# Econometrics - FIN403:
# Practical Assignment 3

Due on Thursday December 23, 2021

**Girolamo Vurro (341816), Filippo Rondoni (338506)**

# Contents

**Part 1** **3**
- Point a) . . . 3
- Point b) . . . 3
- Point c) . . . 4
- Point d) . . . 4
- Point e) . . . 5
- Point f) . . . 6
- Bonus question . . . 7

# Part 1

## Point a)

We report the plot that we obtained in figure 1. We can observe that the series does not appear stationary, but it is not completely clear so we have to investigate further. Give to the fact that there are downwards peaks, we might think that the the series steers away from its mean which supports the idea of *non stationarity*. Furthermore, we plotted the regression line of `log(vehicle sales)` on time and we noticed its upwards slope, thus, we could hypothesize that there could be a trend but it is still difficult conclude whether it is stationary or not.

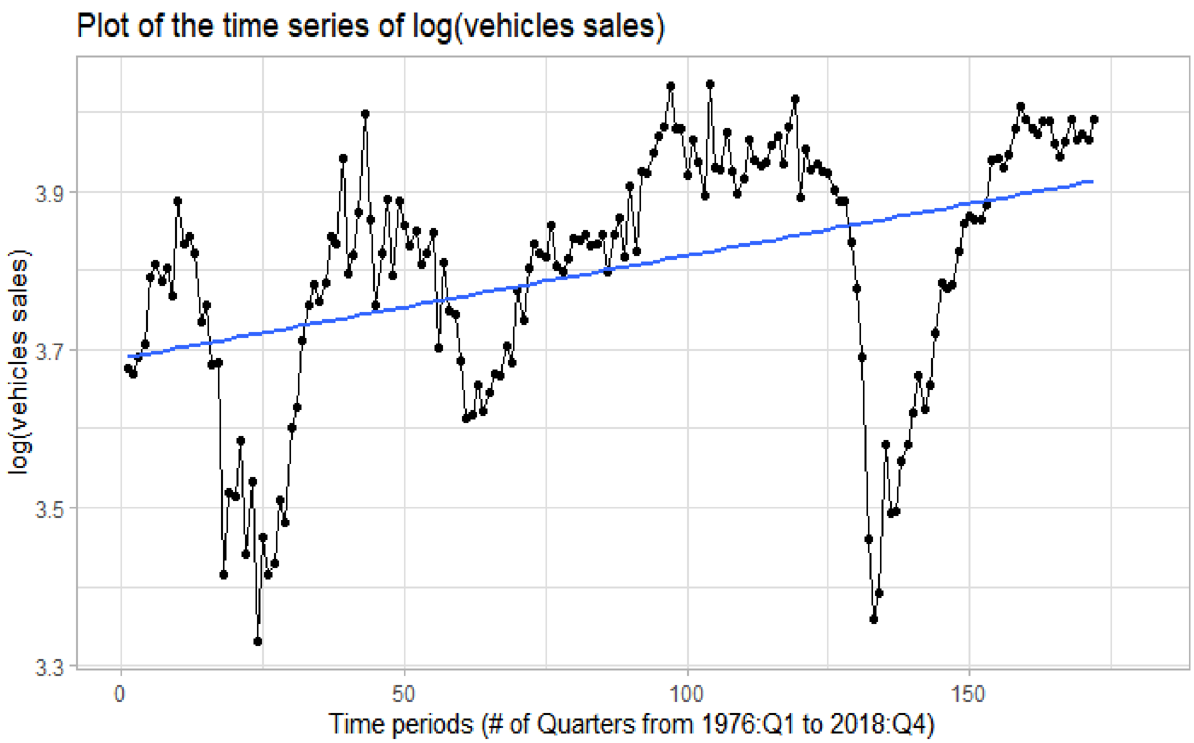


Figure 1: Plot of the time series of the dataset in logs

## Point b)

After having computed the Augmented Dickey-Fuller test on the AR(p):

$$\Delta Y_t = \delta + \kappa t + \pi Y_{t-1} + c_1 \Delta Y_{t-1} + ... + c_{p-1} \Delta Y_{t-p+1} + \epsilon_t.$$

with $p = 0, 2, 4, \ldots, 12$ lags, its results are shown in the table 7 in the bottom. We obtained that the *p-values* of all the computations except the ones obtained for 6 and 12 lags (that are smaller than $\alpha = 0.05$), are bigger than 0.05 and therefore we did not reject the null hypothesis of unit root and no time trend for 5/7 of the computations. For the ones with 6 and 12 lags, as the *p-value* is strictly smaller than 0.05, so we could reject the null hypothesis. However as in the majority of the computations we did not reject the null hypothesis, it is not unlikely that the model might have a unit root neither a time trend.

From this conclusion we could also say that, as hypothesized in the **point a)**, there is not a statistical evidence that the time series is stationary.

From the theory we know that if a time series has a unit root (and we did not reject the hypothesis), then, by taking the first differences we would obtain a stationary series.

## Point c)

After plotting the ACF and PACF we made the following observations (figure 2):

1. The autocorrelations tail off gradually, indeed only after high number of lags (11) they become statistically not significant.
2. The partial autocorrelations become statistically not significant after only two lags (which can be interpreted as the cutoff, i.e. p = 2).

After these observations we could conclude that the clues provided by ACF and PACF plots suggest that we adopt an AR(2) model which presents autocorrelations and partial autocorrelations plot very affine to those that we observe, (as we have also seen in class).

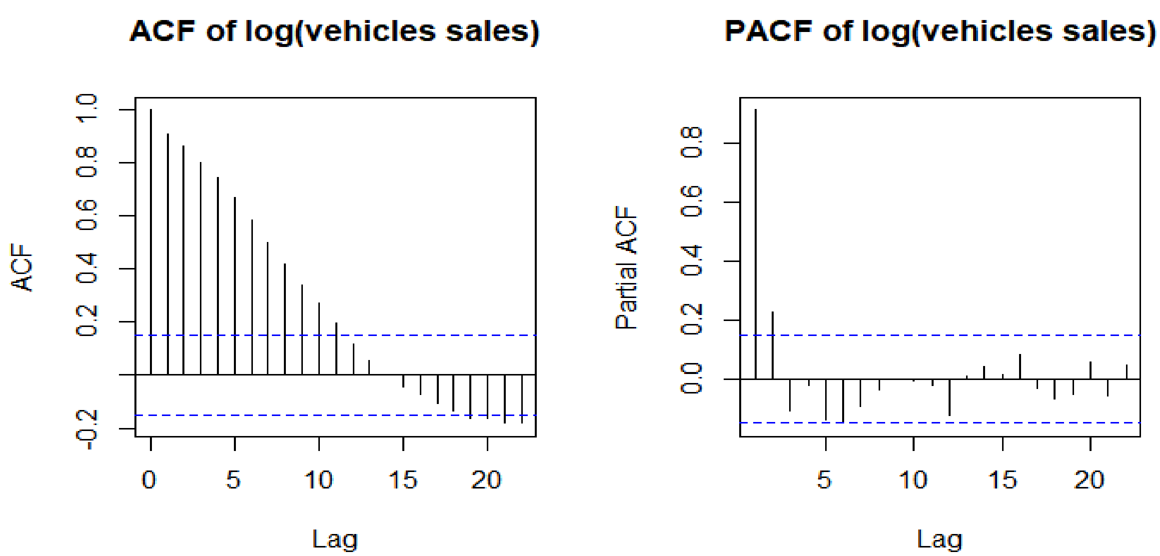


Figure 2: ACF and PACF plots of $log(vehicle\ sales)$

## Point d)

In this section we proceeded following the steps that we report with the aim of finding the best model for the time series of `log(vehicles sales)`.

- We first estimated the our candidate model namely AR(2) or ARIMA(2,0,0)

$$log(vehicles\ sales)_t = \delta + \theta_1 log(vehicles\ sales)_{t-1} + \theta_2 log(vehicles\ sales)_{t-2} + \epsilon_{t-1}$$

  and we report the estimated coefficients in the first column of the table 8.

  Before going on with the diagnostics we made the following consideration. From the theory we know that the condition for a time series to be stationary is that its characteristics roots "lie outside the unit cirle" or equivalently the *inverse* of the roots have to "lie within the unit cirle". However, given that we said that it is likely that the time series has a unit root from the tests that we performed in **point b)**, here we plotted the inverse of characteristics roots for this model in figure 3.

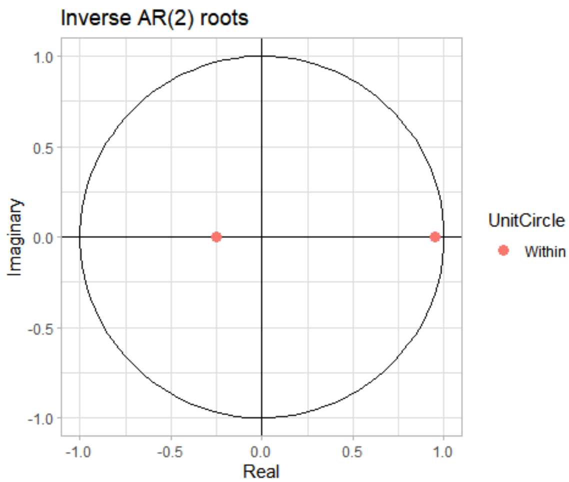


Figure 3: Inverse characteristics roots for the model AR(2)

As we can see from the plot in figure 3 both the inverse of the characteristic roots are contained in the unit circle. However, one of them is very close to the border and therefore it is not unlikely that we would have a unit root as we also stated in **point b)**. Furthermore, we also checked the *sum of of the autoregressive coefficients* **(SARC)**: $|\sum_{i=1}^{2} \hat{\theta}_i| \simeq 0.937$ which is smaller than 1 but still high, so we still consider that what we stated in **point b)** may be valid.

- Subsequently, we performed the diagnostics of our candidate model ARMA(2,0,0) to assess its adequacy, namely:

  1. **Overfitting**, where we estimated ARMA(2,0,1), ARMA(3,0,0) and we reported the resulting estimators of the two models in table 8 respectively in the second and third column. As we can notice both the "overfitted" models produce estimators of the additional coefficients that are not statistically significant, therefore our candidate model passes the first diagnostic (also, having checked the information criteria we saw that our candidate model had the lowest AIC and BIC if compared to the overfitted models).
  2. **Residual analysis**, in which we performed the **Ljung-Box** test first with 6 lags and then with 12 lags, and it produced the following results:

Table 1: Results of residual analysis

| $\chi^2_{K-p-q}$ | Degree of freedom | *p-value* | Method |
|---|---|---|---|
| 9.363 | 4 | 0.053 | Box-Ljung test (with 6 lags) |
| 13.507 | 10 | 0.197 | Box-Ljung test (with 12 lags) |

When we use 6 lags we do not reject the null hypothesis that the errors are not autocorrelated. However, we need to be cautious with the model AR(2) because its *p-value* is 0.053 and it is very close to the threshold of 0.05.

On the other hand, when we used 12 lags, we also do not reject the null hypothesis that the residuals are serially uncorrelated, but this time, as the *p-value* is 0.197, we are more confident.

Therefore, our candidate model passed successfully all the diagnostics and we will utilize to estimate the predictions.

## Point e)

In order to predict quarterly vehicle sales over 2019:Q1-2020:Q4 we use our model our preferred model AR(2). We obtain the following values reported in the table. Notice that the prediction obtained with our model are in logs so we had to transform them back into levels to get the values in the table 2.

Table 2: Quarterly prediction of vehicle sales over 2019:Q1-2020:Q4

| 2019:Q1 | 2019:Q2 | 2019:Q3 | 2019:Q4 | 2020:Q1 | 2020:Q2 | 2020:Q3 | 2020:Q4 |
|---|---|---|---|---|---|---|---|
| 53.204 | 52.872 | 52.418 | 52.025 | 51.646 | 51.291 | 50.955 | 50.639 |

## Point f)

We then build a 95% confidence interval for this prediction, and plot it together with the point forecast, and with the actually realized values. The confidence intervals are reported in the table 3.

Table 3: $log(vehicles\ sales)$ forecast over 2019:Q1-2020:Q4

| Quarters | Forecast | S.E. | Lower Bound | Upper Bound |
|---|---|---|---|---|
| 2019:Q1 | 3.972 | 0.061 | 3.853 | 4.092 |
| 2019:Q2 | 3.966 | 0.074 | 3.820 | 4.112 |
| 2019:Q3 | 3.957 | 0.086 | 3.788 | 4.127 |
| 2019:Q4 | 3.950 | 0.096 | 3.762 | 4.137 |
| 2020:Q1 | 3.943 | 0.103 | 3.740 | 4.145 |
| 2020:Q2 | 3.936 | 0.110 | 3.720 | 4.151 |
| 2020:Q3 | 3.929 | 0.115 | 3.703 | 4.155 |
| 2020:Q4 | 3.923 | 0.120 | 3.687 | 4.158 |

We report the plot that we obtained in figure 4. The confidence intervals at 80% and 95% are marked in blue and light blue; while the projection line is marked in dark blue and the true data in red.

Apparently the recent pandemic due to COVID may be the cause of unexpected lower levels of the realized sales, with the evident drop in the second quarter of 2020 which is clearly seen in the figure 4 where the red line goes below the 95% confidence interval boundary. However, we know that such a drop is supposed to happen with 5% probability, but overall we can conclude that the realized values lie within our 95% confidence interval so our forecast predicts quite well the decrease of sales.

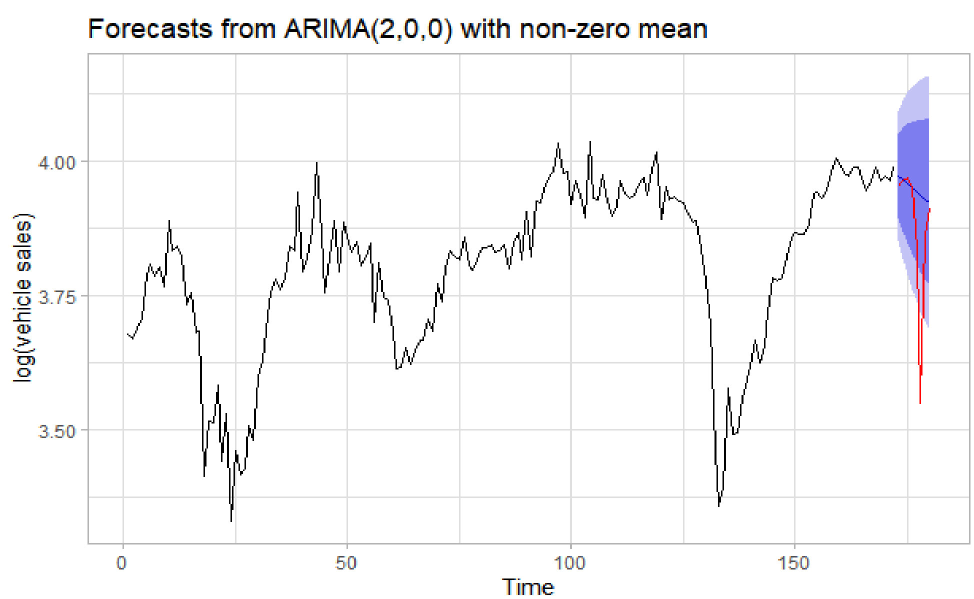


Figure 4: Predictions for $log(vehicle\ sales)$ and confidence intervals

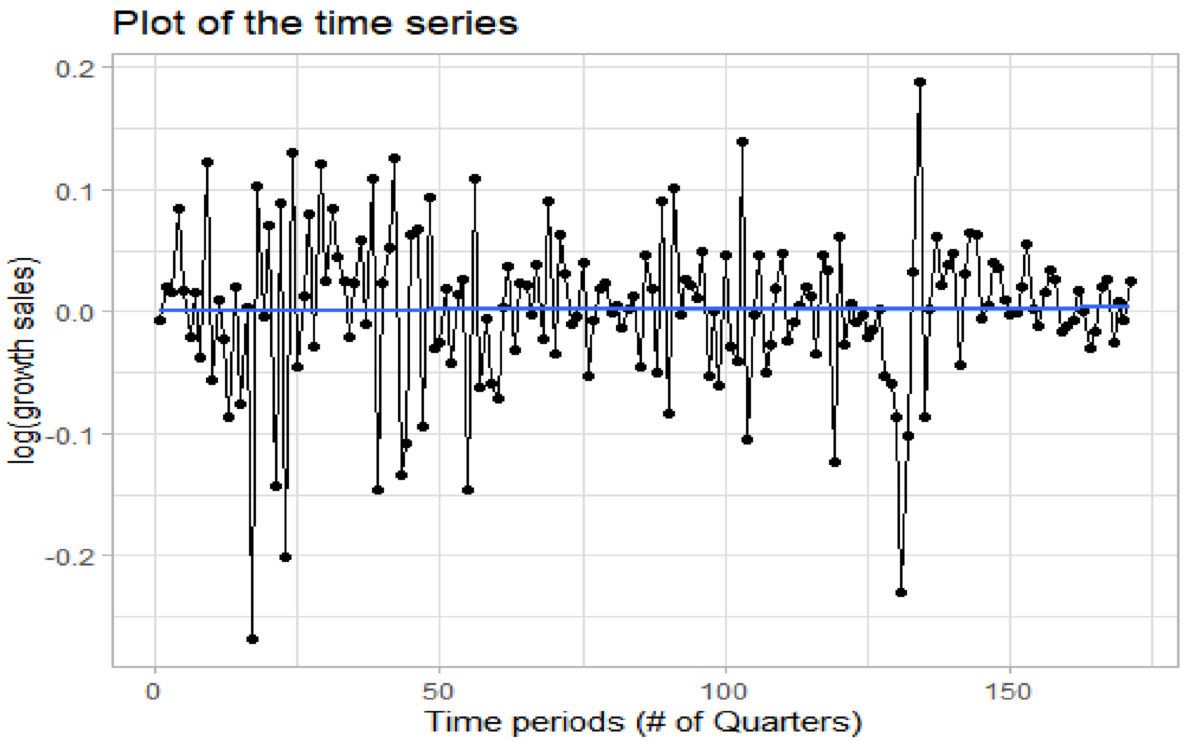


Figure 5: $\Delta log(vehicle\ sales)$ time series

## Bonus question

As we said in the second point, the fact that we did not reject the null hypothesis the we have a unit root led us to the intuition that (as we also saw in the theory) we could take the first differences of the logs of vehicle sales and obtain stationarity (if we consider the assumption that it has one unit root).

Then we proceeded as follows:

- After we plotted the series of the first differences of `log(vehicle sales)` we obtained the plot in figure 5 which includes the linear regression of $\Delta$`log(vehicle sales)` on time.
- Thus, we checked that our assumption of stationary holds throughout the Augmented D-F test. But, as from the series we notice that it does not suggest for a time trend, we only focused our attention of the output of the model that does not allow for a time trend, neither for a drift, same as our models that will be shown later.

  We noticed that we always reject the null hypothesis of non stationarity for every lag (up to 12) (the *p-values* are always strictly smaller than 0.05). So we could be quite confident in saying that the integrated of order one series is indeed stationary, as we expected.
- In order to decide which would be a possible candidate for our model, we recomputed the ACF and PACF reported in figure 6.

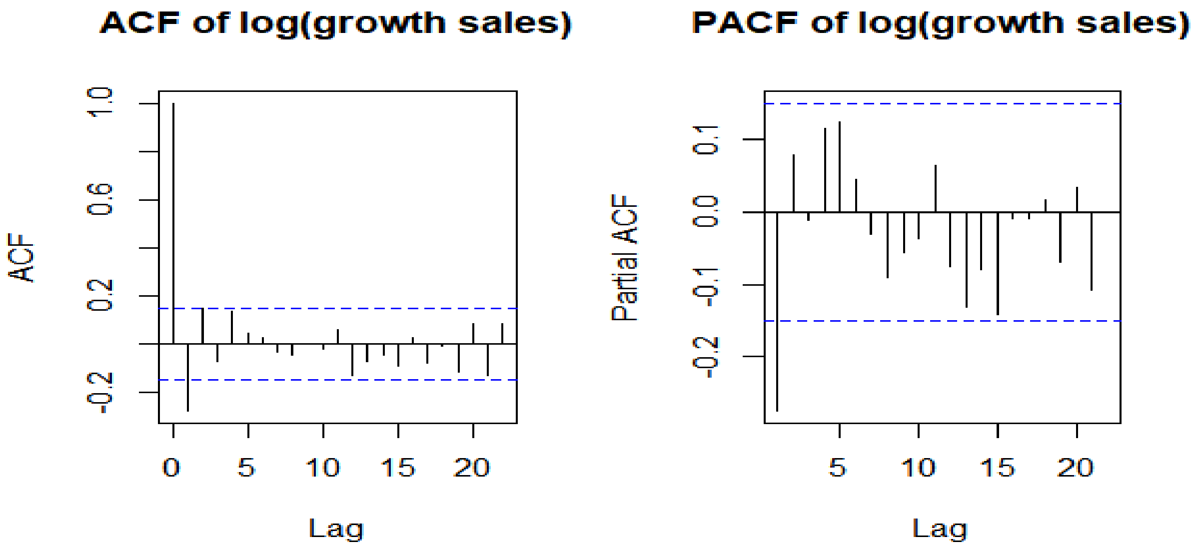


Figure 6: ACF and PCF for the differentiated series

- From the two graphs we can see that only the first lag of ACF and PACF is significant, so we infer that $p, q \leq 1$ and we can consider three possible candidate models ARIMA(1,1,0), ARIMA(0,1,1) ARIMA(1,1,1). Provided that we do not add a time trend in the original model, as in **point b)** we did not reject the hypothesis that we do not have a time trend, we estimated the following models after the first differences (where $Y_t = log(vehicle\ sales)_t$):

$$Y_t = \delta + \theta Y_{t-1} + \epsilon_t$$
$$Y_t - Y_{t-1} = \delta + \theta Y_{t-1} + \epsilon_t - (\delta + \theta Y_{t-2} + \epsilon_{t-1})$$
$$\Delta Y_t = \theta \Delta Y_{t-1} + \Delta \epsilon_t$$

Similar passages to get the other two candidate models:

$$\Delta Y_t = \Delta \epsilon_t + \alpha \Delta \epsilon_{t-1}$$

and

$$\Delta Y_t = \theta \Delta Y_{t-1} + \Delta \epsilon_t + \alpha \Delta \epsilon_{t-1}$$

- We estimated the three models and obtained their coefficients of ARIMA(1,1,0), ARIMA(0,1,1) and ARIMA(1,1,1) respectively in the first, second and third column of table 9. It is possible to see how the coefficient relative to the ARIMA(1,1,0) and ARIMA(0,1,1) are sadistically significant, whereas ARIMA(1,1,1) model (which would be one overfitted model in common for both the previous models), produces less statistically significant estimates (in particular $\hat{\theta_1}$ looses its significance and $\hat{\alpha_1}$ is no more significant). Therefore, we excluded this third model and perform the diagnostics on the remaining two candidate models.

- When performing the diagnostics we proceeded as follows:

  1. **Overfitting**, where we estimated the models respective overfitted models ARIMA(2,1,0) and ARIMA(0,1,2) of our remaining candidates and reported our estimates in the fourth and fifth column of table 9. For the other two overfitted models namely ARIMA(2,1,0) and ARIMA(0,1,2) we obtained respectively that the additional parameters are in both cases not statistically significant. From this we can say that both the proposed models passe the first test therefore we can proceed with the second one.

2. **Residual analysis**, where we performed the **Ljung-Box** test. As it is visible in first two rows of table 4 we do not reject the null hypothesis that the errors are not serially autocorrelated in both the candidate models with the 6 lags. However, the *p-value* for the statistic relative to ARIMA(0,1,1) is closer to the threshold 0.05.

   Also with 12 lags we do not reject the null hypothesis for both the models, however, we have again that the *p-value* for the statistic relative to ARIMA(0,1,1) is closer to the 0.05 threshold.

   This could already suggest us which model we should choose.

Table 4: Results of residual analysis

| $\chi^2_{K-p-q}$ | Degree of freedom | *p-value* | Method | Model |
|---|---|---|---|---|
| 7.105 | 5 | 0.213 | Box-Ljung test (with 6 lags) | ARIMA(1,1,0) |
| 9.723 | 5 | 0.083 | Box-Ljung test (with 6 lags) | ARIMA(0,1,1) |
| 12.598 | 11 | 0.320 | Box-Ljung test (with 12 lags) | ARIMA(1,1,0) |
| 14.983 | 11 | 0.183 | Box-Ljung test (with 12 lags) | ARIMA(0,1,1) |

- However, provided that both model passed the test, we compared the two models based on the information criteria, we saw that ARIMA(1,1,0) has better AIC and BIC, thus we chose it as our preferred model.

  Furthermore we checked if ARIMA(1,1,0) it is indeed stationary (no unit roots) also evaluating its inverse characteristic root which lies in the unit circle (figure 7).

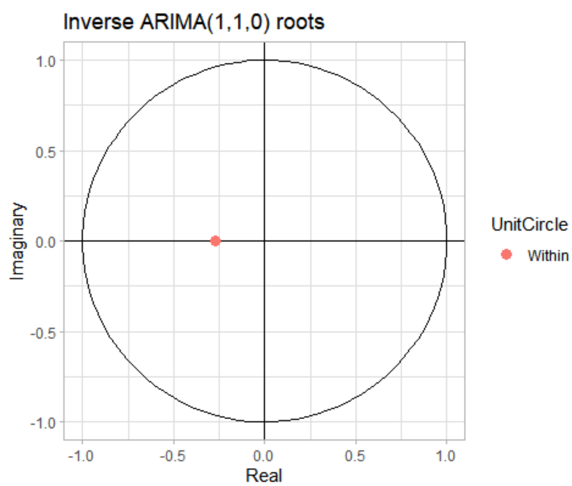


Figure 7: Inverse characteristics roots for the model ARIMA(1,1,0)

- Finally we applied our preferred model to forecast the *level* of `log(vehicle sales)` which results are shown in table 6 (and in levels in table 5) and compared with the realized data in graph 8 where we see again the only realized value outside the 95 % confidence interval is the one corresponding to the second quarter of 2020.

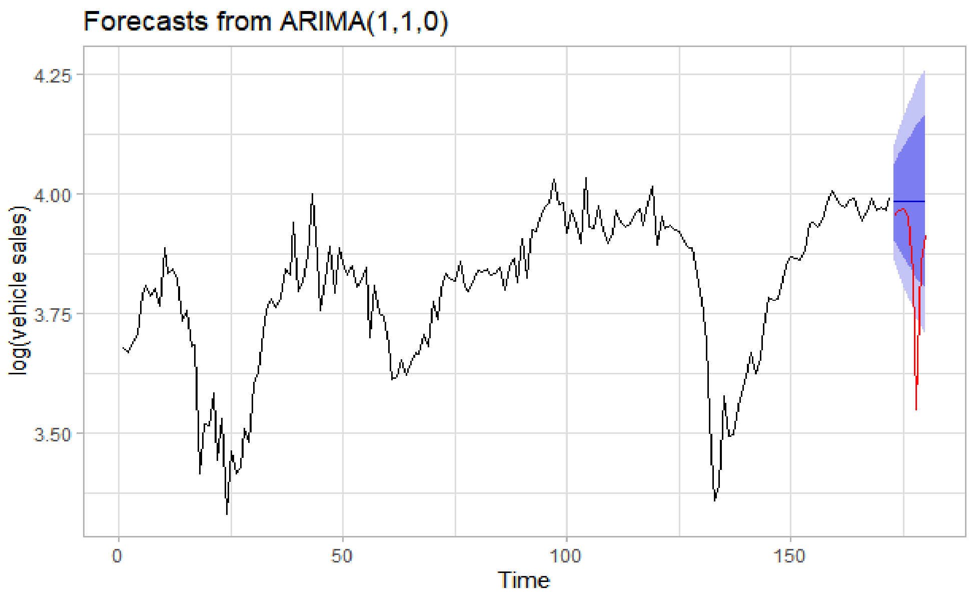


Figure 8: New predictions for $log(vehicle\ sales)$ and confidence intervals

Table 5: Quarterly new prediction of vehicle sales over 2019:Q1-2020:Q4

| 2019:Q1 | 2019:Q2 | 2019:Q3 | 2019:Q4 | 2020:Q1 | 2020:Q2 | 2020:Q3 | 2020:Q4 |
|---|---|---|---|---|---|---|---|
| 53.789 | 53.889 | 53.862 | 53.869 | 53.867 | 53.868 | 53.868 | 53.868 |

Table 6: $log(vehicles\ sales)$ new forecast over 2019:Q1-2020:Q4

| Quarters | Forecast | S.E. | Lower Bound | Upper Bound |
|---|---|---|---|---|
| 2019:Q1 | 3.983 | 0.062 | 3.862 | 4.104 |
| 2019:Q2 | 3.985 | 0.076 | 3.835 | 4.135 |
| 2019:Q3 | 3.985 | 0.091 | 3.806 | 4.163 |
| 2019:Q4 | 3.985 | 0.103 | 3.783 | 4.187 |
| 2020:Q1 | 3.985 | 0.114 | 3.761 | 4.208 |
| 2020:Q2 | 3.985 | 0.124 | 3.742 | 4.227 |
| 2020:Q3 | 3.985 | 0.133 | 3.724 | 4.245 |
| 2020:Q4 | 3.985 | 0.142 | 3.707 | 4.262 |

Table 7: ADF outputs with respective *p-values*

| lag | ADF | *p-value* |
|---|---|---|
| 0 | -2.950 | 0.179 |
| 2 | -2.455 | 0.384 |
| 4 | -2.842 | 0.224 |
| 6 | -3.613 | 0.034** |
| 8 | -3.394 | 0.057 |
| 10 | -3.322 | 0.069 |
| 12 | -3.542 | 0.040** |

*Note:* *p<0.1; **p<0.05; ***p<0.01

Table 8: Results of the models ARMA(2,0,0), ARMA(2,0,1), ARMA(3,0,0)

| | *Dependent variable:* | | |
|---|---|---|---|
| | $log(vehicles\ sales)$ | $log(vehicles\ sales)$ | $log(vehicles\ sales)$ |
| | (1) | (2) | (3) |
| $\hat{\theta}_1$ | 0.696*** | 0.335 | 0.722*** |
| | (0.074) | (0.211) | (0.075) |
| $\hat{\theta}_2$ | 0.241*** | 0.572*** | 0.315*** |
| | (0.074) | (0.191) | (0.090) |
| $\hat{\alpha}_1$ | | 0.386 | |
| | | (0.241) | |
| $\hat{\theta}_3$ | | | −0.107 |
| | | | (0.076) |
| Intercept | 3.806*** | 3.806*** | 3.806*** |
| | (0.067) | (0.063) | (0.061) |
| Observations | 172 | 172 | 172 |
| Log Likelihood | 236.099 | 237.043 | 237.088 |
| $\sigma^2$ | 0.004 | 0.004 | 0.004 |
| Akaike Inf. Crit. | −464.197 | −464.087 | −464.175 |

*Note:* *p<0.1; **p<0.05; ***p<0.01

Table 9: Results of ARIMA(1,1,0), ARIMA(0,1,1), ARIMA(1,1,1), ARIMA(2,1,0), ARIMA(0,1,2)

| | *Dependent variable:* | | | | |
|---|---|---|---|---|---|
| | $\Delta log(v.sales)$ | $\Delta log(v.sales)$ | $\Delta log(v.sales)$ | $\Delta log(v.sales)$ | $\Delta log(v.sales)$ |
| | (1) | (2) | (3) | (4) | (5) |
| $\hat{\theta}_1$ | $-0.272^{***}$ | | $-0.566^{**}$ | $-0.250^{***}$ | |
| | (0.073) | | (0.221) | (0.076) | |
| $\hat{\alpha}_1$ | | $-0.223^{***}$ | 0.322 | | $-0.256^{***}$ |
| | | (0.065) | (0.255) | | (0.077) |
| $\hat{\theta}_2$ | | | | 0.080 | |
| | | | | (0.076) | |
| $\hat{\alpha}_2$ | | | | | $0.112^{*}$ |
| | | | | | (0.065) |
| Observations | 171 | 171 | 171 | 171 | 171 |
| Log Likelihood | 233.513 | 232.279 | 234.152 | 234.063 | 233.749 |
| $\sigma^2$ | 0.004 | 0.004 | 0.004 | 0.004 | 0.004 |
| Akaike Inf. Crit. | −463.026 | −460.559 | −462.303 | −462.125 | −461.497 |
| *Note:* | | | | | $^{*}$p<0.1; $^{**}$p<0.05; $^{***}$p<0.01 |