Algorithmic problem solving

17ECSE309

Legendre's formula

[l*uh*-**zhahn**-der]

By-
Abhijeet.P.G
01FE15BEC003

Legendre's formula gives an expression for the exponent of the largest power of a prime p that divides the factorial $n!$.

$$\nu_p(n!) = \sum_{i=1}^{\infty} \left\lfloor \frac{n}{p^i} \right\rfloor$$

Where , p is prime and p<n

Every p’th number is divisible by p in {1, 2, 3, 4, ….. n}. Therefore in n!, there are ⌊n/p⌋ numbers divisible by p.Similarly, there are ⌊n/(p2)⌋ numbers divisible by p and so on.

Example:

For $n = 6$, one has $6! = 720 = 2^4 \cdot 3^2 \cdot 5^1$. The exponents $\nu_2(6!) = 4, \nu_3(6!) = 2$ and $\nu_5(6!) = 1$ can be computed by Legendre's formula as follows:

$$\nu_2(6!) = \sum_{i=1}^{\infty} \left\lfloor \frac{6}{2^i} \right\rfloor = \left\lfloor \frac{6}{2} \right\rfloor + \left\lfloor \frac{6}{4} \right\rfloor = 3 + 1,$$

$$\nu_3(6!) = \sum_{i=1}^{\infty} \left\lfloor \frac{6}{3^i} \right\rfloor = \left\lfloor \frac{6}{3} \right\rfloor = 2,$$

$$\nu_5(6!) = \sum_{i=1}^{\infty} \left\lfloor \frac{6}{5^i} \right\rfloor = \left\lfloor \frac{6}{5} \right\rfloor = 1.$$

Applications

1)Legendre's formula can be used to prove kummer’s theorem.

2)It follows from Legendre's formula that the *p*-adic exponential function has radius of convergence p-1/(p-1) .

Reference

- Legendre, A. M. (1830), *Théorie des Nombres*, Paris: Firmin Didot Frères.
- Moll, Victor H. (2012), *Numbers and Functions*, American Mathematical Society, ISBN 978-0821887950, MR 2963308, page 77
- Leonard Eugene Dickson, *History of the Theory of Numbers*, Volume 1, Carnegie Institution of Washington, 1919, page 263.