Week 15

Worksheet 14, week of December 3rd, 2018

Computations.

1. $*$

(1) *Find the Fourier coefficients a_0, a_1, b_1, a_2, b_2 of the function*

$$f(x) = \begin{cases} 1 & x \in [0, 2\pi] \\ 0 & otherwise \end{cases}$$

(2) *Can you find formulas for a_k and b_k?*

Solution. We use the formulas for the Fourier transform and the fact that integrating sin or cos over a full period yields zero, to get:

$$a_0 = \frac{1}{2\pi}\int_0^{2\pi} f(x)dx = \frac{1}{2\pi}\int_0^{2\pi} dx = 1,$$

$$a_k = \frac{1}{\pi}\int_0^{2\pi} f(x)\cos(kx)dx = \frac{1}{\pi}\int_0^{2\pi} \cos(kx)dx = 0,$$

and,

$$b_k = \frac{1}{\pi}\int_0^{2\pi} f(x)\sin(kx)dx = \frac{1}{\pi}\int_0^{2\pi} \sin(kx)dx = 0$$

□

2. *Decide if the following matrices are positive definite, negative definite or indefinite*

(1) $A = \begin{bmatrix} 2 & -1 & -1 \\ -1 & 2 & -1 \\ -1 & -1 & 2 \end{bmatrix}$

(2) $*$ $B = \begin{bmatrix} 2 & -1 & -1 \\ -1 & 2 & 1 \\ -1 & 1 & 2 \end{bmatrix}$

(3) $C = \begin{bmatrix} 0 & 1 & 2 \\ 1 & 0 & 1 \\ 2 & 1 & 0 \end{bmatrix}^2$

Solution. We use the leading minors test on the first 2.

(1) $det[2] = 2$, $det \begin{bmatrix} 2 & -2 \\ -1 & 2 \end{bmatrix} = 3$, but $det \begin{bmatrix} 2 & -1 & -1 \\ -1 & 2 & -1 \\ -1 & -1 & 2 \end{bmatrix} = 0$ so A is not definite.

(2) $det[2] = 2 > 0$, $det \begin{bmatrix} 2 & -2 \\ -1 & 2 \end{bmatrix} = 3$, and $det \begin{bmatrix} 2 & -1 & -1 \\ -1 & 2 & -1 \\ -1 & -1 & 2 \end{bmatrix} = 6$. They are all positive so B is positive definite.

(3) Note that the columns of the matrix shown are independent, since the matrix is symmetric this means its eigenvalues are non-zero real numbers. Since C is the square of that matrix, its eigenvalues are squares of the previous eigenvalues and therefore are all positive. C is Positive definite.

Tutoring Room hours: Monday 4-8pm, Tuesday 6-8pm, Wednesday 5-7, Thursday 4-6
Piazza: https://piazza.com/class/j65464jkd255s
REMEMBER: No lecture on Friday

$\square$

3. *Consider the matrix* $A = \begin{bmatrix} 1 & b \\ b & 9 \end{bmatrix}$ *for* $b \in \mathbb{R}$.

(1) *For which numbers* b *is the matrix positive definite?*
(2) *Factor* $A = LDL^T$ *in terms of* b*, when* b *is in the range for positive definiteness.*
(3) *Find the minimum value of* $\frac{1}{2}(x^2 + 2bxy + 9y^2) - y$ *for* b *in this range.*

Solution. (1) We check the eigenvalues (there are more effective ways to check positive definiteness but we will need eigenvalues in part (2) anyways so...). The characteristic polynomial is $p(\lambda) = (1-\lambda)(9-\lambda) - b^2 = \lambda^2 - 10\lambda + 9 - b^2$. Using the quadratic formula we get rational-conjugate eigenvalues $\lambda_1 = 5 + \sqrt{16+b^2}$ and $\lambda_2 = 5 - \sqrt{16+b^2}$. From here, the one at risk of being negative is λ_2. We need the second term to be less than 5, so the argument of the square root should be less than 25. This is achieved by picking $b \in (-3, 3)$.

(2) We already have the eigenvalues so we compute the eigenvectors.

$$A - \lambda_1 I = \begin{bmatrix} -4 - \sqrt{16+b^2} & b \\ b & 4 - \sqrt{16+b^2} \end{bmatrix}$$

so the nul space equation gives $(-4 - \sqrt{16+b^2})x_1 = bx_2 = 0$ which produces the eigenvector $\mathbf{v}_1 = \begin{bmatrix} b \\ 4 + \sqrt{16+b^2} \end{bmatrix}$. By way of rational conjugates we know $\mathbf{v}_2 = \begin{bmatrix} b \\ 4 - \sqrt{16+b^2} \end{bmatrix}$. After normilizing we obtain

$$L = \begin{bmatrix} \frac{b}{\sqrt{b^2+(4+\sqrt{16+b^2})^2}} & \frac{b}{\sqrt{b^2+(4+\sqrt{16-b^2})^2}} \\ \frac{4+\sqrt{16+b^2}}{\sqrt{b^2+(4+\sqrt{16+b^2})^2}} & \frac{4-\sqrt{16+b^2}}{\sqrt{b^2+(4-\sqrt{16+b^2})^2}} \end{bmatrix}, D = \begin{bmatrix} 5 + \sqrt{16+b^2} & 0 \\ 0 & 5 - \sqrt{16+b^2} \end{bmatrix}$$

(3) Let q denote the quadratic form in question. To find the critical point we want to make the gradient zero. $q_x = x + by$, $q_y = bc = x + 9y - 1$. So the critical point occurs at the solution of the system $A \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$, which is $y = \frac{1}{9-b^2}$, $x = \frac{-b}{9-b^2}$. Since A is positive definite, we know the critical point is a minimum.

$\square$

4. *Consider the quadratic form* $q(x, y, z) = 3x^2 + 10xy + 3y^2 + z^2$

(1) *Find the symmetric matrix* A *for* q.
(2) *Find the eigenvalues and eigenvectors of* A.
(3) *Give a formula for* q *in the coordinates of the vector* $\begin{bmatrix} x \\ y \\ z \end{bmatrix}$ *in the eigenbasis of* A.
(4) *Is* q *positive definite, negative definite or indefinite?*

Solution. (1) Since the only term containing z is teh z^2 term, we can quickly see that

$$A = \begin{bmatrix} 3 & 5 & 0 \\ 5 & 3 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

(2) The characteristic polynomial is $p(\lambda) = (\lambda^2 - 6\lambda - 16)(1 - \lambda) = (\lambda - 1)(\lambda - 8)(\lambda + 2)$, so the eigenvalues are $\lambda_1 = 1, \lambda_2 = -2, \lambda_3 = 8$. To find eigenvectors we compute the eigenspace of each eigenvalue.

$$A - \lambda_1 I = \begin{bmatrix} 2 & 5 & 0 \\ 5 & 2 & 0 \\ 0 & 0 & 0 \end{bmatrix},$$

so $v_1 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$.

$$A - \lambda_2 I = \begin{bmatrix} 5 & 5 & 0 \\ 5 & 5 & 0 \\ 0 & 0 & 3 \end{bmatrix},$$

so $v_2 = \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}$.

$$A - \lambda_3 I = \begin{bmatrix} -5 & 5 & 0 \\ 5 & -5 & 0 \\ 0 & 0 & -7 \end{bmatrix},$$

so $v_2 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$

(3) Normalizing the vectors before we get the orthonormal eigenbasis $\mathcal{B} = \left\{ \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} -1/\sqrt{2} \\ 1/\sqrt{2} \\ 0 \end{bmatrix}, \begin{bmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \\ 0 \end{bmatrix} \right\}$

In order to write q in the coordinates of $\mathcal{B}$, we need to write the vector $\begin{bmatrix} x \\ y \\ z \end{bmatrix}$ in those coordinates. This is done by multiplying by the change of basis matrix $I_{\mathcal{B}\mathcal{E}} = B^{-1} = B^T$. We compute $B^T \mathbf{x} = \begin{bmatrix} z \\ \frac{y-x}{\sqrt{2}} \\ \frac{x+y}{\sqrt{2}} \end{bmatrix}$. We then get

$$q(x, y, z) = z^2 - 2(\frac{y-x}{\sqrt{2}})^2 + 8(\frac{x+y}{\sqrt{2}})^2.$$ 从哪来的这货?

(4) Since q has both negative and positive coefficients, it follows that it is indefinite. □

5. ∗ *Consider the matrix* $A = \begin{bmatrix} 0 & -1 \\ 1 & 2 \end{bmatrix}$.

(1) *List the eigenvalues and eigenvectors of A. Is A diagonalizable?*
(2) *Check that $B = A^T A$ is diagonalizable (Hint: spectral theorem). Find an orthogonal matrix $U = \begin{bmatrix} \boldsymbol{u}_1 & \boldsymbol{u}_2 \end{bmatrix}$ and diagonal matrix $D = \begin{bmatrix} \sigma_1^2 & 0 \\ 0 & \sigma_2^2 \end{bmatrix}$ such that $B = UDU^T$.*
(3) *Check that $C = AA^T$ is diagonalizable. Find an orthogonal matrix $V = \begin{bmatrix} \boldsymbol{v}_1 & \boldsymbol{v}_2 \end{bmatrix}$ and diagonal matrix $\widetilde{D} = \begin{bmatrix} \mu_1^2 & 0 \\ 0 & \mu_2^2 \end{bmatrix}$ such that $B = V\widetilde{D}V^T$.*
(4) *What is the relation between D and $\widetilde{D}$?*
(5) *Is $U = V$?*
(6) *Compute UDV^T and compare it to A.*

Solution. (1) The characteristic polynomial is $-\lambda(2-\lambda) + 1 = (\lambda - 1)^2$. Hence we get a single eigenvalue $\lambda = 1$. $A - I = \begin{bmatrix} -1 & -1 \\ 1 & 1 \end{bmatrix}$ so there is only one independent eigenvector. For instance $\begin{bmatrix} -1 \\ 1 \end{bmatrix}$. Therefore A is not diagonalizable.

(2) $B = A^TA = \begin{bmatrix} 1 & 2 & 2 & 5 \end{bmatrix}$. It is not hard to check the eigenvalues are the rational conjugates $3+2\sqrt{2}$ and $3-2\sqrt{2}$. The corresponding eigenvectors are $\begin{bmatrix} 1 \\ 1+\sqrt{2} \end{bmatrix}$, and $\begin{bmatrix} 1 \\ 1-\sqrt{2} \end{bmatrix}$. Normalizing those we obtain

$$\mathbf{u}_1 = \begin{bmatrix} \frac{1}{\sqrt{4+2\sqrt{2}}} \\ \frac{1+\sqrt{2}}{\sqrt{4+2\sqrt{2}}} \end{bmatrix}, \mathbf{u}_2 = \begin{bmatrix} \frac{1}{\sqrt{4-2\sqrt{2}}} \\ \frac{1-\sqrt{2}}{\sqrt{4-2\sqrt{2}}} \end{bmatrix}$$

(3) $C = AA^T = \begin{bmatrix} 1 & -2 \\ -2 & 5 \end{bmatrix}$. The same process will yield the same eigenvalues as for B but the eignevectors come out to be different. In particular

$$\mathbf{v}_1 = \begin{bmatrix} \frac{-1}{\sqrt{4+2\sqrt{2}}} \\ \frac{1+\sqrt{2}}{\sqrt{4+2\sqrt{2}}} \end{bmatrix}, \mathbf{v}_2 = \begin{bmatrix} \frac{-1}{\sqrt{4-2\sqrt{2}}} \\ \frac{1-\sqrt{2}}{\sqrt{4-2\sqrt{2}}} \end{bmatrix}$$

(4) We can indeed see that $D = \tilde{D}$.
(5) $U \neq V$
(6) Doing teh triple matrix product we can see that $UDV^T = A$.

□

6. *Let* $A = \begin{bmatrix} 1 & 4 \\ 2 & 8 \end{bmatrix}$

(1) *Compute the eigenvalues* σ_1^2, σ_2^2 *of* AA^T.
(2) *Compute the eigenvectors* $\boldsymbol{u}_1, \boldsymbol{u}_2$. *Check that they are orthonormal.*

Solution. (1) $AA^T = \begin{bmatrix} 17 & 34 \\ 34 & 68 \end{bmatrix}$. The characteristic polynomial is $\lambda^2 - 85\lambda$ so the eigenvalues are $\sigma_1^2 = 85$ and $\sigma_2^2 = 0$.

(2) By computing the null space of $A - 85I$, one can check that $\mathbf{u}_1 = \begin{bmatrix} \frac{1}{\sqrt{5}} \\ \frac{2}{\sqrt{5}} \end{bmatrix}$. We can also see that in the original matrix, the second column is twice the first one. So the a normalized eigenvector corresponding to $\sigma_2^2 = 0$ is $\mathbf{u}_2 = \begin{bmatrix} \frac{-2}{\sqrt{5}} \\ \frac{1}{\sqrt{5}} \end{bmatrix}$.

□

Testing your understanding.

7. *True or False? Provide reasoning.*

(1) *If A is a symmetric matrix then it is diagonalizable and all its eigenvalues are real.*
(2) *If A is a diagonalizable matrix whose eigenvalues are all real then A is symmetric.*
(3) *If A has an orthonormal eigenbasis then A is symmetric.*
(4) *If f is a quadratic form whose associated matrix is positive definite then f has a maximum at the origin.*
(5) *If f is a quadratic form whose associated matrix is negative definite then f has a maximum at the origin.*
(6) *A symmetric matrix is always definite*
(7) *A matrix whose eigenvalues are all non-negative is positive definite*

Solution. (1) True. This is the spectral theorem.
(2) False. Take any strictly triangular matrix with real numbers on the diagonal.
(3) True. If A is such a matrix, then it has a diagonalization of the form $A = QDQ^T$. Then $A^T = (QDQ^T)^T = (Q^T)^T D^T Q^T = QDQ^T = A$.
(4) False, Positive definite quadratic form has a minimum.

(5) True. Negative definite quadratic form has a maximum.
(6) False. Pick a diagonal matrix with both negatives and positives in the main diagonal.
(7) False. They could be zero.

□

Application.

8 (Discrete Cosine Transform and MP3). *Consider the matrix*

$$C := \sqrt{\frac{1}{2}} \begin{bmatrix} \frac{1}{\sqrt{2}} & \cos\frac{\pi}{8} & \cos\frac{2\pi}{8} & \cos\frac{3\pi}{8} \\ \frac{1}{\sqrt{2}} & \cos\frac{3\pi}{8} & \cos\frac{6\pi}{8} & \cos\frac{9\pi}{8} \\ \frac{1}{\sqrt{2}} & \cos\frac{5\pi}{8} & \cos\frac{10\pi}{8} & \cos\frac{15\pi}{8} \\ \frac{1}{\sqrt{2}} & \cos\frac{7\pi}{8} & \cos\frac{14\pi}{8} & \cos\frac{21\pi}{8} \end{bmatrix}.$$

(1) *Check that the columns of C form an orthonormal basis of $\mathbb{R}^n$. Call this basis $\mathcal{C}$.*
(2) *Let $\mathbf{v}$ in $\mathbb{R}^n$. Someone tells you that using $\mathcal{C}$ you can express $\mathbf{v}$ in terms of a sum of cosine functions oscillating at different frequencies. Explain why!*
(3) *What is $I_{\mathcal{C},\mathcal{E}}$ and how can it be used for frequency analysis of an audio signal?*
(4) *When compressing audio signals, small high-frequency components can be discarded. Using this observation, develop a lossy compression algorithm for audio signals!*

Solution. (1) We first check columns are orthonormal. Define the columns of the matrix C to be

$$C = [\mathbf{u}_0 \quad \mathbf{u}_1 \quad \mathbf{u}_2 \quad \cdots \quad \mathbf{u}_{n-1}].$$

so that the element of each vector is given by

$$u_{l,k} = \alpha(k) \cos\left(\frac{(2l-1)k\pi}{2n}\right),$$

where

$$\alpha(0) = \sqrt{\frac{1}{n}} \quad \text{and} \quad \alpha(k) = \sqrt{\frac{2}{n}}, \quad k = 1, \cdots, n-1.$$

This means that

$$\mathbf{u}_k = \begin{bmatrix} \cos(\frac{k\pi}{2n}) \\ \cos(\frac{3k\pi}{2n}) \\ \vdots \\ \cos(\frac{(2n-1)k\pi}{2n}) \end{bmatrix}.$$

Take for instance $n = 4$, then

$$\mathbf{u}_0 = \frac{1}{2}\begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}, \quad \mathbf{u}_1 = \frac{1}{\sqrt{2}}\begin{bmatrix} \cos(\frac{\pi}{8}) \\ \cos(\frac{3\pi}{8}) \\ \cos(\frac{5\pi}{8}) \\ \cos(\frac{7\pi}{8}) \end{bmatrix}, \quad \mathbf{u}_2 = \frac{1}{\sqrt{2}}\begin{bmatrix} \cos(\frac{\pi}{4}) \\ \cos(\frac{3\pi}{4}) \\ \cos(\frac{5\pi}{4}) \\ \cos(\frac{7\pi}{4}) \end{bmatrix} = \frac{1}{2}\begin{bmatrix} 1 \\ -1 \\ -1 \\ 1 \end{bmatrix}, \quad \mathbf{u}_3 = \frac{1}{\sqrt{2}}\begin{bmatrix} \cos(\frac{3\pi}{8}) \\ \cos(\frac{9\pi}{8}) \\ \cos(\frac{15\pi}{8}) \\ \cos(\frac{21\pi}{8}) \end{bmatrix}.$$

The orthogonality follows by direct verification, for instance,

$$\begin{aligned} \mathbf{u}_1 \cdot \mathbf{u}_2 &= \frac{1}{\sqrt{2}} \cos\frac{\pi}{8} \times \frac{1}{\sqrt{2}} \cos\frac{\pi}{4} + \frac{1}{\sqrt{2}} \cos\frac{3\pi}{8} \times \frac{1}{\sqrt{2}} \cos\frac{3\pi}{4} \\ &+ \frac{1}{\sqrt{2}} \cos\frac{5\pi}{8} \times \frac{1}{\sqrt{2}} \cos\frac{5\pi}{4} + \frac{1}{\sqrt{2}} \cos\frac{7\pi}{8} \times \frac{1}{\sqrt{2}} \cos\frac{7\pi}{4} = 0, \end{aligned}$$

and the same for the other dot products, e.g. $\mathbf{u}_0 \cdot \mathbf{u}_2 = 0$. To prove the general n case, one uses the trigonometric identity. Let m, k and n be integers, then one has

$$\sum_{\ell=1}^{n} \cos\left(\frac{\pi}{2n}(2\ell-1)k\right) \cos\left(\frac{\pi}{2n}(2\ell-1)m\right) = \begin{cases} n, & \text{for } k = m = 0, \\ \frac{n}{2}, & \text{for } k = m \neq 0, \\ 0, & \text{otherwise.} \end{cases}$$

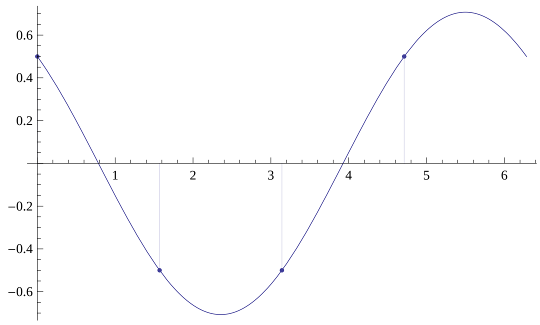
To prove this, one re-writes the above as

$$\frac{1}{2}\sum_{\ell=1}^{n}\left(\cos\frac{\pi}{2n}(2\ell-1)(k-m)+\cos\frac{\pi}{2n}(2\ell-1)(k+m)\right).$$

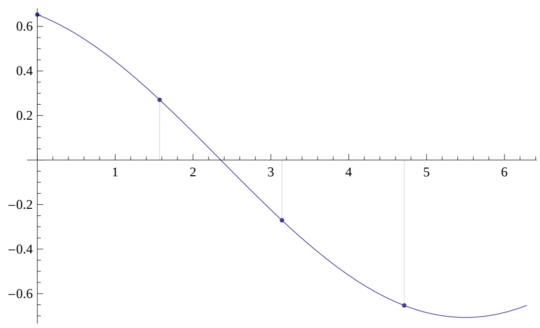
If $k \neq m$, then these functions are cosines (graph it!) and their sums over an interval number of half-cycles are zero. On the other hand, if $k = m \neq 0$, then the sum of the second term evaluates to zero (since it is a cosine) while the first term is $\cos(0) = 1$ and summed over n terms and divided by two so that the end product is $\frac{n}{2}$. Finally, if $k = m = 0$, the sum is n. This proves orthonormality.
(2) Since $\mathcal{C}$ is a basis of $\mathbb{R}^n$, we can write each vector v as a linear combination of $\mathbf{u}_0, \ldots \mathbf{u}_{n-1}$. Observe that The $\mathbf{u}_k$ is a sample of n points on the graph of the function

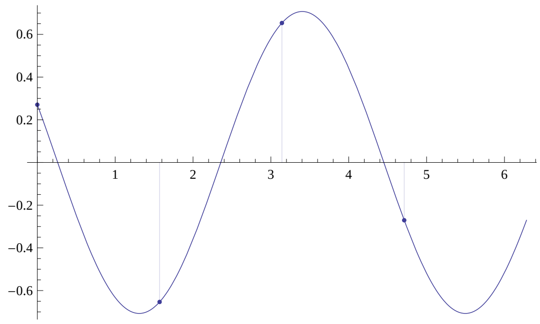
$$f_{k,n}(x) := \sqrt{\frac{2}{n}}\cos(\frac{k}{2}x+\frac{k\pi}{2n}).$$

Indeed, the entries of $\mathbf{u}_k$ are the values of $f_{k,n}$ at $0, \frac{2\pi}{n}, \ldots, \frac{(n-1)2\pi}{n}$. Thus we can think of the vector $\mathbf{u}_k$ as an approximation of the cosine function fk, n. For example, consider $n = 4$ and $k = 2$. Then the vector $\mathbf{u}_2$ contains value of $f_{2,4}$ evaluated at $0, \pi/2, \pi$ and $3/2\pi$. Below is the graph of $f_{2,4}$ including the four data points.



Observe that when we vary k, we get a wave with a different frequency. Here are the graphs for $k = 1$ and $k = 3$:





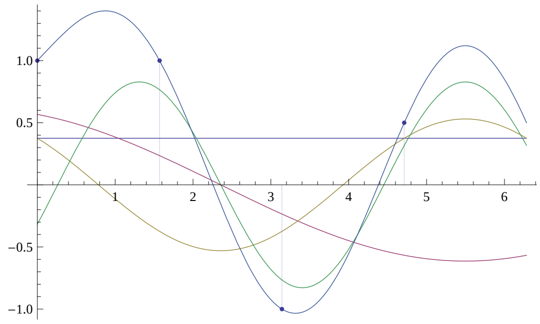
Let us express a vector $\mathbf{v} = \begin{bmatrix} 1 \\ 1 \\ -1 \\ 0.5 \end{bmatrix}$ in $\mathbb{R}^4$ as a linear combination of $\mathbf{u}_0, \ldots \mathbf{u}_4$. One can calculate that

$$\begin{bmatrix} 1 \\ 1 \\ -1 \\ 0.5 \end{bmatrix} = 0.75\mathbf{u}_0 - 0.867837\mathbf{u}_1 + 0.75\mathbf{u}_2 - 1.17126\mathbf{u_3}.$$

Set

$$f_{\mathbf{v}}(x) = 0.75f_{0,4} - 0.867837f_{1,4} + 0.75f_{2,4} - 1.17126f_{3,4}.$$

Then the entries of $\mathbf{v}$ are precisely the values of $f_{\mathbf{v}}$ at $0, \frac{\pi}{2}, \pi, \frac{3\pi}{2}$. In the graph below, the blue curve is $f_{\mathbf{v}}$. The other four curves are $0.75f_{0,4}$, $-0.867837f_{1,4}$ and $0.75f_{2,4}$ and $-1.17126f_{3,4}$.



(3) Since $\mathcal{C}$ is orthonormal, $I_{\mathcal{C},\mathcal{E}}$ is simply C^T. Thus multiplying a vector $\mathbf{v}$ in $\mathbb{R}^n$ by C^T, calculates $\mathbf{v}_{\mathcal{C}}$. Thus if $\mathbf{v}_{\mathcal{C}} = \begin{bmatrix} z_0 \\ \vdots \\ z_{n-1} \end{bmatrix}$, then

$$\mathbf{v} = z_0\mathbf{u}_0 + \cdots + z_{n-1}\mathbf{u}_{n-1}.$$

Now think of $\mathbf{v}$ as an audio signal. So each entry of $\mathbf{v}$ corresponds to the amplitude of the audio signal at a given time. As observed above each $\mathbf{u_i}$ corresponds to cosine wave of a different frequency. Thus each z_i tells how much a signal of the corresponding frequency contributes to $\mathbf{v}$.

(4) Start with $\mathbf{v}$. Calculate $\mathbf{v}_{\mathcal{C}}$. Drop entries corresponding to higher frequency waves that are smaller than given threshold ε. Or: if you are willing to drop higher frequency components all together, take $m < n$ and project $\mathbf{v}$ onto $\mathrm{Span}(\mathbf{u}_0, \ldots, \mathbf{u}_m)$. (There are more tricks to it, but this is the basic mathematical idea behind it. See the Wikipedia article on data compression for more information.) $\square$