强化训练

§2.3. $x_n=(1+\frac{1}{n^2})(1+\frac{2}{n^2})\cdots(1+\frac{n}{n^2})$

同取对数 $\ln x_n=\sum_{i=0}^{n}\ln(1+\frac{i}{n^2})$

根据不等式 $\underbrace{\frac{x}{1+x}<\ln(1+x)<x}$ $(x>0)$

证左边: $f(x)=(1+x)\ln(1+x)-x$

$f'(x)=\ln(1+x)+1-1=\ln(1+x)\overset{x>0}{>}0.$

$f(x)>f(0)=0$ 成立.

即 $\sum_{i=0}^{n}\frac{\frac{i}{n^2}}{1+\frac{i}{n^2}}<\sum_{i=0}^{n}\ln(1+\frac{i}{n^2})<\sum_{i=0}^{n}\frac{i}{n^2}$

$\sum_{i=0}^{n}\frac{i}{n^2+i}$ $\Downarrow$ $<\frac{\frac{(1+n)n}{2}}{n^2}=\frac{(1+n)}{2n}$

$\downarrow$ 继续放缩 $\Downarrow n\to\infty$

$\sum_{i=0}^{n}\frac{i}{n^2+n}=\frac{\frac{n(n+1)}{2}}{n(n+1)}=\frac{1}{2}\Rightarrow \frac{1}{2}\Leftarrow\frac{1}{2}$

即 $\lim\limits_{n\to\infty}x_n=e^{\frac{1}{2}}$ #

§2.9. $x_n=\frac{2022}{2023}x_{n-1}+\frac{1}{x_{n-1}^{2022}}$

$x_n=\frac{1}{2023}(2022x_{n-1}+\frac{2023}{x_{n-1}^{2022}})$

$=\underbrace{\frac{x_{n-1}+\cdots x_{n-1}+\frac{2023}{x_{n-1}^{2022}}}{2023}\geqslant(2023)^{\frac{1}{2023}}}$ 有下界

$\frac{x_n}{x_{n-1}}=\frac{2022}{2023}+\frac{1}{x_{n-1}^{2023}}\leqslant\frac{2022}{2023}+\frac{1}{2023}=1$ 单减

巩固提高

§2.4, §2.5, §2.6

第3讲 强化训练

§3.2 $f(x)=\begin{cases}\dfrac{e^{x^2}-1}{x} & x>0\\ x^2 g(x) & x\le 0\end{cases}$

当 $x>0$ 时 $\lim\limits_{x\to0^+} f(x) = \lim\limits_{x\to0^+}\dfrac{e^{x^2}-1}{x} = \dfrac{x^2}{x} = x = 0 = f(0)$

$\therefore$ $f(x)$ 连续

判断可导 $\Leftarrow$ 极限左右相等

$$\lim_{x\to0^+}\frac{f(x)-f(0)}{x} = \frac{\frac{e^{x^2}-1}{x}}{x} = \lim_{x\to0^+}\frac{e^{x^2}-1}{x^2} = 1$$

$$\lim_{x\to0^-}\frac{x^2g(x)}{x} = \lim_{x\to0^-} x g(x) = 0$$

$\neq$ 极限不存在

不可导

#

§3.3 $f(x)=\int_0^1 \ln\sqrt{x^2+y^2}\,dy \quad x\in[0,1]$

$$f(x) = y\cdot\ln\sqrt{x^2+y^2}\Big|_{y=0}^{y=1} - \int_0^1 y\cdot\frac{1}{\sqrt{x^2+y^2}}\cdot\frac{2y}{2\sqrt{x^2+y^2}}\,dy$$

$$= \ln\sqrt{x^2+1} - \int_0^1 \frac{y^2}{x^2+y^2}\,dy$$

$$= \ln\sqrt{x^2+1} - \int_0^1 \frac{x^2+y^2-x^2}{x^2+y^2}\,dy = \ln\sqrt{x^2+1} - 1 + \int_0^1 \frac{1}{1+\left(\frac{y}{x}\right)^2}\,dy$$

$$= \ln\sqrt{x^2+1} - 1 + x\int_0^1 \frac{1}{1+\left(\frac{y}{x}\right)^2}\,d\left(\frac{y}{x}\right)$$

$$= \ln\sqrt{x^2+1} - 1 + x\arctan\frac{y}{x}\Big|_0^1$$

$$= \ln\sqrt{x^2+1} - 1 + x\arctan\frac{1}{x}$$

$f(0)=-1$

$$f'_+(0) = \lim_{x\to0^+}\frac{f(x)-f(0)}{x} = \lim_{x\to0^+}\frac{\ln\sqrt{x^2+1}\cancel{-1}+x\arctan\frac{1}{x}\cancel{+1}}{x} = \frac{\frac{1}{2}\ln(1+x^2)}{x} + \arctan\frac{1}{x}$$

$t\to+\infty$

$$= 0+\frac{\pi}{2} = \frac{\pi}{2}$$

#

§3.5

$dy = A\,dx = f'(x)\,dx = f'(x)\cdot\Delta x$

$0.3 = 3x^2\cdot f'(x^3)\cdot(-0.1)$

$0.3 = 3\cdot f'(-1)\cdot(-0.1)$ $\qquad$ $f'(-1) = -1$

#

§3.12.

※ $y=\sqrt[3]{x}$ 在 $x=0$ 处不可导.

已知 $f'(0)=a$.

§3.13. $f(x+y) = \dfrac{f(x)+f(y)}{1-f(x)f(y)}$ $\Rightarrow$ $f(0) = \dfrac{2f(0)}{1-f^2(0)}$ $\qquad$ $f(0)=0$

$$f'(x) = \lim_{\Delta x\to 0}\frac{f(x+\Delta x)-f(x)}{\Delta x} = \lim_{\Delta x\to 0}\frac{\frac{f(x)+f(\Delta x)}{1-f(x)f(\Delta x)} - f(x)}{\Delta x} = \lim_{\Delta x\to 0}\frac{f(x)+f(\Delta x) - f(x)(1-f(x)f(\Delta x))}{\Delta x(1-f(x)f(\Delta x))}$$

$$= \lim_{\Delta x\to 0}\frac{f(\Delta x)+f^2(x)f(\Delta x)}{\Delta x(1-f(x)f(\Delta x))} = \lim_{\Delta x\to 0}\frac{f(0+\Delta x)-f(0)}{\Delta x}\cdot\lim_{\Delta x\to 0}\frac{1+f^2(x)}{1-f(x)f(\Delta x)}$$

$$= f'(0)\cdot[1+f^2(x)] \qquad f(0)=0.$$

即 $f'(x) = a\cdot[1+f^2(x)]$

$\dfrac{dy}{dx} = a\cdot[1+y^2]$ $\qquad$ $\dfrac{dy}{1+y^2} = a\cdot dx$ $\qquad$ 微分方程

$\arctan y + C = ax$

即 $\arctan y = ax + C$

又 ∵ $f(0)=0$ $\quad$ ∴ $C=0$ $\quad$ ∴ $y = \tan ax$

#

§3.4 巩固提高

4.【解析】 $f'(x_0)=\lim\limits_{\Delta x\to 0}\dfrac{f(x_0+\Delta x)-f(x_0)}{\Delta x}$

$$\Rightarrow \frac{f(x_0+\Delta x)-f(x_0)}{\Delta x}=f'(x_0)+\alpha,\text{其中 }\alpha\to 0(\Delta x\to 0).$$

于是 $f(x_0+\Delta x)=f(x_0)+[f'(x_0)+\alpha]\Delta x$，其中 $\alpha\to 0(\Delta x\to 0)$，分别取 $\Delta x=\alpha_n$ 及 $\Delta x=-\beta_n$，则

$$f(x_0+\alpha_n)=f(x_0)+[f'(x_0)+\alpha_1]\alpha_n,\text{其中 }\alpha_1\to 0(\alpha_n\to 0);$$

$$f(x_0-\beta_n)=f(x_0)-[f'(x_0)+\alpha_2]\beta_n,\text{其中 }\alpha_2\to 0(\beta_n\to 0).$$

于是
$$f(x_0+\alpha_n)-f(x_0-\beta_n)=f'(x_0)(\alpha_n+\beta_n)+\alpha_1\alpha_n+\alpha_2\beta_n,$$

从而
$$\frac{f(x_0+\alpha_n)-f(x_0-\beta_n)}{\alpha_n+\beta_n}=f'(x_0)+\frac{\alpha_1\alpha_n+\alpha_2\beta_n}{\alpha_n+\beta_n},\tag{*}$$

当 $n\to\infty$ 时，$\dfrac{\alpha_1\alpha_n+\alpha_2\beta_n}{\alpha_n+\beta_n}\to 0$，理由如下：

由于
$$0\leqslant\left|\frac{\alpha_1\alpha_n+\alpha_2\beta_n}{\alpha_n+\beta_n}\right|=\left|\frac{\alpha_1\alpha_n}{\alpha_n+\beta_n}+\frac{\alpha_2\beta_n}{\alpha_n+\beta_n}\right|\leqslant\left|\frac{\alpha_1\alpha_n}{\alpha_n+\beta_n}\right|+\left|\frac{\alpha_2\beta_n}{\alpha_n+\beta_n}\right|$$
$$\leqslant\left|\frac{\alpha_1\alpha_n}{\alpha_n}\right|+\left|\frac{\alpha_2\beta_n}{\beta_n}\right|=|\alpha_1|+|\alpha_2|\to 0,$$

这里用到 $\{\alpha_n\}$，$\{\beta_n\}$ 都是正项数列，对（*）式两端取极限，有

$$\lim_{n\to\infty}\frac{f(x_0+\alpha_n)-f(x_0-\beta_n)}{\alpha_n+\beta_n}=f'(x_0)+0=f'(x_0).$$