强化训练

§5.5. $f(x)=\int_0^x \sin^2(\pi t)\,dt-\frac{x^2}{2}$ 驻点

$f'(x)=\sin^2(\pi x)-x=0$ $x_1=0$

$f'(\frac{1}{10})=\sin^2\frac{\pi}{10}-\frac{1}{10}<\frac{\pi^2-10}{100}<0$ $f'(\frac{1}{4})=\sin^2\frac{\pi}{4}-\frac{1}{4}>0$

$x_2\in(\frac{1}{10},\frac{1}{4})$

$f'(\frac{1}{2})=\frac{1}{2}$ $f'(\frac{3}{4})=-\frac{1}{4}$ $x_3\in(\frac{1}{2},\frac{3}{4})$

3个驻点

#

§5.8 $f(x)=\begin{cases} e^{\frac{1}{x}} & x<0 \\ (3-x)\cdot\sqrt{x} & x\geq 0\end{cases}$ 拐点

$\lim\limits_{x\to 0^+} f(x)=\lim\limits_{x\to 0^-} f(x)=0$ 连续

$x\neq 0$ 时

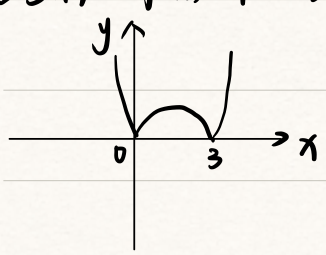
$f'(x)=\begin{cases} -\frac{e^{\frac{1}{x}}}{x^2} & x<0 \\ \frac{3(1-x)}{2\sqrt{x}} & x>0\end{cases}$ $f''(x)=\begin{cases} \frac{(2x+1)e^{\frac{1}{x}}}{x^4} & x<0 \\ \frac{-3(x+1)}{4x\cdot\sqrt{x}} & x>0\end{cases}$

$x=-\frac{1}{2}$ $f''(x)=0$

$x<-\frac{1}{2}$ 凹 $-\frac{1}{2}<x<0$ 凸 $x>0$ 凹 即 $x=-\frac{1}{2}$ / 0 均为拐点

#

§5.14 $f(x)=|x(3-x)|$



$x=0$ 不可导 但为极小值点

#

§5.20. $y=ax^2+bx+c$ $\quad P(1,2)$ $\quad$ 曲率圆 $(x-\frac{1}{2})^2+(y-\frac{5}{2})^2=\frac{1}{2}$

P在曲线上 $\Rightarrow$ $a+b+c=2$

曲率圆在P点曲率 $\quad y_P'=-\left.\frac{x-\frac{1}{2}}{y-\frac{5}{2}}\right|_{\substack{x=1\\y=2}}=1=2a\cdot 1+b.$

曲率 $=\frac{1}{\text{曲率半径}}$

$\frac{|y''|}{(1+y'^2)^{\frac{3}{2}}}=\frac{2a}{(1+1)^{\frac{3}{2}}}=\frac{2a}{2\sqrt{2}}=\frac{1}{\sqrt{2}}\Rightarrow a=2 \quad b=-3 \quad c=3$

§5.29. $e^x-e^y=xy$ 在 $(0,0)$ 的曲率圆.

$e^x-e^y\cdot y'=y+xy'$ $\qquad 1-y'=0$ $\qquad y'=1$

$e^x-e^y(y')^2-e^y\cdot y''=y'+y'+xy''$ $\qquad 1-1-y''=2$ $\qquad y''=-2.$

$k=\frac{|y''|}{(1+y'^2)^{\frac{3}{2}}}=\frac{2}{2^{\frac{3}{2}}}=\frac{2}{2\sqrt{2}}=\frac{1}{\sqrt{2}}\Rightarrow R=\sqrt{2}.$

$y''_{(0)}=-2.\Rightarrow$ 凸

曲率圆向曲线凹的(法线)方向建立

$y'_{(0)}=1\Rightarrow$ 法线 $y=-x\Rightarrow$ 令圆心 $(a,-a)$

$(x-a)^2+(y+a)^2=2$ $\quad$ 过 $(0,0)\Rightarrow a=1$

$\Downarrow$

$(x-1)^2+(y+1)^2=2$

#

§5.33 $\quad r=\cos 2\theta$ $\quad \theta=\frac{\pi}{4}$ 切线方程

$\Downarrow$ 参数

$\begin{cases} x=\cos 2\theta\cos\theta \\ y=\cos 2\theta\sin\theta \end{cases}$ $\qquad \theta=\frac{\pi}{4}\Rightarrow(0,0)$

$\frac{dx}{d\theta}=-2\sin 2\theta\cos\theta+(-\sin\theta)\cos 2\theta$

$\frac{dy}{d\theta}=-2\sin 2\theta\sin\theta+\cos\theta\cos 2\theta.$

$\left.\frac{dy}{dx}\right|_{(0,0)}=\frac{-2\times 1\times\frac{\sqrt{2}}{2}}{-2\times\frac{\sqrt{2}}{2}-\frac{\sqrt{2}}{2}\times 0}=1$ $\qquad \therefore y=x$ $\quad$ #

巩固提高

§5.1 $f(x)=(1+x+\frac{x^2}{2!}+\cdots+\frac{x^n}{n!})e^{-x}$

$f'(x)=(1+x+\frac{x^2}{2!}+\cdots+\frac{x^{n-1}}{(n-1)!})e^{-x}-e^{-x}(1+x+\cdots+\frac{x^n}{n!})$

$=-\frac{x^n}{n!}e^{-x}$

n为偶数 恒减 n为奇数 $x=0$为极大值点.

#

§5.5 $f(x)=\max\limits_{0\le y\le 1}\frac{|x-y|}{x+y+1}$ $0\le x\le 1$

① $0\le y\le x$ $f(x)=\frac{-x-y-1+2x+1}{x+y+1}=-1+\frac{2x+1}{x+y+1}$ $y\nearrow$ $f(x)\downarrow$

$\therefore f(x)_{max}=f(x)|_{y=0}=-1+\frac{2x+1}{x+1}=\frac{x}{x+1}$

② $x<y\le 1$ $f(x)=\frac{y-x}{x+y+1}=\frac{y+x+1-2x-1}{x+y+1}=1-\frac{2x+1}{x+y+1}$ $y\nearrow$ $f(x)\downarrow$

$\therefore f(x)_{max}=f(x)|_{y=1}=\frac{1-x}{x+2}$

$\frac{x}{x+1}\le\frac{1-x}{x+2}$ $0<x\le\frac{\sqrt{3}-1}{2}$

即 $f(x)=\begin{cases}\frac{1-x}{x+2} & [0,\frac{\sqrt{3}-1}{2})\\ \frac{x}{x+1} & [\frac{\sqrt{3}-1}{2},1]\end{cases}$ $\therefore f(\frac{\sqrt{3}-1}{2})$为min $\Rightarrow 2-\sqrt{3}$

$f(0)=f(1)=\frac{1}{2}$为max

#

§5.10 $2f(2+x)+f(2-x)=3+2x+o(x)$

令 $x=0$ $3f(2)=3$ $f(2)=1$

$\frac{2[f(2+x)-f(2)]}{x}-\frac{f(2-x)-f(2)}{-x}=2+\frac{o(x)}{x}$

$2f'(2)-f'(2)=2$ $f'(2)=2$

§5.11 $y=\tan^n x$ 在 $x=\frac{\pi}{4}$ 处切线在 x 轴上截距 x_n，求 $\lim\limits_{n\to\infty} y(x_n)$

$$y'(\tfrac{\pi}{4}) = n\tan^{n-1}x\cdot\sec^2 x\,\Big|_{x=\frac{\pi}{4}} = 2n$$

$$y-1 = 2n(x-\tfrac{\pi}{4}) \Rightarrow x_n = \tfrac{\pi}{4}-\tfrac{1}{2n}$$

$$\lim_{n\to\infty} y(x_n) = \lim_{n\to\infty}\tan^n(\tfrac{\pi}{4}-\tfrac{1}{2n}) = e^{\lim\limits_{n\to\infty} n\ln(\tan(\frac{\pi}{4}-\frac{1}{2n}))} = e^{\lim\limits_{n\to\infty} n[\tan(\frac{\pi}{4}-\frac{1}{2n})-1]}$$

$$= e^{\lim\limits_{n\to\infty}\frac{[\tan(\frac{\pi}{4}-\frac{1}{2n})-\tan\frac{\pi}{4}]}{\frac{1}{n}}} = e^{\lim\limits_{n\to\infty} -\frac{1}{2}\underbrace{\frac{\tan(\frac{\pi}{4}-\frac{1}{2n})-\tan\frac{\pi}{4}}{-\frac{1}{2n}}}}$$ 凑导数定义

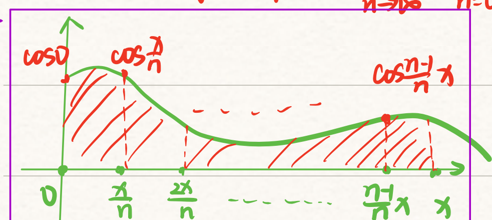
$$= e^{-\frac{1}{2}(\tan x)'|_{x=\frac{\pi}{4}}}$$

$$= e^{-\frac{1}{2}\cdot 2} = e^{-1}$$ #

§5.16 $f(x)=\begin{cases}\lim\limits_{n\to\infty}\frac{1}{n}(1+\cos\frac{x}{n}+\cos\frac{2x}{n}+\cdots+\cos\frac{n-1}{n}x), & x>0\\ 1 & x=0\\ f(-x) & x<0\end{cases}$

当 $x>0$ 时 $f(x)=\lim\limits_{n\to\infty}\frac{1}{n}\sum\limits_{n=0}^{n-1}\cos\frac{i}{n}x = \lim\limits_{n\to\infty}\frac{1}{x}\sum\limits_{i=0}^{n-1}\cos(\frac{i}{n}\cdot x)\,\frac{x}{n}$

*



以左端点y值为高

即 $[0,x]$ 上 $\cos x$ 积分

$$=\frac{1}{x}\int_0^x \cos t\,dt = \frac{\sin x}{x}$$

$$f(-x)=\frac{\sin(-x)}{-x}=\frac{\sin x}{x}$$

即 $f(x)=\begin{cases}\frac{\sin x}{x} & x\neq 0\\ 1 & x=0\end{cases}$ #