Strassen's Matrix Multiplication

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Matrix Multiplication

Input: $A = [a_{ij}], B = [b_{ij}].$
Output: $C = [c_{ij}] = A \cdot B.$

$$\left.\begin{array}{l}\text{Input: } A = [a_{ij}], B = [b_{ij}]. \\ \text{Output: } C = [c_{ij}] = A \cdot B.\end{array}\right\} \quad i, j = 1, 2, \ldots, n.$$

$$\begin{bmatrix} c_{11} & c_{12} & \cdots & c_{1n} \\ c_{21} & c_{22} & \cdots & c_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ c_{n1} & c_{n2} & \cdots & c_{nn} \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix} \cdot \begin{bmatrix} b_{11} & b_{12} & \cdots & b_{1n} \\ b_{21} & b_{22} & \cdots & b_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ b_{n1} & b_{n2} & \cdots & b_{nn} \end{bmatrix}$$

$$c_{ij} = \sum_{k=1}^{n} a_{ik} \cdot b_{kj}$$

Standard algorithm

```
for i ←1 to n
    do for j←1 to n
          do cij←0
          for k←1 to n
                do cij←cij+ aik·bkj
```

$$C_{i,j} = \sum_{k=1}^{N} a_{i,k} b_{k,j}$$

$$\text{Thus } T(N) = \sum_{i=1}^{N} \sum_{j=1}^{N} \sum_{k=1}^{N} c = cN^3 = O(N^3)$$

Divide-and-Conquer

- Divide-and conquer is a general algorithm design paradigm:
  - Divide: divide the input data $\boldsymbol{S}$ in two or more disjoint subsets $\boldsymbol{S}_1$, $\boldsymbol{S}_2$, …
  - Recur: solve the sub problems recursively
  - Conquer: combine the solutions for $\boldsymbol{S}_1$, $\boldsymbol{S}_2$, …, into a solution for $\boldsymbol{S}$
- The base case for the recursion are sub problems of constant size
- Analysis can be done using **recurrence equations**

Matrix Multiplication through Divide and Conquer Approach

1. ***Divide:*** Partition A and B into $(n/2)\times(n/2)$ submatrices. Form terms to be multiplied using $+$ and $-$.
2. ***Conquer:*** Perform 7 multiplications of $(n/2)\times(n/2)$ submatrices recursively.
3. ***Combine:*** Form C using $+$ and $-$ on $(n/2)\times(n/2)$ submatrices.

Divide-and-conquer algorithm

$n \times n$ matrix = 2×2 matrix of $(n/2) \times (n/2)$ submatrices:

$$\begin{bmatrix} r & s \\ t & u \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \cdot \begin{bmatrix} e & f \\ g & h \end{bmatrix}$$

$$C = A \cdot B$$

$$\begin{aligned} r &= ae + bg \\ s &= af + bh \\ t &= ce + dg \\ u &= cf + dh \end{aligned}$$

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$$\begin{bmatrix} r & s \\ t & u \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \cdot \begin{bmatrix} e & f \\ g & h \end{bmatrix}$$

$$C = A \cdot B$$

$$\left.\begin{aligned} r &= ae + bg \\ s &= af + bh \\ t &= ce + dg \\ u &= cf + dh \end{aligned}\right\} \begin{array}{l} 8 \text{ mults of } (n/2) \times (n/2) \text{ submatrices} \\ 4 \text{ adds of } (n/2) \times (n/2) \text{ submatrices} \end{array}$$

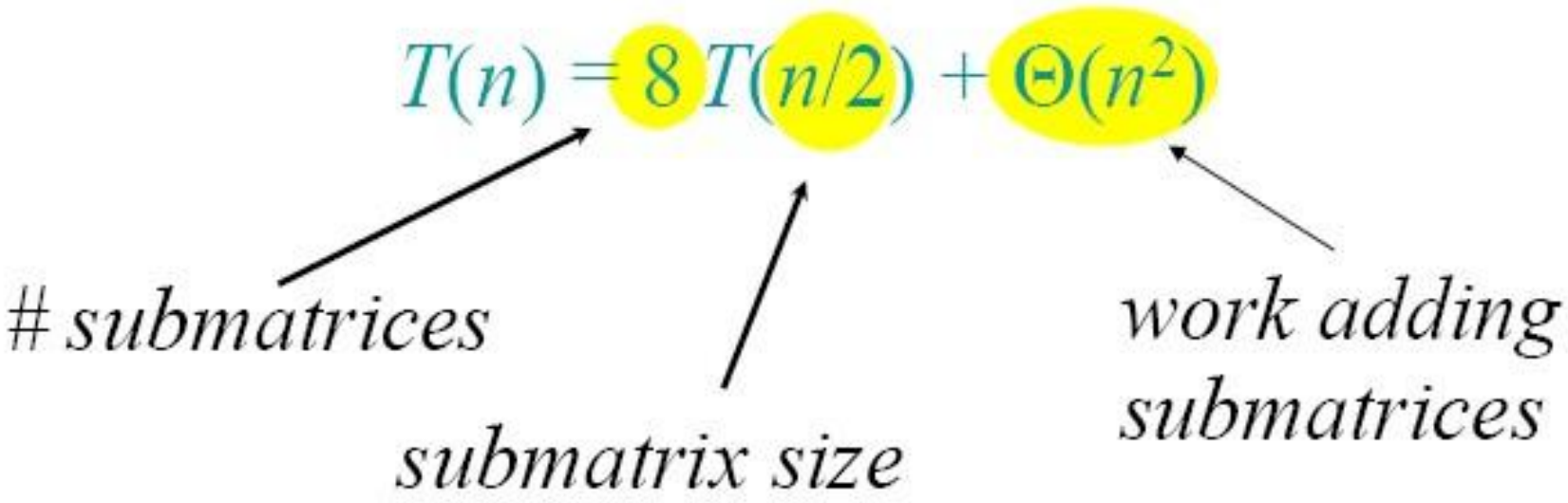
Master theorem

$$T(n) = a\,T(n/b) + f(n)$$

CASE 1: $f(n) = O(n^{\log_b a - \varepsilon})$, constant $\varepsilon > 0$
$\Rightarrow T(n) = \Theta(n^{\log_b a})$.

CASE 2: $f(n) = \Theta(n^{\log_b a} \lg^k n)$, constant $k \geq 0$
$\Rightarrow T(n) = \Theta(n^{\log_b a} \lg^{k+1} n)$.

CASE 3: $f(n) = \Omega(n^{\log_b a + \varepsilon})$, constant $\varepsilon > 0$,
and regularity condition
$\Rightarrow T(n) = \Theta(f(n))$.



$$n^{\log_b a} = n^{\log_2 8} = n^3 \Rightarrow \text{CASE 1} \Rightarrow T(n) = \Theta(n^3).$$

No better than the ordinary algorithm.

Strassen's idea

- Multiply 2×2matrices with only 7 recursive multiplications.
- $P_1 = a \cdot (f - h)$
- $P_2 = (a + b) \cdot h$
- $P_3 = (c + d) \cdot e$
- $P_4 = d \cdot (g - e)$
- $P_5 = (a + d) \cdot (e + h)$
- $P_6 = (b - d) \cdot (g + h)$
- $P_7 = (a - c) \cdot (e + f)$
- **Note:**
- 7mults, 18adds/subs.
- **Note: No reliance on commutative of multiplication!**

$r = P_5 + P_4 - P_2 + P_6$

$s = P_1 + P_2$

$t = P_3 + P_4$

$u = P_5 + P_1 - P_3 - P_7$

$$
\begin{aligned}
P_1 &= a \cdot (f - h) \\
P_2 &= (a + b) \cdot h \\
P_3 &= (c + d) \cdot e \\
P_4 &= d \cdot (g - e) \\
P_5 &= (a + d) \cdot (e + h) \\
P_6 &= (b - d) \cdot (g + h) \\
P_7 &= (a - c) \cdot (e + f)
\end{aligned}
$$

$$
\begin{aligned}
r &= P_5 + P_4 - P_2 + P_6 \\
&= (a + d)(e + h) \\
&\quad + d(g - e) - (a + b)h \\
&\quad + (b - d)(g + h) \\
&= ae + ah + de + dh \\
&\quad + dg - de - ah - bh \\
&\quad + bg + bh - dg - dh \\
&= ae + bg
\end{aligned}
$$

Strassen Algorithm

```
void matmul(int *A, int *B, int *R, int n) {
  if (n == 1) {
    (*R) += (*A) * (*B);
  } else {
    matmul(A, B, R, n/4);
    matmul(A, B+(n/4), R+(n/4), n/4);
    matmul(A+2*(n/4), B, R+2*(n/4), n/4);
    matmul(A+2*(n/4), B+(n/4), R+3*(n/4), n/4);
    matmul(A+(n/4), B+2*(n/4), R, n/4);
    matmul(A+(n/4), B+3*(n/4), R+(n/4), n/4);
    matmul(A+3*(n/4), B+2*(n/4), R+2*(n/4), n/4);
    matmul(A+3*(n/4), B+3*(n/4), R+3*(n/4), n/4);
  }
```

Divide matrices in sub-matrices and recursively multiply sub-matrices

Analysis of Strassen's

- $T(n) = 7T(n/2) + \Theta(n^2)$
- $n^{\log_b a} = n^{\log_2 7} \approx n^{2.81}$
- $\Rightarrow$ CASE 1 $\Rightarrow T(n) = \Theta(n^{\lg 7})$.
- The number 2.81 may not seem much smaller than 3, but because the difference is in the exponent, the impact on running time is significant. In fact, Strassen's algorithm beats the ordinary algorithm on today's machines for $n \geq 32$ or so.

- **Best to date (of theoretical interest only):** $\Theta(n^{2.376\ldots})$.

REFERENCES:-

1.https://en.wikipedia.org/wiki/Strassen_algorithm

2.https://www.geeksforgeeks.org/strassens-matrix-multiplication/