强化训练

§4.7. $y=f\left(\frac{3x-2}{3x+2}\right)$ $f'(x)=\arctan x^2$

$\frac{dy}{dx}=f'\cdot\frac{3(3x+2)-3(3x-2)}{(3x+2)^2}=\frac{12}{(3x+2)^2}\cdot f'$

$x=0$ 时 $\left.\frac{dy}{dx}\right|_{x=0}=f'(-1)\cdot\frac{12}{4}=\arctan 1\cdot 3=\frac{3}{4}\pi.$

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§4.9 $y=-ye^x+2e^y\sin x-7x$ $x=0$ $y=0$

$y'=-y'e^x-ye^x+2e^y\cdot y'\sin x+2e^y\cos x-7$ $x=0$ $y'=-\frac{5}{2}$

$y''=-y''e^x-y'e^x-y'e^x-\cancel{ye^x}+2e^y\cdot(\cancel{y''\sin x}+y'\cos x)+2e^y\cdot \cancel{y'\cdot y'\cdot\sin x}$

$+2e^y\cdot y'\cos x+2e^y\cancel{(\sin x)}$ $x=0$, $y=0$, $y'=-\frac{5}{2}$代入

$y''=-y''-2\cancel{y'}+2\cancel{y'}+2y'$ $\Rightarrow$ $y''=y'=-\frac{5}{2}$

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§4.14 $f(x)=\frac{x}{1-2x^4}$ $f^{(101)}(0)$ $\frac{1}{1-x}=\sum_{n=0}^{\infty}x^n$

$f(x)=\frac{x}{1-2x^4}=x\cdot\frac{1}{1-2x^4}=x\sum_{n=0}^{\infty}(2x^4)^n=\sum_{n=0}^{\infty}2^n\cdot x^{4n+1}$ (其余累加项101阶导为0)

$n=25$ 时 $f^{(101)}(x)=2^{25}\cdot 101!$

或 根据展开式 $f(x)=f(0)+\cdots\cdots\frac{f^{(101)}(0)}{101!}\cdot x^n+\cdots$ # $\frac{f^{(101)}(0)}{101!}=2^{25}.$

§4.21 $f(x)=g'(x)$ $g(x)=\begin{cases}\frac{e^x-1}{x} & x\neq 0\\ 1 & x=0\end{cases}$

$n=0$ $\frac{x^n}{n!}=1$

$\frac{e^x-1}{x}=\frac{1}{x}\left[\sum_{n=0}^{\infty}\frac{x^n}{n!}-1\right]=\sum_{n=1}^{\infty}\frac{x^{n-1}}{n!}$, $x\neq 0$, 且 $\left.\sum_{n=1}^{\infty}\frac{x^{n-1}}{n!}\right|_{x=0}=1$

$\therefore g(x)=\sum_{n=1}^{\infty}\frac{x^{n-1}}{n!}=\sum_{n=0}^{\infty}\frac{x^n}{(n+1)!}$ $f(x)=g'(x)=\sum_{n=1}^{\infty}n\cdot\frac{x^{n-1}}{(n+1)!}=\sum_{n=0}^{\infty}\frac{(n+1)x^n}{(n+2)!}$

同上 展开式 $\frac{f^{(n)}(0)}{n!}=\frac{(n+1)}{(n+2)!}$ $f^{(n)}(0)=\frac{1}{n+2}$ #

巩固提高

§4.2 $\begin{cases} x = \tan t \\ y = \dfrac{u(t)}{\cos t} \end{cases}$ $(1+x^2)^2 \cdot y'' = y$

$\frac{dy}{dt} = \frac{u'(t)\cdot\cos t + \sin t \cdot u(t)}{\cos^2 t}$ $\frac{dx}{dt} = \frac{1}{\cos^2 t}$

$\frac{dy}{dx} = u'(t)\cdot\cos t + u(t)\cdot\sin t$

$\frac{dy^2}{dx^2} = \frac{u''(t)\cos t + \cancel{u'(t)(-\sin t)} + \cancel{u'(t)\sin t} + u(t)\cdot\cos t}{\frac{1}{\cos^2 t}}$

$= u''(t)\cdot\cos^3 t + u(t)\cos^3 t$

代入 $(1+\tan^2 t)^2 \cdot (u''(t) + u(t)) \cdot \cos^3 t = \frac{u(t)}{\cos t}$

$\Downarrow$

$\cancel{\sec^4 t}$ $u''(t) + u(t)$ $\cancel{\cos^4 t} = u(t) \Rightarrow u''(t) = 0$ #

§4.4 $f(x) = (x^2-3x+2)^n \cos\frac{\pi x^2}{16}$

$f(x) = (x-1)^n \cdot (x-2)^n \cdot \cos\frac{\pi x^2}{16}$

$f^{(n)}(x) = C_n^0 \left((x-2)^n\right)^{(n)} \cdot \left((x-1)^n \cdot \cos\frac{\pi x^2}{16}\right)^{(0)}$

只有此时 $f^{(n)}(x)$ 不会出现 $(x-2)$ 因式 有意义

$f^{(n)}(x) = n! \cdot \cos\frac{\pi}{4} = \frac{\sqrt{2}}{2} n!$

$\sin 2x\cos x + \cos 2x \sin x - \sin x$

§4.5(2) $f(x) = \cos 2x \sin x = \frac{1}{2}(\sin 3x - \sin x)$ 积化和差

$\cos\alpha \cdot \sin\beta = \frac{1}{2}(\sin(\alpha+\beta) - \sin(\alpha-\beta))$

$(\text{原式})^{(20)} = \frac{1}{2}(\sin 3x - \sin x)^{(20)} = \frac{1}{2}\cdot\left(3^{20}\cdot\sin(3x+10\pi) - \sin(x+10\pi)\right)$

$= \frac{1}{2}\cdot(3^{20}\sin 3x - \sin x)$

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§4.7. $y=\arcsin x$ $y'=\frac{1}{\sqrt{1-x^2}}$ $y''=\frac{x}{(1-x^2)^{\frac{3}{2}}}$

$$\Downarrow$$

$$(1-x^2)y''-xy'=0$$

莱布尼茨公式

①

$$C_n^0\cdot(1-x^2)y^{(n+2)}+C_n^1\cdot(-2x)\cdot y^{(n+1)}+C_n^2\cdot(-2)\cdot y^{(n)}$$

$$=C_n^0 x y^{(n+1)}+C_n^1\cdot y^{n}$$

$$(1-x^2)y^{(n+2)}-2nxy^{(n+1)}-n(n-1)y^{(n)}=xy^{(n+1)}+ny^{(n)}$$

$$(1-x^2)y^{(n+2)}-(2n+1)xy^{(n+1)}-n^2y^{(n)}=0$$

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②

令上式 $x=0$ $y^{(n+2)}_{(0)}=n^2y^{(n)}$ $y_{(0)}=0 \Rightarrow y^{(2k)}_{(0)}=0$

$$y'_{(0)}=1 \Rightarrow y^{(2k+1)}_{(0)}=(2k-1)^2y^{(2k-1)}_{(0)}=\cdots\ 1^2\cdot y'_{(0)}$$

$$=[(2k-1)!!]^2\quad(k=1,2,3\cdots)$$

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双阶乘表示与此相同奇偶性阶乘