BINOMIAL COEFFICIENT

Binomial Coefficients

- We know that a *binomial* is a polynomial that has two terms.
- To begin, look at the expansion of
- $(x + y)^n$
- for several values of n
- $(x + y)^0 = 1$
- $(x + y)^1 = (x + y)$
- $(x + y)^2 = x^2 + 2xy + y^2$
- $(x + y)^3 = x^3 + 3x^2y + 3xy^2 + y^3$
- $(x + y)^4 = x^4 + 4x^3y + 6x^2y^2 + 4xy^3 + y^4$

Binomial Coefficients

1. The sum of the powers of each term is *n*. For instance, in the expansion of

$$(x + y)^5$$

- the sum of the powers of each term is 5.

$$(x + y)^5 = x^5 + 5x^4y^1 + 10x^3y^2 + 10x^2y^3 + 5x^1y^4 + y^5$$

2. The coefficients increase and then decrease in a symmetric pattern.

- The coefficients of a binomial expansion are called **binomial coefficients.**

Binomial Coefficients

The Binomial Theorem

In the expansion of $(x + y)^n$

$$(x + y)^n = x^n + nx^{n-1}y + \cdots + {}_nC_r x^{n-r} y^r + \cdots + nxy^{n-1} + y^n$$

the coefficient of $x^{n-r} y^r$ is

$${}_nC_r = \frac{n!}{(n - r)!r!}.$$

The symbol

$$\binom{n}{r}$$

is often used in place of ${}_nC_r$ to denote binomial coefficients.

Binomial Expansions

- Sometimes you will need to find a specific term in a binomial expansion.
- Instead of writing out the entire expansion, you can use the fact that, from the Binomial Theorem, the $(r + 1)$th term is
- $\quad {}_nC_r\, x^{n-r}y^r.$

Pascal’s Triangle

Pascal’s Triangle

- There is a convenient way to remember the pattern for binomial coefficients. By arranging the coefficients in a triangular pattern, you obtain the following array, which is called **Pascal’s Triangle.** This triangle is named after the famous French mathematician Blaise Pascal (1623–1662).

1
1 1
1 2 1
1 3 3 1
1 4 6 4 1
1 5 10 10 5 1
1 6 15 20 15 6 1
1 7 21 35 35 21 7 1

4 + 6 = 10

15 + 6 = 21

Pascal’s Triangle

- The first and last number in each row of Pascal’s Triangle is 1. Every other number in each row is formed by adding the two numbers immediately above the number. Pascal noticed that the numbers in this triangle are precisely the same numbers as the coefficients of binomial expansions, as follows.

$$
\begin{array}{ll}
(x+y)^0 = 1 & \text{0th row} \\
(x+y)^1 = 1x + 1y & \text{1st row} \\
(x+y)^2 = 1x^2 + 2xy + 1y^2 & \text{2nd row} \\
(x+y)^3 = 1x^3 + 3x^2y + 3xy^2 + 1y^3 & \text{3rd row} \\
(x+y)^4 = 1x^4 + 4x^3y + 6x^2y^2 + 4xy^3 + 1y^4 & \vdots \\
(x+y)^5 = 1x^5 + 5x^4y + 10x^3y^2 + 10x^2y^3 + 5xy^4 + 1y^5 & \\
(x+y)^6 = 1x^6 + 6x^5y + 15x^4y^2 + 20x^3y^3 + 15x^2y^4 + 6xy^5 + 1y^6 & \\
(x+y)^7 = 1x^7 + 7x^6y + 21x^5y^2 + 35x^4y^3 + 35x^3y^4 + 21x^2y^5 + 7xy^6 + 1y^7 &
\end{array}
$$

Pascal's Triangle

- The top row of Pascal's Triangle is called the *zeroth row* because it corresponds to the binomial expansion

$$(x + y)^0 = 1.$$

- Similarly, the next row is called the *first row* because it corresponds to the binomial expansion

$$(x + y)^1 = 1(x) + 1(y).$$

- In general, the *nth row* of Pascal's Triangle gives the coefficients of $(x + y)^n$.

Algorithm to find the Binomial Coefficient

Optimal Substructure

- The value of C(n, k) can be recursively calculated using following standard formula for Binomial Coefficients.

$$C(n,k) = C(n-1,k-1) + C(n-1,k)$$

$$C(n,0) = C(n,n) = 1$$

Algorithm to find the Binomial Coefficient

C/C++ Implementation

```
#include<stdio.h>
int binomialCoeff(int n, int k){// Returns value of Binomial Coefficient C(n, k)
        if (k==0 || k==n)                              // Base Cases
        return 1;
        return  binomialCoeff(n-1, k-1) + binomialCoeff(n-1, k);      // Recur
}
/* Driver program to test above function*/
int main(){
        int n = 5, k = 2;
        printf("Value of C(%d, %d) is %d ", n, k, binomialCoeff(n, k));
        return 0;
}
```

REFERENCES

- https://en.wikipedia.org/wiki/Binomial_coefficient
- https://www.geeksforgeeks.org/dynamic-programming-set-9-binomial-coefficient/