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Problem Set 2.

Advanced Derivatives

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Ex 1: Let's call E_t the current value of the firm's equity:

$$E_t = E^Q\left[e^{-r(T-t)} \max(0, V_T - D)\right]$$

The price of the option is the price of a B.S. call on the lognormal stochastic process V with strike D.

We apply the B.S. formula with:

$r = r \qquad S_0 = V_t \qquad q = 0$

$\tau = T - t \qquad \sigma = \sigma$

$$E_t = V_t \mathcal{N}(d_+) - e^{-r(T-t)} \mathcal{N}(d_-) D$$

with

$$d_\pm = \frac{\ln\left(\frac{V_t}{D}\right) + \left(r \pm \frac{\sigma^2}{2}\right)(T-t)}{\sigma\sqrt{T-t}}$$

The value of a call option on the firm's equity with strike K and maturity $\tilde{\tau}$ is

$$c(t, V_t) = V_t N_2\left(a_1, b_1, \sqrt{\frac{\tilde{\tau}-t}{T-t}}\right)$$

$$- De^{-r(T-t)} N_2\left(a_2, b_2, \sqrt{\frac{\tilde{\tau}-t}{T-t}}\right)$$

$$- e^{-r(\tilde{\tau}-t)} K N(a_2)$$

where $N_2(\cdot, \cdot; \rho)$ is the bivariate cumulative distribution of two standard normal variables with correlation ρ.

with

$$a_1 = \frac{\ln\left(\frac{V_t}{V^*}\right) + \left(r + \frac{\sigma^2}{2}\right)(\tilde{\tau} - t)}{\sigma\sqrt{\tilde{\tau}-t}}$$

$$a_2 = a_1 - \sigma\sqrt{\tilde{\tau}-t}$$

$$b_1 = \frac{\ln\left(\frac{V_t}{D}\right) + \left(r + \frac{\sigma^2}{2}\right)(T-t)}{\sigma\sqrt{T-t}}$$

$$b_2 = b_1 - \sigma\sqrt{T-t}$$

where V^* is the value of the firm for which the price of the option on the firm's assets will be equal to K at $\tilde{t}$

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Exercice 2 :

Let's call c_t the value of the call option and τ the time to maturity

$$c_t = E_t^Q\left[e^{-r\tau}(S_T - K)^+\right]$$

Applying ito's lemma:

$$d\log(S_t) = \frac{1}{S_t} dS_{c_t} - \frac{1}{2}\frac{1}{S_t^2} S_t^2 \sigma^2 dt$$

$$+ \left(\log^+ - \log^-\right) dN_t$$

$$= \frac{1}{S_t} dS_{c_t} - \frac{1}{2}\sigma^2 dt + \left[\log\left((1+\gamma)S_t\right) - \log(S_t)\right] dN_t$$

$$= \frac{1}{S_t} dS_{c_t} - \frac{1}{2}\sigma^2 dt + \log(1+\gamma) dN_t$$

$$\log(S_T/S_t) = \left(r - \lambda^Q\gamma - \frac{\sigma^2}{2}\right)(T-t) + \sigma(W_T - W_t)$$
$$+ \log(1+\gamma)(N_T - N_t)$$

$$S_T = S_t\, e^{\left(r - \lambda^Q\gamma - \frac{\sigma^2}{2}\right)(T-t) + \sigma(W_T - W_t)} (1+\gamma)^{N_T - N_t}$$

$$c_t = \sum_{i=0}^{\infty} P(N_T - N_t)\, E_t^Q\left[e^{-r\tau}(S_T - K)^+ \mid N_T - N_t = i\right]$$

$N_T - N_t \sim \mathcal{P}(\lambda(T-t))$

Let $k \in \mathbb{N}$,

$$P(N_T - N_t = k) = \exp(-\lambda(T-t)) \frac{(\lambda(T-t))^k}{k!}$$

$$E_t^Q\left[e^{-r\tau}(S_T - K)^+ / N_T - N_t = k\right]$$

is computed applying the B.S. formula with:

$r = r$, $\tau = \tau$, $S_t = S_t(1+\gamma)^k$,

$k = K$, $q = 0$.

this gives

$$S_t(1+\gamma)^k N(d_+) - Ke^{-r\tau} N(d_-)$$

with
$$d_\pm(k) = \frac{\ln\left(\frac{S_t(1+\gamma)^k}{K}\right) + \left(r \pm \frac{\sigma^2}{2}\right)(T-t)}{\sigma\sqrt{T-t}}$$

Finally we obtain

$$c = \sum_{i=0}^{\infty} \exp(-\lambda(T-t)) \frac{(\lambda\tau)^i}{i!} \cdot \left[S_t (1+\gamma)^i \mathcal{N}(d_+(i)) - e^{-r\tau} K \mathcal{N}(d_-(i)) \right]$$