强化训练

§6.9 $f(a)=f(b)=0$ 证 $\forall x\in(a,b)$, $\xi\in(a,b)$ 使 $f''(\xi)=\dfrac{2f(x)}{(x-a)(x-b)}$

令 $\boxed{k=\dfrac{f(x)}{(x-a)(x-b)}}$ 关键

$F(t)=f(t)-k(t-a)(t-b)$ 则由题 $F(a)=F(x)=F(b)=0$

罗尔定理 $\xi_1\in(a,x)$ $F'(\xi_1)=0$ $\xi_2\in(x,b)$ $F'(\xi_2)=0$

$\xi_3\in(\xi_1,\xi_2)$ $F''(\xi_3)=0$

$\Downarrow$

即 $f''(t)-2k=0$

得证. $f''(\xi_3)=2\dfrac{f(x)}{(x-a)(x-b)}$ #

§6.17 $x\in(0,\frac{\pi}{2})$ 证 $\dfrac{\sin x}{x}>\sqrt[3]{\cos x}$

证: 令 $F(x)=\underline{\sin x\cdot(\cos x)^{-\frac{1}{3}}}-x$ 把三角函数部分放一起，易导

$F'(x)=\cos x\cdot(\cos x)^{-\frac{1}{3}}+\sin x\cdot(-\frac{1}{3})(\cos x)^{-\frac{4}{3}}\cdot(-\sin x)-1$

$=(\cos x)^{\frac{2}{3}}+\sin^2 x\cdot(\frac{1}{3})\cdot(\cos x)^{-\frac{4}{3}}-1$

$=(\cos x)^{\frac{2}{3}}[1+\frac{1}{3}(1-\cos^2 x)\cdot\cos^{-2}x]-1$

$=(\cos x)^{\frac{2}{3}}[1+\frac{1}{3}\cos^{-2}x-\frac{1}{3}]-1=\frac{1}{3}(\cos x)^{-\frac{4}{3}}+\frac{2}{3}(\cos x)^{\frac{2}{3}}-1$

$F''(x)=[-\frac{4}{9}(\cos x)^{-\frac{7}{3}}+\frac{4}{9}(\cos x)^{-\frac{1}{3}}](-\sin x)$

$=\frac{4}{9}(\cos x)^{-\frac{7}{3}}(-\sin x)(-\sin x)^2=\frac{4}{9}(\cos x)^{-\frac{7}{3}}\cdot(\sin x)^3>0$

$\therefore F'(x)>F'(0)=0$ $\therefore F(x)>F(0)$ 得证 #

6.18 证: $e^x+e^{-x}\geq 2x^2+2\cos x \quad -\infty<x<+\infty$

令 $f(x)=e^x+e^{-x}-2x^2-2\cos x$ 偶函数. 只需证 $x\geq 0$ $f(x)\geq 0$

$\Rightarrow f'(x)=e^x-e^{-x}-4x+2\sin x$

$f''(x)=e^x+e^{-x}-4+2\cos x$

$f'''(x)=e^x-e^{-x}-2\sin x$

$f^{(4)}(x)=e^x+e^{-x}-2\cos x\geq 2-2\cos x\geq 0.$

$f(0)=f'(0)=f''(0)=f'''(0)=f^{(4)}(0)=0$

$$f(x)=\underset{0}{f(0)}+\underset{0}{f'(0)}\cdot x+\underset{0}{\frac{f''(0)}{2!}}x^2+\underset{0}{\frac{f'''(0)}{3!}}x^3+\frac{f^{(4)}(\xi)}{4!}x^4\geq 0$$

$\therefore f(x)\geq 0.$ 得证 #

§6.19. $f'(a)=f'(b)=0$ 证: $\xi\in(a,b)$ $|f''(\xi)|\geq\frac{4}{(b-a)^2}|f(b)-f(a)|$

泰勒. 凑出

$$f(x)=f(a)+\frac{f'(a)}{2}(x-a)+\frac{f''(\xi_1)}{2}(x-a)^2 \quad \xi_1\in(a,x) \quad ①$$

$$f(x)=f(b)+\frac{f'(b)}{2}(x-b)+\frac{f''(\xi_2)}{2}(x-b)^2 \quad \xi_2\in(b,x) \quad ②$$

② - ① 得 $0=f(b)-f(a)+\frac{f''(\xi_2)}{2}(x-b)^2-\frac{f''(\xi_1)}{2}(x-a)^2.$

令 $x=\frac{a+b}{2}$ 代入 $\quad f(b)-f(a)+\frac{(a-b)^2}{4}\left(\frac{f''(\xi_2)}{2}-\frac{f''(\xi_1)}{2}\right)=0$

$$\frac{4|f(b)-f(a)|}{(b-a)^2}=\frac{1}{2}(f''(\xi_2)-f''(\xi_1))\leq\frac{1}{2}\left[|f''(\xi_2)|+|f''(\xi_1)|\right]$$

令 $|f''(\xi)|$ 为 $\max\{f''(\xi_1),f''(\xi_2)\}$

则 $\frac{4|f(b)-f(a)|}{(b-a)^2}\leq\frac{1}{2}\cdot 2\cdot|f''(\xi)|=|f''(\xi)|$

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巩固提高

§6.1 $f(a)=f(b)=0.\quad f''(x)+\cos f'(x)=e^{f(x)}$

1. (C) 【解析】由题设知，$f(x)$ 在$[a,b]$上连续，故必有最大值M与最小值m，若 $\exists x_0\in(a,b)$，使得 $f(x_0)>0$，则 $f(x)$ 的最大值 $M>0$，记最大值点为 ξ，则 $f(\xi)=M>0$，且由费马定理有 $f'(\xi)=0$. 代入题设方程，有 $f''(\xi)=e^{f(\xi)}-\cos f'(\xi)=e^M-1>0$，则 $x=\xi$为极小值点，矛盾，故 $f(x)\leqslant 0, x\in(a,b)$.

同理，若 $\exists x_1\in(a,b)$，使得 $f(x_1)<0$，则 $f(x)$ 的最小值 $m<0$，记最小值点为 η，则 $f(\eta)=m<0$，且由费马定理有 $f'(\eta)=0$. 代入题设方程，有 $f''(\eta)=e^{f(\eta)}-\cos f'(\eta)=e^m-1<0$，则$x=\eta$为极大值点，亦矛盾，故 $f(x)\geqslant 0, x\in(a,b)$.

综上所述，$f(x)\equiv 0, x\in[a,b]$.

§6.3 $f(a)=f(b)=0$ 　证 $2[f(\xi_i)]^2+f(\xi_i)f''(\xi_i)=0\quad(i=1,2)$

$F(x)=f^2(x)f'(x)$

§6.4 $|f(x)|\leq 1,\quad 0<|f''(x)|\leq 2,\quad x\in[0,+\infty)$ 　证：$|f'(x)|\leq 2\sqrt{2}$.

泰勒 　$f(x+h)=f(x)+f'(x)\cdot h+\frac{f''(\xi)}{2!}h^2.\quad \xi\in(x,x+h)$

$|f'(x)|=\frac{1}{h}[f(x+h)-f(h)]-\frac{h}{2}f''(\xi)$

$\leq\frac{1}{h}[|f(x+h)|+|f(h)|]+\frac{h}{2}|f''(\xi)|\leq\frac{2}{h}+h$

当 $h=\sqrt{2}$ 时 　$f'(x)\leq 2\sqrt{2}$ 　成立

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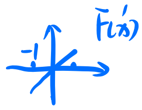
§6.7 $F(x)=\int_{-1}^{1}|x-t|e^{-t^2}dt-\frac{1}{2}(e^{-1}+1)$

$F'(x)=\int_{-1}^{x}e^{-t^2}dt-\int_{x}^{1}e^{-t^2}dt\ \overset{\text{用 } t=-u \text{ 代换}}{=\!=}\ \int_{-1}^{x}e^{-t^2}dt+\int_{-x}^{-1}e^{-u^2}du=\int_{-x}^{x}e^{-t^2}dt$

$=2\int_{0}^{x}e^{-t^2}dt$

$F''(x)=e^{-x^2}+e^{-x^2}=2e^{-x^2}>0$



$F'(0)=0$

$\therefore\ [-1,0)\ F(x)\downarrow\qquad (0,1]\ F(x)\uparrow$

$$F(-1)=\int_{-1}^{1}\underbrace{te^{-t^2}}_{\text{奇}\ \text{为}0}+\int_{-1}^{1}e^{-t^2}dt-\frac{1}{2}(e^{-1}+1)$$

$-e^{-t^2}$

$$=0+2\int_0^1 e^{-t^2}-\frac{1}{2}(e^{-1}+1)>2\int_0^1 e^{-t}dt-\frac{1}{2}(e^{-1}+1)=\frac{3}{2}-\frac{5}{2}e^{-1}>0$$

$$F(0)=\int_{-1}^{1}|t|e^{-t^2}dt-\frac{1}{2}(e^{-1}+1)=2\int_0^1 te^{-t^2}dt-\frac{1}{2}(e^{-1}+1)$$

$$=-e^{-1}+1-\frac{1}{2}e^{-1}-\frac{1}{2}=-\frac{3}{2}e^{-1}+\frac{1}{2}<0$$

$$F(1)=\int_{-1}^{1}e^{-t^2}dt-\int_{-1}^{1}te^{-t^2}dt-\frac{1}{2}(e^{-1}+1)$$

$$=2\int_0^1 e^{-t^2}dt-0-\frac{1}{2}(e^{-1}+1)>2\int_0^1 e^{-t}dt-\frac{1}{2}(e^{-1}+1)>0$$

$\therefore$ $(-1,0)$ 与 $(0,1)$ 上均一根　　2个根 #

§6.8. $f(x)=3x^2+\frac{1}{x^2}-\frac{18}{25}$　　求 $f(x)=x$ 零点

$$g(x)=f(x)-x=3x^2+\frac{1}{x^2}-x-\frac{18}{25}$$

$g'(x)=6x-\frac{2}{x^3}-1$　　$g'(\frac{1}{2})=-14<0$　　$g'(1)=3>0$

$g''(x)=6+\frac{6}{x^4}>0$　　$\therefore$ $(\frac{1}{2},1)$ 有一个 $g'(x)$ 零点、x_0

即 $g(x_0)$ 为极小值点、　　$x_0\in(\frac{1}{2},1)$

$$\therefore g(x)_{min}=g(x_0)=3x_0^2+\underbrace{\frac{1}{x_0^2}-x_0}_{\text{在}(\frac{1}{2},1)\text{处大于}0}-\frac{18}{25}>3\cdot(\frac{1}{2})^2-\frac{18}{25}=\frac{3}{100}>0.$$

即 $g(x)$ 无零点、 #

§6.9. $|f(x)|\le 1$，$f^2(0)+[f'(0)]^2=4$　　证 $(-2,2)$ 上有 ξ，$f(\xi)+f''(\xi)=0$

拉　$f(0)-f(-2)=2f'(\xi_1)$.　　$\xi_1\in(-2,0)$

$f(2)-f(0)=2f'(\xi_2)$　　$\xi_2\in(0,2)$.

$|f'(\xi_1)|=\frac{|f(0)-f(-2)|}{2}\le\frac{|f(0)|+|f(2)|}{2}=1$　　同理 $|f'(\xi_2)|\le 1$

令 $\varphi(x)=f^2(x)+[f'(x)]^2$ 则 $\varphi(\xi_1)\le 2$ $\varphi(\xi_2)\le 2$.

$\because$ $\varphi(0)=4$. 设 ξ 处取 $(\xi_1,\xi_2)\subset(-2,2)$ 上的 $\varphi(x)_{max}$.

则必有 $\varphi(\xi)\ge 4$ 且 $\varphi'(\xi)=0$

$$\Downarrow$$

$$2f'(x)[f(x)+f''(x)]=0$$

由于 $|f(x)|\le 1$ $\varphi(\xi)\ge 4$ $\therefore$ $f'(\xi)\ne 0$ $\Rightarrow$ $f(\xi)+f''(\xi)=0$

得证.