强化训练

1. $a>0$，积分 $I_1=\int_0^{\frac{\pi}{2}}\frac{\cos x}{1+x^a}dx$ $\quad I_2=\int_0^{\frac{\pi}{2}}\frac{\sin x}{1+x^a}dx$ 则

$I_1-I_2=\int_0^{\frac{\pi}{2}}\frac{\cos x-\sin x}{1+x^a}dx$ ← 分段在 $x=\frac{\pi}{4}$ 上

$=\int_0^{\frac{\pi}{4}}\frac{\cos x-\sin x}{1+x^a}dx+\underline{\int_{\frac{\pi}{4}}^{\frac{\pi}{2}}\frac{\cos x-\sin x}{1+x^a}dx}$ 令 $x=\frac{\pi}{2}-t$

$=\int_0^{\frac{\pi}{4}}\frac{\cos x-\sin x}{1+x^a}dx+\int_0^{\frac{\pi}{4}}\frac{\sin t-\cos t}{1+(\frac{\pi}{2}-t)^a}dt$

$=\int_0^{\frac{\pi}{4}}\underbrace{(\cos x-\sin x)}_{>0}\Big(\underbrace{\frac{1}{1+x^a}-\frac{1}{1+(\frac{\pi}{2}-x)^a}}_{>0}\Big)dx.$ $\quad\therefore I_1-I_2>0$

$0<x<\frac{\pi}{4}\quad \frac{\pi}{4}<\frac{\pi}{2}-x<\frac{\pi}{2}$

3. $\lim\limits_{n\to\infty}\int_{-1}^{2}(\arctan nx)^3dx=?$

令 $nx=u$ $\quad n\,dx=du$ $\quad$ 原式 $=\underline{\int_{-1}^{1}\arctan^3(nx)\,dx}+\int_1^2\arctan^3 nx\,dx$ 奇 0.

$=\frac{1}{n}\lim\limits_{n\to\infty}\int_n^{2n}(\arctan u)^3du$

积分中值定理 $\quad=\frac{1}{n}\cdot\lim\limits_{n\to\infty}\cdot n\cdot\arctan^3(\xi)\quad n\leq\xi\leq 2n$

$=(\frac{\pi}{2})^3=\frac{\pi^3}{8}$

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5. $f(x)$ 在 $[a,b]$ 上连续且单调增，证存在 $\xi\in(a,b)$

$$\int_a^b f(x)dx=f(a)(\xi-a)+f(b)(b-\xi)$$

证：$F(x)=f(a)(x-a)+f(b)(b-x)$

$F(a)=f(b)(b-a)\quad F(b)=f(a)(b-a).\quad f(a)\leq f(x)\leq f(b)$

$\Downarrow$

$F(b)=f(a)(b-a)\leq\int_a^b f(x)dx\leq f(b)(b-a)=F(a)$

$\because$ 介值定理 $\exists\,\xi\in(a,b)$ 使 $F(\xi)=\int_a^b f(x)dx$

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4. 同5题思路，不作说明。

7. 证明: $\frac{2}{\sqrt[4]{e}} \le \int_0^2 e^{x^2-x}dx \le 2e^2$

令 $f(x)=e^{x^2-x}$ $\quad f'(x)=e^{x^2-x}(2x-1)$.

$\therefore$ 在 $[0,2]$ 上时 $x=\frac{1}{2}$ 有 min 为 $e^{-\frac{1}{4}}$, $f(0)<f(2)$ $\therefore$ max 为 e^2.

有 $\quad e^{-\frac{1}{4}} \le e^{x^2-x} \le e^2$

$$2e^{-\frac{1}{4}} \le \int_0^2 e^{x^2-x} \le 2e^2$$

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8. $f(x)$ 在 $[a,b]$ 上连续且单调增加

证: $\int_a^b \left(\frac{b-x}{b-a}\right)^n f(x)dx \le \frac{1}{n+1}\int_a^b f(x)dx \quad (n\in N)$

原式等价于 $\quad (n+1)\int_a^b (b-x)^n f(x)dx \le (b-a)^n\int_a^b f(x)dx$

令 $F(x)=(n+1)\int_x^b (b-t)^n f(t)dt-(b-x)^n\int_x^b f(t)dt$

$$F'(x)=\underline{(n+1)(-)(b-x)^n f(x)}-n(b-x)^{n-1}(-1)\int_x^b f(t)dt+\underline{(b-x)^n f(x)}$$

$$=-n(b-x)^n f(x)+n(b-x)^{n-1}\underline{\int_x^b f(t)dt}$$ 积分中值定理

$\int_x^b f(t)dt=f(\xi)\cdot(b-x) \qquad \xi\in(x,b)$

$$F'(x)=-n(b-x)^n f(x)+n(b-x)^{n-1}\cdot f(\xi)\cdot(b-x)$$

$$=n\underbrace{(b-x)^n}_{\vee\atop 0}\big(\underbrace{f(\xi)-f(x)}_{\vee\atop 0}\big)>0.$$

$\therefore \quad F(a)>F(b)=0$ 得证

9. f(x)在[a,b]上连续，且f(x)>0，证：$\ln\left[\frac{1}{b-a}\int_a^b f(x)dx\right] \geq \frac{1}{b-a}\int_a^b \ln f(x)dx$

令 $A=\frac{1}{b-a}\int_a^b f(x)dx$ 即证：$\ln A \geq \frac{1}{b-a}\int_a^b \ln f(x)dx$

$$\ln A(b-a) \geq \int_a^b \ln f(x)dx$$

即 $\int_a^b \ln A\,dx \geq \int_a^b \ln f(x)dx \Rightarrow \int_a^b(\ln f(x)-\ln A)dx \leq 0$

$\ln(1+x)\leq x$ $\quad \ln f(x)-\ln A = \ln\left[1+\frac{f(x)}{A}-1\right] \leq \frac{f(x)}{A}-1$

原式 $\leq \int_a^b\left(\frac{f(x)}{A}-1\right)dx = \frac{1}{A}\int_a^b f(x)dx-(b-a)=0$

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3. f(x)在[0,1]上有连续的导数，证明：

$$\int_0^1|f(x)|dx \leq \max\left\{\int_0^1|f'(x)|dx,\ \left|\int_0^1 f(x)dx\right|\right\}$$

① f(x)在[0,1]上无零点，则 $|f(x)|$ 恒大于0

有 $\int_0^1|f(x)|dx = \left|\int_0^1 f(x)dx\right| \leq \max\left\{\int_0^1|f'(x)|dx,\ \left|\int_0^1 f(x)dx\right|\right\}$

② f(x)在[0,1]上有零点，设零点为c

$f(x)=f(x)-f(c)=\int_c^x f'(t)dt \quad x\in[0,1]$

常数

$\int_0^1|f(x)|dx = \int_0^1\left|\int_c^x f'(t)dt\right| \leq \int_0^1\boxed{\left(\int_0^1|f'(t)|dt\right)}dx = \int_0^1|f'(t)|dt$

积分绝对值 ≤ 绝对值的积分

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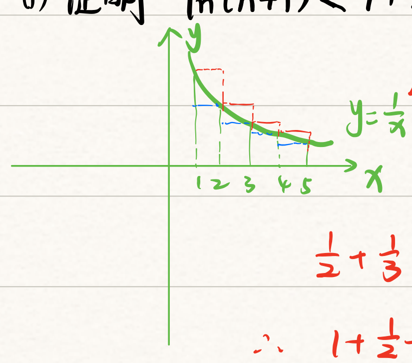
4. (1) 证明 $\ln(n+1)<1+\frac{1}{2}+\frac{1}{3}+\cdots+\frac{1}{n}<1+\ln n$.



$\int_k^{k+1}\frac{1}{x}dx<\frac{1}{k}<\int_{k-1}^{k}\frac{1}{x}dx$

$\ln(1+n)=\int_1^{n+1}\frac{1}{x}dx<\sum_{i=1}^{n}\frac{1}{i}$

$\frac{1}{2}+\frac{1}{3}+\cdots+\frac{1}{n}<\int_1^n\frac{1}{x}dx$

∴ $1+\frac{1}{2}+\frac{1}{3}+\cdots+\frac{1}{n}<1+\ln n$.

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