Pure Mathematics
Advanced Level

"Once your soul has been enlarged by a truth, it can never return to its original size."
-Blaise Pascal

Notes By

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1 | Series

Sequences

Consider the following set of numbers:

$$2, 4, 6, 8, \ldots$$
$$1, 2, 4, 8, 16, \ldots$$
$$4, 9, 16, 25, \ldots$$

In each of the above cases, the numbers are written in a particular order and there is a clear rule for obtaining the next number and as many numbers in the list.

The above are all examples of sequences where a sequence is a set of terms in a defined order with a rule for obtaining the terms.

Summation Series

When the terms of a sequence are added, a summation series is formed:

$2+4+6+8+10+\ldots$ $1+2+4+8+16+\ldots$ $4+9+16+25+36+\ldots$	are all examples of series.

A series can be finite or infinite, where a finite series consists of a fixed number of terms, whereas an infinite series has an infinite number of terms.

Considering the follwing series,

$$1+\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\frac{1}{16}+\ldots$$

we can notice that the general term of this series is $\frac{1}{2^r}$. The general term of a series is not unique, it depends on the initial value of r. Thus, the general term $\frac{1}{2^{r-1}}$ also corresponds to the above series but we take the initial value of r to be 1.

A summation series can be defined in a concise way using the Greek letter Σ denoting the *summation of terms*. The above series may be expressed as

$$\sum_{r=0}^{\infty} \frac{1}{2^r}$$

which is equivalent to

$$\sum_{r=1}^{\infty} \frac{1}{2^{r-1}} \quad r \in \mathbb{Z}$$

Arithmetic Progressions

Definition

An arithmetic progression is a sequence of numbers starting with term a, in which successive terms are obtained by adding the same constant, denoted by d, reffered to as the **common difference**.

General term

Let us consider the general A.P with first term a and common difference d:

$$a + (a + d) + (a + 2d) + (a + 3d) + (a + 4d) + (a + 5d) + ...$$

By observing the coefficient of d and the position of the term, we can conclude that the general term can be obtained by the equation:

$$T_n = a + (n - 1)d$$

Sum of the first n terms

Considering an A.P. with n terms, let the first term be a, the common difference to be d and the last term to be l

$$\begin{aligned}
S_n &= a + (a + d) + (a + 2d) + (a + 3d) + (a + 4d) + (a + 5d) + ... + (l - d) + l \\
\implies S_n &= l + (l - d) + (l - 2d) + (l - 3d) + (l - 4d) + (l - 5d) + ... + (a + d) + a \\
\implies 2S_n &= n(2a + l) \\
\implies S_n &= \frac{n}{2}(2a + l) \\
&= \frac{n}{2}(2a + (n - 1)d)
\end{aligned}$$

Geometric progressions

Definition

A geometric progression is a sequence of numbers starting with term a, in which successive terms are obtained by multiplying the same constant, denoted by r, reffered to as the **common ratio**.

General term

Let us consider a general G.P. with first tirm a and a common ratio r:

$$a + ar + ar^2 + ar^3 + ar^4 + ar^5 + ... + ar^n - 1 + \dots$$

By observing the exponent of r and the position of the term we can conclude that the general term can be obtained by the equation:

$$\mathrm{T_n} = ar^{n-1}$$

Sum of the first n terms

Considering a G.P. with first term a, common ratio d and it's sum denoted by S_n:

$$\begin{aligned}
S_n = a+ &ar + ar^2 + ar^3 + ar^4 + ar^5 + \dots + ar^{n-1} + \dots \\
rS_n = &\quad ar + ar^2 + ar^3 + ar^4 + ar^5 + \dots + ar^n + \dots \\
S_n - rS_n = &\quad a - ar^n \\
(1-r)S_n = &\quad a(1-r^n) \\
S_n = &\quad \frac{a(1-r^n)}{1-r} \quad \square
\end{aligned}$$

Convergence of a series

For any[1] given G.P., we can always add infinitely many times more terms to the end of the series, but what hapens to it's sum? Given that $|*|\, r = 1$, the sum will always get infinitesimally closer to a definite number.

Considering the G.P. $\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \dots$ we can observe that if you keep on adding terms and summing them, the sum will approach 1. In mathematical language we say

$$\text{as } n \to \infty \,,\; S_n = 1$$

or

$$\lim_{n\to\infty} S_n = 1$$

[1]A.P's by definition **do not** converge.

Binomial theorem

The binomial theorem states that

$$(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^k y^{n-k}$$

Maclaurin Series

Derivation

Let $f(x)$ be nay function of x and suppose that $f(x)$ can be expanded as a series of ascending powers of x and that this series can be differentiated $w.r.t.x$

$$f(x) \equiv a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + ... + a_r x^r$$

where a_n are constants to be found

Thus, inputting 0 into $f(x)$ returns:

$$\boxed{f(0) = a_0}$$

Differentiating $f(x)$ $w.r.t.x$:

$$f'(x) \equiv a_1 + 2a_2 x + 3a_3 x^2 + 4a_4 x^3 + \cdots + r a_r x^{r-1} + \cdots$$

Inputting 0 into $f'(x)$:

$$\boxed{f'(0) = a_1}$$

Differentiating $f'(x)$ $w.r.t.x$:

$$f''(x) \equiv 2a_2 + 6a_3 x + 12a_4 x^2 + \cdots + (r-1)(r)a_r x^{r-2} + \cdots$$

Inputting 0 into $f''(x)$:

$$\boxed{f''(0) = 2a_2}$$

Differentiating $f''(x)$ $w.r.t.x$:

$$f'''(x) \equiv 6a_3 + 24a_4 x + \cdots + (r-2)(r-1)(r)a_r x^{r-3} + \cdots$$

Inputting 0 into $f'''(x)$:

$$\boxed{f'''(0) = (2)(3)a_3}$$

$$\vdots$$

By the above calculation we can conclude that:

$$\boxed{a_r = \frac{f^r(x)}{r!}}$$

Considering all of the above:

$$f(x) \equiv f(0) + f'(0)x + \frac{f''(0)x^2}{2!} + \frac{f'''(0)x^3}{3!} + \cdots + \frac{f^r(0)x^r}{r!} + \cdots$$

$$\therefore \quad f(x) \equiv \sum_{r=1}^{\infty} \frac{f^r(x)}{r!}$$

This is known as Maclaurin's Theorem, and can be obtained if and only if $f^r(0) \in \mathbb{R}$. In the following examples we use Maclaurin's Theorem to obtain the series expansion of some standard equations. The range of validity of each expansion is left as an exercise to the reader.

Examples

Ex. 1. *Express e^x as a series expansion using the Maclaurin theorem.*

Let $f(x) = e^x$

$$\begin{aligned} f(x) =& e^x \quad \Rightarrow \quad f(0) = 1 \\ f'(x) =& e^x \quad \Rightarrow \quad f'(0) = 1 \\ f''(x) =& e^x \quad \Rightarrow \quad f''(0) = 1 \\ f'''(x) =& e^x \quad \Rightarrow \quad f'''(0) = 1 \end{aligned}$$

$$\therefore \quad e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \ldots + \frac{x^r}{r!} + \ldots$$

Ex. 2. *Express* $\cos x$ *as a series expansion using the Maclaurin theorem.*

$$\begin{aligned} f(x) =& \ \cos(x) \quad \Rightarrow \quad f(0) = 1 \\ f'(x) =& \ \text{-}\sin(x) \quad \Rightarrow \quad f'(0) = 1 \\ f''(x) =& \ \text{-}\cos(x) \quad \Rightarrow \quad f''(0) = 1 \\ f'''(x) =& \ \sin(x) \quad \Rightarrow \quad f'''(0) = 1 \end{aligned}$$

$$\therefore \quad \cos(x) = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \ldots + (-1)^r \times \frac{x^{2r}}{2r!} + \ldots$$

The above expansion justifies the fact that when x is very small and thus high powers of x may be neglected, then: $\boxed{\cos x \approx 1 - \frac{x^2}{2}}$

Ex. 3. *Express* $\ln(1+x)$ *as a series expansion using the Maclaurin theorem.*

$$\begin{aligned}
f(x) &= \ln(1+x) &&\Rightarrow && f(0) = 0\\
f'(x) &= (x+1)^{-1} &&\Rightarrow && f'(0) = 1\\
f''(x) &= -(1+x)^{-2} &&\Rightarrow && f''(0) = -1\\
f'''(x) &= 2(1+x)^{-3} &&\Rightarrow && f'''(0) = 2
\end{aligned}$$

$$\therefore \quad \ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \ldots + (-1)^{r+1} \times \frac{x^r}{r} + \ldots$$

Ex. 4. *Expand* $\arcsin(x)$ *up to the term in* x^3*. By putting* $x = \frac{1}{2}$*, find an approximate value for* π

$$\begin{aligned}
f(x) &= \arcsin(x) &&\Rightarrow && f(0) = 0\\
f'(x) &= (1-x^2)^{\frac{-1}{2}} &&\Rightarrow && f'(0) = 1\\
f''(x) &= x(1-x^2)^{\frac{-3}{2}} &&\Rightarrow && f''(0) = 0\\
f'''(x) &= 3x(1-x^2)^{\frac{-5}{2}} + (1-x^2)^{\frac{-3}{2}} &&\Rightarrow && f'''(0) = 1
\end{aligned}$$

$$\therefore \quad \arcsin(x) = x + \frac{x^3}{3!} + \ldots$$

Putting $x = \frac{1}{2}$

$$f\left(\frac{1}{2}\right) = \frac{\pi}{6}$$

$$\Longrightarrow \qquad \pi \approx 6\left(\frac{1}{2} + \frac{1}{81}\right)$$

$$\Longrightarrow \qquad \pi \approx \frac{83}{27}$$

Expanding compound functions using standard functions

Ex. 1. Expand *a)* $\dfrac{e^{2x}+e^{-3x}}{e^x}$ *b)* $\ln\left(\dfrac{1-2x}{(1+2x)^2}\right)$ as series of ascending powers of x up to the term in x^4. Give the general term in each case and the range of values of x for which each expansion is valid.

a)

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \dots$$

$$e^{-3x} = 1 + (-3)x + \frac{(-3x)^2}{2!} + \frac{(-3x)^3}{3!} + \frac{(-3x)^4}{4!} + \dots$$

$$\therefore \quad e^x + e^{-3x} = \left(1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!}\right) + \left(1 + (-3)x + \frac{(-3x)^2}{2!} + \frac{(-3x)^3}{3!} + \frac{(-3x)^4}{4!}\right)$$

$$= 2 - 2x + \frac{10x^2}{2!} - \frac{26x^3}{3!} + \frac{89x^4}{4!}$$

b)

$$\ln\left(\frac{1-2x}{(1+2x)^2}\right) = \ln(1-2x) - 2\left(\ln(1+2x)\right)$$

Consider $\ln(1-2x)$:

$$\ln\left(1+(-2x)\right) = -2x + -2x^2 - \frac{8x^3}{3} - 4x^4 + \dots + \frac{(-1)^{r-1}(-2x)^r}{r} + \dots$$

Consider $\ln(1+2x)$:

$$\ln\left(1+2x\right) = 2x + -2x^2 + \frac{8x^3}{3} - 4x^4 + \dots + \frac{-2(-1)^{r-1}(2x)^r}{r} + \dots$$

$$\therefore \quad \ln\left(\frac{1-2x}{(1+2x)^2}\right) = \left(-2x + -2x^2 - \frac{8x^3}{3} - 4x^4\right) - 2\left(2x + -2x^2 + \frac{8x^3}{3} - 4x^4\right)$$

$$= -6x + 2x^2 - 8x^3 + 4x^4$$

Range of Validity:

$$\frac{(-1)^{r-1}(-2x)^r}{r} - \frac{2(-1)^{r-1}(2x)^r}{r}$$

$$= \frac{(-1)^{r-1}(-1)^r(2x)^r + 2(-1)^r(2x)^r}{r}$$

$$= \frac{(-1)^{2r-1}(2x) + 2(-1)^r(2x)^r}{r}$$

$$= \frac{\left((-1)^{2r-1} + 2(-1)^r\right)(2x)^r}{r}$$

$$= \frac{(-1+2(-1)^r(2x)^r)}{r}$$

$$= \frac{2^r(2(-1)^r-1)x^r}{r}$$

Ex. 2. Expand $\ln\left(\frac{x+1}{x}\right)$ as series of ascending powers of x up to the term in x^4. Give the general term in each case and the range of values of x for which each expansion is valid.

$$f(x)\ln\left(\frac{x+1}{x}\right) = \ln\left(1+\frac{1}{x}\right)$$

$$\begin{aligned} f(x) &= \ln(1+\frac{1}{x}) &\Rightarrow\quad& f(0)=0 \\ f'(x) &= (x+1)^{-1} &\Rightarrow\quad& f'(0)=1 \\ f''(x) &= -(1+x)^{-2} &\Rightarrow\quad& f''(0)=-1 \\ f'''(x) &= 2(1+x)^{-3} &\Rightarrow\quad& f'''(0)=2 \\ &= \tfrac{1}{x}-\tfrac{1}{2x^2}+\tfrac{1}{3x^3}-\tfrac{1}{4x^4}+\ldots+\tfrac{(-1)^{r+1}}{r} \end{aligned}$$

Ex. 3. Expand $\sin^2 x$ using Maclaurin's series up to x^4

$$sin^2(x) \equiv \frac{1-\cos(2x)}{2}$$

Consider $\cos(2x)$:

$$= 1-\frac{(2x)^2}{2!}+\frac{(2x)^4}{4!}-\cdots+\frac{(-1)^r(2x)^{2r}}{(2r)!}+\cdots$$

$$= 1-2x^2+\frac{2x^4}{3}-\cdots+\frac{(-1)^r(2x)^{2r}}{(2r)!}+\cdots$$

$$\therefore \quad \sin^2(x) \equiv \frac{1}{2}\left(1-(1-2x^2+\frac{2x^4}{3}-\cdots+\frac{(-1)^r(2x)^{2r}}{(2r)!}+\cdots)\right)$$

$$= \frac{1}{2}\left(1-1+2x^2-\frac{2x^4}{3}+\cdots+\frac{(-1)^r(2x)^{2r}}{(2r)!}+\cdots\right)$$

$$= x^2-\frac{x^4}{3}+\cdots+\frac{(-1)^{r+1}(2x)^{2r}}{(2r)!}+\cdots$$

Ex. 4. Given $e^{2x} \cdot \ln 1 + ax$ find possible values for p and q.

Consider e^{2x}:

$$e^{2x} = 1 + 2x + \frac{(2x)^2}{2!} + \frac{(2x)^3}{3!} + \cdots + \frac{x^r}{r!} + \cdots$$

Consider $\ln(1 + ax)$:

$$\ln(1 + ax) = ax - \frac{(ax)^2}{2} + \frac{(ax)^3}{3} - \cdots + \frac{(-1)^{r+1}x^r}{r} + \cdots$$

$$\therefore \quad e^{2x} \cdot \ln(1 + ax) = \left(1 + 2x + 2x^2 + \frac{4x^3}{3}\right)\left(ax - \frac{a^2x^2}{2} + \frac{a^3x^3}{3}\right)$$

$$= ax - \frac{a^2x^2}{2} + \frac{a^3x^3}{3} + 2ax^2 - 2a^2x^3$$

$$= ax - \left(\frac{a^2}{2} + 2a\right)x^2 + \left(\frac{a^3}{3} - 2a^2\right)x^3$$

$$\therefore \qquad \left.\begin{aligned} p &= a \\ \frac{a^2}{2} + 2a &= \frac{-3}{2} \\ \frac{a^3}{3} - 2a^2 &= q \end{aligned}\right\}$$

$$p = -3, -1$$

$$q = -27, -\frac{7}{3}$$

Summation of Series

Method 1: Generating differences

Ex. 1. Simplify $f(r) - f(r+1)$, when $f(x) = \frac{1}{r^2}$. Hence, find the sum up to n terms of:

$$\sigma_1 = \frac{3}{1^2 \cdot 2^2} + \frac{5}{2^2 \cdot 3^2} + \frac{7}{3^2 \cdot 4^2} + \dots$$

Simplifying $f(r) - f(r+1)$:

$$f(r) - f(r+1) = \frac{1}{r^2} - \frac{1}{(r+1)^2}$$

$$= \frac{(r+1)^2 - r^2}{r^2(r+1)^2}$$

$$= \frac{2r+1}{r^2(r+1)^2}$$

Generating series and adding quantitatively equivalent terms:

$$\frac{1}{1^2} - \cancel{\frac{1}{2^2}}$$
$$\cancel{\frac{1}{2^2}} - \cancel{\frac{1}{3^2}}$$
$$\cancel{\frac{1}{3^2}} - \cancel{\frac{1}{4^2}}$$
$$\vdots$$
$$\cancel{\frac{1}{n^2}} - \frac{1}{n+1^2}$$

$$\boxed{\therefore \quad \sigma_1 = 1 - \frac{1}{n+1^2}}$$

Ex. 2. If $f(r) = r(r+1)!$ simplify $f(r) - f(r-1)$. Hence sum the series:

$$\sigma_1 = 5 \cdot 2! + 10 \cdot 3! + 17 \cdot 4! + ... + (n^2 = 1)n!$$

$$f(r) - f(r-1) = r(r+1)! - (r-1)r!$$

$$= r(r+1)r! - (r-1)r!$$

$$= r!(r^2 + r - r + 1)$$

$$= r!(r^2 + 1)$$

Generating series and adding quantitatively equivalent terms:

$$\cancel{f(2)} - f(1)$$
$$\cancel{f(3)} - \cancel{f(2)}$$
$$\cancel{f(4)} - \cancel{f(3)}$$
$$\vdots$$
$$\cancel{f(n-1)} - \cancel{f(n-2)}$$
$$f(n) - \cancel{f(n-1)}$$

Ex. 3. If $f(r) = \cos 2r\theta$, simplify $f(r) - f(r+1)$. Hence find $\sin 3\theta + \sin 5\theta + \sin 7\theta$ ____________________

$$\begin{aligned} f(r) - f(r+1) &= \cos(2r\theta) - \cos(2(r+1)\theta) \\ &= -2\sin\left(\frac{2r\theta + (2r+2)\theta}{2}\right) \cdot \sin\left(\frac{2r\theta - 2(r+1)\theta}{2}\right) \\ &= -2\sin(2r\theta + \theta)\sin(-\theta) \\ &= 2\sin(\theta[2r+1])\sin\theta \end{aligned}$$

Generating series and adding quantitatively equivalent terms:

$$\begin{array}{cccc} r=1 & 2\sin(3\theta)\sin(\theta) & = & f(1) \cancel{- f(2)} \\ r=2 & 2\sin(5\theta)\sin(\theta) & = & \cancel{f(2)} \cancel{- f(3)} \\ r=3 & 2\sin(7\theta)\sin(\theta) & = & \cancel{f(3)} \cancel{- f(4)} \\ \vdots & \vdots & = & \vdots \\ r=n & 2\sin(2n+1)\sin(\theta) & = & \cancel{f(n)} - f(n+1) \end{array}$$

$$\begin{aligned} f(1) - f(n+1) &= 2\sin(\theta)\sin(2n+1) \\ &= \frac{\cos(2\theta) - \cos(2\theta(n+1))}{2\sin(\theta)} \\ &= \frac{2\sin(\theta(2r+1))\sin\theta}{2\sin(\theta)} \\ &= \frac{sin((n=1)\theta)\sin(n\theta)}{\sin(\theta)} \end{aligned}$$

Method 2: Using partial fractions

A special case of the previous method can happen to imply a partial fraction decomposition.

Ex. 1. Decompose $\frac{1}{r(r+1)}$. Hence find the sum of

$$\sigma_1 = \frac{1}{1 \cdot 2} + \frac{1}{2 \cdot 3} + \frac{1}{3 \cdot 4} + \cdots$$

Decomposing:

$$\frac{1}{r(r+1)} \equiv \frac{1}{r} - \frac{1}{r-1}$$

Generating series and adding quantitatively equivalent terms:

$$\begin{array}{lccc}
r = 1 & \frac{1}{1} & - & \cancel{\frac{1}{2}} \\
r = 2 & \cancel{\frac{1}{2}} & - & \cancel{\frac{1}{3}} \\
r = 3 & \cancel{\frac{1}{3}} & - & \cancel{\frac{1}{4}} \\
\vdots & \cancel{\vdots} & - & \cancel{\vdots} \\
r = n & \cancel{\frac{1}{n}} & - & \frac{1}{n+1}
\end{array}$$

$$\therefore \quad \sigma_1 = 1 - \frac{1}{n+1}$$

Finding convergent value:

$$1 = \lim_{x \to \infty} \left(1 - \frac{1}{n+1}\right)$$

Ex. 2. Find $\sum_{r=3}^{n} \frac{2}{(r-1)(r+1)}$

Consider $\frac{2}{(r-1)(r+1)}$:

$$\frac{2}{(r-1)(r+1)} \equiv \frac{1}{r-1} - \frac{1}{r+1}$$

$$\therefore \quad \sum_{r=3}^{n} \frac{2}{(r-1)(r+1)} \quad \equiv \quad \sum_{r=3}^{n} \frac{1}{(r-1)} - \frac{1}{(r+1)}$$

Generating series and adding
quantitatively equivalent terms:

Method 3: Using standard results

Method 4: Comparing to standard results

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2 | Permutations and Combinations

Permutations

Consider n objects from which r are to be arranged in a particular order, where $r \leq n$. The number of *permutations* of which r objects from a total of n refers to the number of ways in which these r objects can be arranged, where the order of arrangement matters.

Let us consider 3 letters {A, B, C}. There are 6 possible arrangements, or *permutations*, of these letters, namely:

$$\{\{A, B, C\}, \{A, C, B\}, \{B, A, C\}, \{B, C, A\}, \{C, B, A\}, \{C, A, B\}\}$$

In general, the number of permutations of r objects from a total of n is denoted by ${}^nP_r$ defined as

$${}^nP_r = \frac{n!}{(n-r)!}$$

Ex. 1. Consider the set of letters $\{A, B, C, D, E\}$.
(a) How many of these arrangements start with a vowel?

Ex. 2. Consider the set of numbers $\{1, 2, 3, 4, 5, 6\}$.
(a) In how many ways can a 4 digit number be formed from the above set?
(b) How many of these numbers are even?
(c) How many of these numbers are greater than 3000?

Ex. 3. Consider the set of numbers $\{1, 2, 3, 4, 5, 0\}$.
(a) How many 3 digit numbers can be formed?
(b) How many of these numbers are even?
(c) How many of these numbers are greater than 400?
(d) How many even numbers can be formed?

Permutations with Identical Objects

The above method, however, does not suffice in the case that we have identical objects. Suppose we have the set $\{S, E_1, E_2\}$. For every time t the repeated element is present in the given set, we have to divide the total we have to divide by t!

Ex. 1. In how many ways can the letters of the word ‘MALTA’ be arranged?

Ex. 2. In how many ways can the letters of the word ‘ILLUSTRATIONS’ be arranged?

Since the letters $\{I, L, S, T\}$ are repeated twice, the total possibilities have to be divided by the factorial of the number of each recurring letter. (*i.e., divide by* 2! *for the two ‘I’s, by* 2! *for the two ‘L’s ...*)

Circular Permutations

In a particular field of mathematics referred to as group theory, a cyclic permutation is a permutation of the elements of some set X which maps the elements of some subset S of X to each other in a cyclic fashion, while fixing all other elements of X. In other words, this is the number of ways in which a set can be permuted whilst omitting identical cycles.

Ex. 1. In how many ways can 6 people be seated at a round table?

Ex. 2. In how many ways can 4 couples be arranged around a table?
(a) In how many of these arrangements are all the males separated? (*i.e., no male sits next to another*)

Ex. 3. In how many ways can 4 couples be seated such that every one person sits next to their partner?

Combinations

A combination is defined as one of the selections of r objects from a total of n, where order does not matter. The number of total combinations is denoted by ${}^nC_r$, where

$${}^nC_r = \frac{n!}{(n-r)!\,r!}$$

Ex. 1. In how many ways can a committe of 8 people be chosen from a group of 17 candidates?

$${}^{17}C_8 = 24310 \text{ selections}$$

Ex. 2. In how many ways can a team of 5 players be chosen from a group of 15 students?
b) In how many ways can the selection if the eldest student is to be selected?
c) In how many ways can the selection if the twins in the group must not be seperated?

Ex. 3. In how many ways can 16 people be divided into 2 groups of
a) 9 people and 7 people respectively?
b) 8 people each?

Ex. 4. In how many ways can 18 people be divided into 3 groups of 6 people each.

Further Permutations and Combinations

Ex. 1. Find the number of permutations of the word 'ACQUAPANNA'. How many of these arrangements
a) haveo the letters '**A**' appear adjacent?
b) have the letters '**A**' appear seperated?
c) have the letters '**CQ**' appear adjacent?
d) have the letters '**CQ**' appear seperated?

Ex. 2. Find the number of permutations of the word 'ELECTRICITY'. How many of these arrangements
a) have adjacent '**E**'s?
b) **start** and **finish** with the same letter?
c) have the word '**CITY**' appear in them?
d) have the letters '**CITY**' together?

Distribution of Objects

Ex. 1. Grandma has 10 identical 1 euro coins to distribute among her 3 grandchildren. In how many eays can she distribute the coins?

a) In how many of these possibilities is the oldest grandchild the only one to end up with no coin at all?

This problem can be simplified to the number of ways these 12 symbols (€ or |) to 1

Ex. 2. Twelve identical sweets are to be distributed to 4 children such that all children recieve at least one sweet. In how many ways can this be done?