Quantum Physics and the Weird Implications of Bell's Inequality

In this packet we are going to try to explore some of the interesting physics that governs the behavior of quantum systems. Many of you may have heard of quantum mechanics. You may have heard that quantum physics dictates that things behave differently when they're very tiny. Or maybe you've heard some popular science version of uncertainty. Does quantum mechanics tell us that our universe isn't deterministic? Does it tell us that a cat maybe dead and alive? What does quantum physics tell us really? In this packet, we're going to explore a small fragment of what quantum mechanics is really about. And no, we're not going to need any expensive equipment, or a complicated laboratory to understand and observe this. We're going to do it all sitting right here.

In particular, we're going to have a look at the interesting behavior of light.

Hopefully, by the end of this packet, you'll all gain an appreciation for why quantum mechanics is certainly different from regular physics, and that quantum physics isn't just some weird mumbo-jumbo, but instead it's something that you can learn and begin to understand in this short camp itself!

To appreciate and understand what goes on in quantum physics under the hood, we're going to have to build some mathematical background before we can talk about the physics that comes into play. We're going to discuss two topics, Linear Algebra, and Trigonometry.

1 Trigonometry

Since one of our main object of study in this packet is going to be light, we're fundamentally going to be studying the nature of waves. And studying waves naturally brings us to the study of trigonometry. In this section, we're going to use Greek letters, such as θ to represent angles, and we're going to be interested in two trigonometric functions, i.e., $\sin(\theta)$ and $\cos(\theta)$. Both of these functions are closely related to circles, and that's how we're going to study them.

We start off by considering a "coordinate plane" (see Fig. 1). A coordinate plane is nothing by a plane with an origin, an x-axis, a y-axis, where we signify any point on the plane with its x-coordinate and its y-coordinate, in the form of the pair (x, y).

Let us now draw a *unit circle* on this plane, i.e., a circle with radius one having it's center at the origin of the coordinate plane.

Consider the point A on the unit circle that is at an angle θ with respect to the x-axis (See Fig. 2). What are the coordinates of this point in terms of the lengths that have been marked in Fig. 2 (i.e., OA, OB, OC, AC, and, AB)? Now, can we write these coordinates in terms of only the angle θ? That is, given that the distance OA is 1,

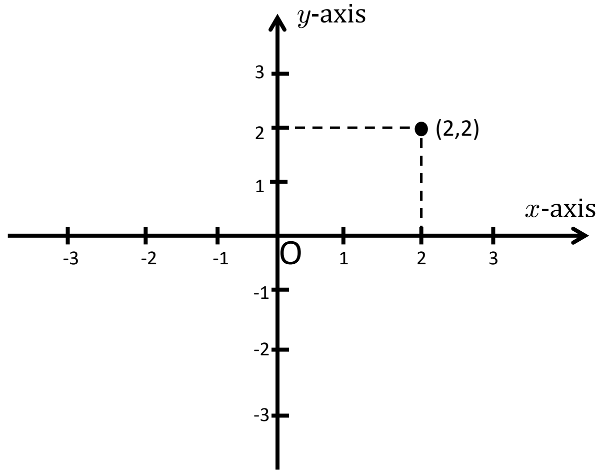

Figure 1: Definition of a coordinate plane. Points are denoted by their positions with respect to the x-axis and y-axis

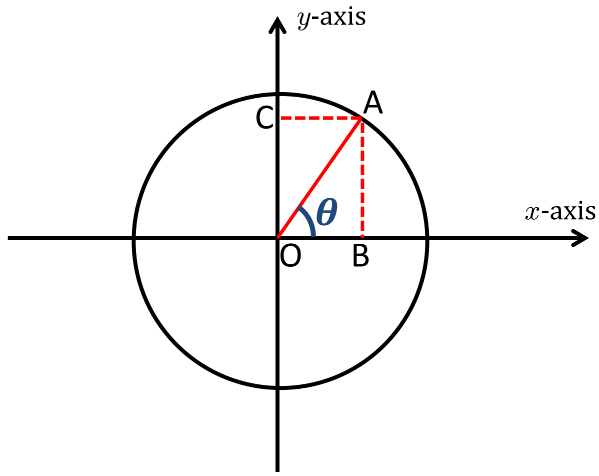

Figure 2: The unit circle is fundamental to the definition of the trigonometric functions $\sin\theta$ and $\cos\theta$.

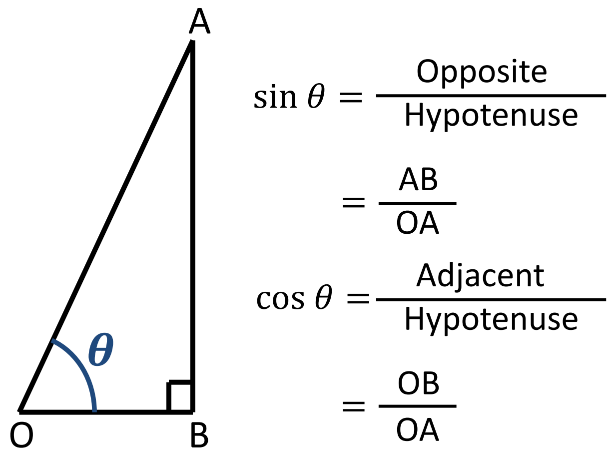

Figure 3: The definitions of trigonometric functions with respect to the sides of a right-angled triangle

and angle $\angle AOB = \theta$, what is the length AB and AC? Well, quite simply, we define $\sin(\theta) = AB$, where whenever A is below B, we represent that as a negative length. Similarly, we define $\cos(\theta)$ as the length AC (with the interpretation that whenever A is to the left of C it is a negative length). Thus, purely by definition, the coordinates of the point A are $(\cos\theta, \sin\theta)$.

These definitions are closely related to the ratios of side lengths in right-angled triangles, as can be seen in Fig. 3.

For those of you familiar with Pythagoras Theorem, you would know that for the right-angled triangle OAB, the following relation holds true:

$$OA^2 = OB^2 + AB^2. \tag{1}$$

Exercise

What does this mean in relation to $\sin\theta$ and $\cos\theta$? Write the lengths in terms of the above expressions to arise at an identity relating the sine and cosine of any angle.

Following the definition of $\sin\theta$ and $\cos\theta$ from a circle, what happens when we consider θ larger than $360°$? Let's take an example of $\theta = 400°$. What does this mean? Well, when the angle is larger than $360°$ you can think of it wrapping around the circle more than once. So in making an angle of $400°$, it's like going one full time around the circle, and then making an additional angle of $400° - 360° = 40°$.

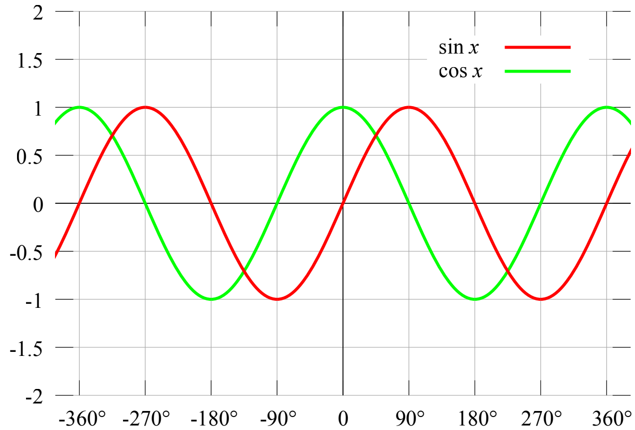

Figure 4: Graphs of $\sin\theta$ versus θ and $\cos\theta$ versus θ. Note how these functions are periodic, i.e., the pattern in the graph appears to be repetitive. This image has been taken from `commons.wikimedia.org`

Exercise

- If I tell you that $\sin(40°) = 0.64$, what is $\sin(400°)$? What is $\cos(400°)$? Hint: Use the relation derived in the previous exercise to calculate $\cos(400°)$
- Similarly, if I tell you that $\sin(550°) = -0.17$, what can you say about the $\sin(190°)$? What about $\cos(190°)$?

Verify these answers online or using your calculator.

Let's do another example so that we're comfortable with this. If $\cos(750°) = 0.87$, what can we say about the sine and cosine of other angles? Note that 750° is greater than $2 \times 360°$. So, we can think of this as making two wraps around the circle, and having a leftover of $750° - 2 \times 360° = 30°$.

Exercise

Using the above information, i.e., that $750° - 2 \times 360° = 30°$, and that $\cos(750°) = 0.87$, calculate $\sin(30°)$ and $\cos(30°)$.

This notion of "wrapping around" is fundamental to the idea of *periodicity* of the functions $\sin\theta$ and $\cos\theta$. What this means is nothing but the idea that every time θ crosses 360°, the values for both of these functions will repeat. This can be seen in Fig. 4 where both of these functions have been plotted. See how the functions have a a repetitive behavior, and look like ripples or waves. This periodicity of functions happens to be fundamentally related to the idea that we use these functions to talk about waves, but we won't quite be able to go into the details of all of these connections in this packet.

Having already introduced the coordinate plane, let's use these ideas to take a quick leap to vector algebra.

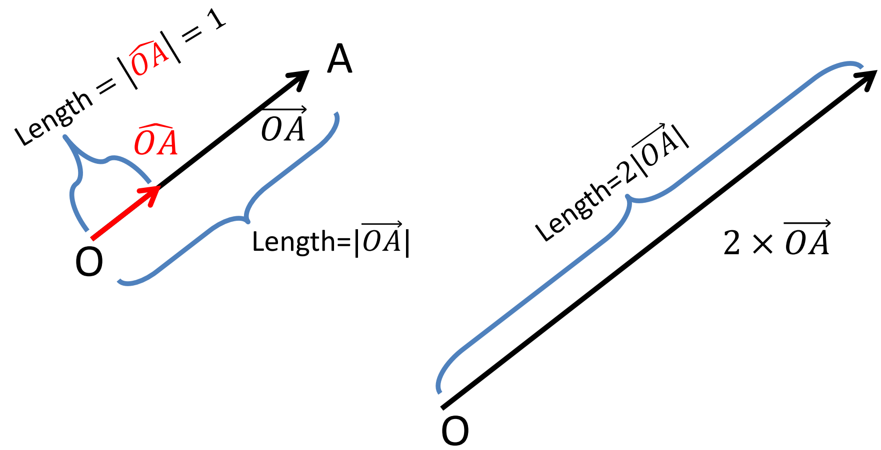

Figure 5: Definition of vectors, their unit vectors and their magnitudes. Also shown is the effect of scalar multiplication on a vector. See text for details

2 Vector Algebra

Instead of thinking of the point A as just a point on the coordinate plane, instead we think of it as a "vector" from the origin to point A, as represented by the arrow in Fig. 6, and denoted by $\overrightarrow{OA}$. What is a vector really? It's nothing but a quantity that has both a magnitude and a direction. So in this case, the magnitude is the length OA, and the direction is the direction that the arrow is pointing in, i.e., from O to A. In general, the magnitude of vector $\vec{v}$ is represented as $|\vec{v}|$; and the direction is represented by $\hat{v}$, which is a vector in the same direction as $\vec{v}$ having a magnitude of 1, i.e., $|\hat{v}| = 1$.

Now, we know how numbers add: for example $2 + 3 = 5$ (These regular numbers are known as scalars). How do these vectors add? They follow what is known as the *parallelogram* rule. In Fig. 6(a), $\overrightarrow{OC} + \overrightarrow{OB} = \overrightarrow{OA}$. In this packet we will usually be interested in the addition of perpendicular vectors, in which case the parallelogram OCAB will just be a rectangle as shown in Fig. 6(b).

Thus, following this parallelogram rule, we can see in Fig. 6(b) that vector $\overrightarrow{OA} = \overrightarrow{OB} + \overrightarrow{OC}$, where $\overrightarrow{OB}$ is along the x-axis, and $\overrightarrow{OC}$ is along the y-axis. If $\angle AOB = \theta < 90°$, then AOB is a right-angled triangle. We can use the defintion of sine and cosine for right angled triangles to then write $|\overrightarrow{OB}| = |\overrightarrow{OA}| \cos\theta$ and $|\overrightarrow{OC} = |\overrightarrow{OA}| \sin\theta$. Following a similar construction, *any* vector can be written as the sum of two vectors, one that is parallel to the x-axis, and the other parallel to the y-axis. Writing a vector as the sum of two such vectors is called a 'decomposition' of $\overrightarrow{OA}$ onto the x-axis and y-axis.

Is this the only way to write the vector $\overrightarrow{OA}$ as the sum of two vectors? No! For the same vector $\overrightarrow{OA}$ let us consider another pair of axes, i.e., the x'-axis and the y'-axis. This will be similarly decomposed into $\overrightarrow{OA} = \overrightarrow{OB'} + \overrightarrow{OC'}$, where $\overrightarrow{OB'}$ is along the x'-axis, and $\overrightarrow{OC'}$ is along the y'-axis. The magnitudes of these components will now be given by $|\overrightarrow{OB'}| = |\overrightarrow{OA}| \cos\theta'$ and $|\overrightarrow{OC'} = |\overrightarrow{OA}| \sin\theta'$. Thus $\overrightarrow{OA} = \overrightarrow{OB} + \overrightarrow{OC} = \overrightarrow{OB'} + \overrightarrow{OC'}$.

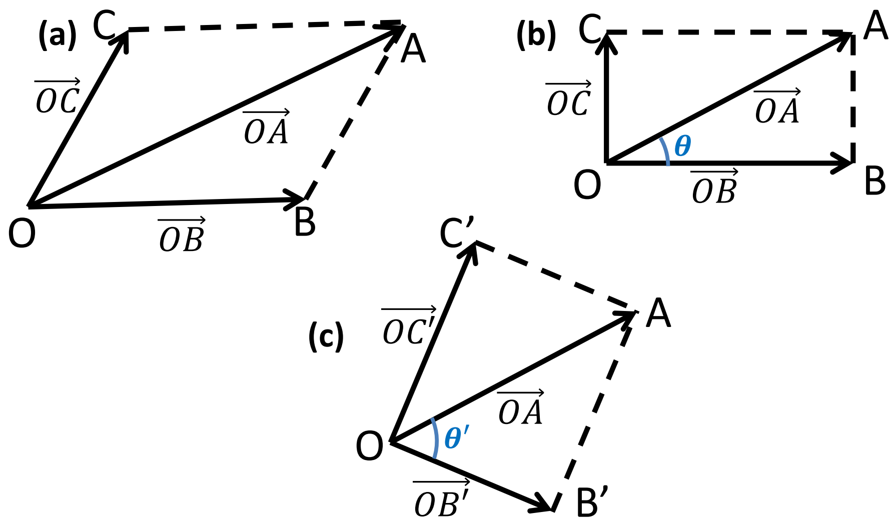

Figure 6: Details for how vectors add. In (a), OCAB is a parallelogram. In this packet we will mostly be interested in cases for when we will add perpendicular vectors, so the parallelogram is a rectangle, as in (b). Note that vector $\vec{OA}$ is the same in (b) and (c), and has been represented as the sum of two vectors in two different ways in (b) and (c). In (b), think of the x-axis and the y-axis as being along $\vec{OB}$ and $\vec{OC}$ respectively. Similarly, in (c), think of the x'-axis and the y'-axis as being along $\vec{OB'}$ and $\vec{OC'}$ respectively

Another property that vectors follow is that they can be multiplied by a scalar. As seen in Fig. 6, the effect of multiplying $\overrightarrow{OA}$ by a factor a results in a new vector with the same direction as OA, but with a magnitude increased by a factor of a. This for any vector $\vec{v}$, we have

$$\vec{v} = |\vec{v}|\hat{v} \tag{2}$$

Thus, following this observation, any vector that is along the x-axis can be represented as scalar multiple of the unit vector along the x-axis. We denote this unit vector by $|\rightarrow\rangle$, which we also write as

$$|\rightarrow\rangle, \text{ or alternately, } \begin{pmatrix} 1 \\ 0 \end{pmatrix}. \tag{3}$$

Thus, any vector with magnitude a whose direction is along the x-axis can be written as

$$a|\rightarrow\rangle, \text{ or alternately, } \begin{pmatrix} a \\ 0 \end{pmatrix}. \tag{4}$$

For example, if we want a vector with length 4 units along the x-axis, it can be denoted simply as $4|\rightarrow\rangle$.

We can make a similar definition for vectors whose directions are along the y-axis, by defining the unit vector along the y-axis as

$$|\uparrow\rangle, \text{ or alternately, } \begin{pmatrix} 0 \\ 1 \end{pmatrix}. \tag{5}$$

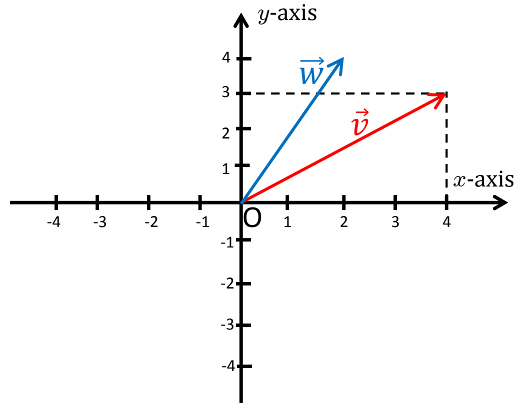

Figure 7: Vectors $\vec{v}$ and $\vec{w}$ as relevant to the discussion in the text

Thus, any vector with magnitude b whose direction is along the y-axis can be written as

$$b|{\uparrow}\rangle, \text{ or alternately, } \begin{pmatrix} 0 \\ b \end{pmatrix}. \tag{6}$$

For example, if we want a vector with length 3 units along the y-axis, it can be denoted simply as $3|{\uparrow}\rangle$.

Thus, following the above observations we note that we can write any vector $\overrightarrow{OA}$ as

$$\overrightarrow{OA} = a|{\rightarrow}\rangle + b|{\uparrow}\rangle, \text{ or alternately, } \begin{pmatrix} a \\ b \end{pmatrix}. \tag{7}$$

For example, what does the sum of $4|{\rightarrow}\rangle$ and $3|{\uparrow}\rangle$ look like? The sum of the two vectors is just some vector $\vec{v}$ which can be written as

$$\mathbf{??}\vec{v} = 4|{\rightarrow}\rangle + 3|{\uparrow}\rangle, \tag{8}$$

which is equivalent to

$$\begin{pmatrix} 4 \\ 3 \end{pmatrix}. \tag{9}$$

Representing a vector as a pair of numbers stacked vertically in brackets as has been done above is referred to as it's column vector representation. When we represent a vector in terms of $|{\rightarrow}\rangle$ and $|{\uparrow}\rangle$, this is described as representing a vector in the *basis* of $|{\rightarrow}\rangle$ and $|{\uparrow}\rangle$. We will also use the terminology that the component of $\vec{v}$ along $|{\rightarrow}\rangle$ is 3, and the component of $\vec{v}$ along $|{\uparrow}\rangle$ is 4. This vector $\vec{v}$ has been shown in Fig. 7.

Exercise

In Fig. 7, let θ_v be the angle that $\vec{v}$ makes with the x-axis. Use the definition of $\sin\theta$ and $\cos\theta$ in Fig. 3 to calculate $\sin\theta_v$ and $\cos\theta_v$

Writing a vector in either of these two representations also makes it easy to compute the magnitude (i.e., the length) of the vectors. To do this, we will use a simple application of Pythagoras' Theorem. Consider the vector $\vec{v}$ shown in Fig. 7, whose equation is given in Eq. (**??**). Given the lengths of the vector along the two axes, how do we calculate the length of the vector? Well, following Pythagoras' Theorem, we can say that

$$|\vec{v}|^2 = 3^2 + 4^2. \tag{10}$$

Thus, the value of $|\vec{v}| = \sqrt{3^2 + 4^2} = 5$. Using this idea, calculate the magnitude of the general vector $\overrightarrow{OA}$ written in Eq. (7).

Let us see how the scalar multiplication we talked about earlier appears in these representations of the vector. What is $2\vec{v}$? Well, as shown in Fig. 6, it is a vector in the same direction as $\vec{v}$, with double the magnitude. How do we write it? We do it in the following way

$$2\vec{v} = 2 \times (4|\rightarrow\rangle + 3|\uparrow\rangle) = 8|\rightarrow\rangle + 6|\uparrow\rangle. \tag{11}$$

Or, alternately,

$$2\vec{v} = 2 \times \begin{pmatrix} 4 \\ 3 \end{pmatrix} = \begin{pmatrix} 8 \\ 6 \end{pmatrix}. \tag{12}$$

Calculate the magnitude of $2\vec{v}$ to verify that it is, indeed, twice the length of vector $\vec{v}$. Thus, this leads the the simple, but important conclusion, which was essentially a part of our definition of scalar multiplication, that $|2\vec{v}| = 2|\vec{v}|$. Or, in general

$$|a\vec{v}| = a|\vec{v}| \tag{13}$$

for any vector $\vec{v}$ and any scalar a.

Exercise

- Look at the other vector, $\vec{w}$ shown in Fig. 7. Write it in terms of $|\rightarrow\rangle$ and $|\uparrow\rangle$. Also write it in a column vector representation.
- We have already calculated the magnitude (i.e., the length) of $\vec{v}$ above. What is the magnitude of $\vec{w}$?
- Similar to how we wrote $2\vec{v}$ earlier, write $4\vec{w}$ in terms of $|\rightarrow\rangle$ and $|\uparrow\rangle$. Calculate it's magnitude and verify that the magnitude is 4 times the magnitude of $\vec{w}$.

What are the directions of these two vectors? As mentioned earlier, often in talking about vectors, by direction of the vector we simply mean the corresponding unit vector.

Exercise

What is the unit vector in the direction of $\vec{v}$? Hint: Use Eq. (2). Similarly, what is a unit vector in the direction of $\vec{w}$?

Exercise

- On Fig. 7, draw the vector $\vec{u} = (-1)|\rightarrow\rangle + 2|\uparrow\rangle$. Calculate its magnitude and direction. Repeat the same for a vector $\vec{r} = 3|\rightarrow\rangle - 2|\uparrow\rangle$.
-

Consider a vector $\vec{x}$ that has magnitude of 5 units, and is at an angle of 30° with the x-axis.

Exercise

Make a rough sketch of the coordinate plane, and draw this vector on the basis of this information. Can we use our knowledge of trigonometry to write $\vec{x}$ in terms of $|\rightarrow\rangle$ and $|\uparrow\rangle$? What is the unit vector in this direction? (i.e., the vector with unit magnitude making an angle 30° with the x-axis).

In general, say a unit vector makes an angle θ with the x-axis. We will denote this unit vector as $|\theta\rangle$. Can you write it in the $|\rightarrow\rangle$ and $|\uparrow\rangle$ basis?

Exercise

$$|\theta\rangle = \tag{14}$$

This is a very simple, but important result. Let us now consider a vector with magnitude a which is at an angle θ with respect to the x-axis, i.e., at an angle theta with respect to $|\rightarrow\rangle$. This vector would simply be $a|\theta\rangle$. Following your answer to the above exercise, this vector would be written as

$$a|\theta\rangle = a\cos\theta|\rightarrow\rangle + a\sin\theta|\uparrow\rangle. \tag{15}$$

Verify that the above vector has a magnitude a as we initially wanted. Thus, the component of a vector with magnitude a, along a unit vector at an angle θ to the vector, is $a\cos\theta$. Keep this in mind, we'll definitely use this later!

Ask your team leader if you have any questions at this point, and make sure that the whole team is together at this point before moving ahead! We're going to take a step away from mathematics and get into some physics instead!

3 What is Light?

Most of you have probably seen light before (Sorry for the terrible joke :P). But what is light really? In this packet we're going to discuss two different ways to talk about light.

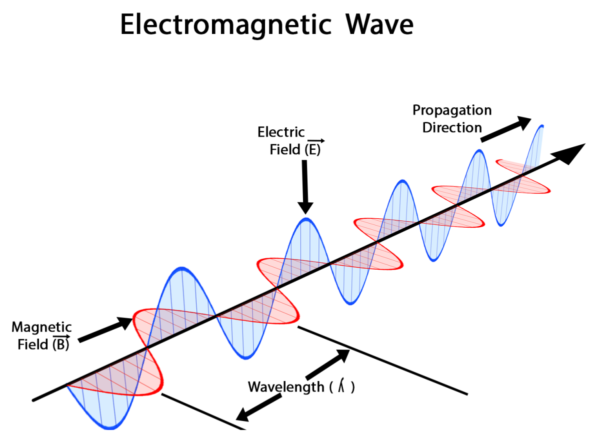

Figure 8: A ray of light is an 'electromagnetic wave', i.e., each ray of light consists of rapidly oscillating electric and magnetic fields, which oscillate in synchronization as shown here. This image has been taken from `commons.wikimedia.org`

One of the great scientific debates that has extended across several centuries has been the nature of light. Newton originally described light consisting of 'corpuscles', which were thought to be like small balls of light that were very small masses that travelled in straight lines. Newton used this theory to successfully describe reflection and refraction. Around the same time, a competing theory of the wave nature of light was put forth by Christiaan Huygens, and seemed necessary to explain the nature of light as it passed through certain crystals. The wave theory of light was further boosted by the study of electricity, magnetism and the development of what we now know as Maxwell's equations. This almost cemented the wave theory of light, until the two great physicists Albert Einstein and Max Planck. Einstein demonstrated that light must behave like a small packet of energy, almost like a particle rather than a wave. We will be looking at Planck's work in a moment in Sec. 3.2, but broadly he demonstrated that the energy of light must be *quantized*, i.e., it cannot attain any continuously varying value, but rather can only attain discrete values

What does this mean? For example, the speed of a car is a continuously varying quantity. It can take any value (up to a limit, which is hopefully the speed limit :P). You can continuously increase the speed of a car from zero, to 50 miles per hour, while crossing every value in between. An examples of a quantized value, which only attains discrete values is say, the number of spoons you may have at home. You can have 2 spoons, or five spoons, or 10 spoons, but you cant have 0.5 spoons, or 3.75 spoons. Another example is say, the number of strands of hair on your head. It's a large number, but it is still only an integer. In fact, it's so large, that if you remove only a few strands, it would appears as if the total 'amount' of hair on your head changed by a very tiny amount, almost as if it changed continuously, but it is definitely a discrete number.

Firstly , light as 'electromagnetic waves', and secondly light as 'photons'.

3.1 Electromagnetic waves

While we will not need many details about light as an electromagnetic wave for the purpose of this packet, let us take a quick overview of what this means. As the name suggests, there are three important things to think about: Electricity, Magnetism and Waves.

Electricity:

This is something that you are probably familiar with. Electricity is the physical phenomena associated with charges and the motion of charges. For example, electricity in your phone makes your phone work by pushing electrons around, and electricity in a light bulb makes the light bulb glow by pushing electrons through it. The 'Electric Field' is a vector field (i.e., a vector at every point in space) that quantifies how much these electrons get pushed at what points in space (i.e., the vector tells you in what way an electron would move because of electric forces). We will not go into the details of this in this packet.

Magnetism:

This is the physical phenomena associated with magnets. Similar to electric fields described above, the 'magnetic field' is the vector field that tells you in what way a magnet would move because of magnetic forces, with vectors at each point effectively telling you how a small magnet would align itself.

Waves:

Waves are just periodic 'ripples' of some sort. You could have a wave in the ocean, where you you're thinking about actual water waves. You could have waves on a string, which is what happens when you take a long piece of rope and rapidly move one end up and down. Or, you could have waves of electric and magnetic fields, where the electric and magnetic fields just rapidly keep oscillating. What does that mean? We'll explain that in a moment, but for now just think of waves as being a periodic, repetitive change in something.

The dynamics of Electric and Magnetic fields is described by a set of equations called Maxwell's Equations. While we do not need to understand what the equations are, or what they mean, there is one important result that we will use from these equations:

When you solve Maxwell's equations the solutions of the equation give waves of electric and magnetic fields. What that means, is that Maxwell's equations predict that if you have some initial oscillation of electric and magnetic fields, then these oscillations will continue to persist, and will actually spread. These oscillations of electric and magnetic fields look like what has been shown in Fig. 8 . In these waves, note that the electric and magnetic fields wave in directions that are perpendicular to the direction of propagation, i.e., the direction in which these oscillations are moving. When the Electric field is at its maximum value, then so is the Magnetic Field, and when the Electric field is at zero so is the magnetic field.

The remarkable thing is, that this propagation of oscillating electric and magnetic fields is light! This is not obvious, and was not known for several decades until the 1860s. While we won't have time to explain this further, this is something that you can assume is true: all the light that you see consists of oscillations of quantities called 'Electric fields', and oscillations of quantities called 'Magnetic fields'. These quantities oscillate in directions that are perpendicular to the direction of light. Thus, for example, if you consider light that is hitting this sheet of paper perpendicular to it, then at the surface of the paper there is a tiny electric field that is oscillating in the plane of the paper in some direction. For the most part, in this packet we are only going to be interested in the *direction* that this electric field will have.

In this packet we will mostly only look at the behavior of the electric field and will not show or discuss the behavior of the magnetic field.

(See `https://commons.wikimedia.org/wiki/File:EM-Wave.gif` for an animation.)

Looking at Fig. 8 and Fig. 4 might lead you to believe that there is a similarity or connection between the sine and cosine functions and the nature of light waves. That is indeed correct. In fact, if we assume for simplicity that the x-axis of our figure is along the direction of the electric field (Note from our previous discussion of the x and x' axes that we can arbitrarily choose the axes in whatever orientation we want, we just have to remember to think of the components in that 'basis'), then the electric field can indeed be written as

$$\overrightarrow{E} = E_x \sin(\omega t)|{\rightarrow}\rangle + 0|{\uparrow}\rangle, \tag{16}$$

for some constant ω. In general, we will call the direction of oscillation of the electric field associated with a light ray to be its 'polarization'. Since the electric field is oscillating in the horizontal direction, we will call such a light wave to be horizontally 'polarized'. As one might expect, a vertically polarized light wave can be written as

$$\overrightarrow{E_1} = 0|{\rightarrow}\rangle + E_y \sin(\omega t)|{\uparrow}\rangle. \tag{17}$$

How do you get other polarizations of light? We can simply add these two waves up! For example, if we consider the wave

$$\overrightarrow{E_2} = E_x \sin(\omega t)|{\rightarrow}\rangle + E_y \sin(\omega t)|{\uparrow}\rangle, \tag{18}$$

then in what direction is it polarized? To answer that, we can write the electric field as

$$\overrightarrow{E} = (E_x|{\rightarrow}\rangle + E_y|{\uparrow}\rangle)\sin(\omega t). \tag{19}$$

Defining a new vector $\overrightarrow{E_0}$ as

$$\overrightarrow{E_0} = E_x|{\rightarrow}\rangle + E_y|{\uparrow}\rangle = \begin{pmatrix} E_x \\ E_y \end{pmatrix}, \tag{20}$$

we can write

$$\overrightarrow{E} = \overrightarrow{E_0}\sin(\omega t). \tag{21}$$

Thus, we can see that the electric field is oscillating along the vector given by $\overrightarrow{E_0}$. Thus, the light wave is polarized in the direction parallel to $\overrightarrow{E_0}$.

In this way, any light ray has a particular polarization. What that means is the following: For any light ray, you can think of it as looking like Fig. 8, with the electric field being in some direction. Whatever the direction of the electric field is, that is the polarization of the light.

In everyday life, most sources of light that you see are *unpolarized* light. What that means, is that most sources of light that you see are usually a mixture of light in all sorts of polarizations in all directions. There are some sources of light that do have a more marked polarization, but we'll get into that in more detail in our section on Polarizers.

A result that follows from the mathematical theory of waves is that the energy contained in the wave is proportional to the square of the amplitude. This is true in not only electromagnetic waves, but also material waves (such as waves on a string, etc.) We will denote this as

$$\text{Energy} = k|\overrightarrow{E_0}|^2, \tag{22}$$

for an electromagnetic wave given by $\vec{E} = \overrightarrow{E_0}\sin(\omega t)$. This Energy relates directly to the intensity of the light. I.e., if we look at two light sources, and the electric fields corresponding to one of the light sources is twice that of the other, then the first will appear 4 times as intense as the first.

This ties in wonderfully with the superposition of waves that we talked about a moment earlier.

Exercise

- Consider the electromagnetic wave with the electric field given by Eq. (16). What is the energy of that wave?
- Consider the electromagnetic wave with the electric field given by Eq. (17). What is the energy of that wave?
- Now, what is the energy of the wave that is made by the sum of the two waves, given by Eq. (18)? How does this energy relate to the sum of the energies of each of the waves?

What you should be able to show, from the above three exercises, is that when you add the two electromagnetic waves, then the energy of the added wave is equal to the sum of the energies of the two waves.

Let us do a couple of exercises, to makes sure we understand the implications of the square dependence in Eq. (22)

Exercise

- By what factor should the electric field of an electromagnetic wave be increased by to get light of triple the original intensity?
- By what factor should the electric field of an electromagnetic wave be decreased by to get light of half the original intensity?

3.2 Photons

What is a photon now? When we looked at light as a wave, we described it as oscillating electric (and magnetic) fields.

It turns out, that light can also be described in a different way. Light can be described as consisting of a stream of 'photons', which are like a small packet of energy. Each photon carries a certain amount of energy that is dependent on the color of the light. While this concept is extremely well established by experimental research which can be done in undergraduate laboratories, it will be a little hard to give a demonstration of this fact here. This will however be an important point as we go ahead, and is central of the quantum nature of light, so let us go into it in a bit of detail.

In the context of electromagnetic waves, given any value of energy, we can consider an electromagnetic wave with the appropriate amplitude to obtain a wave of the given energy. This can be seen easily following Eq. (22). It can be observed, however, that this is not the case! Planck showed that the energy of light only appears as an integer multiple of a certain value (that is dependent on the frequency of the light; in this entire packet we will only light with a single frequency, so we can ignore this dependence on frequency).

$$\text{Energy} = ne_0, \tag{23}$$

for some positive integer n. (If you wish to include the dependence on frequency e_0 is written as $\hbar\omega$, where $\hbar$ is a constant known as Planck's constant, and ω is the frequency of the light and is the same ω constant that appeared in our wave equations for light earlier). What are the implications of this statement in terms of the allowed energies? What is the minimum allowed energy of light? What is the minimum value of n? (Note that n cannot be zero! That would correspond to zero energy, zero photons, and hence zero light)

Why only integer multiples? Einstein (in explaining an effect called the photoelectric effect) argued that light is made up of these individual packets of energy called 'photons'. He argued that each packet has a fixed (frequency dependent) energy (e_0 above), and a beam of light consists of a large number of such photons travelling. Since the number of photons can only be an integer, the energy of a beam of light was quantized. What does this mean? This means that you can't have light of any intensity. Only of particular quantized values. So you can't have light with an energy of $5.5e_0$ or $13.2e_0$ or $3.1415e_0$.

In the context of waves of light, polarization was defined in terms of the *orientation* of the electric field vector that comprised the electromagnetic wave. How does polarization make sense in the photon picture? While it may appear that photons might be incapable of describing polarization of photons, it turns out that each individual photon has it's own polarization, which can be described by some polarization state unit vector $|P\rangle$. So, for example, in this language a photon polarized in the x direction can be represented by $|{\rightarrow}\rangle$. In practice, you can think of the polarization vector as a unit vector pointing in the direction of the electric field corresponding to the light, just as earlier. In this case, however, the polarization vector is an inherent property of the photon.

So, which is light? Is it a wave, with a continuously varying energy? It usually appears like that when we consider it in everyday life, doesn't it? It appears that you can light of continuously varying intensities, no? Well, one photon is a *very* small amount of light. To put it in perspective, the flashlight on your table emits around 10^{19} photons per second. That's about 10 billion billion photons per second. So adding or removing one photon reduces the intensity by a *tiny* fraction, so it appears as if it changes continuously.

So if photons can explain everything, is light a photon then? Nope. Oh, is it a wave then? Nope. It's *both*. Now, this probably comes as a surprise, and you won't be alone in that. It turns out that both of these descriptions are valid descriptions of light, and in some cases one is more appropriate than the other, and in other cases it is the other way around. A reasonable intuition to keep is that if you are thinking of tiny amounts of light, with energies corresponding to just a handful of photons, then the photon description is the reasonable one. But, if you're thinking of large amounts of light, similar to the lights we see everyday, the wave description is quite reasonable. However, in some sense you can think of large amounts of light as behaving like a dense stream of photons. Now, this is already quite weird, but this isn't going to be the focus of our discussion in this packet. In this discussion, we're going to look at another kind of weirdness, something which you should hopefully be able to follow along, and actually have some understanding about.

What all theoretical understanding of light are we going to require? We're going to assume the following things going ahead:

Light can classically be thought of as being described by waves

Light can also be thought of as comprising of photons. For the purpose of our discussion it will suffice to assume that **light is actually really photons, but it can be described by waves too.**

Photons are quantized. You can have 5 photons, or 208 photons, but you can't have 0.5 photons, or 1.25 photons.

4 Polarizers

Polarizing filters are tools that allow you to produce and examine polarized light. They are the fundamental technology behind 'Polaroid sunglasses', that are particularly effective at reducing glare from reflected light. Each lens in a pair of polaroid sunglasses consists of a weak polarizing filter.

On your table are a few pieces of what appear to be small plastic squares Some of these are a special type of material which are called polarizing filters (also knows as polarizers).

What's on your table?

Play around with all of the plastic squares. Are they all just regular pieces of tinted plastic, or is there something more about them? Can you distinguish between which ones are just regular tinted plastic and which ones are polarizing filters? Which ones are which? Can you try to describe how one is different from the other?

Confirm with your team leader that you have correctly identified the polarizing filters.

So what is a polarizer, really? A polarizer is a material that only allows light of a certain polarization to pass through. Each polarizer has a fixed axis that chooses the direction of polarization that it allows to pass through.

One way to quickly figure out the axis is to look for a source of polarized light. If you have a light whose polarization direction you know, you can then look at it through your polarizer, and see what orientation of the polarizer allows for the most light to pass through. In that orientation the axis of the polarizer will then be parallel to the polarization of the light.

While most light around you is unpolarized, there are some sources of polarized light. Look at your laptop or mobile screens, for example. However, it is usually unclear what the direction of polarization of the light is. One source of partially polarized light is reflected light. Look at reflections from any flat horizontal surface. The reflected light will generally also be horizontally polarized. You can then use this to figure out what the axis of your polarizers are. (Hint: These polarizers are such that the axis of each polarizer is parallel to one of the sides of the polarizer.)

4.1 Polarizers and electric fields

As described earlier, a polarizer only allows light with a certain polarization to pass through. Let's try to make this statement a bit more quantitative, so we can actually use this idea to evaluate something. Consider a light wave with the following electric field.

$$\vec{E}^{initial} = E_0(0.6|\rightarrow\rangle + 0.8|\uparrow\rangle)\sin(\omega t). \tag{24}$$

What is the direction of polarization of this light? It's just the unit vector in the direction along which the electric field is varying, i.e., $0.6|\rightarrow\rangle + 0.8|\uparrow\rangle$. As is apparent from this notation, this electric field vector can be written as the sum of two electric fields in the following way

$$\vec{E}^{initial} = 0.6E_0|\rightarrow\rangle\sin(\omega t) + 0.8E_0|\uparrow\rangle\sin(\omega t). \tag{25}$$

Thus, what is the electric field along the x-axis? It's just $0.6E_0\sin(\omega t)$. So, if we were to place a polarizer that only allows polarization parallel to the x-axis to pass through, and shine light of the form of Eq. (25) through it, what is the electric field of the light after having passed through the polarizer? It will be given by

$$\vec{E}^{final} = 0.6E_0|\rightarrow\rangle\sin(\omega t). \tag{26}$$

Following Eq. (22), what is the intensity of the light after passing through the polarizer as compared to the intensity of the light before the polarizer? Following the notation of Eq. (22), the energy of the electromagnetic wave $\vec{E} = \vec{E_0}\sin(\omega t)$ is equal to $k|\vec{E_0}|^2$, i.e., k times the square of the magnitude of the vector $\vec{E_0}$. Thus, the energy of the incident wave before the polarizer is given by

$$\text{Energy}^{initial} = k|E_0(0.6|\rightarrow\rangle + 0.8|\uparrow\rangle)|^2 \tag{27}$$

We have earlier seen how to calculate the magnitude of a vector given in terms of $|\rightarrow\rangle$ and $|\uparrow\rangle$. Verify that $|0.6|\rightarrow\rangle + 0.8|\uparrow\rangle| = 1$, hence giving

$$\text{Energy}^{initial} = k(E_0)^2. \tag{28}$$

Similarly calculate Energy^{final} to obtain

$$\text{Energy}^{final} = 0.36k(E_0)^2. \tag{29}$$

Thus, the intensity of light after passing through the polarizer is 36% that of the light that was incident on the polarizer.

Exercise

Let us repeat the above calculation for another wave incident on a polarizer. Consider a light ray with an electric field given by

$$\vec{E}^{initial} = E_0(0.3|\rightarrow\rangle + 0.95|\uparrow\rangle)\sin(\omega t). \tag{30}$$

Consider a polarizer that is aligned such that it only allows polarization parallel to the x-axis to pass through, i.e., the axis of the polarizer is parallel to $|\uparrow\rangle$. What is the electric field of the light after passing through the polarizer? What was the intensity of the light before passing through the polarizer? What was the intensity of light after passing through the polarizer? By what fraction did the intensity change?

Repeat these calculations for a polarizer with axis that is parallel to the y-axis.

Let us do another exercise. Let us construct an example of the electric field from a light wave that is polarized at an angle θ with respect to the x-axis. Recall from Eq. (15) that a vector with length a at an angle θ with respect to the x-axis is given by $a\cos\theta|{\rightarrow}\rangle + a\sin\theta|{\uparrow}\rangle$. Thus, if we want an electric field that is polarized along an angle θ, we can similarly write it as

$$\vec{E} = E_0(\cos\theta|{\rightarrow}\rangle + \sin\theta|{\uparrow}\rangle). \tag{31}$$

Exercise

What is the energy of the wave given above in Eq. (31)? If we let this light pass through a polarizer that is oriented along the x-axis, what is the fraction of the energy of the wave that passes through the polarizer?

What you should get is that for light that is polarized at an angle θ with respect to the polarizer, the fraction of light that passes through is $(\cos\theta)^2$.

Note that this answer only depends on the angle between the polarization of the light and the polarizer. It does *not* depend on the orientation or angle of the polarizer with respect to any other frame of reference. It only depends on the angle between the two. This is an important result.

Exercise

To make sure this point we understand this point, calculate the fraction of light that passes through the polarizing filter in the following cases:

- The polarizer is along the y-axis, and the light is polarized along the x-axis
- The polarizer is at an angle of $30°$ with respect to the x-axis, and the light is polarized along the y-axis
- The polarizer is along the x-axis, and the light is also polarized along the x-axis
- The polarizer is along the y-axis, and the electric field corresponding to the light is given by $\vec{E} = E_0(0.25|{\rightarrow}\rangle + 4.3|{\uparrow}\rangle)$
- The polarizer is along the x-axis and the light is polarized at an angle of $45°$ with respect to the x and y axes

Qualitatively explain the results of the first and third calculations above without the use of any equations.

Now, we've been talking about polarized light and what happens when it's incident on a polarizer. How do we get polarized light to start with? As mentioned earlier, most light sources that you see everyday are unpolarized. This includes the flashlight on your table. Given that we have polarizers, the simplest way to obtain polarized light is just going to be to place the polarizer in front of the flash light. Since the polarizer only allows light in one direction to pass through it, the flashlight+polarizer can be thought of has being a source of polarized light, whose polarization direction corresponds to the direction of the polarizer placed on the flashlight.

The discussion in the next section is going to get a little philosophical, and it would be nice if the entire group could discuss these issues together. **Wait until everyone has reached this point in the packet before going ahead.**

4.2 Polarizers, photons and philosophy

Let us look at the last example question in the above list a bit more closely, and use that to enter our discussion of how photons are affected by polarizers. If you performed the calculation correctly, you would observe that the intensity of light after passing through a polarizer at a 45° angle with respect to the polarization of the incident light is half that of the initial intensity. As discussed earlier, in the section on photons, a light beam can be thought of as a stream of photons. Also, as we saw earlier, in this context the intensity of light is just related to the number of photons. Thus, if the polarized light had an intensity corresponding to a stream of N photons per second, what is the corresponding rate of photons after the light passes through the filter? Since the intensity drops to half, and intensity is proportional to the number of photons, we can say that the light after passing through the filter corresponds to a stream of $N/2$ photons per second. So, if we initially had a stream of a 100 billion photons with a polarization along the x axis, then after passing through a polarizer with an axis at an angle of 45°, we would expect around 50 billion photons to pass through. This sounds quite reasonable, and is in fact what can be observed, matching exactly with our understanding from thinking of light as an electromagnetic wave.

Now, let's get to the "problematic" case. What happens when we're passing not a 100 billion photons, but just a single photon through the polarizer? From our calculation from waves, we expect the light passing through the filter to have half the intensity. But, if you have a single photon, how do you have half the intensity of a single photon? You can't! This is because of the fundamental fact we talked about earlier. We can have 1 photon, or 1232 photons, but we can never have 0.5 photons, or 0.3521 photons. Thus either photon passes through, or it does not.

Discuss

What do you think happens in this case when a single photon hits the polarizer? Does it go through or does it not go through? If it does go through, does that make sense? Because we discussed earlier that sending a 100 billion photons should result in around 50 billion photons passing through (You can see with the polarizers on your table too! Clearly the intensity of light does go down, so not all photons are coming through). But in the same way, it does not make sense to say it does not go through, because again, we can see from the polarizers on the table that some light does pass through. What do you think happens? For a single photon, does it go through, or not?

To elaborate on the above discussion a bit, note that 'classical' theories (i.e., theories of physics that describe most of the things we see and interact with in everyday life) are always 'deterministic'. That is, if you know the initial conditions of the system, then you how the system will evolve, and what will happen. For example, if you pick up a ball, and throw it then you can calculate exactly what the trajectory of the ball will be if you know enough about the initial conditions. This is not just an idealized statement to make. In reality a ball's path can have air resistance, or wind blowing or something like that, but if you knew about all of them initially, then you'll still be able to exactly map out the trajectory of the ball.

Discuss:

Can you think of any exceptions to this? Are there any examples that come to mind which you think cannot be predicted exactly by knowledge of the initial conditions?

There are two popular examples that might appear to be exceptions to this rule.

Firstly: what about so-called random events, such as flipping a coin, or rolling a die? It appears that before you flipped the coin, you wouldn't know whether it'll land up as heads or tails, right? Actually, it turns out that this is not the case. Even for a coin, if you carefully measure the physical properties of a coin, and measure the exact force being used to toss the coin, in principle you can exactly calculate whether you'll get a heads or a tails. Yes, this calculation can be a very complicated one (Supplementary reading: Chaos on Wikipedia), but in principle it can be done.

Secondly, free will. If I measured the position and velocity of every particle in your body, and tried to use that to predict what would happen by using the physics to say, for example, that you will raise your hand. You can still choose not to, right? Isn't this a case where you fundamentally can't predict what will happen? This is a discussion which has no clear answer. We don't have the computational power in any super computer in the world to actually be able to do the calculation in any reasonable amount of time, so we don't know if we can predict humans that way. It's quite unlikely that we can, but the solution to this may actually lie in ideas of quantum mechanics. Revisit this question after you have completed the packet.

So what happens for a single photon? The answer, is that quantum mechanics appears to be non-deterministic. I.e., half of the time the photon passes through, and half of the time it doesn't. It's a probabilistic event that happens randomly. While we can't do an experiment with single photons using just a flashlight, this is something that people have experimentally observed. This does answer the question of how when you send a 100 billion photons you only get around 50 billion photons, but it really only raises more fundamental questions! How can something at the fundamental level of particles actually not behave in a definite, deterministic manner?

These allegedly probabilistic behavior is really what people think of when they talk about popular quantum physics, as being 'maybe it will, maybe it won't, who knows? *shrugs shoulders*'. Herein lies an important point which we wish to address through this packet. Is quantum mechanics really a "who knows?!" scenario? *Or*, is it possible that we don't know enough about how things work, and maybe we *could* know? Is there some underlying description of photons which truly decides what happens (like the positions and velocities of the atoms of the coin)? Or is it fundamentally random? Is there some hidden being/god/deity/agent flipping coins? Or was Einstein correct when he said "I, at any rate, am convinced that He [God] does not throw dice."? To put it in technical terms, this fundamental question is the following: Does the physical universe have "hidden variables"?

Hidden variables are underlying properties of a system which we cannot study. Let's say that there is a person standing next to the door of this room. Whenever someone approaches the door, he looks at the person, looks at some rule book that he has, and you observe that sometimes he opens the door, and sometimes he doesn't. With only this much information, it appears that the person is opening the door in a probabilistic fashion. However, there is the "hidden variable" of the rule book that he's using. Maybe the rule book just says that if the person approaching the door is wearing a jacket he should open the door. Or maybe if the person's name ends with a vowel he should open the door. These would be hidden variables. Maybe with further research on the subject you manage to actually get a chance to read the hidden variable. Then, you can incorporate this additional knowledge into your model, and predict whether someone will pass through or not. Or, alternately, maybe you never get a chance to read the rule book. So you still fail to predict when the door will open, but there is some underlying rule that remains hidden that dictates whether the door will open or not.

Coming back to the question of photons, the question of hidden variables is the following: Is there a hidden rule book that we may uncover with future research? Or will this rule book remain hidden forever? It turns out, that for photons and other quantum mechanical systems *there is no rule book*! This is a fundamentally different statement from saying that we will never be able to read the rule book. What we are going to demonstrate in this packet, is that there *cannot* be a rule book that decides if the photons will go through.

If there is no rule book, and the events truly happen randomly, it really means that in some sense Einstein was wrong about this. It's not quite like a God playing dice though. That would be like a hidden rule book. We're going to argue that it's worse that that, and **there is no rule book.**

Now, we've had the above discussion for the particular case of the polarizer being at a $45°$ angle with respect to the polarization of light. These ideas are all true in general. If you consider a polarizer at an angle θ with respect to the polarization of the light, then you can say the following: In the context of waves, using our results from the previous section, the fraction of energy that is transmitted across the polarizer is $(\cos\theta)^2$. In the context of a single photon, the probability that it will pass through the filter is given by $(\cos\theta)^2$. Note that in one case we say that a fraction of the energy goes through, whereas in the second case, with a certain probability the entire energy goes through (after all, a fraction of a photon can't go through, that is not a meaningful concept).

Also, as discussed in the case of electric fields, the light that comes out of the polarizer has a polarization that is parallel to the axis of the polarizer. The same is true for photons. Say you had a photon that was initially horizontally polarized. Then you try to pass it through a polarizer that is oriented at a $45°$ angle to the horizontal. Then, as we discussed the photon will go through some times, and won't go through the other times. When the photon does go through, its new polarization direction is also at a $45°$ angle to the horizontal. What's surprising about all of this is the following. Say I gave you a photon and told you that it was either horizontally polarized, or it had a polarization at a 45° angle to the horizontal. If you pass the photon through the polarizer, and it goes through, you cannot know whether it went through because it was aligned with the polarizer and also had a 45° polarization, or that it was horizontally polarized and went through by chance, because in both cases you end up with one complete photon polarized at a 45° angle. This would not happened with waves! You would be able to measure that the intensity decreased for the case when the light was not aligned with the polarizer. For the case of single photons, this decrease appears as a probability, and hence you can't say for sure what the photon state was. In the language of quantum mechanics, if we pass the photon through a polarizer, we say that we have 'measured' the photon state, because we are trying to perform a measurement of the polarization of the photon to see if if passes through.

Getting back to the polarizers, let's conduct a simple experiment that will distinctly point out the difference between the polarizing filters, and regular tinted plastic. This extremely simple experiment will later lead to our main results, and will eventually hopefully give you a glimpse into the weirdness of quantum physics.

Experiment: Tinted Plastic

Place a piece of tinted plastic in front of the given light source. Note how the intensity of light has dropped. This means that (as expected), the tinted plastic has absorbed some of the light. Place another piece of tinted plastic in front of the first. Note that the intensity drops further. None of this should really be surprising so far, right? Now, place a piece of tinted plastic in between the two pieces of tinted plastic. What happens? The intensity continues to drop. It's all as expected.

Now lets repeat all of this with the polarizers.

Experiment: Polarizers

Place the first polarizer in front of a light source. Note that the intensity goes down. So the polarizer also clearly absorbs some light. After placing the first polarizer, place the other two polarizers in front of it, and play around with the orientations of the other two polarizers with respect to the first. Do you notice anything unusual?

What you should be able to see, is that for some orientations, adding a polarizer between two other polarizers can actually result in a net result of *more* light passing through the three polarizers as compared to just the two of them. (If you cannot see this, then continue ahead anyway. In the next section we will show exactly what orientations lead to this, and then you can try this again.) What is going on?

We'll see that this can be explained quite simply by considering the wave nature of light. After that we'll look into the particle nature of light, and that is going to lead us to some unexpected results!

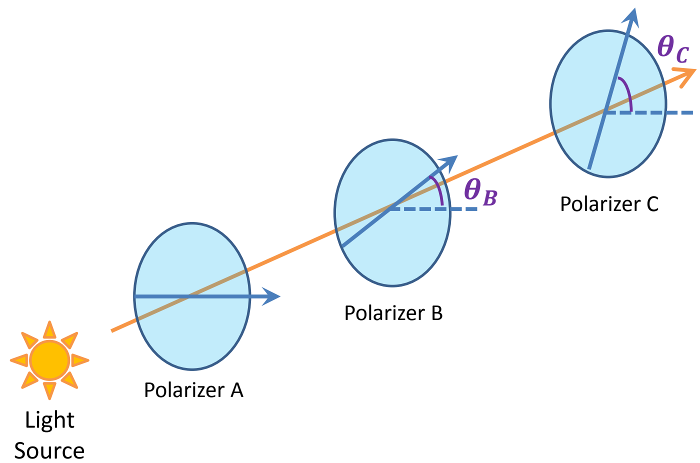

Figure 9: Arrangement of 3 polarizers that we will be considering in Sec. 5 and beyond. The blue arrows marked at each polarizer indicate the polarization axis of the polarizers, i.e., the direction of polarized light that they allow to pass through. Polarizer A has its axis parallel to the horizontal. For polarizers B and C, the angle the axis makes with the horizontal is θ_B and θ_C respectively

5 Polarizers that absorb light, and yet let more light pass through

Consider the particular arrangement of polarizers shown in Fig. 9. The arrows marked denote the axis of polarization of each polarizer, i.e., what direction of polarization of light can pass through it. The first polarizer (A) in front of the light source is going to be thought of as being part of the light source, so we effectively are going to think of the system as being a polarized light source, with the light polarized horizontally, with two polarizers (B and C) placed in front of it.

Let the intensity of the (light source + polarizer A) be I_A. Let us first consider the case where only polarizers A and B are present, with $\theta_B = 45°$, and polarizer C is absent.

Exercise

What is the intensity of light going to be after it passes through both polarizers A and B? Let us call this intensity $I_{AB}(\theta_B = 45°)$. (Go back to Sec. 4.1 to remind yourself how to do this calculation if you do not recall it.)

You calculation should tell you that $I_{AB} < I_A$. You can easily verify this with the polarizers on your table too. This justifies something that you can see by just picking up a polarizer too — a polarizer absorbs light. That's mostly obvious, after all it does appear similar to a tinted sheet of plastic.

Similarly, consider the case where only polarizers A and C are present, with $\theta_C = 90°$, and polarizer B is absent.

> **Exercise**
>
> What is the intensity of light going to be after it passes through both polarizers A and C? Let us call this intensity $I_{AC}(\theta_C = 90°)$.

Verify that this appears to be true using the polarizers on your table. This also clearly establishes that polarizers reduce the intensity of passing light, i.e., $I_{AC} < I_A$.

Now for the magic ☺.

We're going to consider polarizers A, B and C all together, with $\theta_B = 45°$ and $\theta_C = 90°$. Recall that when A and C were used, then no light passed through. The only addition is of inserting polarizer B in between the two polarizers, and as discussed earlier, a polarizer can only absorb light.

> **Experiment: Three polarizers**
>
> Before we perform the calculation, set this up with the polarizers on the table and see what happens. First place polarizers A (with polarization axis horizontal) and C (with polarization axis at an angle $\theta_C = 90°$). Note the negligible amount of light passing through. Now, slide in polarizer B (with polarization axis at an angle $\theta_B = 45°$). Remarkably, you should be able to observe that the intensity of light passing through should go up!

Why did the intensity of light go up? Let's do the calculation to see how this happens.

The intensity after polarizer A is I_A. We have already calculated $I_{AB}(\theta_B = 45°)$. What we now need to calculate is $I_{ABC}(\theta_B = 45°; \theta_C = 90°)$.

Let's approach this question via two smaller questions. What is the direction of polarization of light after passing through polarizer B? Since polarizer B only allows light of polarization parallel to its polarization axis to pass through, the light must be polarized at an angle $\theta_B = 45°$ to the horizontal.

Now, to find the intensity of light that passes through polarizer C, note that the angle between the polarization of the light and the polarizer is 45°. Recalling what we calculated earlier, the fraction of light passing though only depends on this angle. Have a look at the end of Sec. 4.1 to remind yourself about this point if it is not clear to you. Using this, do the following exercise:

> **Exercise**
>
> What is the fraction of light that passes through polarizer C, i.e., what is $I_{ABC}(\theta_B = 45°; \theta_C = 90°)/I_{AB}(\theta_B = 45°)$? Hence, what is $I_{ABC}(\theta_B = 45°; \theta_C = 90°)/I_A$? Is it larger or smaller than $I_{AC}(\theta_C = 90°)$?

If you do the calculation right, you should reach the remarkable result that the intensity of light actually increases by inserting the polarizer B, despite polarizers only absorbing light!

Polarizers	✓✓	✓✗
A A	100%	0%
A B	85%	15%
B C	85%	15%
A C	50%	50%

Table 1: In the first column we present pairs of polarizers. The second column represents the probability that a photon passing through the first polarizer also passes through the second, while the third column represents the probability that a photon passing through the first polarizer does not pass through the second

Exercise

Can you repeat the same calculation for other values of θ_B and θ_C?

- What happens when $\theta_B = 30°$ and $\theta_C = 60°$?
- What about the general case or arbitrary angles? Can you express the final intensity in terms of I_A, θ_B and θ_C?
- (Extra question to think about and calculate if you have time: what if we had three polarizers after A? each at angles increasing successively by 30°, so that the last polarizer is vertical? What about 5 polarizers each at angles successively increasing by 18°, so that the last polarizer is still vertical?)

Let's take a particular case now, which we will use and stick to until the end of the packet. Consider $\theta_B = 22.5°$ and $\theta_C = 45°$? As earlier, the polarization after (light source + polarizer A) is along the horizontal axis, i.e, $|{\rightarrow}\rangle$, and has an intensity I_A. What is the intensity after polarizer B, i.e., I_{AB}? What is the polarization after polarizer B? What is the intensity after polarizer C, i.e., I_{ABC}? What is the polarization after polarizer C? Lastly, if polarizer B was not there, what would the intensity by after polarizer C, i.e., I_{AC}? What would the polarization be in that case?

Make sure you completely understand this scenario going ahead. Because it is going to be important, I'm going to give the answers to the questions about the intensities here. Don't read ahead if you don't want to read the answers.

You should calculate $I_{AB} \approx 0.85$, $I_{AC} = 0.5$ and $I_{ABC} \approx 0.73$. Since $I_{AB} \approx 0.85$, this means that approximately 15% of light is being absorbed by polarizer B, and another 15% of that is being absorbed by polarizer C. It will be useful to keep in mind that 15% of light is lost each time it passes through a polarizer at an angle of 22.5° with respect to its own polarization. A key point to note, consistent with our above discussion of excess light passing through, is that when the light passes through B, 15% is blocked; after that if it passes through B then another 15% is blocked; yet, if you were to pass it directly through C, then 50% of the light would be blocked, which is a *lot* more than just blocking 15% of the light two times. We're going to represent this important information in a table here, because you're going to want to keep referring back to these numbers:

Also, what this means in light of our earlier discussion regarding photons, is that when the photons hit polarizer B, the pass through with an 85% probability. Then, among the photons which have passed through B, there's again an 85% probability for those photons to pass through C. However, if the photons were to directly approach C, then there would only be a 50% chance for them to pass through. We are going to use these numbers to argue the key point that we mentioned earlier: these number will indicate that there *cannot* be a 'hidden' rule book indicating when a photon can pass through and when it won't.

We will now ask a simple question. Before the polarization of a photon has been 'measured' is there a definite answer for whether the photons will or will not pass through polarizer B? Is there a definite answer for whether or not the photon will pass through C? This definite answer may depend on some hidden rule book that we haven't been able to study, or even may never be able to study. But, for the moment, let's assume that there exists a hidden rule book that determines these answers.

Consider 100 photons that pass through A. For each of these photons, let's assume the rule has individually pre-determined which ones will be allowed through B, and which ones will be allowed through C. For photons that will be allowed through B, we will assume that they have a hidden variable represented by $B✓$. Conversely, if the rule book says that the photon cannot pass through B, we will represent the hidden variable as $B✗$. Similarly if the photon passes through C, we write it as $C✓$, and if it doesn't then $C✗$. Since all of these 100 photons have passed through A, we will assume that they have the hidden variable $A✓$. Thus, among these 100 photons, their hidden variables will be one of the following:

- $A✓, B✓, C✓$
- $A✓, B✓, C✗$
- $A✓, B✗, C✓$
- $A✓, B✗, C✗$

Of the 100 photons, how many would have $A✓, B✓$? This is just the fraction of photons that have passed through A, and will go on to pass through B. From our calculations earlier, the probability of a photon to pass through B is 85%, and hence around 85 photons would have hidden variables $A✓, B✓$. We will write this as $N(A✓, B✓) = 85$, and correspondingly $N(A✓, B✗) = 15$. Of these 85 photons, how many will pass through C? As we saw earlier, around 85% of these photons will pass through, and hence about 73 photons will have $A✓, B✓, C✓$. Thus, $N(A✓, B✓, C✓) = 73$. Also, of these 85 photons with $A✓, B✓$, the remaining 15% will get blocked at C, hence about 15% of 85 = 12 photons will have $A✓, B✓, C✗$. Thus, $N(A✓, B✓, C✗) = 12$.

But, when we only asked the photons whether or not they would pass through C, then we saw that 50% of the times they would pass through. Thus, of the 100 photons which would pass through A, there should be 50 that won't pass through C, i.e., $N(A✓, C✗) = 50$. Take a moment to think for yourself about whether these numbers make any sense. It's probably confusing to think about all of these numbers, so let's take a different example, having nothing to do with photons or polarizers.

Let's assume were thinking of 100 people who like Apples ($A✓$), and are interested in asking whether these people play Baseball (is $B✓$ or $B✗$) and whether they play Chess (is $C✓$ or $C✗$). If $N(A✓, B✓, C✗) = 12$ people play Baseball but not Chess, and if $N(A✓, B✗) = 15$ people don't okay Baseball, is there something that we can say about the number of people that don't play Chess? At the very least, we can say the following.

If you do not play Chess ($A✓, C✗$), then either you do not play Chess and do not play Baseball ($A✓, B✗, C✗$), or you do not play Chess and do play Baseball ($A✓, B✓, C✗$). Thus

$$N(A✓, C✗) = N(A✓, B✗, C✗) + N(A✓, B✓, C✗) \tag{32}$$

However, if the number of people who do not play Baseball, is definitely going to be larger than or equal to the number of people that do not play Baseball and do not play Chess. Thus,

$$N(A✓, B✗) \geq N(A✓, B✗, C✗). \tag{33}$$

And hence,

$$N(A✓, C✗) \leq N(A✓, B✗) + N(A✓, B✓, C✗), \tag{34}$$

i.e., among people who like apples, the number of people who don't play Chess is less than or equal to the number of people who don't play Baseball ($N(A✓, B✗) = 15$ people) plus the number of people who play Baseball, but don't play Chess ($N(A✓, B✓, C✗) = 12$ people). Thus, $N(A✓, C✗) \leq 15 + 12 = 27$, and there can be at most 27 people who don't play Chess but like Apples. This is an examples of a Bell inequality. As you can see, it's just a simple inequality about counting the number of objects in various sets. It's something that must be true, if the questions do you like Apples, do you play Baseball, and do you play Chess have well defined answers. This however, isn't quite going to work for photons.

Going back to the numbers for photons, recall that again $N(A✓, B✗) = 15$ and $N(A✓, B✓, C✗) = 12$. Thus, following Eq. (34), $N(A✓, C✗)$ should be less than or equal to 27. However, if you go back to see what this value was, it was 50, which is certainly not less than or equal to the 27 photons we expected!!! It seems, surprisingly, that thinking about whether the photons have a definite answer to whether or not the photon passes through B leads to a fundamental contradiction! There is just no possible values for all of these numbers to be for them to add up mathematically, while still making sense from the physics perspective.

For simplicity in what comes up next, we're going to note that in classical settings, $N(A✓, B✓, C✗) \leq N(A✓, B✓, C✗) + N(A✗, B✓, C✗) = N(B✓, C✗)$. We can use this to then re-write the Bell inequality as

$$N(A✓, C✗) \leq N(A✓, B✗) + N(B✓, C✗). \tag{35}$$

This is clearly true in the classical case (such as the example of Apples, Baseball and Chess), and it is clearly violated in the quantum case, as can be seen by just reading off the values from Table 1 above.

So now we have conclusively proven that there cannot be any hidden variables or rule book that is being followed to decide whether or not the photons go through. Not only can it not be that we can't *see* the rule book, but in fact there can't *be* a rule book. The events are fundamentally random. It's not even that there's a God playing dice, because, as we said, even dice aren't fundamentally random.

...... Or, have we? Is there no way out? There is *one* possible way in which this hidden variable nonsense can add up. Note that we assumed that the hidden variables of any photon, say $A✓, B✓, C✗$ remain constant after passing through the filters. But, what if these hidden variables changed after passing through the filters? That is, that while the photon initially had the hidden variable state $A✓, B✓, C✗$, saying that it would pass through B, but then after passing through polarizer B the hidden variable state changes to $A\checkmark, B\checkmark, C\checkmark$ and then hence passes through C. If hidden variables can change, then that changes the analysis that we just did! Now, the implications for hidden variables changing would imply that the rule book not only depends on the photon, but in some sense it also depends on some memory of where the photon's been. But independent of that, there is a method that can be used to show that this won't work either. The method, is to use the remarkable idea of '*entanglement*'.

(Before we get into the next section, I should specify that the section is optional. In the next section we will describe why it does not matter that hidden variables change after passing through polarizers. If you can believe this directly, then you can skip this section. I will not be able to give table-top experimental justification for these arguments, so they're going to be a bit hand-wavy. If you do not want to get into the details of entanglement, but are interested in the final results of the discussion, start reading the next section from the bold paragraph onwards.)

6 Entangled photons and pairs of socks

One way to try to figure out if these changing hidden variables are a problem is to try and calculate the probabilities in Table 1 not by sending photons through the polarizers one at a time, but in some sense sending them through the polarizers simultaneously. What does that mean though?

Let us say that two photons are 'entangled' if the photons behave identically to any measurement. I.e., if I take two entangled photons and try to pass both of them through a polarizer at some angle independent of each other, they will either both go through, or both not go through. When photons are entangled this is true even though the passing through a polarizer is a probabilistic event. For example, say we had two photons, P1 and P2, that were entangled and were vertically polarized. The polarization of this pair of photons is generally represented as

$$|\uparrow_{P1}\uparrow_{P2}\rangle, \text{ or sometimes, } |\uparrow\rangle_{P1}|\uparrow\rangle_{P2} \tag{36}$$

Say we gave photon P1 to you, and photon P2 to your friend on another table, and both of you pass it through a polarizer that is oriented at a 45° angle to the vertical. Now, we know for a single photon that the probability of it passing through a polarizer aligned in this fashion is 0.5. Say you measure that photon P1 has gone through the polarizer. Then, on your friends table, whenever he/she passes photon P2 through a polarizer at the same angle, then he/she will *also measure that the photon went through the polarizer*. If you measured that photon P1 did not go through, then your friend will also see photon P2 will not go through. Now, I know being able to create photons that have this spooky entanglement itself is quite weird, after all how does photon P2 know what happened to photon P1? Do they communicate in some unknown way?

Not quite. Let's take an analogy to make the idea a bit easier to grasp. Let's think of socks. (Search for Bertlmann socks on Google for more information.)

I have a variety of pairs of socks in my drawer. I have red pairs, blue pairs, green pairs, purple pairs and black pairs. One day when I walk past you, with my socks hidden underneath my trousers, but you manage to get a glimpse of my left sock, and you see that it is purple. Now, whenever you manage to sneak a peak at my right sock, you already know that it's going to be purple. Did one sock have to communicate with another to make this happen?

Say on some other day I came in wearing socks, and I took them off and I took two paper bags and put one sock in each bag. One bag I give to my astronaut friend, who travels all the way to Andromeda Galaxy, and I give the other bad to Sarah and Cara. They open the bag to see a smelly green sock. Now, even if my astronaut friend opens his paper bag at almost the same time as Sarah and Cara, he will still see a green sock. There was no communication between the two socks. They were created in a manner that paired them up, so to say.

It should be noted though, that this kind of a description does not quite work *exactly* for describing photons. This is really more of a hidden variable version of entanglement, where the photons were made so that their hidden variables were identical. But, photon entanglement doesn't require hidden variables, but going into the details will only make this packet longer than it needs to be.

In the above example though, there was a reason for suggesting to take one sock to the Andromeda Galaxy. In physics, one of the fundamental laws that exist is called 'locality'. What it means, is that if I do something at one location, then the effect of what I did can only travel at a finite speed. For example, if I wanted to send my astronaut friend some information, the fastest method for me to do that is by sending the information in a beam of light, which travels only at the *speed of light*, which is about 670 million miles per hour. So, if he is, say 670 million miles away, then he can only get that information an hour after I send it. So not only does the green sock here not have to communicate with the green sock that he as to decide, but moreover it can't communicate, because there wouldn't be enough time to.

People have conducted such experiments where they explicitly measure two entangled photons at times such that information could not have travelled between the two, and yet for whatever angle the polarizers may be at, as long as they are the same, then the outcome will be the same. This itself is quite remarkable, and could easily make for the material to another quantum mechanics packet :P

What's important to keep in mind going forward, is that entangled photons behave identically without having to communicate with each other .... or they somehow communicate faster than the speed of light. Let's assume that physics doesn't break the speed of light at any point, and that these photons really don't and can't communicate with each other, and let's go forward.

Okay, knowing how entangled photons behave, how can we make sure that these hidden variables aren't affecting changing after passing through a polarizer? What is done is that a large number of pairs of entangled photons $\{(P_1, P_1'), (P_2, P_2'), (P_3, P_3'), \ldots\}$ are generated. These photons have random polarizations to begin with. Give photons $P_1, P_2, P_3, \ldots$ to Alice, and give photons $P_1', P_2', P_3', \ldots$ to Bob. Then both Alice and Bob independently do the following:

First they throw a die to generate a "random" number between 1 and 6.

- If the outcome is 1 or a 2, they pass the photon through polarizer A, i.e., a polarizer with a horizontal polarization;
- If the outcome is 3 or a 4, they pass the photon through polarizer B, i.e., a polarizer whose polarization axis is at 22.5° with respect to the horizontal

- If the outcome is 5 or a 6, they pass the photon through polarizer C, i.e., a polarizer whose polarization axis is at 45° with respect to the horizontal

Then, they note down which polarizer they used and whether the photon went through or not. Thus, for the photons $P_1, P_2, P_3, \ldots$, Alice has a list that looks something like

$$A\checkmark, B\text{✗}, A\text{✗}, C\checkmark, B\text{✗}, B\checkmark, C\text{✗}\ldots \tag{37}$$

Similarly, for the photons $P_1', P_2', P_3', \ldots$, Bob has a list that looks something like

$$C\text{✗}, A\text{✗}, B\text{✗}, A\text{✗}, A\checkmark, B\checkmark, C\text{✗}\ldots \tag{38}$$

Then, Alice and Bob meet to combine their data to obtain for the pairs of photons $(P_1, P_1'), (P_2, P_2'), (P_3, P_3'), \ldots$ a list of data that would look like

$$(A\checkmark, C\text{✗}), (B\text{✗}, A\text{✗}), (A\text{✗}, B\text{✗})(C\checkmark, A\text{✗}), (B\text{✗}, A\checkmark), (B\checkmark, B\checkmark), (C\text{✗}, C\text{✗}), \ldots \tag{39}$$

We can then use this data to construct a table similar to Table 1, by asking the following questions: From the data, look at all pairs of photons (P_i, P_i'), such that P_i passed through polarizer A. For what fraction was the corresponding entangled pair P_i' also measured to pass through A? Well, from our definition of entanglement, this should be 100%. What fraction did not pass through A? Again, by definition, 0%.

Next, look at all pairs of photons (P_i, P_i'), such that P_i passed through polarizer A. For what fraction was the corresponding entangled pair P_i' also measured to pass through B? Experimentally, this turns out to be 85%, and hence the fraction that does not pass through B is 15%. (This can also be argued out from the math in a similar fashion to how we calculated the 85% for the photons passing through one polarizer after the other, but that will be beyond the scope of the packet for now. However, do keep in mind that this can be calculated using very similar theory to what we have used so far).

We can ask similar question for polarizers B and C, and then for polarizers A and C. Each time, we will observe that the percentages that we measure for passing the entangled pairs of photons through two polarizers at different locations will exactly match those in table 1. But, as we've already seen in Eq. (35), the percentages in Table 1 just can't be true in a hidden variable theory! What's remarkable, is that Alice and Bob can be as far away as they want when making their measurements. Earlier we argued that the hidden variable changed due to one polarizer, affecting the photons behavior in front of the next polarizer. Now, the only way for that to happen would be that the photons would have to communicate faster than the speed of light!

Let's summarize what we've done. We started by assuming that photons have hidden variables, i.e, some rule book (that is inaccessible to us) according to which they decide whether or not they will go through a polarizer. Then, we said that either hidden variables cannot exist, or hidden variables change after going through polarizers. Now, in this entanglement version of the polarizer experiments, we see that either hidden variables cannot exist, or photons can communicate with each other faster than the speed of light!

Which of these is true? Both are ideas that are held on to strongly by physicists.

It's convenient to assume that our physical world is deterministic, and that there is no fundamental unpredictability. It's the assumption that photons have some deep underlying state that is hidden, that maybe some day we will be able to probe and understand. In physics, this idea is called '*realism*'.

Physicists also like to believe that effects only travel at finite speeds. Things only travelling as fast as the speed of light is a foundational idea in the field or Relativity, and is important to how we think of things. We don't want our calculations of a particle over here to have to depend on every other particle in the universe, but rather only on nearby particles. In physics, this idea is called '*locality*'.

The violation of Bell's inequality tells us that we can't have both. Physics can't have both realism and locality. Physicists will have to give up one of these two fundamental ideas.

Either the universe does not have realism, or it does not have locality. And simple counting arguments and ideas from your polarized sunglasses can tell you that.

7 Epilogue

Several experiments have been conducted in careful laboratories and with precise experimental setups to verify everything that we have talked out above. While the math behind it all is certainly quite sound and fairly simple and was first argued out by John Bell in 1964, it took until 2015 for the first 'loop-hole free' experimental verification of all of these numbers.

Congratulations if you've made it all the way to the end of the packet! I hope I've managed to give you some insight about why quantum mechanics is so weird. Not only are things random, but they are also fundamentally random in a way that is completely different compared to our every day experiences with probabilities. And yet, all of this can be probed just sitting on a table. I would highly recommend you to see the following two YouTube videos, which lay out all of the ideas in this packet in a condensed form, without which I would have been unable to write this packet.

- `https://youtu.be/MzRCDLre1b4`, titled "Some light quantum mechanics (with minutephysics)", on a channel called '3Blue1Brown'
- `https://youtu.be/zcqZHYo7ONs`, titled "Bell's Theorem: The Quantum Venn Diagram Paradox", on a channel titled 'minutephysics'

Watching both of these two videos will help you get a much better understanding of these topics!