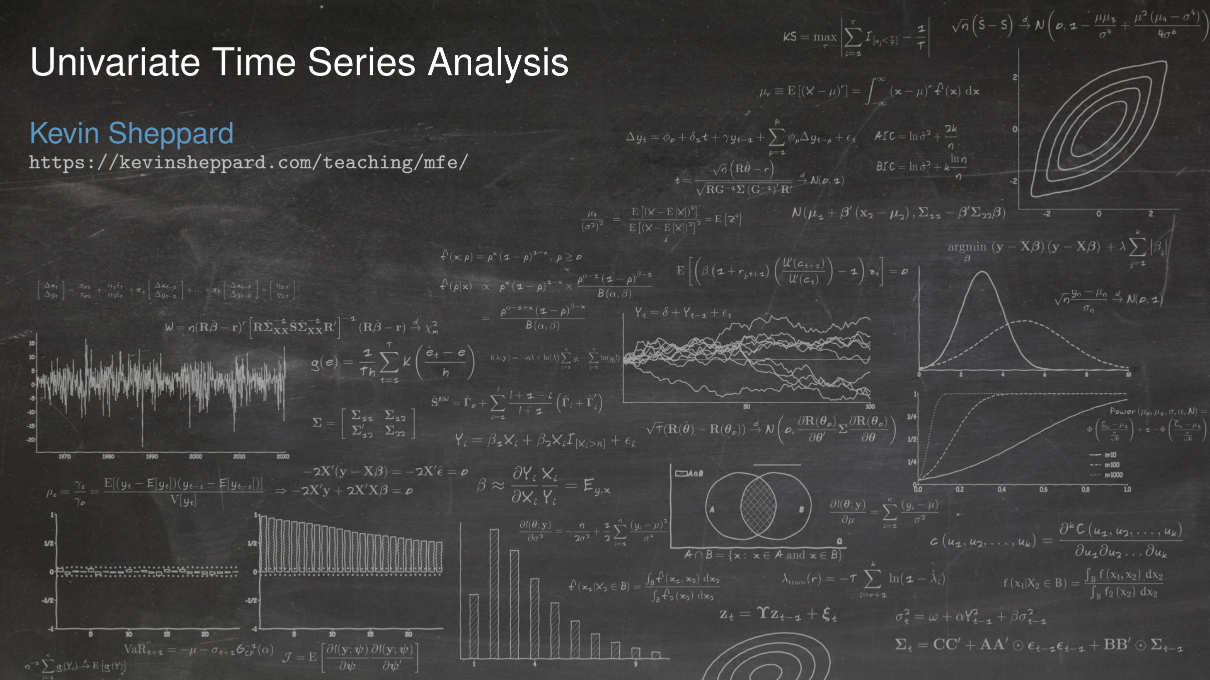

Univariate Time Series Analysis

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https://kevinsheppard.com/teaching/mfe/

Time Series Analysis

- Introduction to Time Series Analysis
- Key Concepts in Time Series Analysis
- Autoregressive Moving-Average Processes
- Properties of ARMA Processes
- Autocorrelations and Partial Autocorrelations
- Estimating Autocorrelations and Partial Autocorrelations
- Parameter Estimation
- Model Building
- Forecasting
- Forecast Evaluation
- Nonstationary Time Series
- Random Walks, Unit Roots and Stochastic Trend
- Non-linear Models

Hilary 2021 Teaching Structure

- Viewing pre-recorded content is **mandatory** before the lecture
  - Restricted to less than 2 hours per week
  - Prerecorded videos are as short as possible and limited to a single topic
- **Alternatively** read the corresponding section of the course notes
- Lecture focuses on application and problems
- Expanded review section
  - Review sections appear at content section breaks
  - Key concepts, questions and problems
  - Solutions to review problems covered in detail
- Two sets of office hours on Wednesdays
  - 8.00-9.00 and 16.30-17.30 (UK Local Time)
  - Weeks 0 to 9

Stochastic Processes

Univariate Time Series Analysis

Stochastic Processes

Definition (Stochastic Process)

A stochastic process is a collection of random variables $\{Y_t\}$ defined on a common probability space indexed by a set $\mathcal{T}$ usually defined as $\mathbb{N}$ for discrete time processes or $[0, \infty)$ for continuous time processes.

Basic Example: An i.i.d. time series

$$Y_t \overset{\text{i.i.d.}}{\sim} N(0, 1)$$

More Complex Examples

- Random Walk

$$Y_t = Y_{t-1} + \epsilon_t,\ \epsilon_t \overset{\text{i.i.d.}}{\sim} N\left(0, \sigma^2\right)$$

- ARMA(1,1)

$$Y_t = \phi_1 Y_{t-1} + \theta\epsilon_{t-1} + \epsilon_t$$

  - Series focuses on ARMA
- GARCH(1,1)

$$Y_t \sim N(0, \sigma_t^2)$$

$$\sigma_t^2 = \omega + \alpha Y_{t-1}^2 + \beta\sigma_{t-1}^2$$

  - GARCH and other non-linear processes later
- Ornstein-Uhlenbeck Process

$$Y\left(t\right) = e^{-\beta t} Y\left(0\right) + \sigma \int_0^t e^{-\beta(t-s)} \mathrm{d}W\left(s\right)$$

Key Concepts

Stochastic Process

Questions

- What are the requirements for a sequence of random variables to be a stochastic process?
- Are cross-sectional random variables indexed by i a stochastic process?
- Are the observations of stochastic processes always regularly spaced in time?

Autocovariance

Univariate Time Series Analysis

Autocovariance

Definition (Autocovariance)

The autocovariance of a covariance stationary scalar process $\{Y_t\}$ is defined

$$\gamma_s = \mathrm{E}\left[(Y_t - \mu)(Y_{t-s} - \mu)\right]$$

where $\mu = \mathrm{E}\left[Y_t\right]$. Note that $\gamma_0 = \mathrm{E}\left[(Y_t - \mu)(Y_t - \mu)\right] = \mathrm{V}\left[Y_t\right]$.

- Covariance of a process at different points in time
- Otherwise identical to usual covariance

Stationarity

Univariate Time Series Analysis

Stationarity

The future resembles the past

Key concept

- Stationarity is a statistically meaningful form of regularity
- First type:

Definition (Covariance Stationarity)

A stochastic process $\{Y_t\}$ is covariance stationary if

$$\begin{aligned}
\mathrm{E}\left[Y_t\right] &= \mu && \text{for } t = 1, 2, \ldots \\
\mathrm{V}\left[Y_t\right] &= \sigma^2 < \infty && \text{for } t = 1, 2, \ldots \\
\mathrm{E}\left[(Y_t - \mu)(Y_{t-s} - \mu)\right] &= \gamma_s && \text{for } t = 1, 2, \ldots, s = 1, 2, \ldots, t-1
\end{aligned}$$

- *Unconditional* mean, variance and autocovariance do *not* depend on time

Stationarity

Second type (stronger):

Definition (Strict Stationarity)

A stochastic process $\{Y_t\}$ is strictly stationary if the joint distribution of $\{Y_t, Y_{t+1}, \ldots, Y_{t+h}\}$ only depends only on h and not on t.

- *Entire joint distribution* does not depend on time.
- Examples of stationary time series:
  - i.i.d. : Always strict, covariance if $\sigma^2 < \infty$
  - i.i.d. sequence of t_2 random variables, strict only
  - Multivariate normal, both
  - AR(1): $Y_t = \phi_1 Y_{t-1} + \epsilon_t$, covariance if $|\phi_1| < 1$ and $\mathrm{V}[\epsilon_t] < \infty$, strict is ϵ_t is i.i.d.
  - ARCH(1): $Y_t \sim N(0, \sigma_t^2)$,$\sigma_t^2 = \omega + \alpha Y_{t-1}^2$ both if $\alpha < 1$.

What processes are not stationary?

Nonstationary time series

- Seasonalities, Diurnality, Hebdomadality: $Y_t = \mu + \beta I_{[\text{Quarter}(t) = Q1]} + \epsilon_t$
  - $\mathrm{E}[Y_t]$ is different in Q1 than in other quarters
- Time trends: $Y_t = t + \epsilon_t$
  - $\mathrm{E}[Y_t] = t$
- Random walks: $Y_t = Y_{t-1} + \epsilon_t$
  - $\mathrm{V}[Y_t] = t\sigma^2$
- Processes with structural breaks: $Y_t = \mu_1 + \epsilon_t$ if $t < 1974$, $Y_t = \mu_2 + \epsilon_t$, $t \geq 1974$.
  - $\mathrm{E}[Y_t] = \mu_1 + (\mu_2 - \mu_1)(1 - I_{t<1974})$

Ergodicity

Univariate Time Series Analysis

Ergodicity

- Measure of "asymptotic independence"

Theorem (Ergodic Theorem)

If $\{Y_t\}$ is ergodic and the r^{th} moment μ_r is finite, then $T^{-1}\sum_{t=1}^{T} Y_t^r \xrightarrow{p} \mu_r$.

- Asymptotic independence ensures that averages that use points far apart in time converge to their expected value
- Example of a nonergodic process:

$$Y_t = \mu + \epsilon_t$$

  - $\mu \sim N(0,1)$ and $\epsilon_t \overset{\text{i.i.d.}}{\sim} N(0,1)$
  - $\mathrm{E}[Y_t] = 0$
  - $T^{-1}\sum_{t=1}^{T} Y_t \xrightarrow{p} \mu \neq 0$
  - μ has a permanent effect on all Y_t

Key Concepts

Covariance Stationarity, Strict Stationarity, Ergodicity

Questions

- Why is stationarity important when modeling and forecasting a time series?
- What is the difference between strict and covariance stationarity?
- Why does asymptotic independence help to ensure that a LLN will apply?
- What are the four main sources of non-stationarity in a time series?

Problems

1. Why are the two processes below non-stationary when $\epsilon_t \stackrel{\text{i.i.d.}}{\sim} N\left(0, \sigma^2\right)$?
   a. $Y_t = 0.3t + \epsilon_t$
   b. $Y_t = 0.7 + 0.2 I_{[t>2020]} + \epsilon_t$.

White Noise

Univariate Time Series Analysis

White noise

Essential Building Block of Time Series

Definition (White Noise)

A process $\{\epsilon_t\}$ is known as white noise if

$$\begin{aligned} \mathrm{E}\left[\epsilon_t\right] &= 0 && \text{for } t = 1, 2, \ldots \\ \mathrm{V}\left[\epsilon_t\right] &= \sigma^2 < \infty && \text{for } t = 1, 2, \ldots \\ \mathrm{E}\left[\epsilon_t \epsilon_{t-j}\right] &= 0 && \text{for } t = 1, 2, \ldots, \; j \neq 0 \end{aligned}$$

- Not necessarily independent
  - ARCH(1) process $Y_t \sim N(0, \sigma_t^2)$, $\sigma_t^2 = \omega + \alpha Y_{t-1}^2$
  - Variance is dependent, mean is not

Linear Time Series Processes

Univariate Time Series Analysis

Linear Time-series Processes

Standard tool of time-series analysis

- *Linear* time series process can always be expressed as

$$Y_t = \delta_t + Y_0 + \sum_{i=0}^{t} \theta_i \epsilon_{t-i}$$

  - Linear in the errors
  - δ_t is a purely deterministic process
  - $\{\epsilon_t\}$ is a White Noise process
- Example of non-linear processes
  - GARCH(1,1)

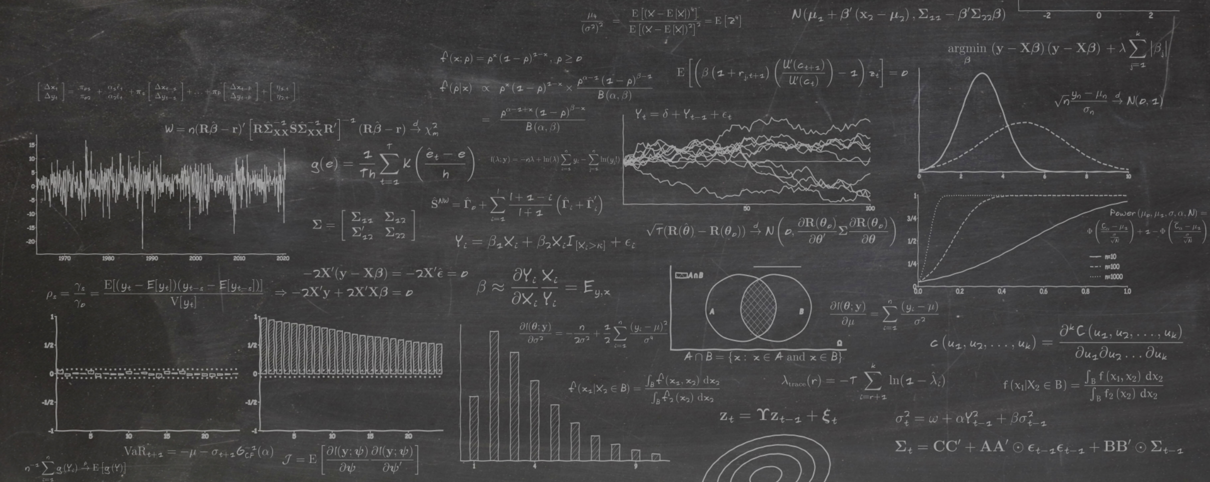
$$Y_t \sim N(0, \sigma_t^2)$$
$$\sigma_t^2 = \omega + \alpha Y_{t-1}^2 + \beta \sigma_{t-1}^2$$

  - Threshold Autoregression

$$Y_t = \phi_s Y_{t-1} + \epsilon_t, \;\; \phi_s = 1 \text{ if } L < Y_{t-1} < U \text{ otherwise } 0.9$$

Autoregressive-Moving Average Processes

Univariate Time Series Analysis



ARMA Processes

- Inclusive class of all linear time-series processes

Definition (Autoregressive-Moving Average Process)

An Autoregressive Moving Average process with orders P and Q, abbreviated ARMA(P,Q), has dynamics which follow

$$Y_t = \phi_0 + \sum_{p=1}^{P} \phi_p Y_{t-p} + \sum_{q=1}^{Q} \theta_q \epsilon_{t-q} + \epsilon_t$$

where ϵ_t is a white noise process with the additional property that $E_{t-1}[\epsilon_t] = 0$.

- ARMA(1,1)

$$Y_t = \phi_1 Y_{t-1} + \theta_1 \epsilon_{t-1} + \epsilon_t$$

Special case: Moving Average

- ARMA family compromises two sub-classes

Definition (Moving Average Process of Order Q)

A Moving Average process of order Q, abbreviated MA(Q), has dynamics which follow

$$Y_t = \phi_0 + \sum_{q=1}^{Q} \theta_q \epsilon_{t-q} + \epsilon_t$$

where ϵ_t is white noise series with the additional property that $E_{t-1}[\epsilon_t] = 0$.

- 1st order Moving Average (MA(1))

$$Y_t = \phi_0 + \theta_1 \epsilon_{t-1} + \epsilon_t$$

- Simplest non-degenerate time series process

Special cases of ARMA processes: Autoregression

- Other sub-class of ARMA

Definition (Autoregressive Process of Order P)

An Autoregressive process of order P, abbreviated AR(P), has dynamics which follow

$$Y_t = \phi_0 + \sum_{p=1}^{P} \phi_p Y_{t-p} + \epsilon_t$$

where ϵ_t is white noise series with the additional property that $E_{t-1}\left[\epsilon_t\right] = 0$.

- 1^{st} order Autoregression (AR(1))

$$Y_t = \phi_0 + \phi_1 Y_{t-1} + \epsilon_t$$

Conditional Moments

Univariate Time Series Analysis

Moments and Autocovariances

$$Y_t = \phi_0 + \phi_1 Y_{t-1} + \epsilon_t$$

- *Unconditional* Mean

$$\mathrm{E}\left[Y_t\right]$$

- *Unconditional* Variance

$$\gamma_0 = \mathrm{V}\left[Y_t\right]$$

- Autocovariance

$$\gamma_s = \mathrm{E}\left[\left(Y_t - \mathrm{E}\left[Y_t\right]\right)\left(Y_{t-s} - \mathrm{E}\left[Y_{t-s}\right]\right)\right]$$

- *Conditional* Mean

$$\mathrm{E}_t[Y_{t+1}] = \mathrm{E}[Y_{t+1}|\mathcal{F}_t]$$

- *Conditional* Variance

$$\mathrm{V}_t[Y_{t+1}] = \mathrm{E}_t[(Y_{t+1} - \mathrm{E}_t[Y_{t+1}])^2]$$

Review

Linear Time Series Processes

Key Concepts

White Noise, Linear Stochastic Process, Autoregression, Moving Average, ARMA, Conditional Moment

Questions

- Is White Noise covariance stationary?
- Is White Noise homoskedastic?
- Is an i.i.d. sequence White Noise?
- Is an i.i.d. normal sequence White Noise?
- In what sense is a linear process *linear*?
- Why are linear processes important in the context of covariance stationary time series?
- What is the difference between a conditional and an unconditional moment?
- What is the difference between an AR and an MA model?

Moments of an AR(1) Process

Univariate Time Series Analysis

How to work with ARMA processes: AR(1)

The MA(∞) Representation

$$Y_t = \phi_0 + \phi_1 Y_{t-1} + \epsilon_t$$

- Use backward substitution (assume $|\phi_1| < 1$)

$$\begin{aligned}
Y_t &= \phi_0 + \phi_1 Y_{t-1} + \epsilon_t \\
&= \phi_0 + \phi_1(\phi_0 + \phi_1 Y_{t-2} + \epsilon_{t-1}) + \epsilon_t \\
&= \phi_0 + \phi_1\phi_0 + \phi_1^2 Y_{t-2} + \phi_1\epsilon_{t-1} + \epsilon_t \\
&= \phi_0 + \phi_1\phi_0 + \phi_1^2(\phi_0 + \phi_1 Y_{t-3} + \epsilon_{t-2}) + \phi_1\epsilon_{t-1} + \epsilon_t \\
&= \phi_0 \sum_{j=0}^{\infty} \phi_1^j + \sum_{i=0}^{\infty} \phi_1^i \epsilon_{t-i} \\
&= \frac{\phi_0}{1-\phi_1} + \sum_{i=0}^{\infty} \phi_1^i \epsilon_{t-i}
\end{aligned}$$

- $\lim_{s\to\infty} \sum_{i=0}^{s} \phi_1^i = 1/(1-\phi_1)$

Properties of an AR(1)

$$\begin{aligned}
\mathrm{E}[Y_t] &= \mathrm{E}\left[\frac{\phi_0}{1-\phi_1} + \sum_{i=0}^{\infty} \phi_1^i \epsilon_{t-i}\right] \\
&= \frac{\phi_0}{1-\phi_1} + \sum_{i=0}^{\infty} \phi_1^i \mathrm{E}\left[\epsilon_{t-i}\right] \\
&= \frac{\phi_0}{1-\phi_1} + \sum_{i=0}^{\infty} \phi_1^i 0 \\
&= \frac{\phi_0}{1-\phi_1}
\end{aligned}$$

- In general AR(P): $\mathrm{E}[Y_t] = \frac{\phi_0}{1-\phi_1-\phi_2-\ldots-\phi_P}$
- Only sensible if $\phi_1 + \phi_2 + \ldots + \phi_P < 1$
- Variance can be shown in same manner
  - AR(1): $\mathrm{V}[Y_t] = \frac{\sigma^2}{1-\phi_1^2}$
  - AR(P): $\mathrm{V}[Y_t] = \frac{\sigma^2}{1-\rho_1\phi_1-\rho_2\phi_2-\ldots-\rho_P\phi_P}$
    - ρs are autocorrelations

Autocovariance of an AR(1)

$$
\begin{aligned}
\mathrm{E}\left[(Y_t - \mathrm{E}[Y_t])(Y_{t-s} - \mathrm{E}[Y_{t-s}])\right] &= \mathrm{E}\left[\left(\sum_{i=0}^{\infty} \phi_1^i \epsilon_{t-i}\right)\left(\sum_{j=0}^{\infty} \phi_1^j \epsilon_{t-s-j}\right)\right] \\
&= \mathrm{E}\left[\left(\underbrace{\sum_{i=0}^{s-1} \phi_1^i \epsilon_{t-i}}_{\text{After} t-s} + \underbrace{\sum_{k=s}^{\infty} \phi_1^k \epsilon_{t-k}}_{t-s \text{ and later}}\right)\left(\sum_{j=0}^{\infty} \phi_1^j \epsilon_{t-s-j}\right)\right] \\
&= \phi_1^s \frac{\sigma^2}{1-\phi_1^2}
\end{aligned}
$$

- Full details in notes
- The autocovariance *function*

$$
\gamma_s = \phi_1^{|s|} \left\{ \frac{\sigma^2}{1-\phi_1^2} \right\}
$$

- Autocovariance declines geometrically with the lag length
- Requires $\phi_1^2 < 1$ to exist
  - Same condition as the mean

Stationarity of AR Processes

Univariate Time Series Analysis

Stationarity of ARMA processes

- Primarily interested in covariance stationarity
- Stationarity depends on parameters of *AR* portion
- AR(0) or finite order MA: always stationary
- AR(1) or ARMA(1,Q): $Y_t = \phi_1 Y_{t-1} + \text{MA} + \epsilon_t$
  - $|\phi_1| < 1$
- AR(P) or ARMA(P,Q) $Y_t = \phi_1 Y_{t-1} + \phi_2 Y_{t-2} + \ldots + \phi_P Y_{t-P} + \text{MA} + \epsilon_t$
- Rewrite $Y_t - \phi_1 Y_{t-1} - \phi_2 Y_{t-2} - \ldots - \phi_P Y_{t-P} = \text{MA} + \epsilon_t$
- Easy to determine using the characteristic equation and corresponding characteristic roots

The characteristic equation

Definition (Characteristic Equation)

Let Y_t follow a P$^{\text{th}}$ order linear difference equation

$$Y_t = \phi_0 + \phi_1 Y_{t-1} + \phi_2 Y_{t-2} + \ldots + \phi_P Y_{t-P} + x_t$$

which can be rewritten as

$$\begin{aligned} Y_t - \phi_1 Y_{t-1} - \phi_2 Y_{t-2} - \ldots - \phi_P Y_{t-P} &= \phi_0 + x_t \\ (1 - \phi_1 L - \phi_2 L^2 - \ldots - \phi_P L^P) Y_t &= \phi_0 + x_t \end{aligned}$$

The characteristic equation of this process is

$$z^P - \phi_1 z^{P-1} - \phi_2 z^{P-2} - \ldots - \phi_{P-1} z - \phi_P = 0$$

- Key is in the forming of the characteristic equation and its roots
- L is known as “lag operator”

Characteristic roots

Definition (Characteristic Root)

Let

$$z^P - \phi_1 z^{P-1} - \phi_2 z^{P-2} - \ldots - \phi_{P-1} z - \phi_P = 0$$

be the characteristic polynomial associated with some P^{th} order linear difference equation. The P characteristic roots, $c_1, c_2, \ldots, c_P$ are defined as the solution to this polynomial

$$(z - c_1)(z - c_2) \ldots (z - c_P) = 0.$$

- The roots are c_1, c_2, ..., c_P
- AR(P) or ARMA(P,Q) is covariance stationary if $|c_j| < 1$ for all j
- If complex, $|c_j| = |a_j + b_j i| = \sqrt{a^2 + b^2}$ (complex modulus)

Characteristic roots example

- Difficult to determine by inspection

Example 1

$$Y_t = .1Y_{t-1} + .7Y_{t-2} + .2Y_{t-3} + \epsilon_t$$

- Characteristic equation

$$z^3 - .1z^2 - .7z^1 - .2$$

- Roots: 1, $-.5$, and $-.4 \Rightarrow$ nonstationary

Example 2

$$Y_t = 1.7Y_{t-1} - .72Y_{t-2} + \epsilon_t$$

- Characteristic equation

$$z^2 - 1.7z^1 + .72$$

- Roots: $.9$ and $.8 \Rightarrow$ stationary

Review

Properties or ARMA Models

Key Concepts

Backward Substitution, Characteristic Equation, Characteristic Root

Questions

- What role so the MA component play in determining stationarity?
- What is the key condition for stationarity of an ARMA model?
- What is complex modulus and why is it needed?

Problems

1. Which of the models listed below are covariance stationary?
   a. $Y_t = 1.8Y_{t-1} - 0.8Y_{t-2} + \epsilon_t$
   b. $Y_t = 0.4 - 0.75Y_{t-1} - 0.25Y_{t-2} + \epsilon_t$
   c. $Y_t = 10 + \sum_{j=1}^{100} 0.01Y_{t-j} + \epsilon_t$
2. Write the ARMA(1,1) $Y_t = \phi_1 Y_{t-1} + \theta_1 \epsilon_{t-1} + \epsilon_t$ as a function of $\epsilon_t, \epsilon_{t-1}, \epsilon_{t-2}, \ldots, \epsilon_{t-h}$ and Y_{t-h} using backward substitution.
3. Use backward substitution to write the model $Y_t = -0.5\epsilon_{t-1} + \epsilon_t$ as an AR(∞) using the relationship that $Y_{t-1} = -0.5\epsilon_{t-2} + \epsilon_{t-1}$ implies $\epsilon_{t-1} = Y_{t-1} + 0.5\epsilon_{t-2}$.

Autocorrelations and Partial Autocorrelations

Univariate Time Series Analysis

Autocorrelations and the ACF

- Autocorrelations are a key element of model building

Definition (Autocorrelation)

The autocorrelation of a covariance stationary scalar process is defined

$$\rho_s = \frac{\gamma_s}{\gamma_0}$$

where $\gamma_s = \mathrm{E}\left[(Y_t - \mu)(Y_{t-s} - \mu)\right]$.

- Measures the correlation of a process at different points in time
- AR(1):

$$\rho_s = \phi_1^s$$

- One of two possibilities
  - Decay geometrically if $0 < \phi_1 < 1$
  - Oscillate and decay $-1 < \phi_1 < 0$

Partial Autocorrelations (PACF)

- Partial Autocorrelation is the other key element of model building
- More complicated than autocorrelations:
- Regression interpretation of s$^{\text{th}}$ partial autocorrelation:

$$Y_t = \phi_0 + \phi_1 Y_{t-1} + \phi_2 Y_{t-2} + \ldots + \phi_{s-1} Y_{t-s+1} + \varphi_s Y_{t-s} + \epsilon_t$$

- φ_s is the s$^{\text{th}}$ partial autocorrelation
  - Population (not sample) value of φ_s
- AR(1):

$$\varphi_s = \begin{cases} \phi_1^{|s|} \text{ for s} = -1, 0, 1 \\ 0 \text{ otherwise} \end{cases}$$

- Partial autocorrelation function maps the parameters of a process to the s$^{\text{th}}$ autocorrelation, $\varphi(s)$

Autocorrelations Structure of ARMA Processes

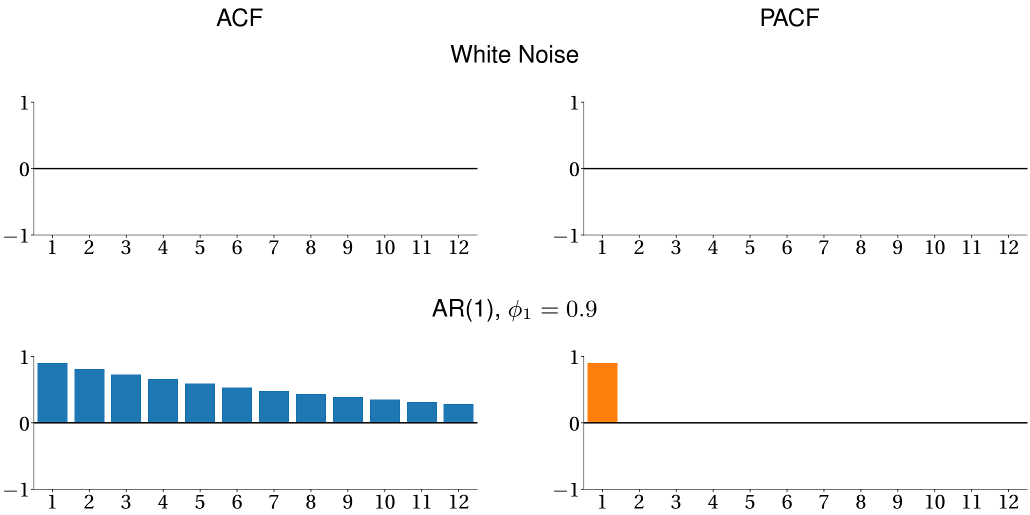
Univariate Time Series Analysis

Using the ACF and PACF to categorize processes

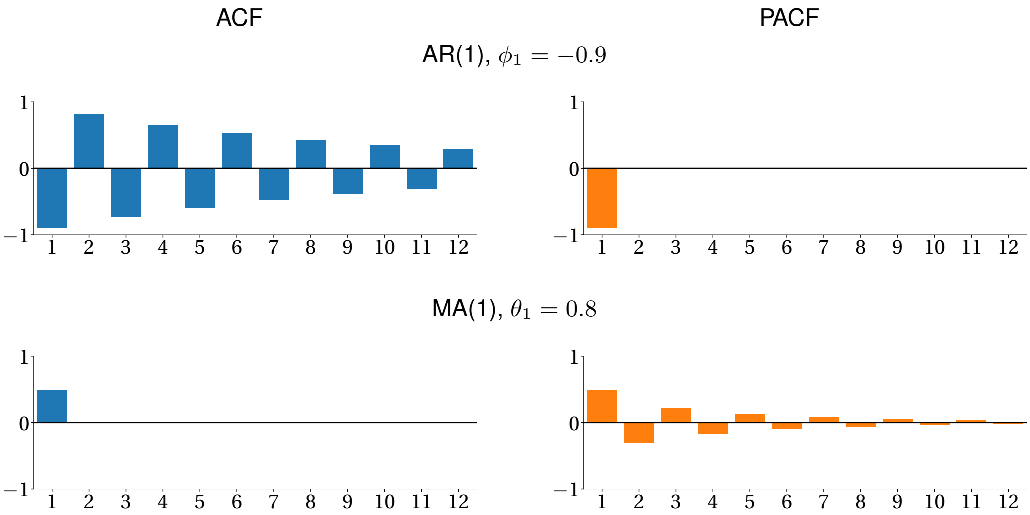
- ACF and PACF are useful when choosing models

Process	ACF	PACF
White Noise	All 0	All 0
AR(1)	$\rho_s = \phi_1^s$	0 beyond lag 2
AR(P)	Decays toward zero exponentially	Non-zero through lag P, 0 thereafter
MA(1)	$\rho_1 \neq 0$, $\rho_s = 0$, $s > 0$	Decays toward zero exponentially
MA(Q)	$\rho_s \neq 0 \; s \leq Q$, $\rho_s = 0$, $s > Q$	Decays toward zero exponentially, possible oscillating
ARMA(P,Q)	Exponential Decay	Exponential Decay

Autocorrelation for ARMA processes



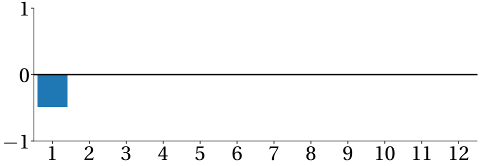
Autocorrelation for ARMA processes

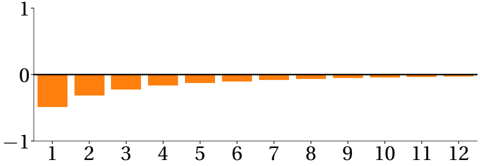


Autocorrelation for ARMA processes

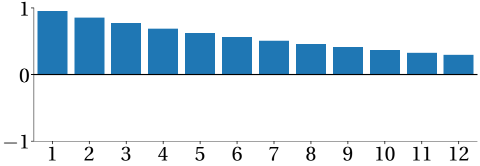
ACF | PACF

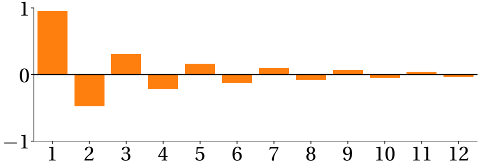
MA(1), $\theta_1 = -0.8$





ARMA(1,1), $\phi_1 = 0.9$, $\theta_1 = -0.8$

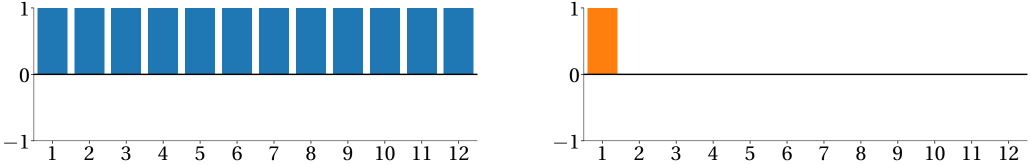




Autocorrelation for ARMA processes

ACF PACF

Random Walk, $Y_t = Y_{t-1} + \epsilon_t$



Review

Autocorrelation and Partial Autocorrelation

Key Concepts

Autocorrelation, Partial Autocorrelation

Questions

- What is the difference between the h-lag autocorrelation and the h-lag partial autocorrelation?
- When are the autocorrelation and partial autocorrelation always the same for any DGP?
- What shape would you expect in the ACF and PACF of an AR(3)?
- What shape would you expect in the ACF and PACF of an MA(12)?

Problems

1. What is the ACF and PACF of an AR(1) $Y_t = \phi_1 Y_{t-1} + \epsilon_t$?
2. What is the ACF of an MA(2) $Y_t = \theta_1 \epsilon_{t-1} + \theta_2 \epsilon_{t-2} + \epsilon_t$?

Estimating Autocorrelations and Partial Auocorreations

Univariate Time Series Analysis

Sample ACF and PACF

- Sample autocorrelations

$$\hat{\rho}_s = \frac{\sum_{t=s+1}^{T} Y_t^* Y_{t-s}^*}{\sum_{t=1}^{T} Y_t^{*2}} = \frac{\hat{\gamma}_s}{\hat{\gamma}_0}$$

  - $Y_t^* = Y_t - \bar{Y}$ where $\bar{Y} = T^{-1} \sum_{t=1}^{T} Y_t$

- Some prefer the small-sample-size corrected version

$$\hat{\rho}_s = \frac{\sum_{t=s+1}^{T} Y_t^* Y_{t-s}^*}{\sqrt{\sum_{t=s+1}^{T} Y_t^{*2} \sum_{t=1}^{T-s} Y_t^{*2}}}.$$

- Sample partial autocorrelations
  - Run regression to estimate $\hat{\varphi}_s$

$$Y_t = \phi_0 + \phi_1 Y_{t-1} + \phi_2 Y_{t-2} + \ldots + \varphi_s Y_{t-s} + \epsilon_t$$

- More efficient ways to compute PACF using Yule-Walker (see notes)

Testing Autocorrelations and Partial Auocorreations

Univariate Time Series Analysis

Testing autocorrelations and partial ACs

- Inference on autocorrelations:

$$
\begin{aligned}
\mathrm{V}[\hat{\rho}_s] &= T^{-1} && \text{for } s = 1 \\
&= T^{-1}(1 + 2\sum_{j=1}^{s-1} \hat{\rho}_j^2) && \text{for } s > 1
\end{aligned}
$$

- Standard t-stats

$$\frac{\hat{\rho}_s}{\sqrt{\mathrm{V}[\hat{\rho}_s]}} \overset{A}{\sim} N(0,1).$$

- Inference on partial autocorrelations:

$$\mathrm{V}[\hat{\varphi}_s] \approx T^{-1}$$

- Standard t-stats

$$T^{\frac{1}{2}}\hat{\varphi}_s \overset{A}{\sim} N(0,1)$$

Testing multiple autocorrelations

- Testing multiple autocorrelations: Ljung-Box Q, $H_0 : \rho_1 = \ldots = \rho_s = 0$

$$Q = T(T+2) \sum_{k=1}^{s} \frac{\hat{\rho}_k^2}{T-k} \sim \chi^2_s$$

- Note: Not heteroskedasticity robust, use LM test for serial correlation

Definition (LM test for serial correlation)

Under the null, $\mathrm{E}[Y_t^* Y_{t-j}^*] = 0$ for $1 \leq j \leq s$. The LM-test for serial correlation is constructed by defining the score vector $\mathbf{s}_t = Y_t^* \left[Y_{t-1}^* \, Y_{t-2}^* \ldots Y_{t-s}^* \right]'$,

$$LM = T \bar{\mathbf{s}}' \hat{\mathbf{S}}^{-1} \bar{\mathbf{s}} \xrightarrow{d} \chi^2_s$$

where $\bar{\mathbf{s}} = T^{-1} \sum_{t=1}^{T} \mathbf{s}_t$, $\hat{\mathbf{S}} = T^{-1} \sum_{t=1}^{T} \mathbf{s}_t \mathbf{s}_t'$ and $Y_t^* = Y_t - \bar{Y}$ where $\bar{Y} = T^{-1} \sum_{t=1}^{T} Y_t$.

Review

Sample Autocorrelations and Partial Autocorrelations

Key Concepts

Sample Autocorrelation, Sample Partial Autocorrelation, Ljung-Box Test, LM Test for Serial Correlation

Questions

- What is the asymptotic distribution of estimated autocorrelations and partial autocorrelations?
- Where does the rule-of-thump $2/\sqrt{T}$ come from when plotting sample autocorrelations?
- What is the difference between the Q-test and an LM test for serial correlation?
- If you computed a sample autocorrelation in Excel using the `correl` function by copying and shifting a variable by h places, would you get the usual sample autocorrelation estimator?

Parameter Estimation

Univariate Time Series Analysis

Conditional MLE

- Conditional MLE assuming distribution of $Y_t|Y_{t-1}, \epsilon_{t-1}, Y_{t-2}, \epsilon_{t-2}, \ldots$ is $N\left(0, \sigma^2\right)$
- If $\epsilon_{t-1}, \epsilon_{t-2}, ..., \epsilon_{t-Q}$ are observable, identical to least squares

$$\underset{\boldsymbol{\phi},\boldsymbol{\theta}}{\operatorname{argmin}} \sum_{t=P+1}^{T} \left(Y_t - \phi_0 - \phi_1 Y_{t-1} - \ldots - \phi_P Y_{t-P} - \theta_1 \epsilon_{t-1} - \ldots - \theta_Q \epsilon_{t-Q}\right)^2$$

- Ignore distribution of $Y_1, \ldots Y_P$ in fit
  - Finite sample effects, asymptotically irrelevant
- If $\epsilon_{P-1}, \ldots, \epsilon_{P-Q}$ are observable, can recursively compute $\epsilon_P \ldots, \epsilon_T$ for a set of parameters $\boldsymbol{\phi},\boldsymbol{\theta}$
- Overcome missing initial shocks by assuming $\epsilon_{P-1} = \ldots = \epsilon_{P-Q} = 0$

Ordinary Least Squares

- If $Q = 0$, conditional MLE simplifies

$$\underset{\boldsymbol{\phi}}{\operatorname{argmin}} \sum_{t=P+1}^{T} \left(Y_t - \phi_0 - \phi_1 Y_{t-1} - \ldots - \phi_P Y_{t-P}\right)^2$$

- Conditional MLE is identical to OLS
- Inference is identical
- Use classical or White's covariance estimator as appropriate
- Can also incorporate deterministic terms such as time trends while maintaining simplicity of OLS

Exact MLE

- Define the vector of data

$$\mathbf{y} = [Y_1, Y_2, \ldots, Y_{T-1}\, Y_T]'$$

- $\mathbf{\Gamma}$ be the T by T covariance matrix of $\mathbf{y}$

$$\mathbf{\Gamma} = \begin{bmatrix} \gamma_0 & \gamma_1 & \gamma_2 & \gamma_3 & \cdots & \gamma_{T-2} & \gamma_{T-1} \\ \gamma_1 & \gamma_0 & \gamma_1 & \gamma_2 & \cdots & \gamma_{T-3} & \gamma_{T-2} \\ \gamma_2 & \gamma_1 & \gamma_0 & \gamma_1 & \cdots & \gamma_{T-4} & \gamma_{T-3} \\ \vdots & \vdots & \vdots & \vdots & \ldots & \vdots & \vdots \\ \gamma_{T-2} & \gamma_{T-3} & \gamma_{T-4} & \gamma_{T-5} & \cdots & \gamma_0 & \gamma_1 \\ \gamma_{T-1} & \gamma_{T-2} & \gamma_{T-3} & \gamma_{T-4} & \cdots & \gamma_1 & \gamma_0 \end{bmatrix}$$

- The joint likelihood of $\mathbf{y}$

$$f(\mathbf{y}|\boldsymbol{\phi}, \boldsymbol{\theta}, \sigma^2) = (2\pi)^{-\frac{T}{2}} |\mathbf{\Gamma}|^{-\frac{T}{2}} \exp\left(-\frac{\mathbf{y}'\mathbf{\Gamma}^{-1}\mathbf{y}}{2}\right)$$

- Log-likelihood

$$l(\boldsymbol{\phi}, \boldsymbol{\theta}, \sigma^2; \mathbf{y}) = -\frac{T}{2}\ln(2\pi) - \frac{T}{2}\ln|\mathbf{\Gamma}| - \frac{1}{2}\mathbf{y}'\mathbf{\Gamma}^{-1}\mathbf{y}$$

Review

Parameter Estimation

Key Concepts

Conditional Maximum Likelihood, Exact Maximum Likelihood

Questions

- How are missing initial innovations addressed in conditional MLE?
- What is the key advantage of exact MLE over conditional MLE?
- When does conditional MLE reduce to OLS?
- How is the autocovariance matrix computed in exact MLE?

Model Building

Univariate Time Series Analysis

Model building the Box-Jenkins way

- Model building is similar to cross-section regression
- Can use same techniques
  - General to Specific or Specific to General
  - Information criteria: AIC, BIC
- Box-Jenkins is dominant methodology, 2-steps
  - Identification: Use ACF and PACF to choose model
  - Estimation: Estimate model and do diagnostic checks
- Two principles
  - Parsimony
  - Invertibility

Strategies

- General to Specific
  - Fit largest specification
  - Drop regressor with largest p-value
  - Refit
  - Stop if all p-values indicate significance using a size of α
    - α is the econometrician's choice
- Specific to General
  - Fit all specifications with a single variable
  - Retail variable with smallest p-value
  - Extend this model adding on additional variables one at a time
  - Stop if the p-values of all excluded variables are larger than α

Information Criteria

- Information Criteria
  - Akaike Information Criterion (AIC)

$$AIC = \ln \hat{\sigma}^2 + k\frac{2}{T}$$

  - Schwartz (Bayesian) Information Criterion (SIC/BIC)

$$BIC = \ln \hat{\sigma}^2 + k\frac{\ln T}{T}$$

- Both have versions suitable for likelihood based estimation
- Reward for better fit: Reduce $\ln \hat{\sigma}^2$
- Penalty for more parameters: $k\frac{2}{T}$ or $k\frac{\ln T}{T}$
- Choose model with smallest IC
  - AIC has fixed penalty $\Rightarrow$ inclusion of extraneous variables
  - BIC has larger penalty if $\ln T > 2$ ($T > 7$)

Model Diagnostics

Univariate Time Series Analysis

Model Diagnostics

- Important to assess whether your model "fits"
  - Are the residuals white noise?
    - Eye-ball test
    - Ljung-Box Q stat or LM serial correlation test of $H_0 : \rho_1 = \ldots = \rho_s = 0$.
    - SACF/SPACF of the residuals
  - Are there any large outliers?
    - Eye-ball test
- What to do if there are problems?
  - Use SPACF/SACF to repeat Box-Jenkins and augment your model with correct dynamics to pick up problem
  - Repeat diagnostics
- Concern: Repeated testing may render critical values misleading

Review

Model Selection

Key Concepts

Invertibility, Parsimony, AIC, BIC

Questions

- How are the ACF and PACF used to identify candidate models?
- How does GtS differ in an ARMA from application to a linear regression?
- Which chooses a larger model, AIC or BIC, and why?
- What property should residuals have from a well specified model?
- What use is the parsimony principle?
- What does invertibility ensure?

The Information Set

Univariate Time Series Analysis

The information set and the law of iterated expectations

- Information set: $\mathcal{F}_t$
- Contains a lot of information!
  - Every time t *measurable* event
  - Observed variables: prices, returns, GDP, interest rates, FX rates
  - Functions of these
  - Excludes variables which are latent: volatility
- Conditional expectation:

$$\mathrm{E}[Y_{t+1}|\mathcal{F}_t]$$

  Conditional Variance

$$\mathrm{V}[Y_{t+1}|\mathcal{F}_t]$$

  - Shorthand $\mathrm{E}_t[Y_{t+1}]$ and $\mathrm{V}_t[Y_{t+1}]$
- Law of Iterated Expectation (LIE):

$$E_t[E_{t+1}[Y_{t+2}]] = E_t[Y_{t+2}]$$

  - Monday's belief about what Tuesday's belief about Wednesday is the same as Monday's belief of Wednesday

Loss Functions

Univariate Time Series Analysis

Forecasting

- A $h-$step ahead forecast, $\hat{Y}_{t+h|t}$, is designed to minimize a loss function
  - MSE: $(Y_{t+h} - \hat{Y}_{t+h|t})^2$
  - MAD: $|Y_{t+h} - \hat{Y}_{t+h|t}|$
  - Quad-Quad: $\alpha_1(Y_{t+h} - \hat{Y}_{t+h|t})^2 + \alpha_2 I_{[Y_{t+h} - \hat{Y}_{t+h|t} < 0]}(Y_{t+h} - \hat{Y}_{t+h|t})^2$
    - Asymmetric if $\alpha_1 \neq \alpha_2$

The MSE Optimal Forecast is the conditional mean

- Let $Y^*_{t+h} = \mathrm{E}_t[Y_{t+h}]$
- Let $\tilde{Y}_{t+h}$ be any other value

$$
\begin{aligned}
\mathrm{E}_t[(Y_{t+h} - \tilde{Y}_{t+h})^2] &= \mathrm{E}_t[\left(\left(Y_{t+h} - Y^*_{t+h}\right) + \left(Y^*_{t+h} - \tilde{Y}_{t+h}\right)\right)^2] \\
&= \mathrm{E}_t[\left(Y_{t+h} - Y^*_{t+h}\right)^2 + 2\left(Y_{t+h} - Y^*_{t+h}\right)\left(Y^*_{t+h} - \tilde{Y}_{t+h}\right) + \left(Y^*_{t+h} - \tilde{Y}_{t+h}\right)^2] \\
&= \mathrm{V}_t[Y_{t+h}] + 2\mathrm{E}_t[\left(Y_{t+h} - Y^*_{t+h}\right)\left(Y^*_{t+h} - \tilde{Y}_{t+h}\right)] + \mathrm{E}_t[\left(Y^*_{t+h} - \tilde{Y}_{t+h}\right)^2] \\
&= \mathrm{V}_t[Y_{t+h}] + 2\left(Y^*_{t+h} - \tilde{Y}_{t+h}\right)\mathrm{E}_t[\left(Y_{t+h} - Y^*_{t+h}\right)] + \mathrm{E}_t[\left(Y^*_{t+h} - \tilde{Y}_{t+h}\right)^2] \\
&= \mathrm{V}_t[Y_{t+h}] + 2\left(Y^*_{t+h} - \tilde{Y}_{t+h}\right)\cdot 0 + \mathrm{E}_t[\left(Y^*_{t+h} - \tilde{Y}_{t+h}\right)^2] \\
&= \mathrm{V}_t[Y_{t+h}] + \left(Y^*_{t+h} - \tilde{Y}_{t+h}\right)^2
\end{aligned}
$$

Forecasting

Univariate Time Series Analysis

Forecasting

- MSE optimal forecast for an AR(1):

$$Y_t = \phi_1 Y_{t-1} + \epsilon_t$$

$$\begin{aligned}
\mathrm{E}_t[Y_{t+1}] &= \mathrm{E}_t[\phi_1 Y_t + \epsilon_{t+1}] \\
&= \phi_1 \mathrm{E}_t[Y_t] + \mathrm{E}_t[\epsilon_{t+1}] \\
&= \phi_1 Y_t + 0
\end{aligned}$$

$$\begin{aligned}
\mathrm{E}_t[Y_{t+2}] &= \mathrm{E}_t[\phi_1 Y_{t+1} + \epsilon_{t+2}] \\
&= \phi_1 \mathrm{E}_t[Y_{t+1}] + \mathrm{E}_t[\epsilon_{t+2}] \\
&= \phi_1 \left(\phi_1 Y_t\right) + 0 \\
&= \phi_1^2 Y_t + 0
\end{aligned}$$

Note: Long-run forecast is always $\mathrm{E}\left[Y_t\right]$ for a covariance stationary process

Forecast Errors

$$
\begin{aligned}
\mathrm{V}_t[Y_{t+1}] &= \mathrm{E}_t\left[\left(Y_{t+1} - \mathrm{E}_t\left[Y_{t+1}\right]\right)^2\right] \\
&= \mathrm{E}_t\left[\left(\phi Y_t + \epsilon_{t+1} - \phi Y_t\right)^2\right] \\
&= \mathrm{E}_t\left[\epsilon_{t+1}^2\right] = \sigma^2 \text{ if homoskedastic}
\end{aligned}
$$

$$
\begin{aligned}
\mathrm{V}_t[Y_{t+2}] &= \mathrm{E}_t\left[\left(Y_{t+2} - \mathrm{E}_t\left[Y_{t+2}\right]\right)^2\right] \\
&= \mathrm{E}_t\left[\left(\phi^2 Y_t + \phi\epsilon_{t+1} + \epsilon_{t+2} - \phi^2 Y_t\right)^2\right] \\
&= \mathrm{E}_t\left[\left(\phi\epsilon_{t+1} + \epsilon_{t+2}\right)^2\right] \\
&= \phi^2 \mathrm{E}_t\left[\epsilon_{t+1}^2\right] + \mathrm{E}_t\left[\epsilon_{t+2}^2\right] = \left(1 + \phi^2\right)\sigma^2 \text{ if homoskedastic}
\end{aligned}
$$

Note: Long-run forecast error variance is always $\mathrm{V}\left[Y_t\right]$ for a covariance stationary process

Key Concepts

Mean Square Error, Conditional Expectation

Questions

- How is the MSE optimal forecast related to the conditional mean? What about the conditional median?
- What is the key principle for producing multi-step forecasts?
- What does the long-run forecast for a covariance stationary time series always converge to? What is the long-run variance of the error?

Problems

1. What are the first three forecasts from the model $Y_t = \phi_0 + \phi_1 Y_{t-1} + \theta_1 \epsilon_{t-1} + \epsilon_t$?
2. What are the first three forecasts errors?
3. What is the variance of the first three forecast errors?

Mincer-Zarnowitz Tests

Univariate Time Series Analysis

Forecast evaluation

Mincer-Zarnowitz regressions

- Objective Forcecast Evaluation

$$Y_{t+h} = \alpha + \beta \hat{Y}_{t+h|t} + \eta_t$$

- $H_0 : \alpha = 0, \beta = 1$, $H_1 : \alpha \neq 0 \cup \beta \neq 1$
  - Use any test: Wald, LR, LM
- Can be generalized to include any variable available when the forecast was produced

$$Y_{t+h} = \alpha + \beta \hat{Y}_{t+h|t} + \boldsymbol{\gamma} \mathbf{x}_t + \eta_t$$

- $H_0 : \alpha = 0, \beta = 1, \boldsymbol{\gamma} = \mathbf{0},$, $H_1 : \alpha \neq 0 \cup \beta \neq 1 \cup \gamma_j \neq 0$
- $\mathbf{x}_t$ *must* be in the time t information set
- Important when working with macro data

Diebold-Mariano Tests

Univariate Time Series Analysis

Relative evaluation: Diebold-Mariano

- Two forecasts, $\hat{Y}^A_{t+h|t}$ and $\hat{Y}^B_{t+h|t}$
- Two losses, $l^A_t = (Y_{t+h} - \hat{Y}^A_{t+h|t})^2$ and $l^B_t = (Y_{t+h} - \hat{Y}^B_{t+h|t})^2$
  - Losses do not need to be MSE
- If equally good or bad, $\mathrm{E}[l^A_t] = \mathrm{E}[l^B_t]$ or $\mathrm{E}[l^A_t - l^B_t] = 0$
- Define $\delta_t = l^A_t - l^B_t$

Relative evaluation: Diebold-Mariano

- Implemented as a t-test that $\mathrm{E}[\delta_t] = 0$
- $H_0 : \mathrm{E}[\delta_t] = 0$, $H_1^A : \mathrm{E}[\delta_t] < 0$, $H_1^B : \mathrm{E}[\delta_t] > 0$
  - Composite alternative
  - Sign indicates which model is favored

$$DM = \frac{\bar{\delta}}{\sqrt{\widehat{\mathrm{V}[\bar{\delta}]}}} = \frac{T^{-1}\sum_{t=1}^{T}\delta_t}{\sqrt{\frac{\hat{\sigma}^2_{NW}}{T}}}$$

- One complication: $\{\delta_t\}$ cannot be assumed to be uncorrelated, so a more complicated variance estimator is required
- Newey-West covariance estimator:

$$\hat{\sigma}^2_{NW} = \hat{\gamma}_0 + 2\sum_{l=1}^{L}\left[1 - \frac{l}{L+1}\right]\hat{\gamma}_l$$

Implementing a Diebold-Mariano Test

$$DM = \frac{\bar{\delta}}{\sqrt{\widehat{\mathrm{V}[\bar{\delta}]}}}$$

Algorithm (Diebold-Mariano Test)

1. *Using the two forecasts,* $\hat{Y}^{A}_{t+h|t}$ *and* $\hat{Y}^{B}_{t+h|t}$*, compute* $\delta_t = l^A_t - l^B_t$
2. *Run the regression*

$$\delta_t = \beta + \eta_t$$

3. *Use a Newey-West covariance estimator (`cov_type="HAC"`)*
4. *T-test* $H_0 : \beta = 0$ *against* $H^A_1 : \beta < 0$*, and* $H^B_1 : \beta > 0$
5. *Reject if* $|t| > C_\alpha$ *where* C_α *is the critical value for a 2-sided test using a normal distribution with a size of* α*. If significant, reject in favor of model A if test statistic is negative or in favor of model B if test statistic is positive.*

Review

Forecast Evaluation

Key Concepts

Objective Forecast Evaluation, Relative Forecast Evaluation, Mincer-Zarnowitz Test, Diebold-Mariano Test, Newey-West Variance Estimator

Questions

- What is the difference between objective and relative forecast evaluation?
- Why is a Newey-West covariance estimator used in Diebold-Mariano test?
- How is rejection of the null in a Newey-West test different from most tests?
- Why is a multi-step forecast be sensitive to a future realization of the time series between the current period and the forecast horizon?
- How is a MZ regression transformed to an Augmented MZ regression?

Nonstationary Time Series

Univariate Time Series Analysis

Nonstationarity defined

- Any series which is not stationary is nonstationary
- Four major types
  - Seasonality
    - Only slightly problematic
    - Can often be analyzed using standard tools and Box-Jenkins
  - Deterministic trends: growth over time
    - Linear
    - Polynomial
    - Exponential
  - Random walks or unit roots
  - Structural breaks

Deterministic trends

- Trending series can be decomposed

$$Y_t = \text{deterministic trend} \ + \ \text{stationary component} \ + \ \text{noise}$$

- Two major types
  - Polynomial

$$Y_t = \phi_0 + \delta_1 t + \delta_2 t^2 + \ldots + \delta_s t^s + \epsilon_t$$

    - Linear (important special case)

$$Y_t = \phi_0 + \delta_1 t + \epsilon_t$$

  - Exponential

$$\ln Y_t = \phi_0 + \delta_1 t + \epsilon_t$$

- Solution is to detrend
  - Detrended series is a stationary process
  - Standard model building on residuals
  - Can directly include time trends in ARMA models

The Lag Operator

Univariate Time Series Analysis

The Lag Operator

- The Lag Operator is a useful tool in time series
- Simplifies expressing complex models with seasonal dynamics
- Key properties
  1. $LY_t = Y_{t-1}$
  2. $L^2Y_t = LY_{t-1} = L(LY_t) = Y_{t-2}$
  3. $L^aL^b = L^{(a+b)}$
  4. $Lc = c$ where c is a constant

Seasonality

Univariate Time Series Analysis

Seasonality

- Seasonality is technically a form of non-stationarity
  - Mean explicitly depends on the quarter, month, day or minute
- Three types:

Definition (Seasonality)

Data are said to be seasonal if they exhibit a non-constant deterministic pattern on an annual basis.

Definition (Hebdomadality)

Data which exhibit day-of-week deterministic effects are said to be hebdomadal.

Definition (Diurnality)

Data which exhibit intra-daily deterministic effects are said to be diurnal.

Seasonality

- Simpler to think of processes with seasonality as having two models
  - Short-run AR and MA dynamics
  - Seasonal AR and MA dynamics
- Model building is standard with these two goals in mind
- Also consider seasonal deterministic terms
  - Seasonal dummy variables
  - Seasonal Fourier series

ARMA Modeling of Seasonality

Univariate Time Series Analysis

ARMA Modeling of Seasonality

Four Components

- Observation AR

$$\left(1-\phi_1 L\right) Y_t=\phi_0+\epsilon_t$$

- Seasonal AR

$$\left(1-\phi_s L^s\right) Y_t=\phi_0+\epsilon_t$$

- Observation MA

$$Y_t=\phi_0+\left(1+\theta_1 L^1\right) \epsilon_t$$

- Seasonal MA

$$Y_t=\phi_0+\left(1+\theta_s L^s\right) \epsilon_t$$

- Combined Model

$$\left(1-\phi_1 L\right)\left(1-\phi_s L^s\right) Y_t=\left(1+\theta_1 L^1\right)\left(1+\theta_s L^s\right) \epsilon_t$$

$$\begin{aligned} Y_t = & \phi_0+\phi_1 Y_{t-1}+\phi_s Y_{t-s}-\phi_1 \phi_s Y_{t-s-1} \\ & +\theta_1 \epsilon_{t-1}+\theta_s \epsilon_{t-s}+\theta_1 \theta_s \epsilon_{t-s-1}+\epsilon_t \end{aligned}$$

ARMA Modeling of Seasonality

Four Components

- Generalizes to higher orders of each term
- Known as SARIMA$(p, 0, q) \times (P, 0, Q, s)$
- Imposes restrictions on parameters due to multiplication of terms
- Can estimate unrestricted equivalent

$$Y_t = \phi_0 + \phi_1 Y_{t-1} + \phi_s Y_{t-s} + \phi_{s+1} Y_{t-s-1} + \theta_1 \epsilon_{t-1} + \theta_s \epsilon_{t-s} + \theta_{s+1} \epsilon_{t-s-1} + \epsilon_t$$

- Can test $H_0 : \phi_{s+1} = \phi_1 \phi_s \cap \theta_{s+1} = \theta_1 \theta_s$

Review

Seasonality

Key Concepts

Seasonality, Lag Operator, SARIMA, Deterministic Trend, Exponential Trend

Questions

- How can seasonality be modeled in an ARMA model?
- Define diurnality, hebdomadality and seasonality.
- What are seasonal determinist terms and how do they differ from seasonal AR and MA terms?
- What is an exponential trend?
- What do the orders in a SARIMA mean?
- How could a standard AR be used to model a time series with a seasonal AR component?

Random Walks, Unit Roots and Stochastic Trends

Univariate Time Series Analysis

Stochastic trends

- Stochastic trends are similar to deterministic trends
  - Dominant feature of a process

$$Y_t = \text{stochastic trend} + \text{stationary component} + \text{noise}$$

- Most common stochastic trend is a unit root
- There are others (generally non-linear)
- Removed using stochastic detrending (differencing)
  - Meaningfully different that deterministic detrending

Short-run Dynamics in a Unit Root process

- Unit root processes, in the long-run, behave like random walks
- In the short run, can have stationary dynamics

$$Y_t = \phi_1 Y_{t-1} + \phi_2 Y_{t-2} + \phi_3 Y_{t-3} + \epsilon_t$$

- If this process contains a unit root, $\phi_1 + \phi_2 + \phi_3 = 1$
- Can see the SR dynamics by differencing

$$\begin{aligned}
Y_t &= \phi_1 Y_{t-1} + \phi_2 Y_{t-2} + \phi_3 Y_{t-2} - \phi_3 Y_{t-2} + \phi_3 Y_{t-3} + \epsilon_t \\
Y_t &= \phi_1 Y_{t-1} + \phi_2 Y_{t-2} + \phi_3 Y_{t-2} - \phi_3 \Delta Y_{t-2} + \epsilon_t \\
Y_t - Y_{t-1} &= (\phi_1 + \phi_2 + \phi_3 - 1) Y_{t-1} - (\phi_2 + \phi_3) \Delta Y_{t-1} - \phi_3 \Delta Y_{t-2} + \epsilon_t \\
\Delta Y_t &= \pi_1 \Delta Y_{t-1} + \pi_2 \Delta Y_{t-2} + \epsilon_2
\end{aligned}$$

What's the problem with unit roots?

- Unit roots cause a number of problems
  - Exploding variance: $\mathrm{V}[Y_t] = t\sigma^2$
  - Parameter estimates converge at different rates
  - Hypothesis tests have non-standard distributions
  - No mean reversion in long-run forecasts
  - Spurious regression
- Crucial to understand whether a process is stationary or contains a unit root
- Often has large economic consequences
  - PPP
  - Covered interest rate parity
  - Carry trades

Testing for Unit Roots

Univariate Time Series Analysis

Testing for unit roots

- Dickey-Fuller looks like a standard t-test

$$Y_t = \phi_1 Y_{t-1} + \epsilon_t$$

- $H_0 : \phi_1 = 1$, $H_1 : \phi_1 < 1$
- Impose the null

$$\begin{aligned} Y_t - Y_{t-1} &= \phi_1 Y_{t-1} - Y_{t-1} + \epsilon_t \\ \Delta Y_t &= (\phi_1 - 1) Y_{t-1} + \epsilon_t \\ \Delta Y_t &= \gamma Y_{t-1} + \epsilon_t \end{aligned}$$

- New $H_0 : \gamma = 0$, $H_1 : \gamma < 0$
- Test with t-stat
- Augmented Dickey Fuller (ADF) captures short run dynamics as well

$$\Delta Y_t = \gamma Y_{t-1} + \rho_1 \Delta Y_{t-1} + \rho_2 \Delta Y_{t-2} + \ldots + \rho_P \Delta Y_{t-P} + \epsilon_t$$

- Lags of ΔY_{t-1} needed to ensure $\epsilon_t \sim WN\left(0, \sigma^2\right)$, also reduce variance of residuals

The problem

- t-stat is no longer asymptotically normal
- Requires Dickey-Fuller distribution
  - Most software packages contain the correct critical value
- Many processes with unit roots also contain deterministic components
- Asymptotic distribution depends on choice of model:

$$\Delta Y_t = \gamma Y_{t-1} + \sum_{p=1}^{P} \phi_p \Delta Y_{t-p} + \epsilon_t \qquad \text{(No trend)}$$

$$\Delta Y_t = \delta_0 + \gamma Y_{t-1} + \sum_{p=1}^{P} \phi_p \Delta Y_{t-p} + \epsilon_t \qquad \text{(Constant, linear in } Y_t\text{)}$$

$$\Delta Y_t = \delta_0 + \delta_1 t + \gamma Y_{t-1} + \sum_{p=1}^{P} \phi_p \Delta Y_{t-p} + \epsilon_t \qquad \text{(Constant, quadratic in } Y_t\text{)}$$

- More deterministic regressors lower the critical value
- Reject null of unit root if t-stat of γ is ***negative*** and below the critical value

The Role of The Deterministic Terms

- ADF tests include deterministic terms to remove these effects from Y_{t-1}
- Suppose Y_t is a pure time trend process

$$Y_t = \alpha + \beta t + \epsilon_t$$

- The differenced value is

$$\begin{aligned}\Delta Y_t &= \alpha + \beta t + \epsilon_t - \alpha - \beta\left(t-1\right) - \epsilon_{t-1} \\ &= \beta - \epsilon_{t-1} + \epsilon_t\end{aligned}$$

  - MA(1) without a trend
- In an ADF with deterministic regressors

$$\Delta Y_t = \delta_0 + \delta_1 t + \gamma Y_{t-1} + \epsilon_t$$

- The deterministic terms remove determinitic components from Y_{t-1}
- γ depends on

$$\mathrm{Cov}\left[\Delta Y_t, Y_{t-1} - \alpha - \beta\left(t-1\right)\right] = \mathrm{Cov}\left[\beta - \epsilon_{t-1} + \epsilon_t, \epsilon_{t-1}\right] = -\sigma^2$$

- Failing to include the deterministic regressors results in γ that depends on

$$\mathrm{Cov}\left[\Delta Y_t, Y_{t-1}\right] = 0$$

  - Time trend dominates the other components of Y_{t-1}

Important considerations

- Unit root tests are well known for having low power
- Power = 1-Pr(type II)
  - Chance you don't reject when alternative is true
- Some suggestions
  - Use a loose model selection when choosing the number of lags of ΔY_{t-j}, e.g. AIC
  - Be conservative in excluding deterministic regressors.
    - Including a constant or time-trend when absent hurts power
    - Excluding a constant or time-trend when present results in no power
  - More powerful tests than the ADF are available: DF-GLS
  - Visually inspect the data and differenced data
  - Use a general-to-specific search
- Number of differences needed is the *order of integration*
  - Integrated of Order 1 or I(1): Y_t is nonstationary but ΔY_t is stationary
  - I(d): Y_t is nonstationary, $\Delta^j Y_t$ also nonstationary when $j < d$, $\Delta^d Y_t$ is stationary

Seasonal Differencing

Univariate Time Series Analysis

Seasonal Differencing

- Seasonal series should use seasonal differencing

$$\Delta_s Y_t = Y_t - Y_{t-s}$$

- Complete SARIMA$(P, D, Q) \times (P_s, D_s, Q_s, s)$ model
  - D is order of observational difference
  - D_s is order of seasonal difference
  - P and Q are observational AR and MA orders
  - P_s and Q_s are seasonal AR and MA orders
- Special Cases
  - ARMA(P, Q): $D = D_s = P_s = Q_s = 0$
  - ARIMA(P, D, Q): $D_s = P_s = Q_s = 0$
  - SARMA$(P, Q) \times (P_s, Q_s, s)$: $D = D_s = 0$

Review

Unit Roots and Integration

Key Concepts

Unit Root, Integrated Process, I(1), Augmented Dickey-Fuller Test, Seasonal Difference

Questions

- What happens if a relevant deterministic term is omitted in a ADF test?
- What is the effect of including an unnecessary deterministic in an ADF test?
- How should you decide how many lags of the differenced variable to include in an ADF test?
- When should you use seasonal differencing?
- What is the relationship between a random walk and a unit root process?
- What are the consequences of ignoring a unit root when modeling a time series?

Self-Exciting Threshold Autoregression

Univariate Time Series Analysis

Nonlinear Models for the mean

- *Linear* time series process

$$Y_t = Y_0 + \sum_{i=0}^{t} \theta_i \epsilon_{t-i}$$

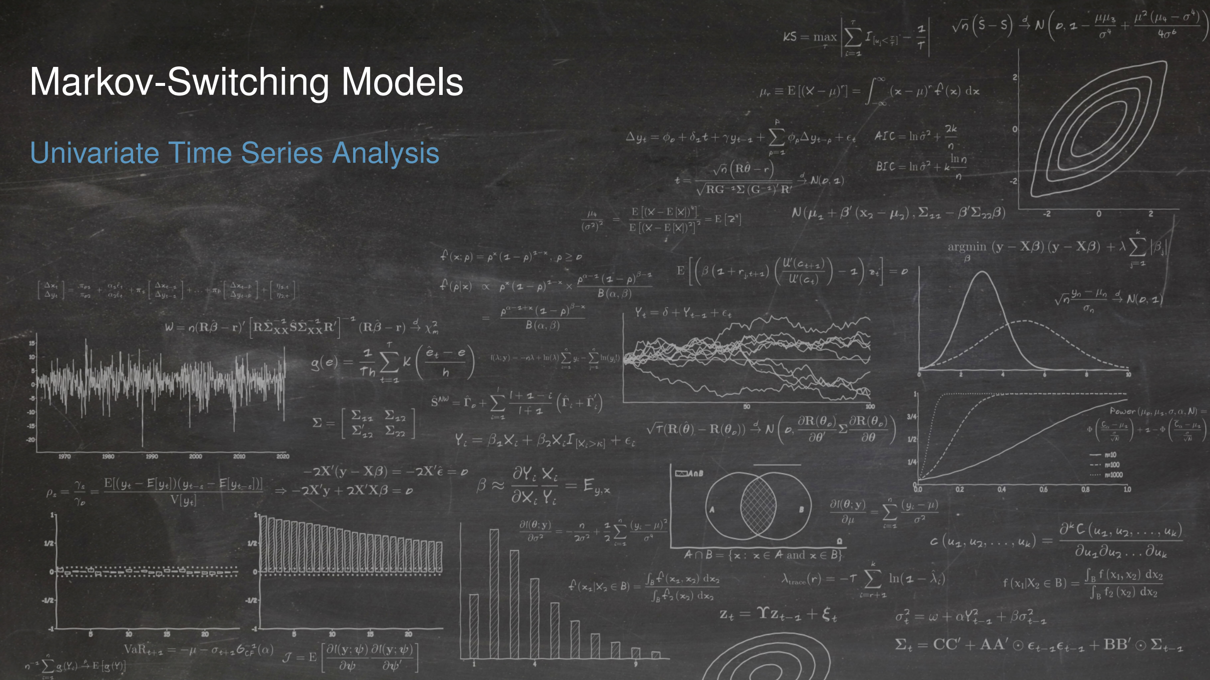
- Alternatives
  - Markov Switching Autoregression (MSAR)
  - Threshold Autoregression (TAR) and Self-exciting Threshold Autoregression (SETAR)
  - Many, many others
- Nonlinear models can capture different dynamics
  - *State-dependent parameters*

$$Y_t = \phi_0^{s_t} + \phi_1^{s_t} Y_{t-1} + \sigma^{s_t} \epsilon_t$$

  - Models differ in how s_t evolves

Markov-Switching Models

Univariate Time Series Analysis



Markov Switching Example

- Two states, H and L

$$Y_t = \begin{cases} \phi^H + \epsilon_t \\ \phi^L + \epsilon_t \end{cases}$$

- States evolve according to a 1st order Markov Chain

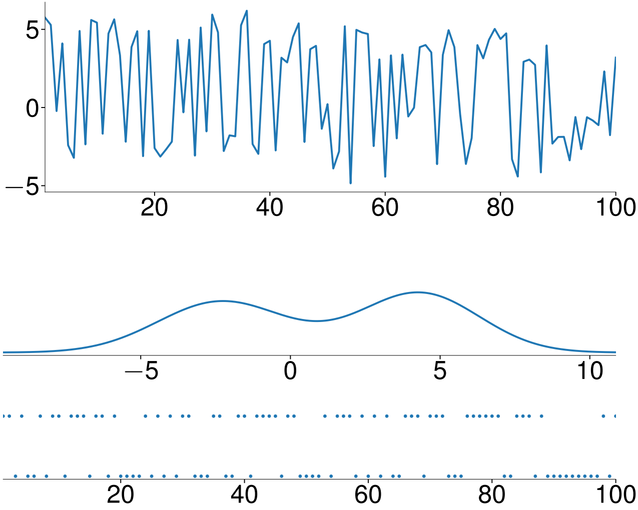
$$\{s_t\} = \{H, H, H, L, L, L, H, L, \ldots\}$$

- Transition Probabilities

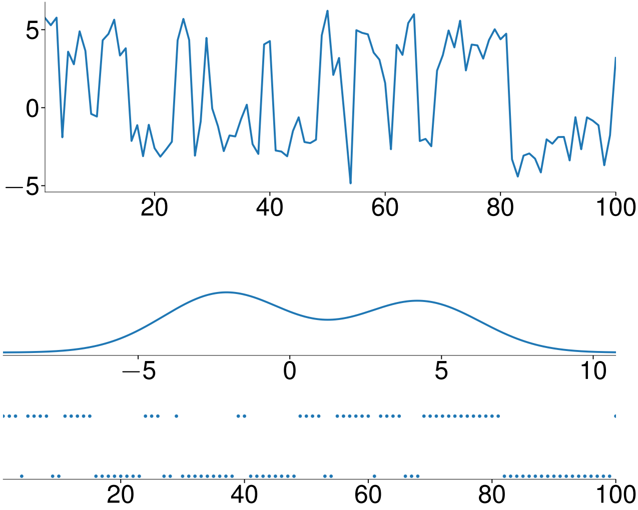
$$\begin{bmatrix} p_{HH} & p_{HL} \\ p_{LH} & p_{LL} \end{bmatrix} = \begin{bmatrix} p_{HH} & 1 - p_{LL} \\ 1 - p_{HH} & p_{LL} \end{bmatrix}$$

  - p_{HH} is the probability $s_{t+1} = H$ given $s_t = H$.
- Model will switch between a high mean state and a low mean state
- Models like this are very flexible and nest ARMA
  - Successful in financial econometrics for asset allocation, volatility modeling, modeling series with business-cycle length patterns: GDP

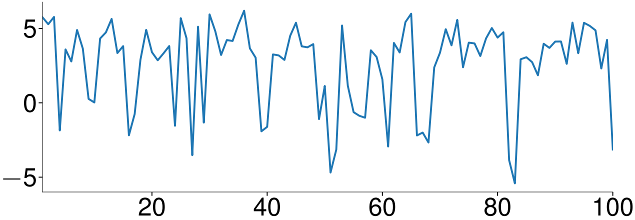
Markov Switching: i.i.d. Mixture

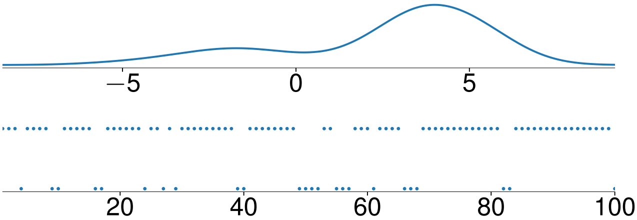


Markov Switching: Symmetric Persistent

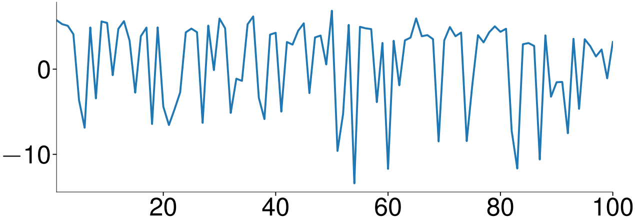


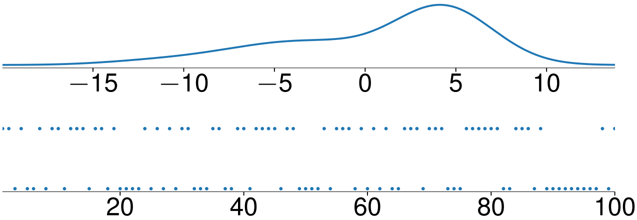
Markov Switching: Asymmetric Persistent





Markov Switching: Different Variances





Key Concepts

Self-exciting Threshold Autoregression, Markov Switching Processe

Questions

- It is always necessary to consider nonlinear models to model covariance stationary time series?
- What advantages might a nonlinear model have over a linear model when modeling a covariance stationary time series?

Review