1n} \\ \vdots \\ a_{mn} \end{bmatrix} = \begin{bmatrix} y_1 \\ \vdots \\ y_{mn} \end{bmatrix}$$

Fig: What happens if we Multiply Matrix A with vector x?

Good to Know!!!

- A tensor is a **multidimensional array** and a **generalization** of the concepts of a **vector and a matrix.**



- Tensors can have many axes, here is a tensor with three axes:

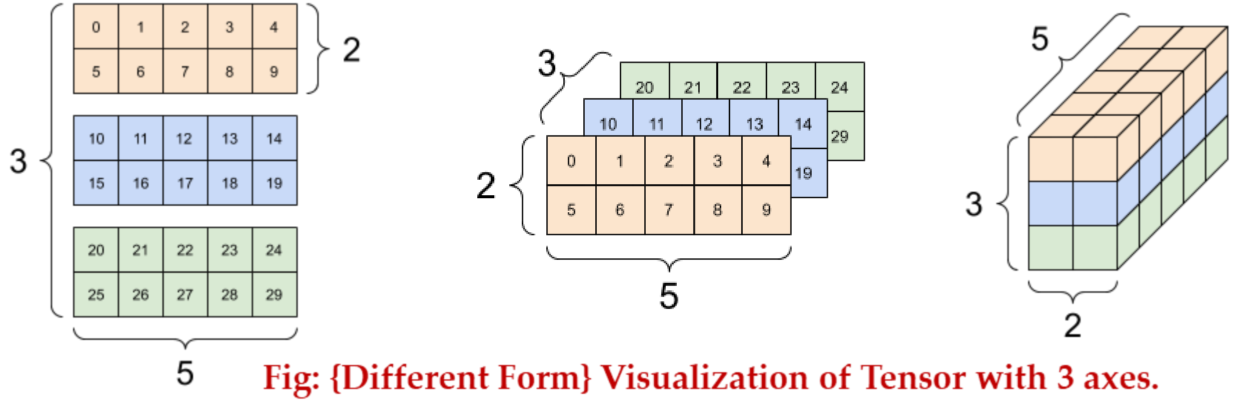
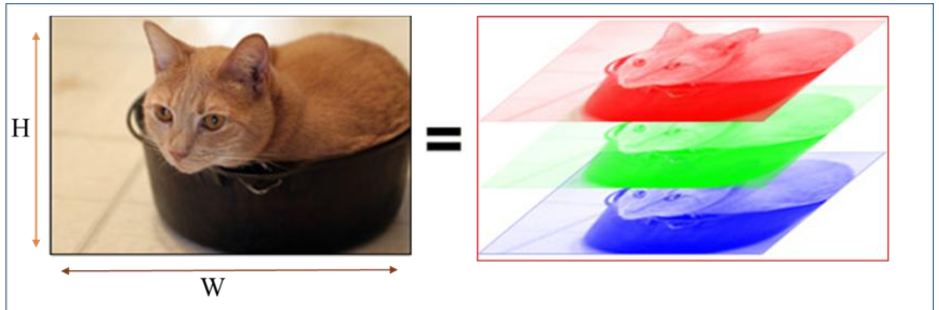


Fig: {Different Form} Visualization of Tensor with 3 axes.

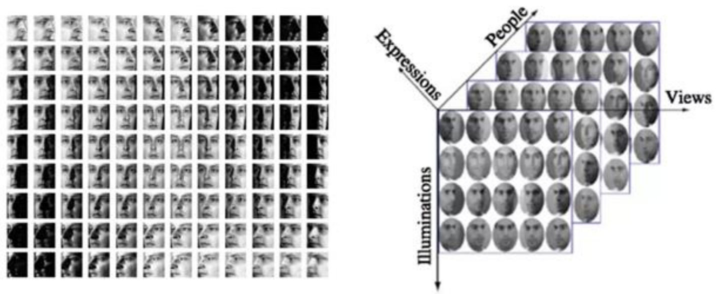
Tensor → Example.

- Tensors in DL are Used to represent an image.
  - image_shape := Height × Width × Color Channel (RGB)



A color image is composed of 3 channels and is 3rd rank tensors.

facial images database is 6th-order tensor



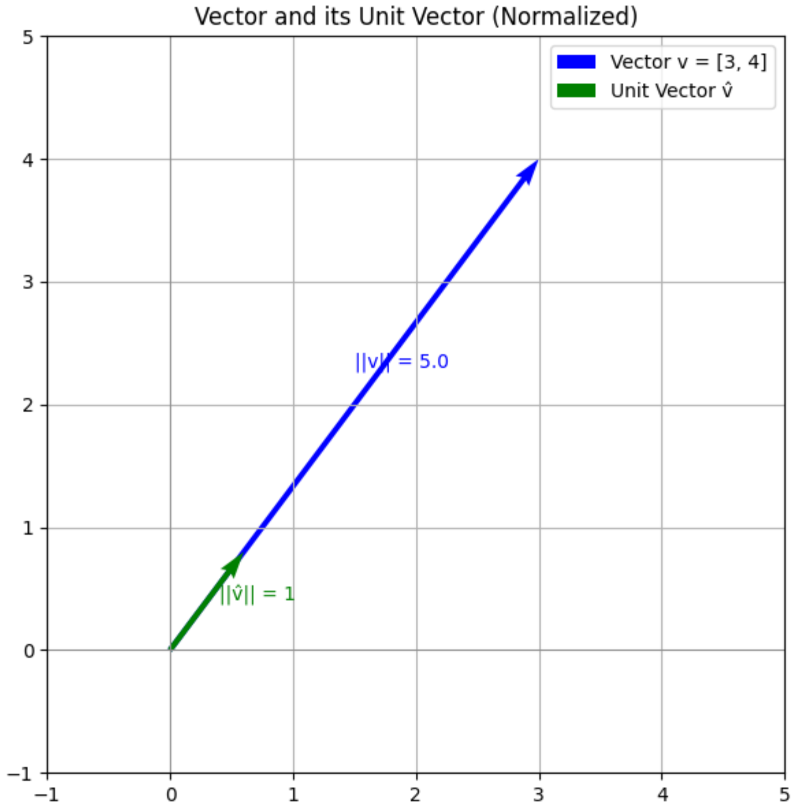
color video is 4th-order tensor

2. The Geometry of Vectors.

{Operations, Linear Dependence, and Basis}

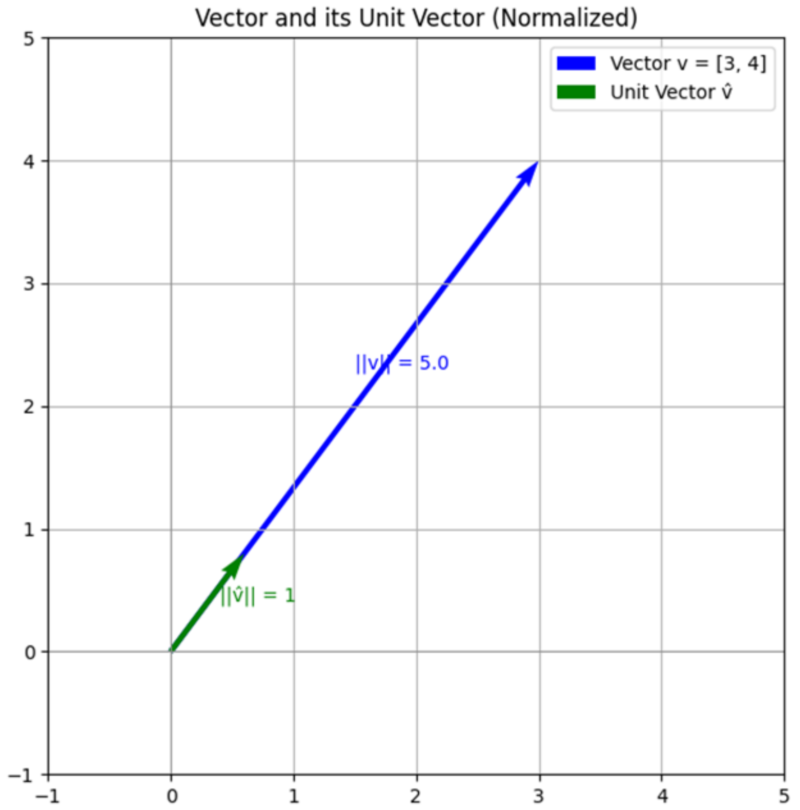
2.1 Norm – "Length" of a vector.

- The **norm** of a **vector v,** often written as **$\|\mathbf{v}\|$,** is a measure of its magnitude or length.
- For vectors in $\mathbb{R}^n$, the most common is the Euclidean norm aka L2 Norm:
  - $\|\mathbf{v}\|_2 = \sqrt{\mathbf{v}_1^2 + \mathbf{v}_2^2 + \cdots + \mathbf{v}_n^2} = \sqrt{\langle \mathbf{v}, \mathbf{v} \rangle}$



2.2 Unit Vector.

- A **unit vector** is any vector **whose norm is exactly 1:**
  - $\|\mathbf{u}\| = 1$
- Why unit vectors matter?
  - They capture direction only, no magnitude.
  - They are building blocks of basis vectors.
  - Useful in normalizations ,projections and angle calculations.
- How to Get a unit vector from any vector – **Normalization?**
  - You can normalize a vector by dividing it by its own norm:
    - $\hat{\mathbf{v}} = \frac{\mathbf{v}}{\|\mathbf{v}\|}$
  - This gives a unit vector in the same direction as v.



2.3 Inner Product (General Concept).

- An inner product is a general mathematical concept that defines a **way to compute an angle – like and length – like** relationship between two vectors.
  - Formally, an inner product on a **vector space V** is a function:
    - $\langle . , . \rangle : \mathbf{V} \times \mathbf{V} \rightarrow \mathbb{R}$
  - that satisfies these properties:
  - **Linearity in the first argument:**
    - $\langle \alpha \mathbf{u} + \mathbf{b}\mathbf{v}, \mathbf{w} \rangle = \alpha \langle \mathbf{u}, \mathbf{w} \rangle + \mathbf{b} \langle \mathbf{v}, \mathbf{w} \rangle$
    - Taking the inner product of a linear combination of vectors u and v with another vector w, is the same as taking the same linear combination of their inner products with w.
  - **Symmetry:**
    - $\langle \mathbf{u}, \mathbf{v} \rangle = \langle \mathbf{v}, \mathbf{u} \rangle$
  - **Positive – definiteness:**
      - $\langle \mathbf{v}, \mathbf{v} \rangle \geq \mathbf{0}$, **and equals 0 only if** $\mathbf{v} = \mathbf{0}$
    - When you take the inner product of a vector with itself, you always get a **non-negative number until and unless itself is the zero vector.**
    - This property ensures that the inner product defines a **valid notion of length** (or norm), because:
      - $\|\mathbf{v}\| = \sqrt{\langle \mathbf{v}, \mathbf{v} \rangle}$

2.3.1 Why Non – Negativity Matters?

- The inner product $\langle \mathbf{v}, \mathbf{v} \rangle$ is used to define the norm (length) of a vector: $\|\mathbf{v}\| = \sqrt{\langle \mathbf{v}, \mathbf{v} \rangle}$
- Now imagine what would happen if $\langle \mathbf{v}, \mathbf{v} \rangle < \mathbf{0}$**:**
  - You'd have to take the square root of a **negative number**.
  - That would **break the geometry** of the space
    - distances and lengths would **become imaginary or undefined in** $\mathbb{R}$**.**
  - Concepts like distance, angle, and orthogonality would stop making sense
    - in the real – valued world of machine learning.
- Therefore:
  - The **positive – definiteness condition** ensures that the **norm or length of any vector is a real**, non – negative number something we can safely interpret geometrically.

2.3.2 Understanding Dot Products.

- **Dot product:**
  - Given two vectors $\mathbf{u}, \mathbf{v} \in \mathbb{R}^n$, the quantity $\mathbf{u}^T\mathbf{v}$, sometimes called the **inner product** or **dot product** of the vectors, is a real number given by:
  - $\mathbf{u}^T\mathbf{v} \in \mathbb{R} = [\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_n] \cdot \begin{bmatrix} \mathbf{v}_1 \\ \mathbf{v}_2 \\ \ldots \\ \mathbf{v}_n \end{bmatrix} = \sum_{i=1}^{n} \mathbf{u}_i \times \mathbf{v}_i$
- It satisfies all the inner product properties - so the dot product is **special case of inner product**
  - but not all **inner products are dot products.**
- **Orthogonal Vectors:**
  - A pair of vectors **u and v** are **orthogonal** if their **dot product** is zero
    - i.e. $\langle \mathbf{u}, \mathbf{v} \rangle = \mathbf{0}.$
  - Notation for a pair of orthogonal vectors is $\mathbf{u} \perp \mathbf{v}$ {i.e. **Vector are perpendicular to each other**}.
  - In the $\mathbb{R}^n$; this is equal to pair of vector forming a $\mathbf{90^0}$ angle.

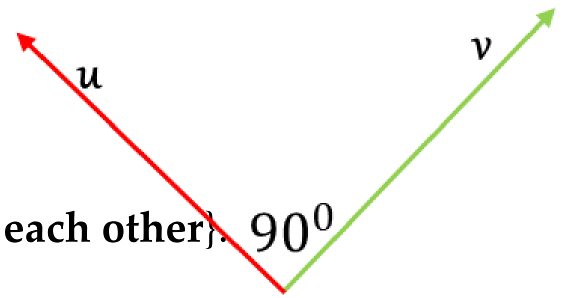
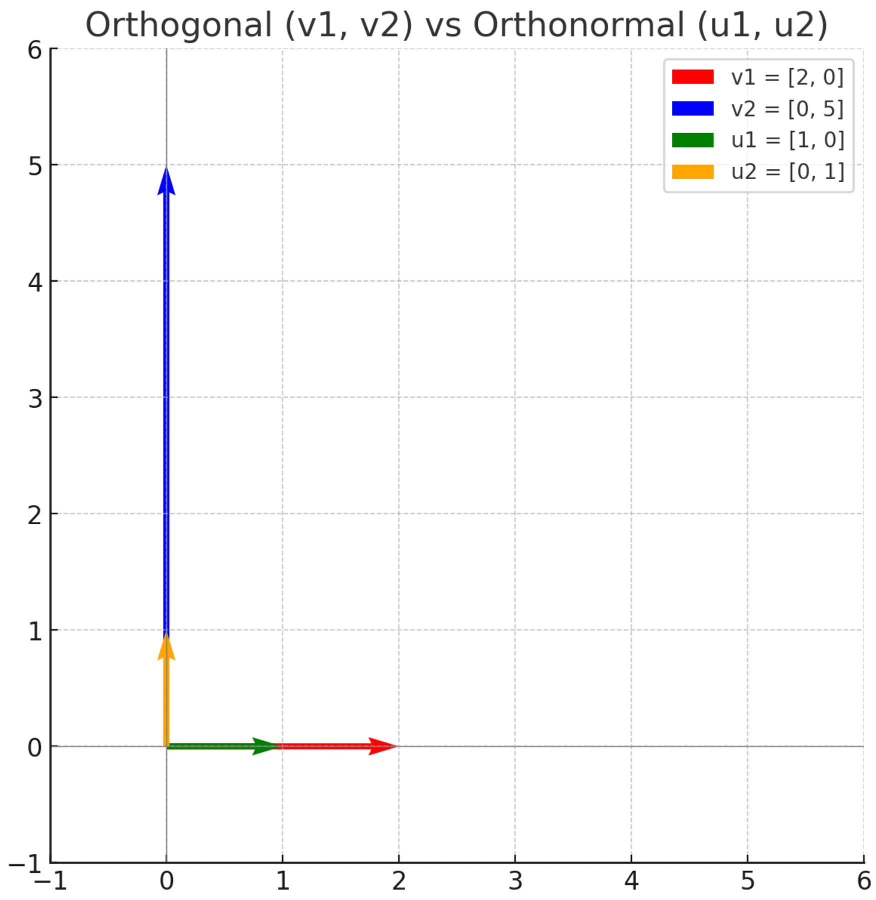


Fig: Orthogonal Vectors

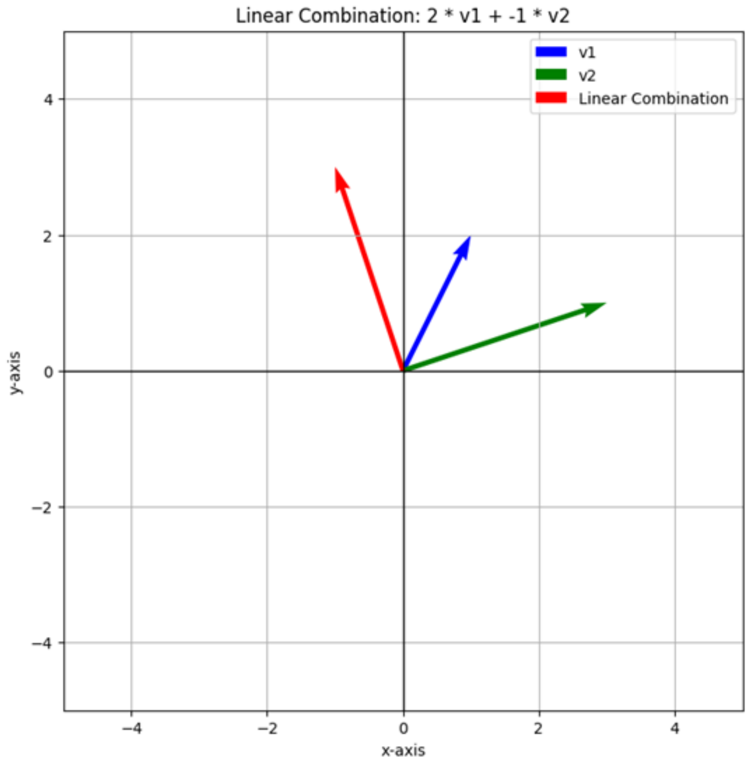
2.3.3 Orthogonal vs. Orthonormal.

- A set of vectors is orthogonal if: $\langle \mathbf{v_i}, \mathbf{v_j} \rangle = \mathbf{0}$ **for all i** $\neq$ **j**
  - They are **perpendicular**, But not **necessarily unit length**.
- **Orthonormal:**
  - A set of vectors is orthonormal if:
    - **They are orthogonal i.e.** $\langle \mathbf{v_i}, \mathbf{v_j} \rangle = \mathbf{0}$ **for all i** $\neq$ **j**
    - **Each vector has unit length:** $\|\mathbf{v_i}\| = \mathbf{1}$ **for all i**
  - **Example in** $\mathbb{R}^2$:
    - **Orthogonal but not orthonormal:**
      - $\mathbf{v_1} = [\mathbf{2}, \mathbf{0}]$
 $\mathbf{v_2} = [\mathbf{0}, \mathbf{5}]$
        - $\langle \mathbf{v_1} \cdot \mathbf{v_2} \rangle = \mathbf{0}$ → **orthogonal**
        - **But** $\|\mathbf{v_1}\| = \mathbf{2}, \|\mathbf{v_2}\| = \mathbf{5}$ → **not unit length**
    - **Orthonormal:**
      - $\mathbf{u_1} = [\mathbf{1}, \mathbf{0}]$ $\quad \mathbf{u_2} = [\mathbf{0}, \mathbf{1}]$
      - **Perpendicular and each has length 1.**



2.4 Linear Combinations of Vectors.

- Idea Combining two or more than two vectors to form a new vector.
- Definition:
  - A vector v is a linear combination of a set of vectors $\mathbf{v_1}, \mathbf{v_2}, \dots, \mathbf{v_n}$, if it can be expressed as:
    - $\mathbf{v} = \mathbf{c_1 v_1} + \mathbf{c_2 v_2} + \cdots + \mathbf{c_n v_n}$
      - where:
        - $\mathbf{c_1}, \mathbf{c_2}, \dots, \mathbf{c_n}$ are scalars (**coefficients**).
        - $\mathbf{v_1}, \mathbf{v_2}, \dots, \mathbf{v_n}$ are vectors in a vector space.
- Example in $\mathbb{R}^2$:
  - Let $\mathbf{v_1} = [1, 2]$ and $\mathbf{v_2} = [3, 1]$,
  - If we take scalars $\mathbf{c_1} = \mathbf{2}$ and $\mathbf{c_2} = \mathbf{-1}$,
    - then their linear combination will
      - produce a new *vector v* in same **vector space**.
      - $\mathbf{v} = \mathbf{2} \times \begin{bmatrix} \mathbf{1} \\ \mathbf{2} \end{bmatrix} + (\mathbf{-1}) \times \begin{bmatrix} \mathbf{3} \\ \mathbf{1} \end{bmatrix}$ ■

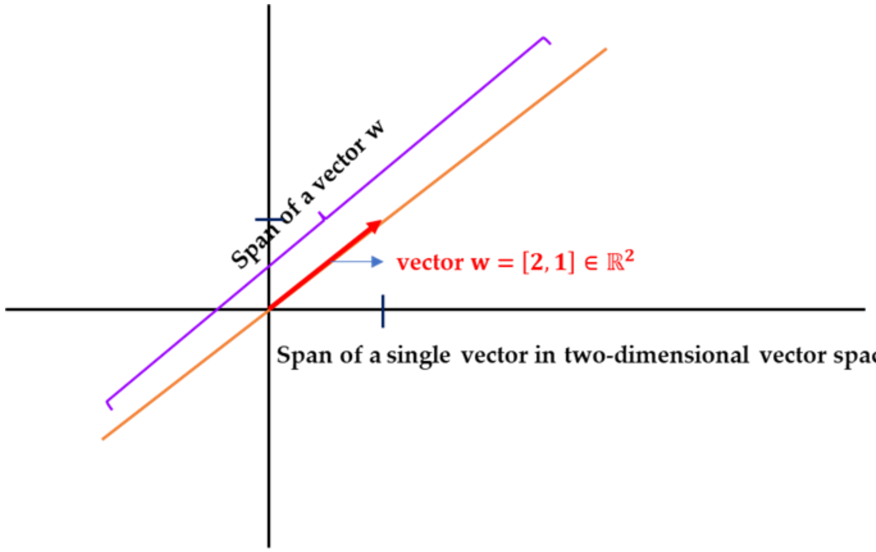


2.5 Span of a Set of vectors.

- **Span** is a consequences of **Linear combination of vectors** and can be thought as a **subset** inside a **vector space** (also known as **vector subspace**).
- A subspace, $\mathbb{S}$ **of real vector space** $\mathbb{R}^{\mathbf{n}}$ is thought of a **flat (having no curvature) surface** with in $\mathbb{R}^{\mathbf{n}}$:
  - is a collection of **all the vectors in** $\mathbb{S}$ which satisfies the following (algebraic) conditions:
    - The ***origin* (0 *vector*)** is contained in $\mathbb{S}$.
    - If vector $\boldsymbol{v}_1$ **and** $\boldsymbol{v}_2$ are in $\mathbb{S}$; then $\boldsymbol{v}_1 + \boldsymbol{v}_2 \in \mathbb{S}$.
    - If $\boldsymbol{v}_1 \in \mathbb{S}$ and $\boldsymbol{\alpha}$ a scalar then $\boldsymbol{\alpha v}_1 \in \mathbb{S}$.
- The **span of a set of vectors** $\{\boldsymbol{v}_1, \boldsymbol{v}_2, \dots, \boldsymbol{v}_n\} \in \mathbb{R}^{\boldsymbol{n}}$ is the **set** of all **possible linear combinations** of **those vectors**. Formally, the span of $\{\boldsymbol{v}_1, \boldsymbol{v}_2, \dots, \boldsymbol{v}_n\}$ is:
  - $\boldsymbol{span}(\boldsymbol{v}_1, \boldsymbol{v}_2, \dots, \boldsymbol{v}_n) = \{\boldsymbol{c}_1\boldsymbol{v}_1 + \boldsymbol{c}_2\boldsymbol{v}_2 + \dots + \boldsymbol{c}_n\boldsymbol{v}_n | \boldsymbol{c}_1, \boldsymbol{c}_2, \dots, \boldsymbol{c}_n \in \mathbb{R}\}$
    - where $\boldsymbol{c}_1, \boldsymbol{c}_2, \dots, \boldsymbol{c}_n$ are **scalar coefficients**.

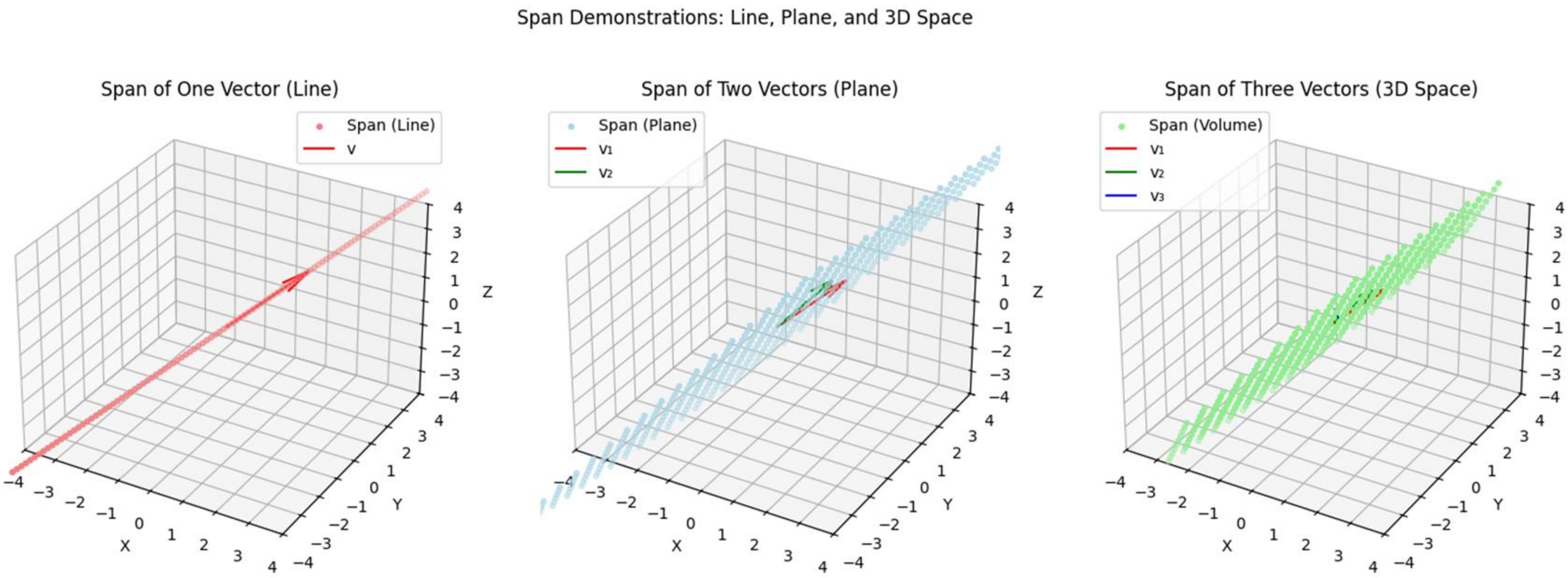
2.5.1 Geometric Interpretation of a Span.

- **The span of one nonzero vector is a line** through the origin in the direction of that vector.
- **The span of two linearly independent vectors is a plane** through the origin.
- **The span of three linearly independent vectors in $\mathbb{R}^3$ is the entire 3D space**, which you can think of as filling a volume (like a cube).



Span of a single vector in two-dimensional vector space is a line.

2.5.1 Geometric Interpretation of a Span.



2.5.2 Span and Learning.

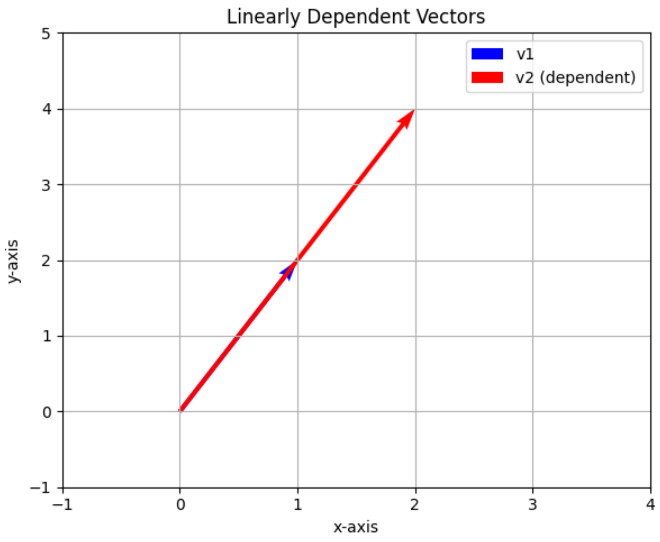
- **Feature Representation and Dimensionality:**
  - Each data point is often represented as a vector in a feature space.
  - The **span of your features** essentially defines the **space where your data lives**.
  - If your features are **linearly dependent** (i.e., don't add new "directions"), your data effectively lies in a **lower-dimensional subspace**.
  - This is why **dimensionality reduction** techniques (like PCA) look for a **smaller spanning set (basis)** that **captures most of the variance**.
- **Model Expressiveness:**
  - Linear models like **linear regression** find **weights** that are **linear combinations of features**.
  - The **predictions** lie in the **span of the feature vectors** (plus bias).
  - Understanding the span helps explain whether your model can represent the target well (e.g., if the target lies in the span of your features).
- **Feature Span and Model Weights:**
  - Your features form a vector space; the model's predictions are linear combinations of these features.
  - When features are **linearly dependent or highly correlated**, the effective **span** of your features is "smaller" than the number of features suggests.
  - This can cause the model weights to become **unstable or non-unique, hurting generalization.**

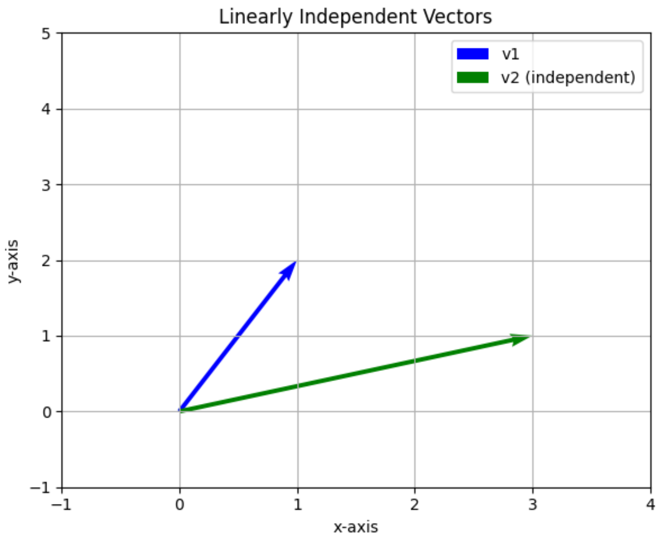
2.5.2 Span and Learning.

- **Regularization and Overfitting:**
  - When features are **highly correlated (dependent),** models **can overfit** by giving **extreme weights** to **redundant features.**
  - Understanding span and linear dependence helps motivate **regularization** techniques like Ridge or Lasso to constrain weights.
  - **Lasso (L1 Regularization):**
    - Pushes some feature coefficients **exactly to zero**, effectively **removing those features** from the model. This reduces the **dimension of the span** of the selected features, creating a smaller subspace that still explains the data well.
  - **Ridge (L2 regularization):**
    - Shrinks the coefficients of correlated features **towards zero but rarely exactly zero**, so it **keeps all features** but reduces their impact.
    - This controls instability caused by overlapping spans without changing the span's dimension.

2.6 Linearly Independent and Dependent Vectors.

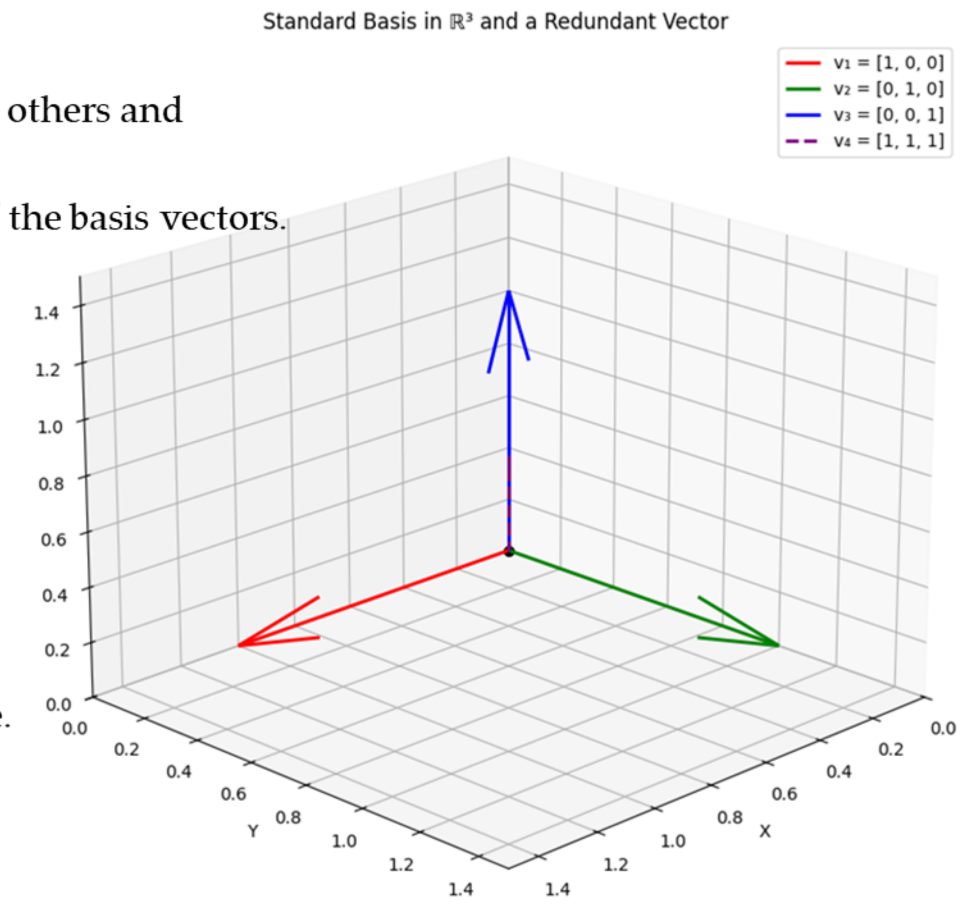
- A set of vectors $v_1, v_2, \ldots, v_n$ in a **vector space** $\mathbb{R}^n$ is:
  - **Linearly dependent** if at least one vector can be written as a linear combination of the others.
    - Mathematically, this means there exists at least one scalars $c_1, c_2, \ldots, c_n$ which is not zero, such that:
      - $\mathbf{c_1 v_1 + c_2 v_2 + \cdots c_n v_n = 0}$; **at least one** $\mathbf{c \neq 0}$.
  - **Linearly Independent** if the only possible solution for above equation is $c_1 = c_2 = \cdots = c_n = 0$, i.e. no vector set can be written as a combination of the others.





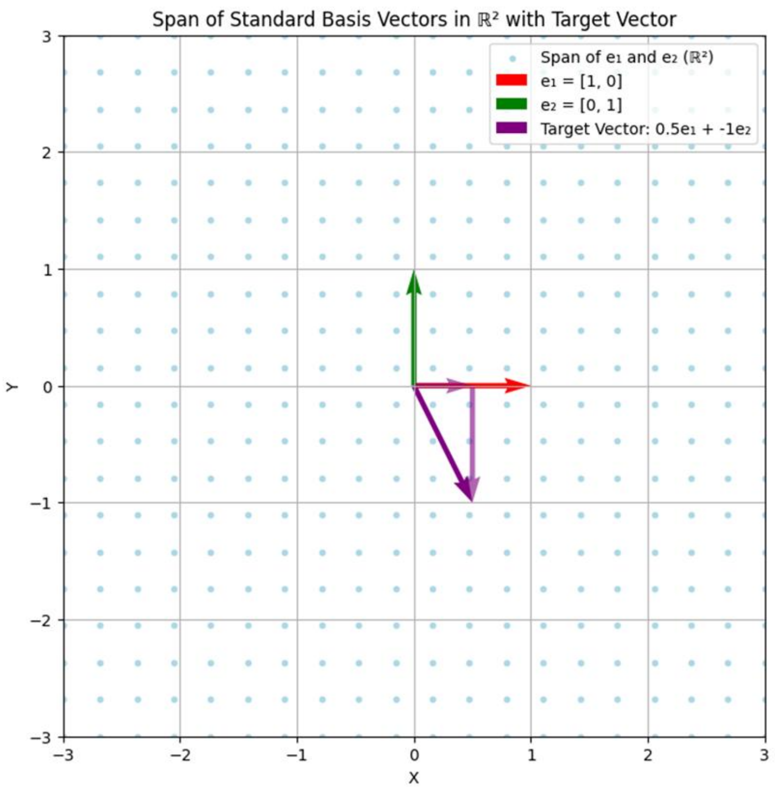
2.7 Basis of a Vector Space.

- Definition:
  - A basis of a **vector space V** is a **set of vectors:** $\{\mathbf{v_1}, \mathbf{v_2}, \dots, \mathbf{v_k}\}$ such that:
    - The vectors **are linearly independent**,
      - i.e. No vector in a basis can be written as a combination of the others and
    - They **span the space V,**
      - i.e. every vector in V can be written as a linear combination of the basis vectors.
  - Basis is the minimal set needed to span the entire vector space.
- **Example in $\mathbb{R}^3$:**
  - Following three vectors forms the basis in $\mathbb{R}^3$ vector space
    - $\mathbf{B} \coloneqq \{\mathbf{v_1} = [1, 0, 0], \mathbf{v_2} = [0, 1, 0], \mathbf{v_3} = [0, 0, 1]\}$ as they are:
    - Linearly independent and
    - There span is all of $\mathbb{R}^3$
  - If you add a fourth vector (e.g. $v_4 = [1,1,1]$),
    - $\mathbf{B} \coloneqq \{\mathbf{v_1} = [1, 0, 0], \mathbf{v_2} = [0, 1, 0], \mathbf{v_3} = [0, 0, 1], \boldsymbol{v_4} = [1, 1, 1]\}$
    - you still span $\mathbb{R}^3$, but now it not minimal, Thus not a basis anymore.



2.7.1 Basis of a Vector Space: Intuition.

- **Intuition:**
  - A basis is like a **"minimal" coordinate system for a space.**
  - It gives you the smallest set of building blocks from which you can construct any vector in the space.
  - Think of:
    - $\left\{\begin{bmatrix}1\\0\end{bmatrix}, \begin{bmatrix}0\\1\end{bmatrix}\right\}$ as the **standard basis** for $\mathbb{R}^2$.
  - Any $\mathbf{x} \in \mathbb{R}^2$ can be written as:
    - $\mathbf{x} = \mathbf{a}\begin{bmatrix}1\\0\end{bmatrix} + \mathbf{b}\begin{bmatrix}0\\1\end{bmatrix}$; **here a, and b are any scalar in** $\mathbb{R}$.

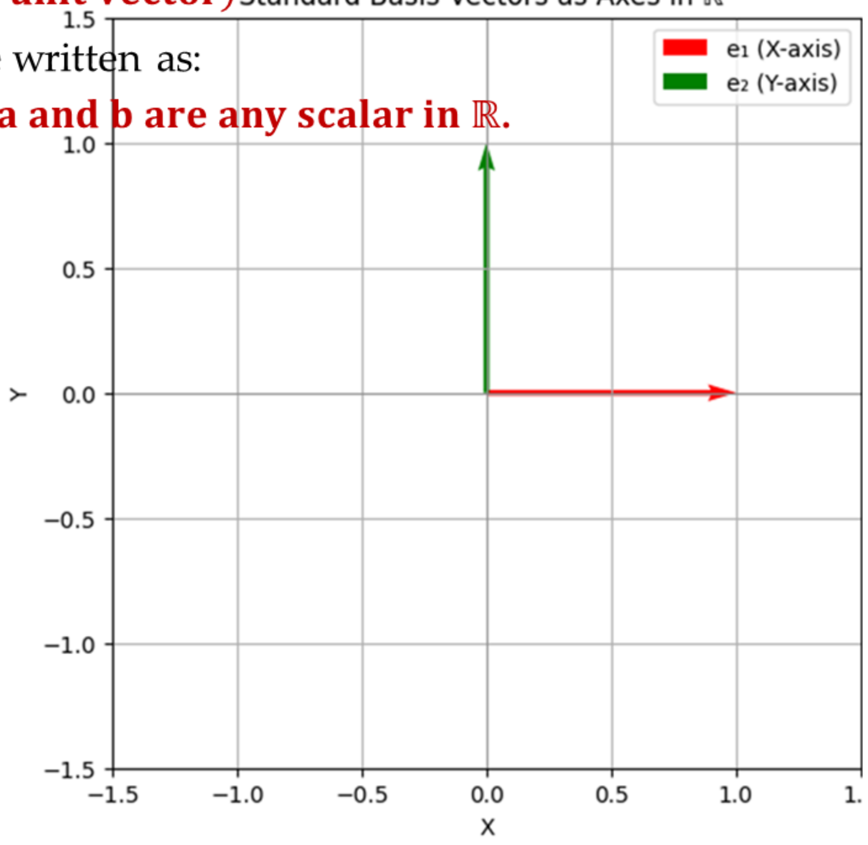


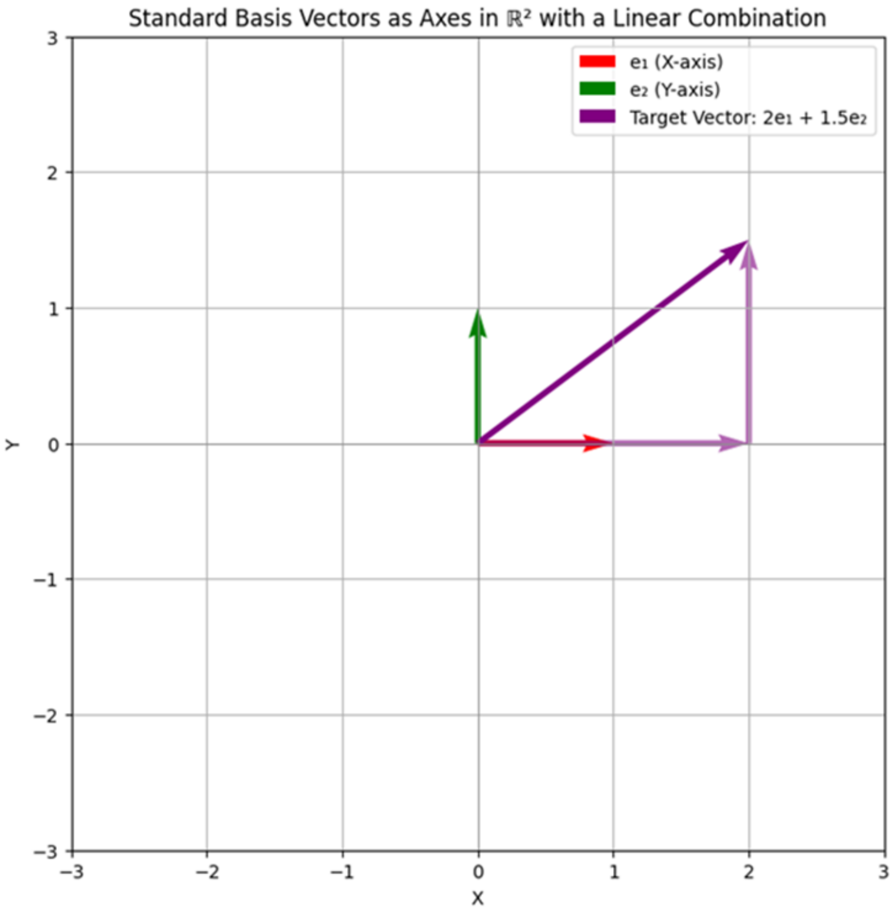
2.7.2 Standard Basis.

- **Standard Basis:**
  - For $\mathbb{R}^n$, the standard basis is: $\{\mathbf{e_1}, \mathbf{e_2}, \dots, \mathbf{e_n}\}$
    - Where:
      - $\mathbf{e_i} = \begin{bmatrix} \mathbf{0} \\ \vdots \\ \mathbf{1\ in\ i^{th} position} \\ \vdots \\ \mathbf{0} \end{bmatrix} \Rightarrow$ **a vector with 1 in position i, 0 elsewhere**
- **Connection to Unit Vectors:**
  - Each standard basis vector e_i is also a unit vector:
    - $\|\mathbf{e_i}\| = \sqrt{\mathbf{1^2 + 0^2 + \cdots + 0^2}} = \mathbf{1}$
    - And they point **along the coordinate axes**.
- So, we say: The standard basis vectors are unit vectors along the coordinate directions of $\mathbb{R}^n$.

2.7.2.1 Standard Basis: Example

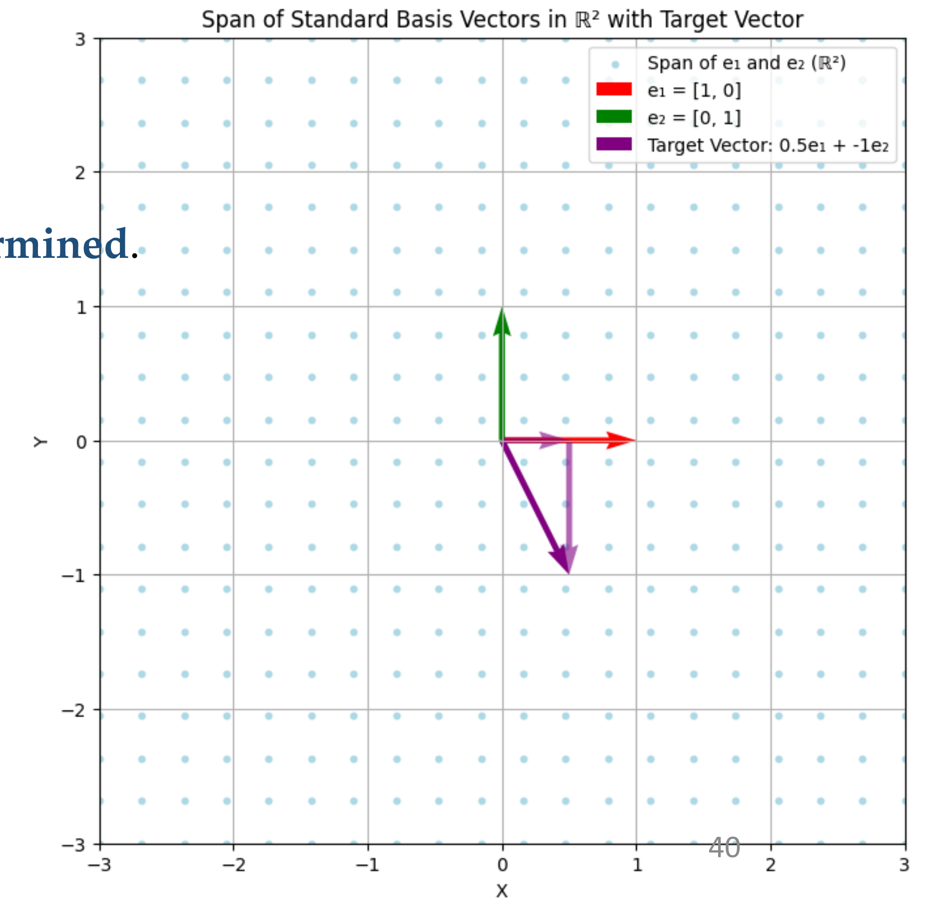
- Standard Basis in $\mathbb{R}^2$:
  - $\mathbf{e_1} = [\mathbf{1}, \mathbf{0}]$ $(\mathbf{x}$ $-$ **axis unit vector**$)$
    $\mathbf{e_2} = [\mathbf{0}, \mathbf{1}]$ $(\mathbf{y}$ $-$ **axis unit vector**$)$
- Any vector $\mathbf{x} \in \mathbb{R}^{\mathbf{2}}$ can be written as:
  - $\mathbf{x} = \mathbf{a}\mathbf{e_1} + \mathbf{b}\mathbf{e_2}$**, here a and b are any scalar in** $\mathbb{R}$**.**





2.7.3 Properties of Basis.

1. **Unique Representation:**
    - Every vector in a vector space can be uniquely represented as a linear combination of the basis vectors.
    - What it means:
        - If $\mathcal{B} = \{\mathbf{v}_1, \dots, \mathbf{v}_n\}$ is a basis of **vector space V**,
        - Then any $\mathbf{x} \in \mathbf{V}$ can be written as:
            - $\mathbf{x} = \alpha_1 \mathbf{v}_1 + \alpha_2 \mathbf{v}_2 + \cdots + \alpha_n \mathbf{v}_n$
                - And the **coefficients $\alpha_1, \alpha_2, \dots, \alpha_n$ are uniquely determined**.
    - Why?
        - Because the **basis vectors are linearly independent**
            - there is **only one way to combine** them to
                - **reach a given vector in the space.**



2.7.3.1 Properties of Basis.

2. **Same number of elements in any Basis:**
   - All bases of a **finite dimensional vector space have the same number of vectors**.
   - What it means:
     - If one **basis of V has n vectors**, then **every basis of V has exactly n vectors**.
     - You cannot have a basis of 2 vectors and another basis of 3 vectors for the same space.
   - Why?
     - Because the number of vectors in a basis corresponds to the dimension of the space.
3. **Dimension of the Vector space:**
   - The **number of vectors** in a basis is called the **dimension of the vector space**.
   - Examples:
     - $$\mathbb{R}^2: \text{dimension} = 2 \ \rightarrow \text{basis has 2 vectors}$$
       $$\mathbb{R}^3: \text{dimension} = 3 \ \rightarrow \text{basis has 3 vectors}$$
       $$\text{A plane in } \mathbb{R}^3: \text{dimension} = 2 \ \rightarrow \text{any basis of that plane will have 2 vectors.}$$
     - We can have a 2-dimensional subspace of $\mathbb{R}^n$ that has a basis of 2 linearly independent vectors.
     - **Cautions:** A basis of a vector space must have **exactly as many vectors as the dimension of the space**.

2.7.3.2 Properties of Basis.

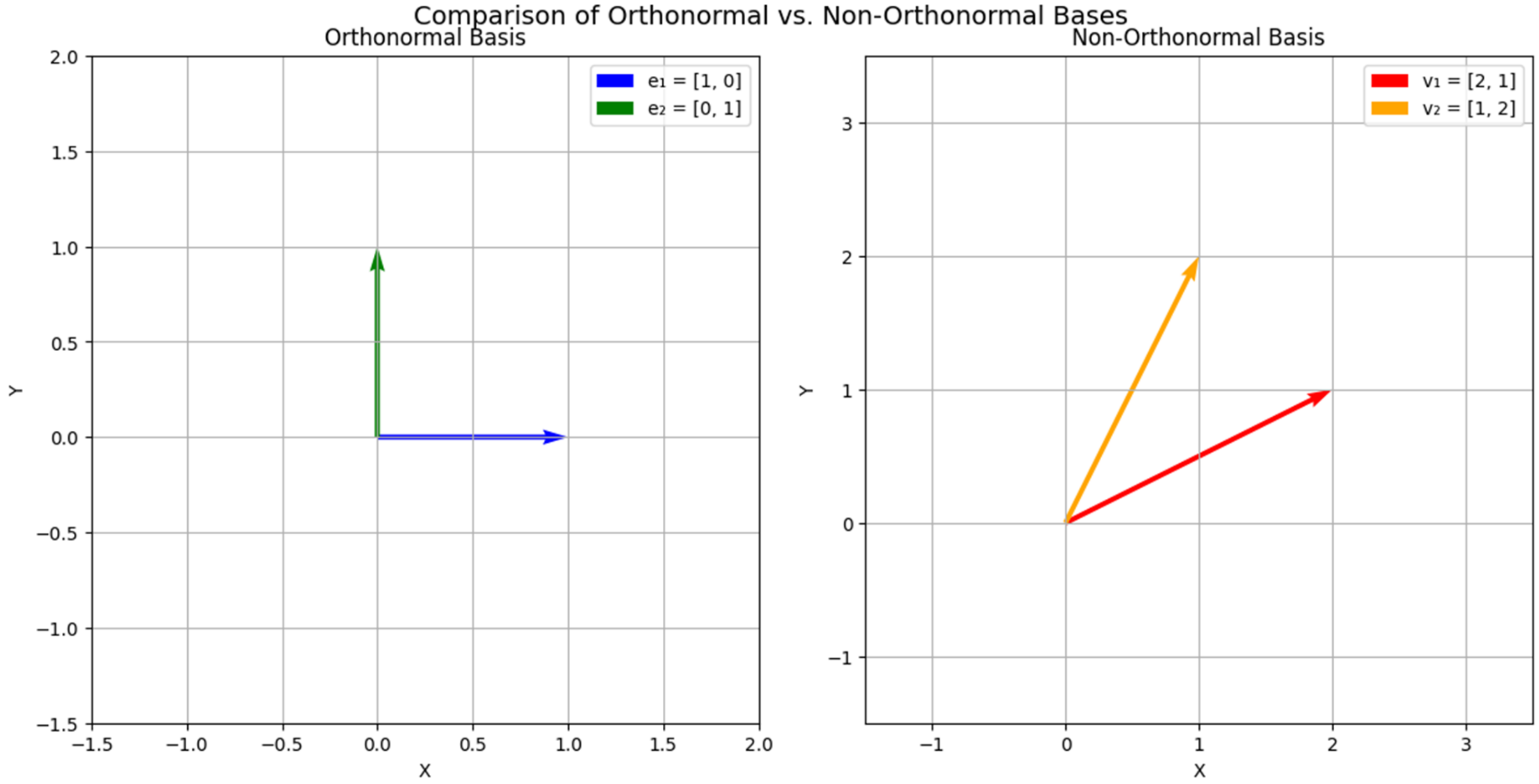
4. **We can have infinitely many bases for the same vector space but all of them must:**
    - Be linearly independent.
    - Span the space.
    - Contain the same number of vectors (equal to the dimension of the space).

- **Example: Multiple Bases in $\mathbb{R}^2$:**
    - **Standard Basis:**
        - $\boldsymbol{\mathcal{B}_1 = \left\{\begin{bmatrix}1\\0\end{bmatrix}, \begin{bmatrix}0\\1\end{bmatrix}\right\}}$
    - Another valid Basis:
        - $\boldsymbol{\mathcal{B}_2 = \left\{\begin{bmatrix}1\\1\end{bmatrix}, \begin{bmatrix}1\\-1\end{bmatrix}\right\}}$
    - Both sets have 2 linearly independent vectors.
    - Both span $\mathbb{R}^2 \rightarrow$ every vector in $\mathbb{R}^2$ can be written as unique linear combination of them.
    - **Thus, both are valid basis.**

2.7.4 Orthogonal Basis.

- An orthogonal basis for a vector space is a basis in which all vectors are mutually perpendicular (orthogonal) to each other.
  - If in addition, all the vectors are of unit length, it becomes an orthonormal basis.
- **Formal Definition:**
  - Let $\mathbf{V}$ be an $\mathbf{n-dimensional}$ **inner product space** (like $\mathbb{R}^n$ with dot product). A set of vectors
    - $\{\mathbf{v_1}, \mathbf{v_2}, \dots, \mathbf{v_n}\} \in \mathbf{V}$ is an **orthogonal basis if :**
      - $\mathbf{v_i} \cdot \mathbf{v_j} = \mathbf{0 \text{ for all } i \neq j}$.
    - **That is each pair of distinct vectors is orthogonal (dot product = 0).**
- **Example in $\mathbb{R}^3$:**
  - The standard basis in $\mathbb{R}^3$: $\mathbf{e_1} = [\mathbf{1}, \mathbf{0}, \mathbf{0}], \mathbf{e_2} = [\mathbf{0}, \mathbf{1}, \mathbf{0}], \mathbf{e_3} = [\mathbf{0}, \mathbf{0}, \mathbf{1}]$
    - is an orthogonal basis and also orthonormal since:
      - **They are mutually perpendicular:** $\mathbf{e_i} \cdot \mathbf{e_j} = \mathbf{0}$ **and**
      - $\|\mathbf{e_i}\| = \mathbf{1}$ **for all** $\mathbf{i}$

Fig: Orthogonal vs Non – Orthogonal basis.

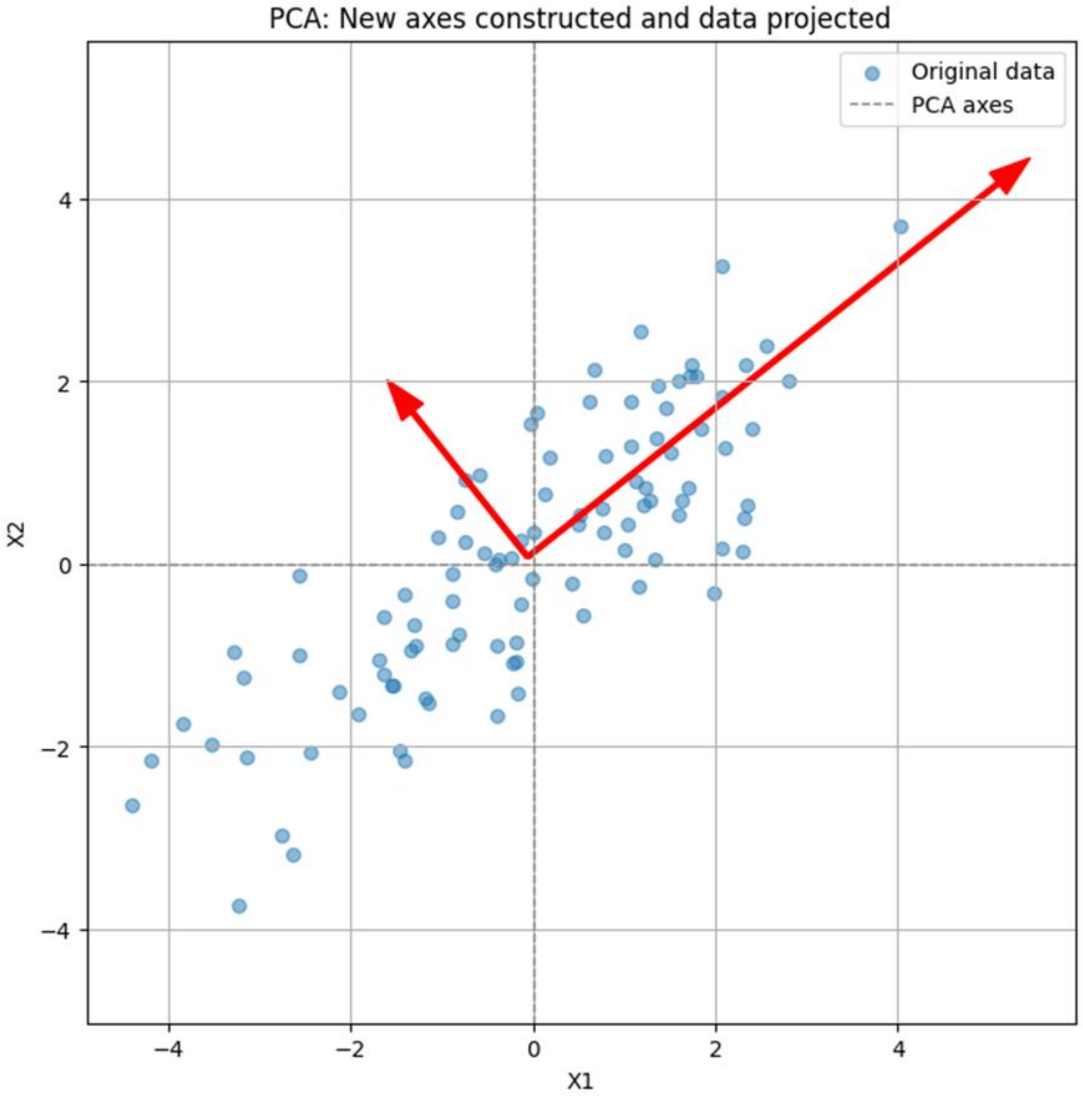


2.8 Linear Models, Linear Combinations and Basis.

- In most classical linear machine learning models, prediction takes the form:
  - $\hat{y} = \mathbf{w}^T\mathbf{x} = \sum_{i=1}^{n} w_i x_i$
    - Each feature $\mathbf{x_i}$ **is weighted by learned parameter** $\mathbf{w_i}$.
  - Implicitly this assumes:
    - **The features are orthogonal i.e. independent.**
    - **Each feature contributes unique information.**
    - **But this assumption does not always hold in real – world data.**
- Correlated Features – Redundant Basis:
  - **When features are correlated:**
    - Two or more features lie in the same direction in the feature space.
    - So, the **feature matrix X has** linearly dependent **columns.**
    - The basis i.e. the feature directions is redundant.
    - The geometry of the model becomes ill – conditioned i.e. numerically unstable.

2.8.1 Linear Models, Linear Combinations and Basis.

- **Solution: change to better basis:**
  - To fix this issue: we can transform the data into a new basis where:
    - Features are uncorrelated (orthogonal)
    - Each axis represents distinct variance.
    - The basis is better aligned with data geometry.
- **How we do that?**
  - We apply **feature extraction techniques** that learn or construct a **new basis** from the data itself.
  - Example: Principal Component Analysis
    - Projects the data onto orthogonal axes (principal components) that maximizes variance.



3. Matrix Algebra.

{Important Matrix Operations.}

3.1 Matrix Determinant.

- **Determinant** of a matrix, denoted by **det(A) or |A|**, is a **real-valued scalar** encoding certain properties of the matrix
  - E.g., for a matrix of size 2×2:

$$\det\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right) = ad - bc$$

  - For larger-size matrices the determinant of a matrix id calculated as

    $\mathbf{det(A)} = \sum_j \mathbf{a_{ij}} (-\mathbf{1})^{\mathbf{i+j}} \mathbf{det}\left(\mathbf{A}_{(\mathbf{i,j})}\right)$

  - In the above, $\mathbf{A}_{(\mathbf{i,j})}$ is a minor of the matrix.
- Properties:
  - $\det(AB) = \det(BA)$
  - $\det(A^{-1}) = \frac{1}{\det(A)}$
  - $\det(A) = 0 \rightarrow A$ is singular i. e. non square matrix.

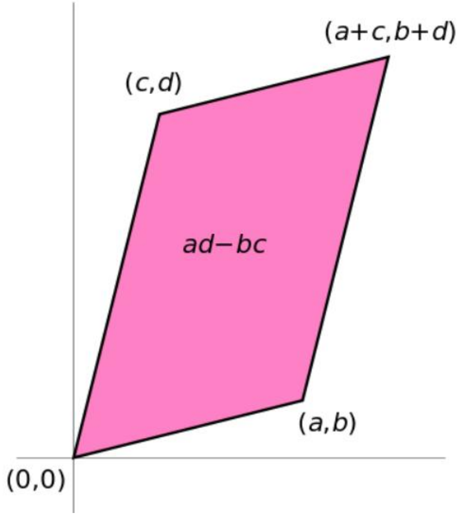
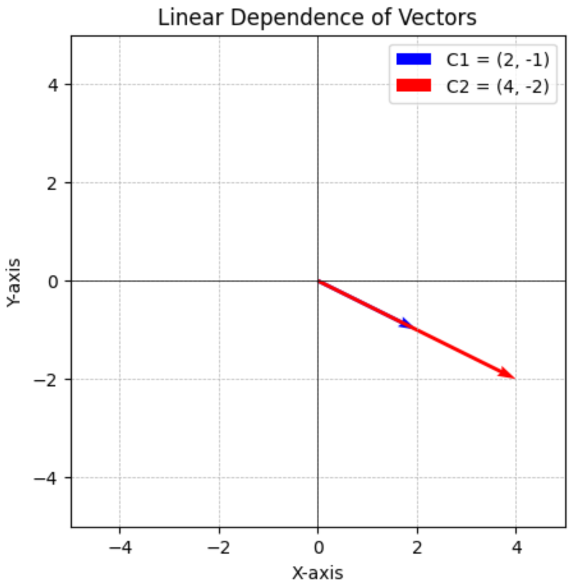
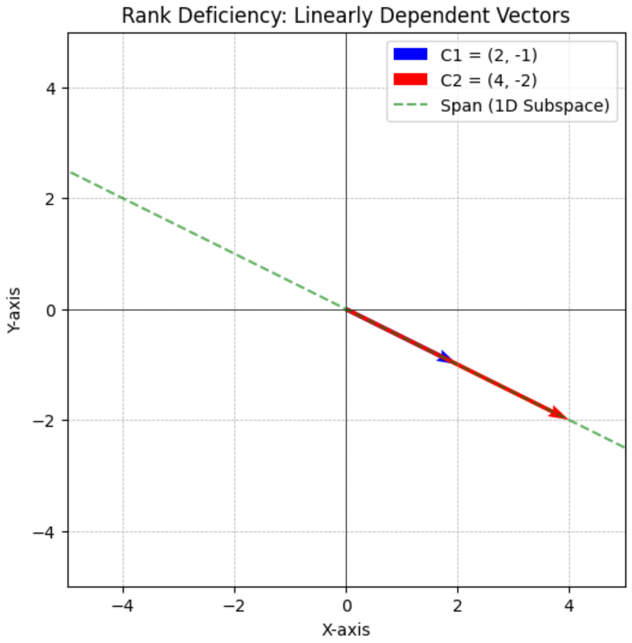


Fig: determinant represents area (or volume) of the parallelogram described by the vectors in the rows of the matrix

3.2 Rank of a Matrix.

- For **m × n matrix** the rank of the matrix is the largest number of linearly independent row or columns.
- For Example:
  - For Matrix; **B** $\coloneqq \begin{bmatrix} 2 & -1 \\ 4 & -2 \end{bmatrix}$ Find the **Rank and Interpret**.
  - **Our Observation:**
    - The second **column** $\boldsymbol{c_2}$ can be written as : $\boldsymbol{c_2 = 2 \times c_1}$
      - Since one **column can be expressed as a multiple of the other**, there is only **one independent column**.
  - Thus, the **rank of B is 1**, meaning it can span only **a 1 D space in** $\mathbb{R}^2$ vector space.
    - All columns of B lie along the same line in $\mathbb{R}^2$
    - Since the **full rank of 2 × 2 matrix in** $\mathbb{R}^2$ **vector space is 2**,
      - B is considered **rank – deficient**.
- **Why it matters?**
  - The matrix cannot invertedly transform $\mathbb{R}^2$.
  - It collapses the 2D space onto a 1D line.
  - This is often a sign of Singularity (determinant = 0), which makes the matrix non – invertible.





3.3 Inverse of a Matrix.

- The inverse of a square matrix A, denoted as A^{-1}, is a matrix that satisfies:
  - $\mathbf{AA^{-1} = A^{-1}A = I}$
    - here **I** is the identity matrix.
- **Conditions for Invertibility:**
  - A **matrix $\mathbf{A_{m\times n}}$** has an **inverse** if and only if :
    - It is a **square matrix $(\mathbf{n \times n})$**.
    - Its **determinant** is **nonzero** i.e. $\mathbf{det(A) \neq 0}$.
    - Its **rank is full**, meaning $\mathbf{rank(A) = n}$.
  - If any of these conditions fail, **the matrix is singular and does not have an inverse**.

3.3.1 Finding inverse of a Matrix.

- If Inverse Exist, we can find the inverse of a Matrix by:
  - **Using Row Reduction:**
    - Row reduction is a method of transforming a matrix into a simpler form (row echelon form – REF) usually the identity matrix for finding the inverse.
    - **REF** can be reached via following valid row operations:
      - Swap two rows
      - Multiply a row by a non zero scalar
      - Add or subtract multiples one to/from another row.
    - It can be done using:
      - Gaussian Elimination: **Transform** the matrix A into REF and then use back – substitution to solve for the inverse.
      - Gauss – Jordan Elimination: **Transform** the matrix A into the **identity matrix** directly, with no need for back substitution.
  - **Using Adjoint (Cofactor) Formula:**
    - Find the inverse of A using the **adjoint** (also called adjugate) **of the matrix**.
      - $\mathbf{A^{-1} = \frac{1}{det(A)} \times adj(A)}$,
      - For 2×2 matrix:
        - $\mathbf{A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}}$;
          - $\mathbf{A^{-1} = \frac{1}{ad - bc} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}}$

3.4 System of Linear Equation.

- A **system of linear equation** is a **collection of one or more linear equations** that share a common **set of variables**. For example:
  - $2x + y = 5$
    $3x + 4y = 6$
- **Types of Systems as per solution behavior:**
  - **Consistent System**: A system that has at least one solution.
    1. **Unique Solution**: Occurs when the system has a single solution.
    2. **Infinite Solutions**: Occurs when the system has many solutions.
  - **Inconsistent System**: A system that has no solution.
- **Types of Systems as per Equation behavior:**
  - **Determined system:** system has exactly as many equations as unknowns (i.e. variables).
    - Usually, exact solution and has **Square Shape**.
  - **Underdetermined system:** System has fewer equations than unknowns.
    - $x + y + z = 1 \rightarrow$ **one equation three varaibles**.
    - Usually, **infinite solution** and has **Wide matrix shape i.e. m < n**.
  - **Overdetermined system:** More equations than unknowns: **m > n.**
    - Usually **no exact solution**, but you can find a **best approximation** using **least squares**.

3.4.1 Solving System of Linear Equations.

- There are different techniques , our interest is **Matrix Method (aka Matrix Inversion Method).**
  - Any system of Linear Equation:
    - $\mathbf{a_{11}x_1 + a_{12}x_2 = b_1}$
      $\mathbf{a_{21}x_1 + a_{22}x_2 = b_2}$
  - can be represented in the form:
    - $\begin{bmatrix} \mathbf{a_{11}} & \mathbf{a_{12}} \\ \mathbf{a_{21}} & \mathbf{a_{22}} \end{bmatrix} \cdot \begin{bmatrix} \mathbf{x_1} \\ \mathbf{x_2} \end{bmatrix} = \begin{bmatrix} \mathbf{b_1} \\ \mathbf{b_2} \end{bmatrix}$ i.e. $\rightarrow$ $\mathbf{Ax = b}$
  - here:
    - $\mathbf{A} \rightarrow$ is a matrix of coefficients with size $m \times n$, m is the number of equations and n is the number of variable.
    - $\mathbf{x} \rightarrow$ **is** a column vector representing the unknown variables with size $\mathbf{n \times 1}$.
    - $\mathbf{b} \rightarrow$ is a column vector representing the constants with size $\mathbf{m \times 1}$.
- The equation can be modified:
  - $\mathbf{A^{-1}Ax = A^{-1}b}$ {Multiplying both side by A^{-1}}
    $\mathbf{Ix = A^{-1}b}$ {I is the identity matrix}
    $\mathbf{x = A^{-1}b}$ ∎ {you know how to find A^{-1}}

3.5 Matrix – Matrix Multiplication.

- Matrix multiplication between $\mathbf{A} \in \mathbb{R}^{\mathbf{n}\times\mathbf{p}}$ and $\mathbf{B} \in \mathbb{R}^{\mathbf{n}\times\mathbf{p}}$ with resultant matrix $\mathbf{C} \in \mathbb{R}^{\mathbf{m}\times\mathbf{p}}$ can be defined as:

$$\boxed{A.B := \begin{bmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{m1} & \cdots & a_{mn} \end{bmatrix} \begin{bmatrix} b_{11} & \cdots & b_{1p} \\ \vdots & \ddots & \vdots \\ b_{n1} & \cdots & b_{np} \end{bmatrix} = \begin{bmatrix} c_{11} & \cdots & c_{1p} \\ \vdots & \ddots & \vdots \\ c_{m1} & \cdots & c_{mp} \end{bmatrix}}$$

$$\boxed{c_{ij} := \sum_{l=1}^{n} a_{il} b_{lj}; \text{ with i=1,...m; and j=1,...,p}}$$

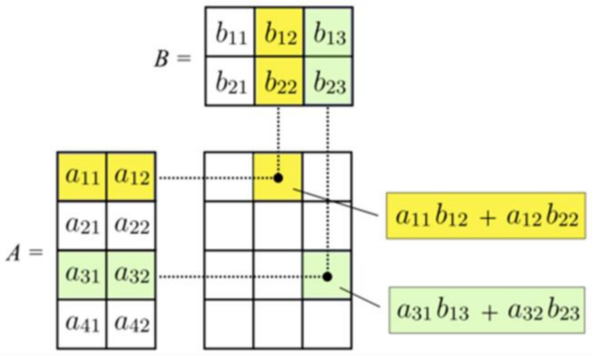


Fig: Schematic representation of Matrix product

Properties of Matrix – Matrix Multiplication:

1. Associativity: $\mathbf{(AB)C = A(BC)}$
2. Associativity with scalar Multiplication: $\boldsymbol{\alpha}\mathbf{(AB) = (}\boldsymbol{\alpha}\mathbf{A)B}$
3. Distributive with sum: $\mathbf{A(B \mp C) = AB \mp BC}$
4. Cautions!! In matrix – matrix multiplication orders matter, it is not commutative i.e. $\mathbf{AB \neq BA}$.

3.6 Matrix – Vector Multiplication.

- Matrix-vector multiplication is an operation between a matrix and a vector that produces a new vector.
- Matrix-vector multiplication equals to taking the dot product of each column n of matrix-A with each element of vector-x resulting in vector y and is defined as:

$$A.X := \begin{bmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{m1} & \cdots & a_{mn} \end{bmatrix} \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} = x_1 \begin{bmatrix} a_{11} \\ \vdots \\ a_{m1} \end{bmatrix} + x_2 \begin{bmatrix} a_{12} \\ \vdots \\ a_{m2} \end{bmatrix} + x_n \begin{bmatrix} a_{1n} \\ \vdots \\ a_{mn} \end{bmatrix} = \begin{bmatrix} y_1 \\ \vdots \\ y_{mn} \end{bmatrix}$$

- Matrix – vector multiplication can be interpreted as taking a **linear combination** of the columns of a matrix A **weighted** by elements of **vector x**.
  - What can be the consequences of such operation?

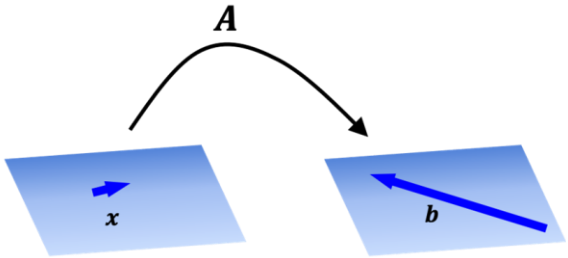
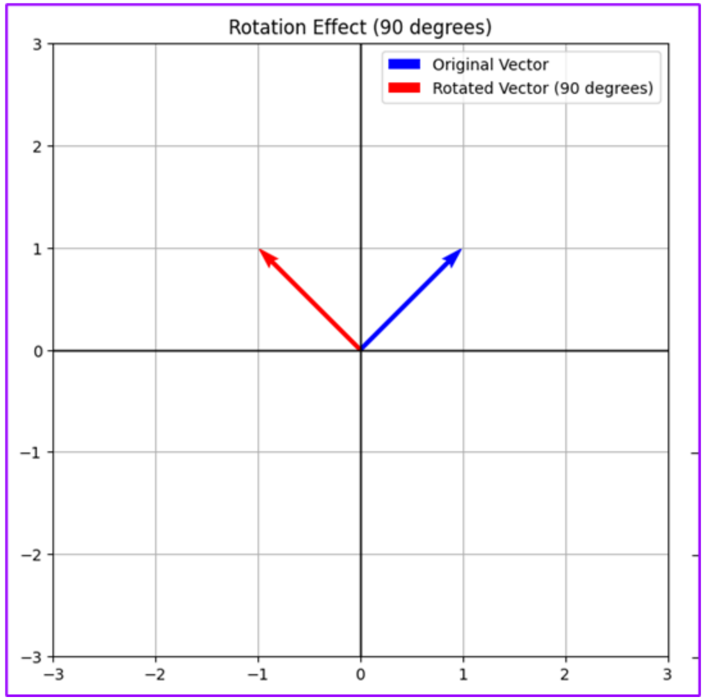


Fig: How my vector will Transformed?

Matrix – vector multiplication can result in:
Change in magnitude **or**,
Change in direction or,
Both changes depending on the matrix involved.

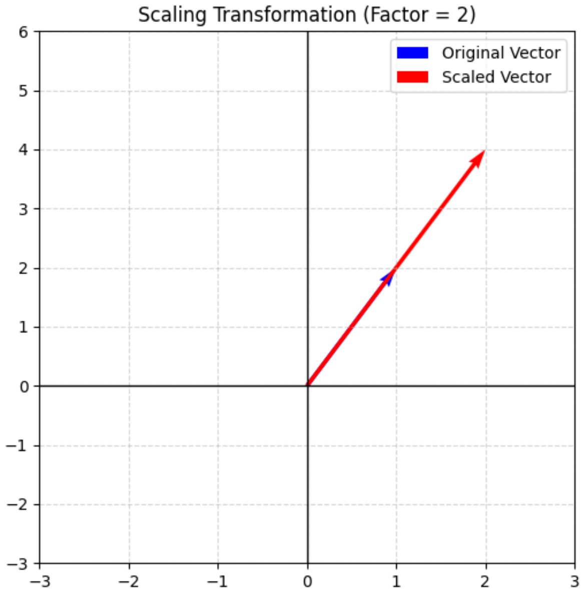
3.6.1 Geometric Interpretation of Matrix – vector Multiplication.

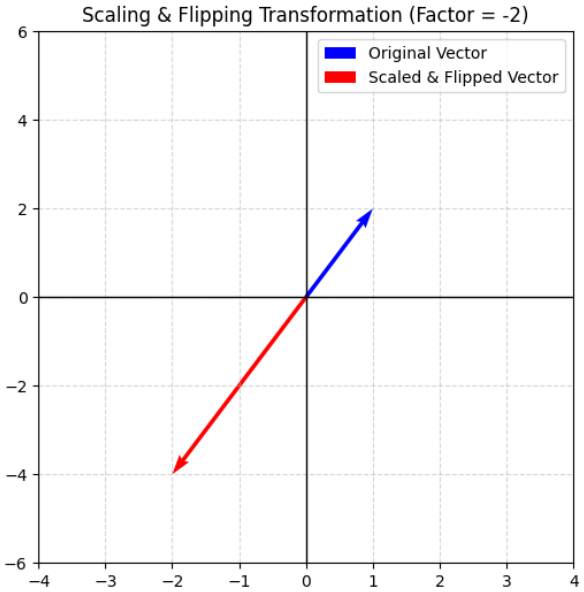
- **Rotation Matrix:**
  - A rotation matrix rotates a vector by a specified angle while preserving its magnitude.
  - **Example**: A 2D rotation matrix that rotates a vector by 90 degrees counterclockwise:
    - $\mathbf{R} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$
  - **Effect**: This matrix rotates the vector without changing its length.
- **Example Calculation**:
  - Given the vector $\mathbf{v} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \rightarrow \mathbf{Rv} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$
  - The **magnitude remains 1**, but the **direction changes** from the **x-axis to the y-axis**.



3.6.2 Geometric Interpretation of Matrix – vector Multiplication.

- **Scaling Matrix:**
  - A scaling matrix increases or decreases the magnitude of the vector without changing their direction.
    - $\mathbf{S} = \begin{bmatrix} \mathbf{k} & \mathbf{0} \\ \mathbf{0} & \mathbf{k} \end{bmatrix}$
  - where $\mathbf{k}$ is **the scaling factor**:
    - If $\mathbf{k} > \mathbf{1}$, the vector is stretched.
    - If $\mathbf{0} < \mathbf{k} < \mathbf{1}$, the vector is compressed.
    - If $\mathbf{k} < \mathbf{0}$, the vector is **flipped** and scaled.
- Example:
  - Given a vector: $\boldsymbol{v} = \begin{bmatrix} \mathbf{1} \\ \mathbf{2} \end{bmatrix}$ and;
  - a scaling matrices
    - $\boldsymbol{i})\ \begin{bmatrix} \mathbf{2} & \mathbf{0} \\ \mathbf{0} & \mathbf{2} \end{bmatrix}$ and $\boldsymbol{ii})\ \begin{bmatrix} \mathbf{-2} & \mathbf{0} \\ \mathbf{0} & \mathbf{-2} \end{bmatrix}$;
  - Applying $\boldsymbol{S}$ **to** $\boldsymbol{v}$:
    - $\mathbf{i})\ \begin{bmatrix} \mathbf{2} & \mathbf{0} \\ \mathbf{0} & \mathbf{2} \end{bmatrix} \cdot \begin{bmatrix} \mathbf{1} \\ \mathbf{2} \end{bmatrix} = \begin{bmatrix} \mathbf{2} \\ \mathbf{4} \end{bmatrix}$
    - $\mathbf{ii})\ \begin{bmatrix} \mathbf{-2} & \mathbf{0} \\ \mathbf{0} & \mathbf{-2} \end{bmatrix} \cdot \begin{bmatrix} \mathbf{1} \\ \mathbf{2} \end{bmatrix} = \begin{bmatrix} \mathbf{-2} \\ \mathbf{-4} \end{bmatrix}$





3.7 Formalizing Matrix – vector Multiplication.

- **Linear Transformation: An overview.**
  - A **linear transformation** is a **function T** that maps vectors from one vector space to another (possibly the same space), preserving two key operations:
  - **Additivity:**
    - $\mathbf{T(u+v) = T(u) + T(v)}$
  - **Homogeneity (Scalar Multiplication):**
    - $\mathbf{T(cv) = cT(v)}$
      - for all vectors u, v, and scalars c.
- Every **linear Transformation T** from $\mathbb{R}^{\mathbf{n}}$ **to** $\mathbb{R}^{\mathbf{m}}$ can be represented by an $\mathbf{m \times n}$ **matrix A**.
  - Applying T to a vector x is the same as multiplying by the matrix:
    - $\mathbf{T(x) = Ax}$
      - Each column of A shows how the transformation acts on a **basis vector** of the input space.
      - Multiplying **A by x** combines these transformed basis vectors weighted by the **coordinates in x.**

Thank You