1 | Differential Equations

First order differential equations

Exact differential equations

When solving *exact differential equations* we recognize that one side of the equation is of the form $u\frac{dv}{dx}+v\frac{du}{dx}$ and hence $\int u\frac{dv}{dx}+v\frac{du}{dx}\,dx$ can be quoted as $uv+k$.

Integrating factor

Consider a first order differential equation that can be written in the form $\frac{dy}{dx}+f(x)y+g(x)$. The LHS of the differential equation ia not yet exact, but suppose that it becomes exact when it is multiplied by a function $I(x)$.

Thus, we have the exact differential equation:

$$I(x)\left(\frac{dy}{dx}+f(x)y=g(x)\right)$$

Which is simplified to:

$$I(x)\frac{dy}{dx}+I(x)f(x)y=I(x)g(x)$$

Comparing the LHS with $v\frac{du}{dx}+u\frac{dv}{dx}$ we have:

$$v=I \qquad\qquad u=y$$

$$\frac{du}{dx}=I(x)f(x) \qquad\qquad \frac{dv}{dx}=\frac{dy}{dx}$$

Which results in:

$$\int\frac{1}{I(x)}\,dI(x)=\int f(x)\,dx$$

$$\implies \ln|I(x)|=\int f(x)\,dx$$

$$\therefore \quad I(x)=e^{\int f(x)\,x} \quad \square$$

Second order differential equations

A second order differential equation is one of the form:

$$a\frac{d^2y}{dx^2} + b\frac{dy}{dx} + cy = f(x)$$

with a general solution

$$y = \text{C.F.} + \text{P.I.}$$

The complementary function

To obtain this part of the solution, the general solution of the quadratic equation $ax^2 + bx + c = 0$ by comparing it to $a\frac{d^2y}{dx^2} + b\frac{dy}{dx} + cy = 0$ when $f(x)$, or the **RHS** $= 0$, and thus the two so called roots can be found.

Roots	General Solution
Two real distinct roots α and β	$y = Ae^{\alpha x} + Be^{\beta x}$
Two real equal roots α and α	$y = (A + Bx)e^{\alpha x}$
Two complex roots $p \pm qi$	$y = e^{px}(A\cos qx + B\sin qx)$

Type 2: $\mathbf{f(x) \neq 0}$

When $f(x) \neq 0$ the general solution of such a differential equation is of the form $y =$ C.F. + P.I., where C.F. is the complimentary function and P.I. is the particular integral.
The C.F. is obtained by finding the general solution of the differential equation $a\frac{d^2y}{dx^2} + b\frac{dy}{dx} + cy = 0$, similarly to the previous case.

The P.I. is any solution of the given differential equation. It depends on the function $f(x)$ and is usually a general form of it.

$\mathbf{f(x)}$	**Trial Solution**
5	$y = k$
$2x + 7$	$y = px + q$
$3x^2 + x - 6$	$y = px^2 + qx + r$
$3e7x$	$y = ke^{7x}$
$3xe^{-2x}$	$y = (px + q)e^{-2x}$
$2\sin 3x + 4\cos 3x$	$y = p\sin 3x + q\cos 3x$
$\cos 8x$	$y = p\sin 8x + q\cos 8x$
$\sin x - 3\cos 2x$	$y = p\sin x + q\cos x + r\sin 2x + t\sin 2x$
$2x + 6e^{4x}$	$y = px + q + ke^{4x}$

The trial solution is to be chosen according to the above table and differentiated twice. These values are then substitued in the given differential equation, so that the unknown constant/s of the trial solution can be found.

The general solution $y = \text{C.F.} + \text{P.I.}$ is then written.

The unknown constants of the C.F. can be found if additional information is given (i.e. $y = 1$ and $\frac{dy}{dx} = 0$ when $x = 0$).

In some cases (failure cases) the trial solution listed in the above table leads to an *inconsistent equation*. This usually happens when the trial solution is included in the complementary function. The correct trial solution is obtained by multiplying the trial solution in the above table by x or x^2. Such trial solutions are usually given by the question.