I — Congruence and Similarity

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Congruence

We say that triangles ABC and XYZ are *congruent* if

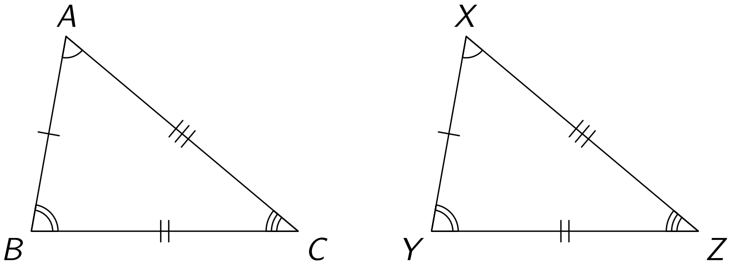
$$BC = YZ,\ CA = ZX,\ AB = XY,\ \angle BAC = \angle YXZ,$$

$$\angle ABC = \angle XYZ,\ \angle ACB = \angle XZY.$$

If all of these conditions are true then we can write

$$\triangle ABC \cong \triangle XYZ.$$

We also know that $|ABC| = |XYZ|$.



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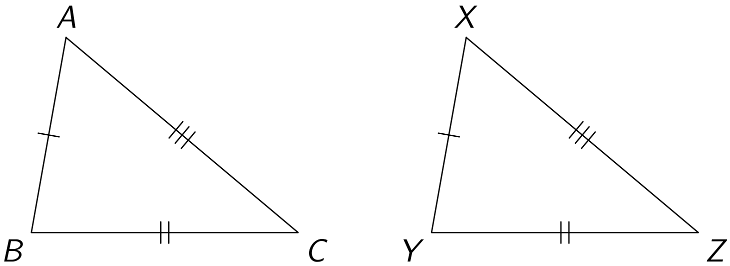
- For instance, we've just shown that if two triangles have the same area, a side of the same length, and the same angle opposite that side then they must be congruent
- Though I have seen that test used in a problem before, what follows are the canonical congruence tests that are more likely to come up

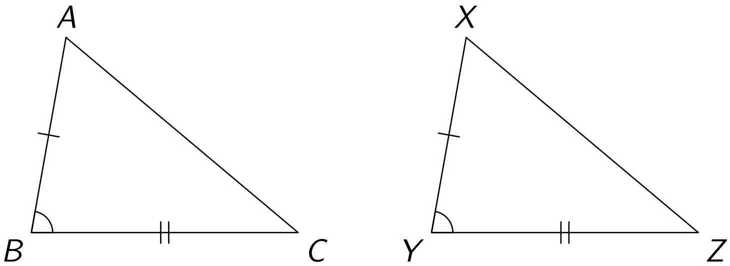
SSS

If $BC = YZ, \; CA = ZX, \; AB = XY$ then $\triangle ABC \cong \triangle XYZ$.



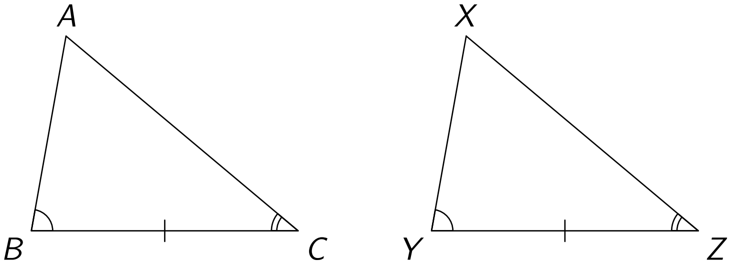
SAS

If $AB = XY,\ \angle ABC = \angle XYZ,\ BC = YZ$ then $\triangle ABC \cong \triangle XYZ$.



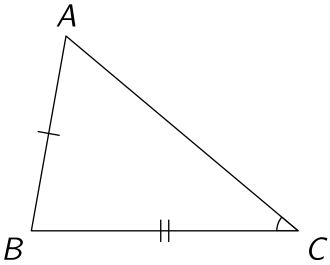
AAS

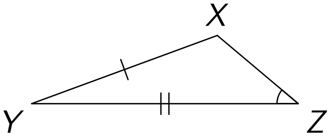
If $\angle ABC = \angle XYZ,\ \angle BCA = \angle YZX,\ BC = YZ$ then $\triangle ABC \cong \triangle XYZ$.



SSA?

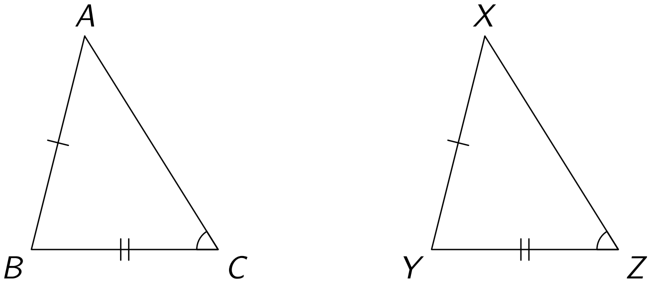
If $AB = XY$, $BC = YZ$, $\angle BCA = \angle YZX$ then you don't necessarily know $\triangle ABC \cong \triangle XYZ$.





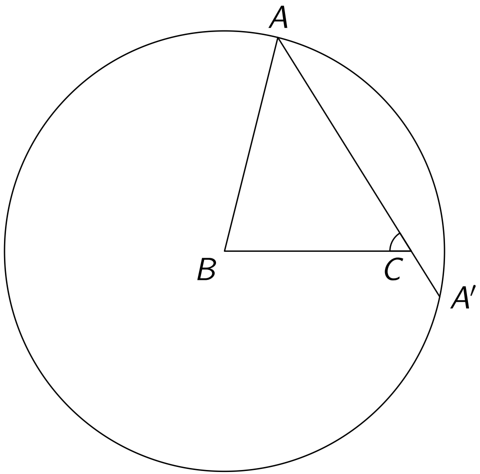
Fixed SSA

If $AB = XY,\ BC = YZ,\ \angle BCA = \angle YZX$ **and** $AB > BC$ then $\triangle ABC \cong \triangle XYZ$.



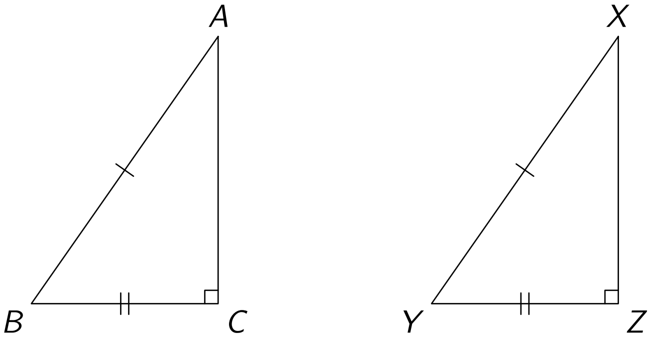
Fixed SSA

Why does this work? The other intersection is on the wrong side of BC.



RHS

This is a special case of fixed SSA.
If $AB = XY, \ BC = YZ, \ \angle BCA = \angle YZX = 90^\circ$ then $\triangle ABC \cong \triangle XYZ$.



Similarity

We say that triangles ABC and XYZ are *similar* if

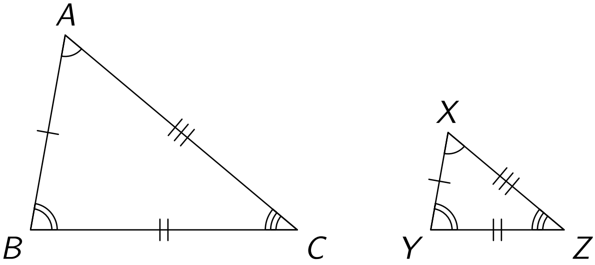
$$\frac{BC}{YZ} = \frac{CA}{ZX} = \frac{AB}{XY} = r,$$

$$\angle ABC = \angle XYZ, \ \angle BCA = \angle YZX, \ \angle CAB = \angle ZXY.$$

If all of these conditions are true then we can write

$$\triangle ABC \sim \triangle XYZ.$$

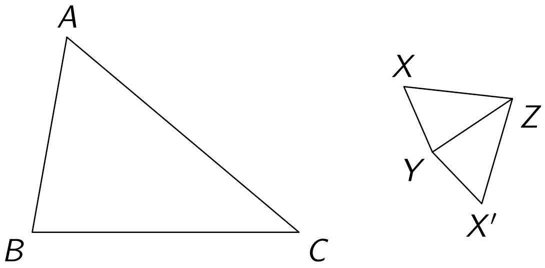
We also know that $\frac{|ABC|}{|XYZ|} = r^2$.



Similarity

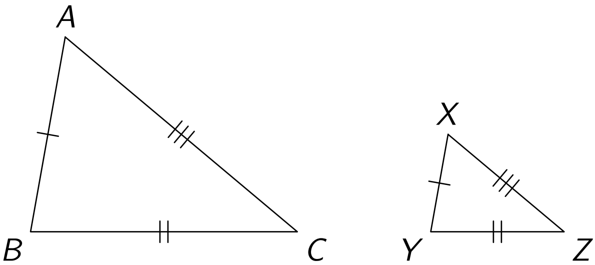
Triangles can be directly similar or oppositely similar. For example, $\triangle ABC$ is directly similar to $\triangle XYZ$ and oppositely similar to $\triangle X'YZ$.

We write: $\triangle ABC \overset{+}{\sim} \triangle XYZ$, $\triangle ABC \overset{-}{\sim} \triangle X'YZ$.



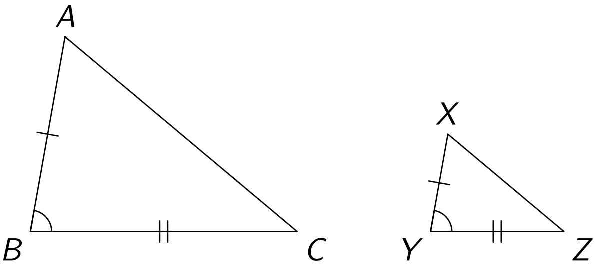
PPP

If $\frac{BC}{YZ} = \frac{CA}{ZX} = \frac{AB}{XY}$ then $\triangle ABC \sim \triangle XYZ$.



PAP

If $\frac{AB}{XY} = \frac{BC}{YZ}$, $\angle ABC = \angle XYZ$ then $\triangle ABC \sim \triangle XYZ$.



AA

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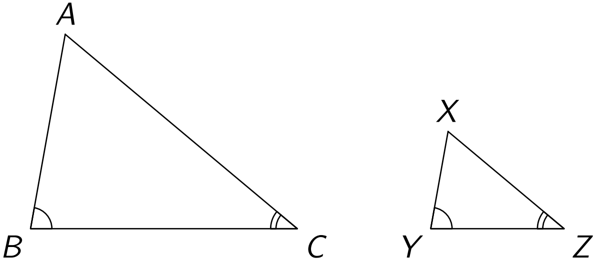

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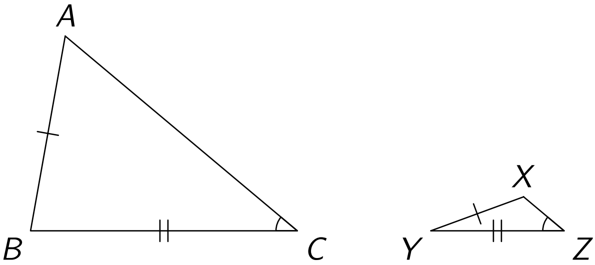

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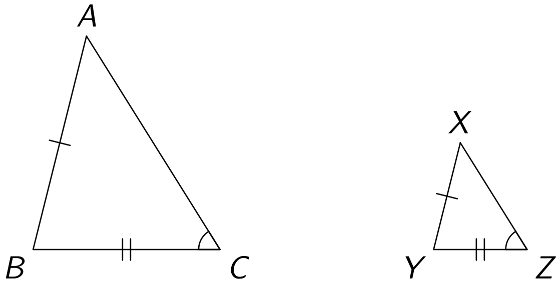

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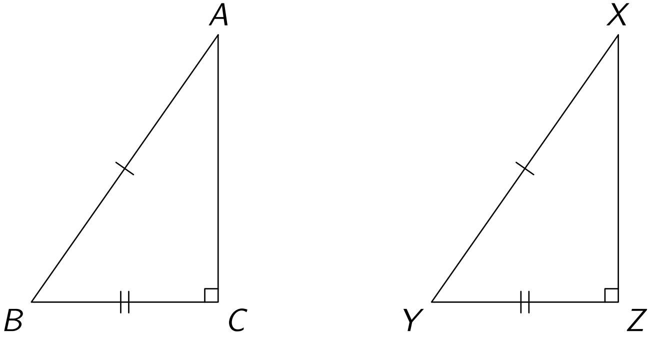

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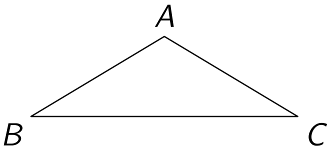

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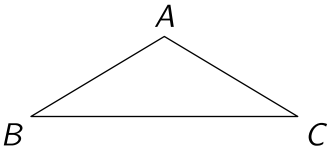

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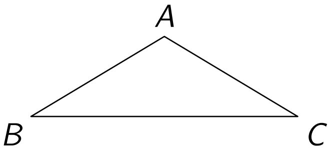

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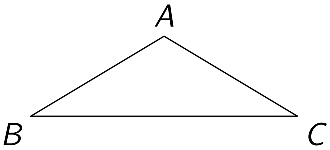

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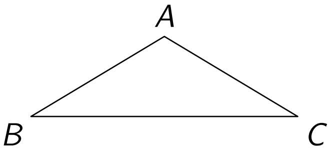

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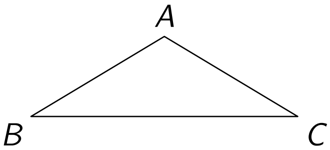

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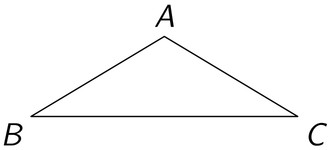

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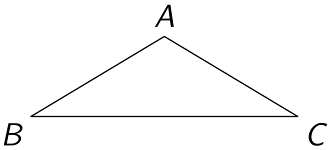

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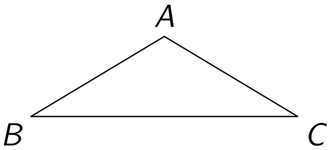

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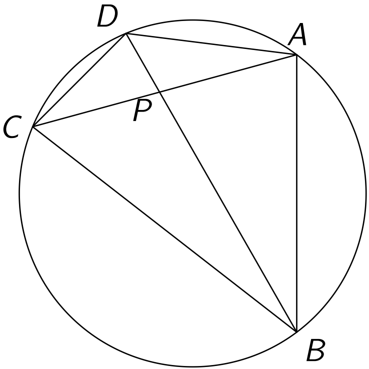

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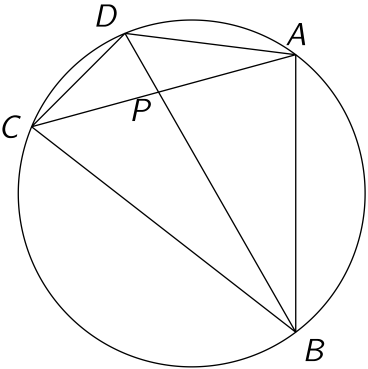

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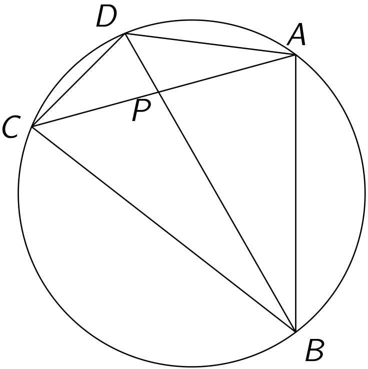

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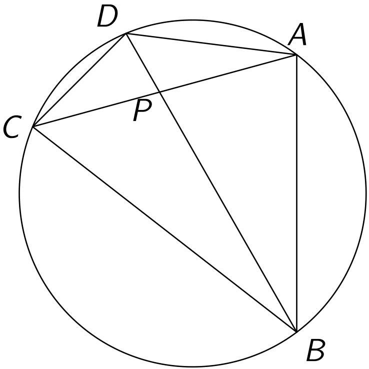

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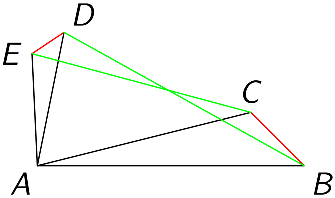
Similar switch

Let ABC and ADE be triangles that are directly similar. Prove that $\triangle ABD$ and $\triangle ACE$ are also directly similar.



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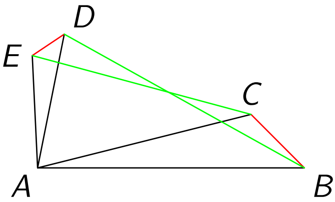

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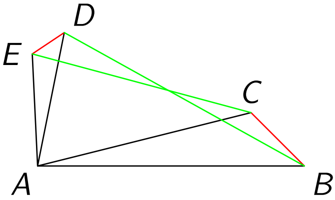

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Ice cream cone

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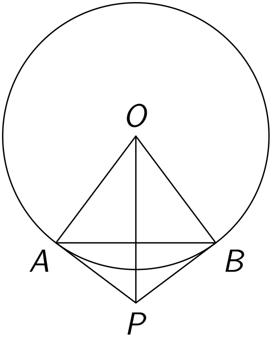
Construction: let O be centre of the circle.



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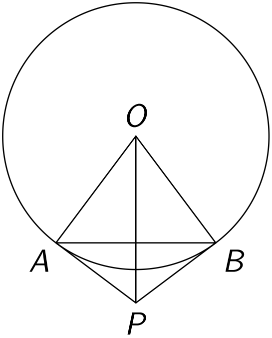

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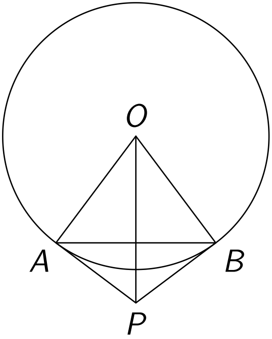
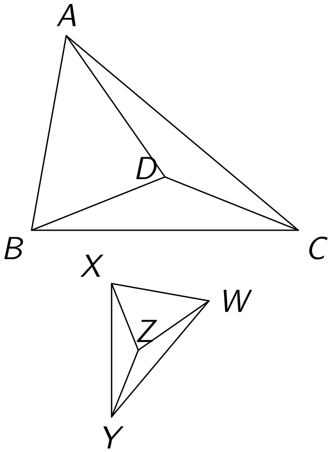


- $\triangle PAO \cong \triangle PBO$
- $PA = PB$

Similar figures

Let points $ABCDWXYZ$ be such that $\triangle ABC \overset{+}{\sim} \triangle WXY$ and $\triangle BCD \overset{+}{\sim} \triangle XYZ$. Prove that $\triangle ABD \sim \triangle WXZ$ and $\triangle ACD \sim \triangle WYZ$.



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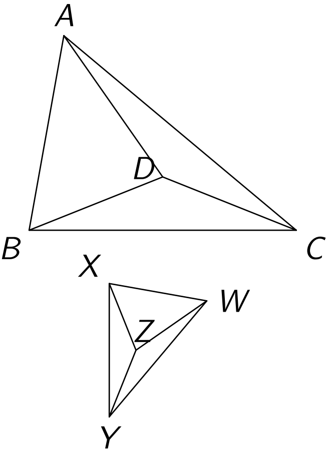

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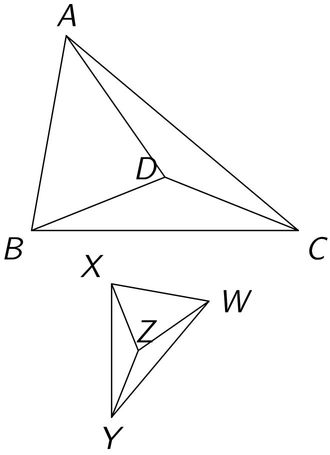

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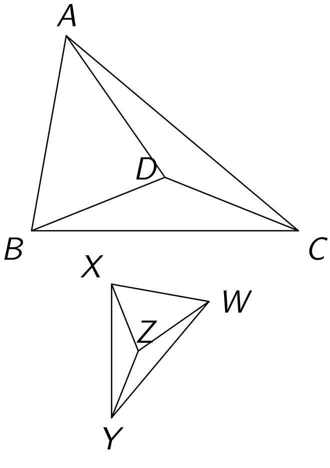

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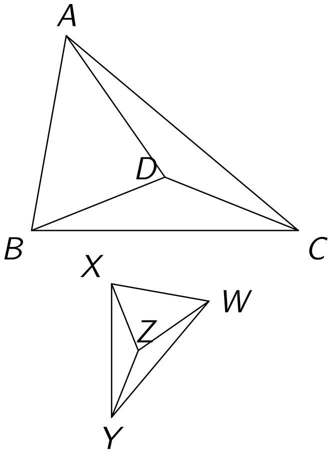

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All of this still works if the triangles are instead oppositely similar.

Menelaus

Let ABC be a triangle. Let X, Y, Z be collinear points on sides BC, CA, AB respectively.
Prove that

$$\frac{AZ}{ZB} \times \frac{BX}{XC} \times \frac{CY}{YA} = -1.$$

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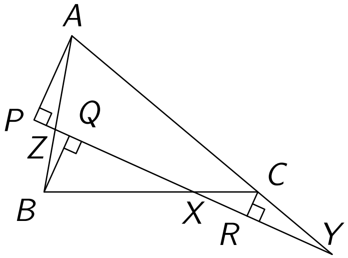
Construction: let P, Q, R be the bases of the perpendiculars from A, B, C to XYZ.



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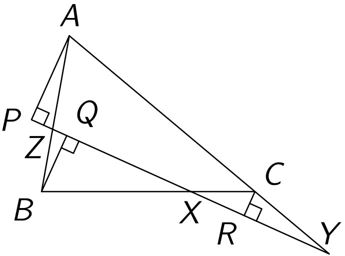

- $\frac{BX}{XC} = \frac{BQ}{CR}$

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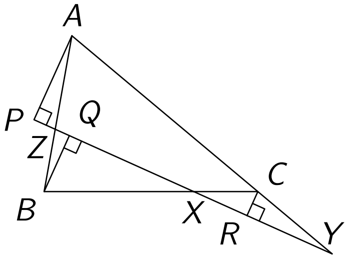

- $\frac{BX}{XC} = \frac{BQ}{CR}$
- $\frac{AZ}{ZB} \times \frac{BX}{XC} \times \frac{CY}{YA} = 1$

Menelaus

Let ABC be a triangle. Let X, Y, Z be collinear points on sides BC, CA, AB respectively.
Prove that

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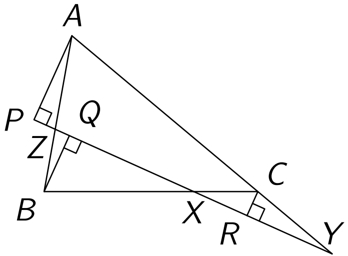

- $\frac{BX}{XC} = \frac{BQ}{CR}$
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- Why is it -1?

Homothetic triangles

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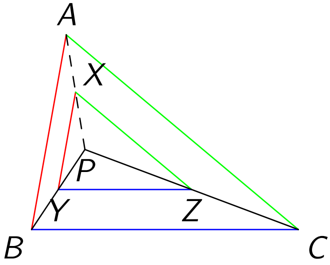

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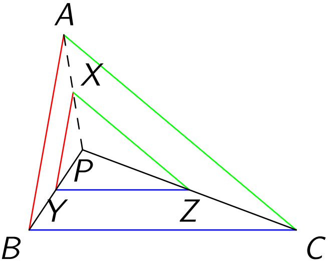

- $ABCP \sim XYZP$

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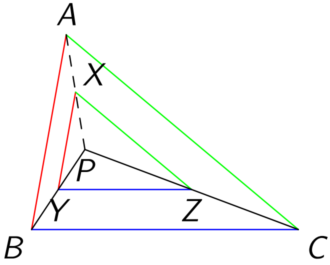

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Let ABC be a triangle such that $\angle BAC = 90^\circ$. Prove that $AB^2 + AC^2 = BC^2$.

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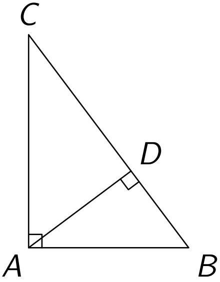

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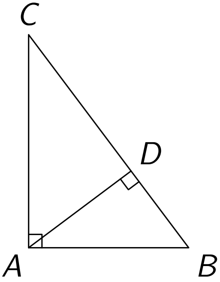

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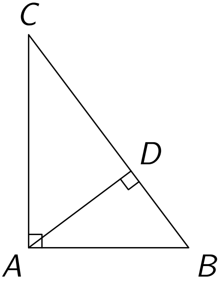

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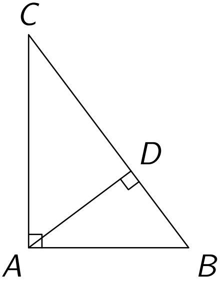

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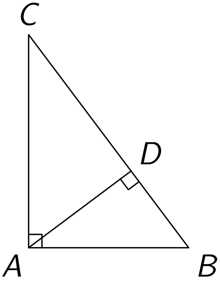

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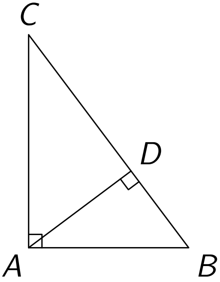

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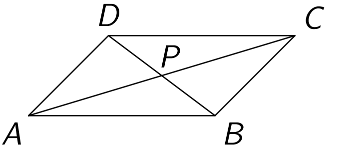
Diagonals of a parallelogram

Let $ABCD$ be a parallelogram with $AB||CD$ and $BC||DA$. Let AC intersect BD at P, Prove that P is the midpoint of AC and of BD.



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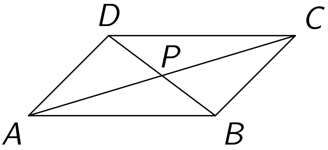

- $\triangle ABC \cong \triangle CDB$

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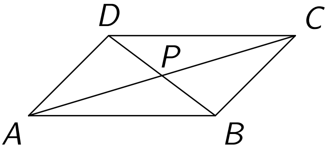

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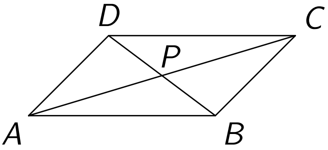

- $\triangle ABC \cong \triangle CDB$
- $\triangle ABP \cong \triangle CDP$
- $PA = PC, \; PB = PD$

Alternate segment switch

Let A, B, C be collinear points, and let D, E be points such that $AD = AE$. Prove that if $\angle ADB = \angle ACD$ then $\angle AEB = \angle ACE$.

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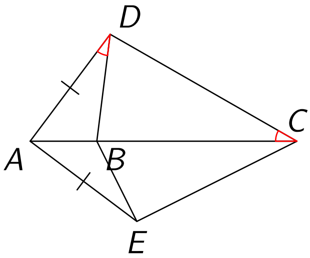

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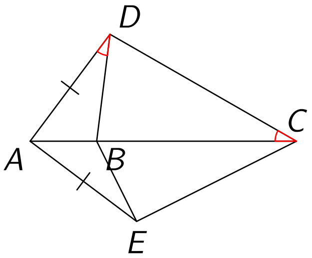

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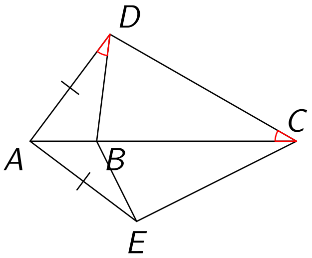

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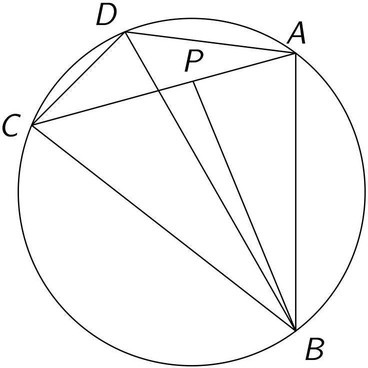

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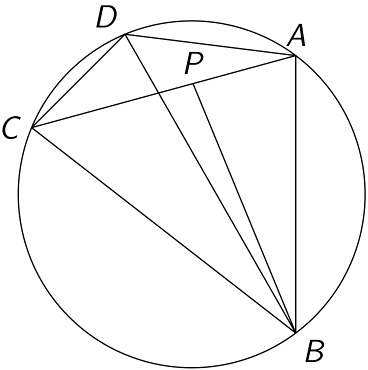

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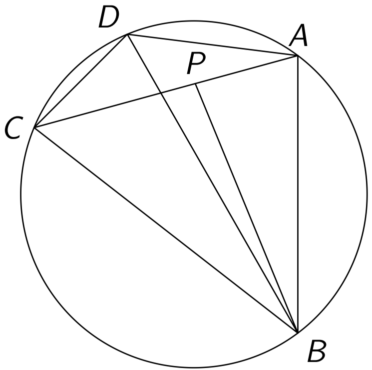

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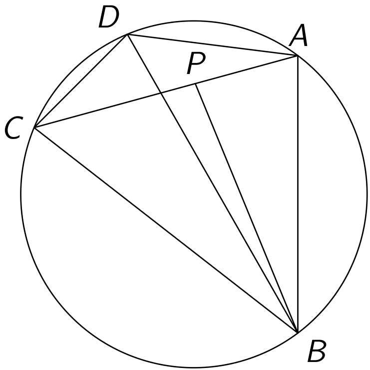

- $\triangle ABP \sim \triangle DBC$
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- $AB \times CD + BC \times AD = AC \times BD$

Diameter of the incircle

Let ABC be a triangle with incentre I and A-excentre I_A. Let the incircle touch BC at D and the A-excircle touch BC at E. Let P be the reflection of D over I.

Prove that P is on AE.

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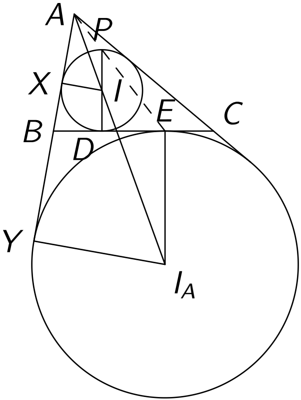

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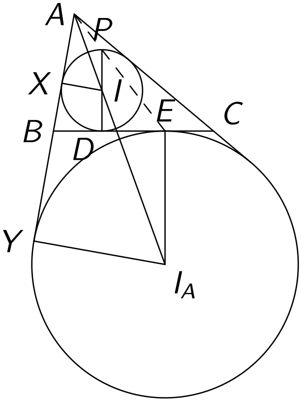

- $AXIP \sim AYI_AE$

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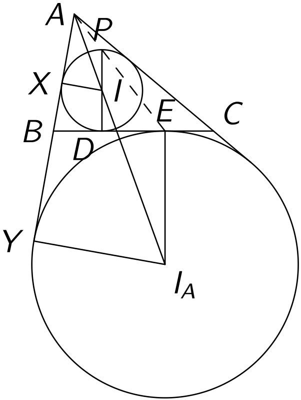

- $AXIP \sim AYI_AE$
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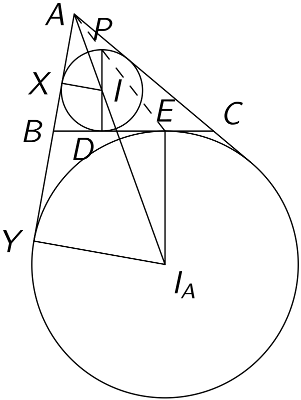

- $AXIP \sim AYI_AE$
- A, P, E collinear
- Alternatively, triangles PIX and EI_AY are homothetic

Symmedian

Let ABC be a triangle with circumcircle Γ. Let the tangents to Γ at B and C intersect at P, and let the midpoint of BC be M. Prove that $\angle PAB = \angle MAC$.

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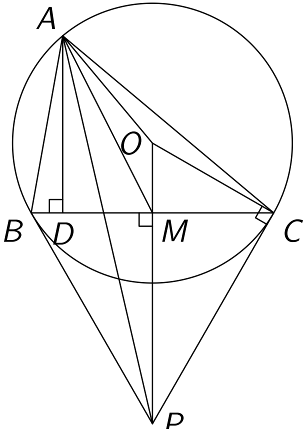

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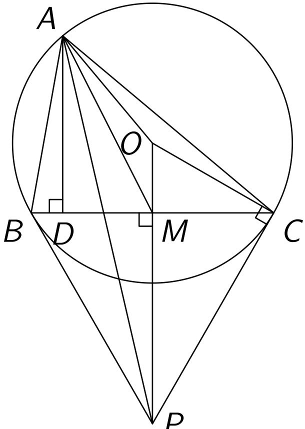

- $\triangle OAM \sim \triangle OPA$

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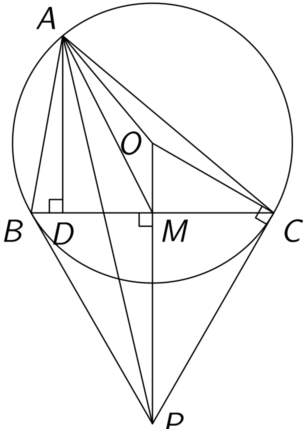

- $\triangle OAM \sim \triangle OPA$
- $\angle OAM = \angle DAP$

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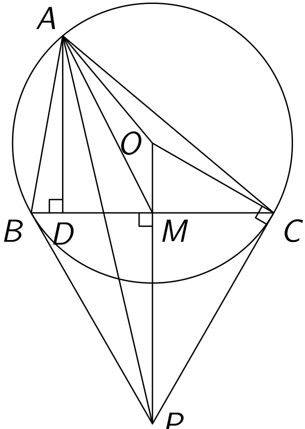

- $\triangle OAM \sim \triangle OPA$
- $\angle OAM = \angle DAP$
- $\angle BAD = \angle OAC$

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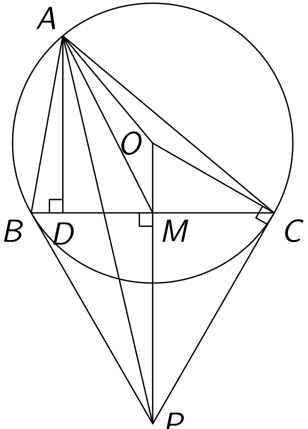

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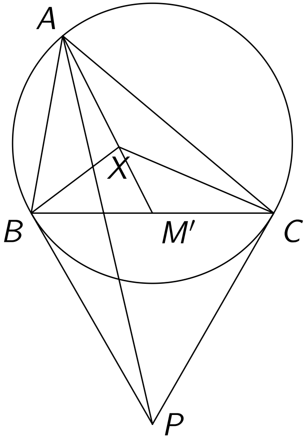

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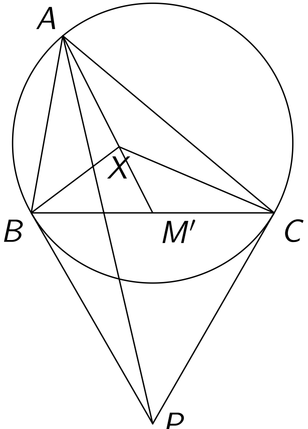

- $\angle M'BA = \angle M'XB$

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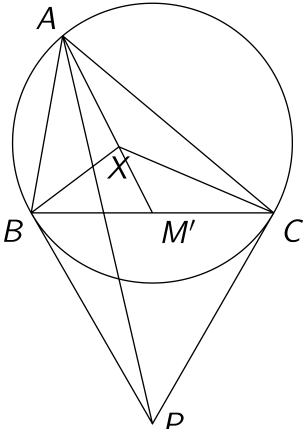

- $\angle M'BA = \angle M'XB$
- $M'B^2 = M'A \times M'X$

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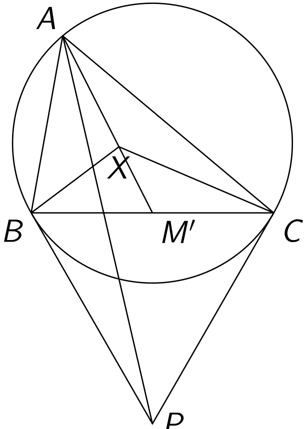

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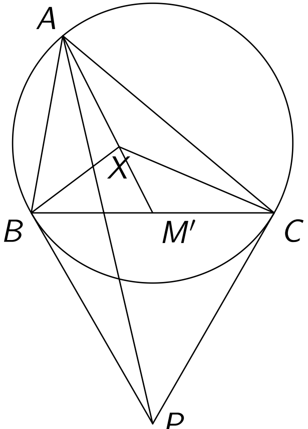

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Harmonic quad

Let ABC be a triangle. Let Q be a point such that the circumcircle of AQB is tangent to AC and the circumcircle of AQC is tangent to AB. Let D be the reflection of A over Q. Prove that $ABCD$ is cyclic and $AB \times CD = BD \times AC$.

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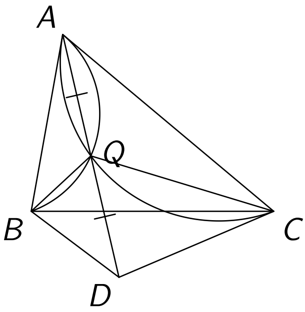

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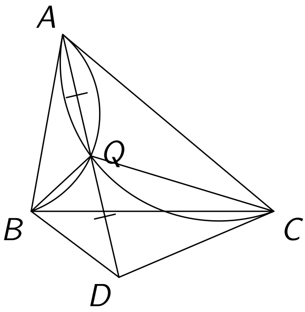

- $\triangle DQB \sim \triangle CQD$

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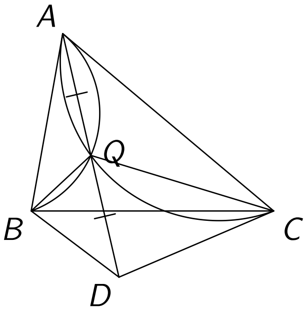

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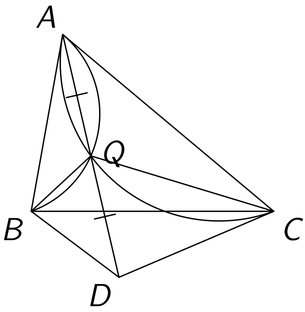

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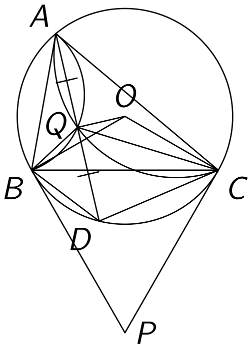

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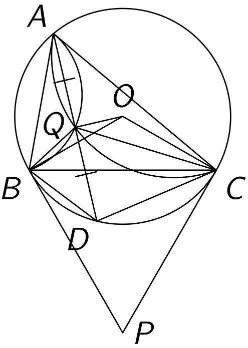

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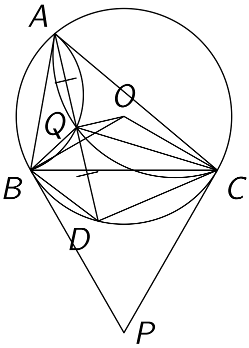

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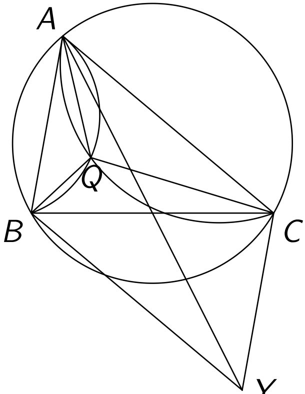

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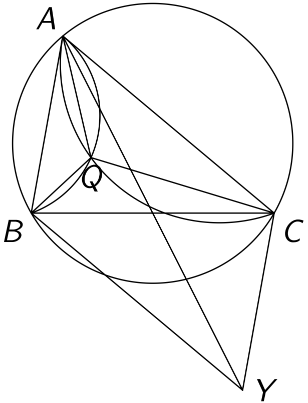

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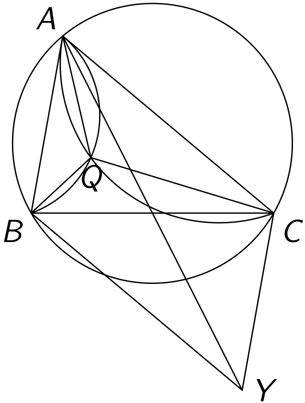

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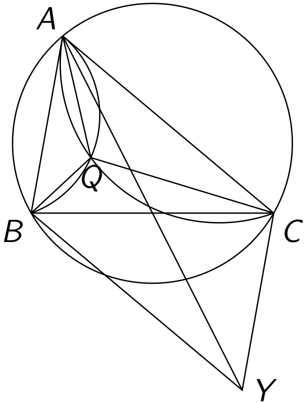

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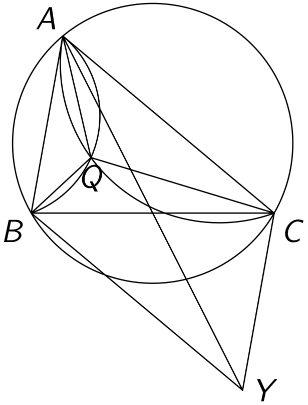

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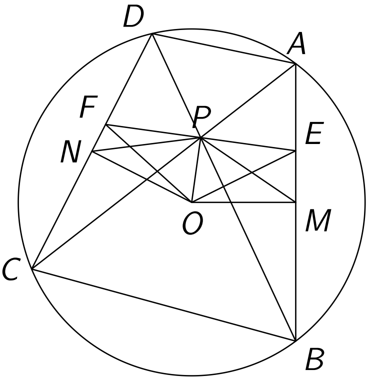

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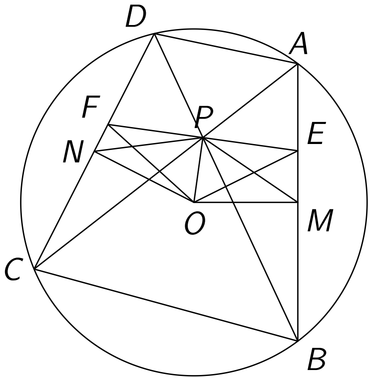

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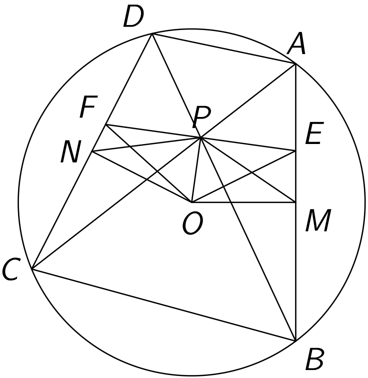

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Generalised Movie Theorem

Let ABC and PQR be directly similar triangles. If X, Y, Z are points such that triangles APX, BQY, CRZ are directly similar, then prove that $\triangle XYZ$ is also directly similar to triangles ABC and PQR.

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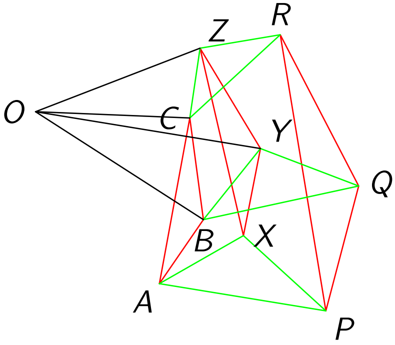

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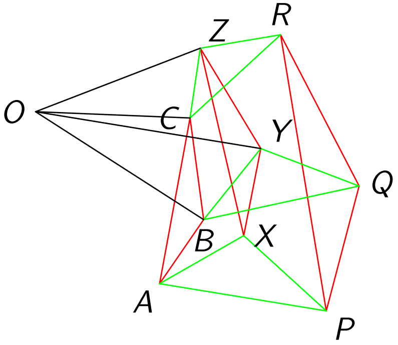

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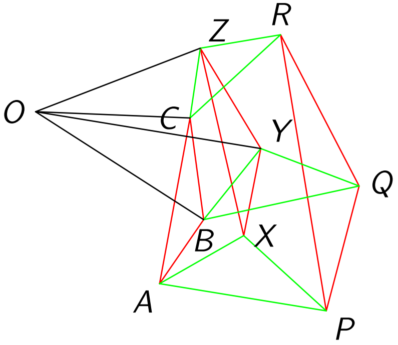

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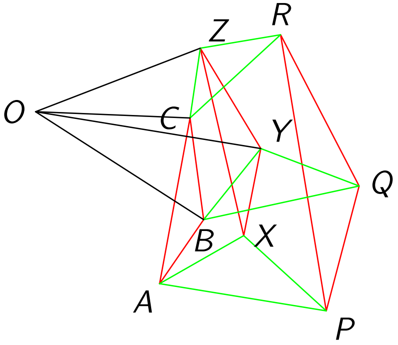

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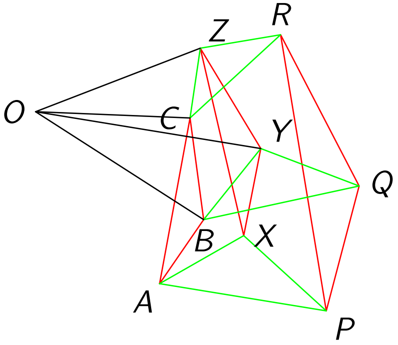

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Note: O is the centre of each of the *spiral similarities* sending a coloured triangle to another of the same colour.