强化训练

§4.7. $y=f\left(\frac{3x-2}{3x+2}\right)$ $f'(x)=\arctan x^2$

$$\frac{dy}{dx}=f'\cdot\frac{3(3x+2)-3(3x-2)}{(3x+2)^2}=\frac{12}{(3x+2)^2}\cdot f'$$

$x=0$ 时 $\left.\frac{dy}{dx}\right|_{x=0}=f'(-1)\cdot\frac{12}{4}=\arctan 1\cdot 3=\frac{3}{4}\pi.$

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§4.9 $y=-ye^x+2e^y\sin x-7x$ $x=0$ $y=0$

$$y'=-y'e^x-ye^x+2e^y\cdot y'\sin x+2e^y\cos x-7 \quad x=0 \quad y'=-\frac{5}{2}$$

$$y''=-y''e^x-y'e^x-y'e^x-\cancel{ye^x}+2e^y\cdot(\cancel{y''\sin x}+y'\cos x)+2e^y\cdot \cancel{y'\cdot y'\cdot\sin x}$$

$$+2e^y\cdot y'\cos x+2e^y(\cancel{\sin x})$$ $x=0,\ y=0,\ y'=-\frac{5}{2}$ 代入

$$y''=-y''-2y'+2y'+2y'\Rightarrow y''=y'=-\frac{5}{2}$$

#

§4.14 $f(x)=\frac{x}{1-2x^4}$ $f^{(101)}(0)$ $\frac{1}{1-x}=\sum_{n=0}^{\infty}x^n$

$$f(x)=\frac{x}{1-2x^4}=x\cdot\frac{1}{1-2x^4}=x\sum_{n=0}^{\infty}(2x^4)^n=\sum_{n=0}^{\infty}2^n\cdot x^{4n+1}$$

(其余累加项101阶导为0)

$n=25$ 时 $f^{(101)}(x)=2^{25}\cdot 101!$

或 根据展开式 $f(x)=f(0)+\cdots\frac{f^{(101)}(0)}{101!}\cdot x^n+\cdots$ # $\frac{f^{(101)}(0)}{101!}=2^{25}.$

§4.21 $f(x)=g'(x)$ $g(x)=\begin{cases}\frac{e^x-1}{x} & x\neq 0\\ 1 & x=0\end{cases}$

$n=0$ $\frac{x^n}{n!}=1$

$$\frac{e^x-1}{x}=\frac{1}{x}\left[\sum_{n=0}^{\infty}\frac{x^n}{n!}-1\right]=\sum_{n=1}^{\infty}\frac{x^{n-1}}{n!},\ x\neq 0,\ \text{且}\ \left.\sum_{n=1}^{\infty}\frac{x^{n-1}}{n!}\right|_{x=0}=1$$

$$\therefore g(x)=\sum_{n=1}^{\infty}\frac{x^{n-1}}{n!}=\sum_{n=0}^{\infty}\frac{x^n}{(n+1)!} \qquad f(x)=g'(x)=\sum_{n=1}^{\infty}n\cdot\frac{x^{n-1}}{(n+1)!}=\sum_{n=0}^{\infty}\frac{(n+1)x^n}{(n+2)!}$$

同上 展开式 $\frac{f^{(n)}(0)}{n!}=\frac{(n+1)}{(n+2)!}$ $f^{(n)}(0)=\frac{1}{n+2}$ #

巩固提高

§4.2 $\begin{cases} x = \tan t \\ y = \dfrac{u(t)}{\cos t} \end{cases}$ $\qquad (1+x^2)^2 \cdot y'' = y$

$$\frac{dy}{dt} = \frac{u'(t)\cdot\cos t + \sin t\cdot u(t)}{\cos^2 t} \qquad \frac{dx}{dt} = \frac{1}{\cos^2 t}$$

$$\frac{dy}{dx} = u'(t)\cdot\cos t + u(t)\cdot\sin t$$

$$\frac{dy^2}{dx^2} = \frac{u''(t)\cos t + \cancel{u'(t)\cdot(-\sin t)} + \cancel{u'(t)\sin t} + u(t)\cdot\cos t}{\frac{1}{\cos^2 t}}$$

$$= u''(t)\cdot\cos^3 t + u(t)\cos^3 t$$

代入 $(1+\tan^2 t)^2 \cdot (u''(t)+u(t))\cdot\cos^3 t = \dfrac{u(t)}{\cos t}$

$$\cancel{\sec^4 t}\quad u''(t)+u(t)\quad \cancel{\cos^4 t} = u(t) \Rightarrow u''(t) = 0$$ #

§4.4 $f(x) = (x^2-3x+2)^n \cos\frac{\pi x^2}{16}$

$$f(x) = (x-1)^n\cdot(x-2)^n\cdot\cos\frac{\pi x^2}{16}$$

$$f^{(n)}(x) = C_n^0\left((x-2)^n\right)^{(n)}\cdot\left((x-1)^n\cdot\cos\frac{\pi x^2}{16}\right)^{(0)}$$

只有此时 $f^{(n)}(x)$ 不会出现 $(x-2)$ 因式 有意义

$$f^{n}(x) = n!\cdot\cos\frac{\pi}{4} = \frac{\sqrt{2}}{2}n!$$

$\sin 2x\cos x + \cos 2x\sin x - \sin x$

§4.5(2) $f(x) = \cos 2x\sin x = \frac{1}{2}(\sin 3x - \sin x)$ 积化和差

$$\cos\alpha\cdot\sin\beta = \frac{1}{2}(\sin(\alpha+\beta) - \sin(\alpha-\beta))$$

$$(原式)^{(20)} = \frac{1}{2}(\sin 3x - \sin x)^{(20)} = \frac{1}{2}\cdot(3^{20}\cdot\sin(3x+10\pi) - \sin(x+10\pi))$$

$$= \frac{1}{2}\cdot(3^{20}\sin 3x - \sin x)$$

#

§ 4.7 $y = \arcsin x$ $y' = \frac{1}{\sqrt{1-x^2}}$ $y'' = \frac{x}{(1-x^2)^{\frac{3}{2}}}$

$$\Downarrow$$

$$(1-x^2)y'' - xy' = 0$$

莱布尼茨公式

①

$$C_n^0 \cdot (1-x^2) y^{(n+2)} + C_n^1 \cdot (-2x) \cdot y^{(n+1)} + C_n^2 \cdot (-2) \cdot y^{(n)}$$

$$= C_n^0 x y^{(n+1)} + C_n^1 \cdot y^n$$

$$(1-x^2) y^{(n+2)} - 2nx y^{(n+1)} - n(n-1) y^{(n)} = x y^{(n+1)} + n y^{(n)}$$

$$(1-x^2) y^{(n+2)} - (2n+1) x y^{(n+1)} - n^2 y^{(n)} = 0$$

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②

令上式 $x=0$ $y^{(n+2)}_{(0)} = n^2 y^{(n)}$ $y_{(0)} = 0 \Rightarrow y^{(2k)}_{(0)} = 0$

$y'_{(0)} = 1 \Rightarrow y^{(2k+1)}_{(0)} = (2k-1)^2 y^{(2k-1)}_{(0)} = \cdots\cdots 1^2 \cdot y'_{(0)}$

$$= [(2k-1)!!]^2 \quad (k = 1, 2, 3 \cdots)$$

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双阶乘表示与此相同奇偶性阶乘