Assorted Number Theory (S)

Andres Buritica

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1. Let a and b be positive integers, and let p be a prime greater than 3. Prove that
$$\binom{ap}{bp} \equiv \binom{a}{b} \pmod{p^3}.$$

2. Let n be a fixed positive integer. Let $\frac{a_1}{b_1}, \ldots, \frac{a_k}{b_k}$ be the rational numbers between 0 and 1 inclusive with denominators at most n, written in increasing order and lowest terms.
   (a) Prove that for each i, $a_{i+1}b_i - a_i b_{i+1} = 1$.
   (b) Prove that the rational number x with smallest denominator such that $\frac{a_i}{b_i} < x < \frac{a_{i+1}}{b_{i+1}}$ is $\frac{a_i + a_{i+1}}{b_i + b_{i+1}}$.
   (c) Which pairs of numbers appear as consecutive b_is?

3. Let p be a prime. What is the sum of all the generators mod p?

4. Let n be a positive integer larger than 1.
   (a) Prove that the product of all primes between $\lceil \frac{n}{2} \rceil$ and n (not including $\lceil \frac{n}{2} \rceil$) is less than 2^n.
   (b) Prove that the product of all primes between 1 and n is at most 4^{n-1}.
   (c) Find some real number c independent of n such that there are at most $\frac{cn}{\log_2 n}$ primes that are at most n.

5. Let n be a positive integer larger than $2^{2^{2^2}}$.
   (a) Let p be a prime.
      - Prove that if $p^k \mid \binom{2n}{n}$ then $p^k < 2n$.
      - Prove that if $2p \leq 2n < 3p$ then $p \nmid \binom{2n}{n}$.
   (b) Prove that
$$\prod_{\substack{p^k \| \binom{2n}{n} \\ p \leq n}} p^k < \binom{2n}{n}.$$
   (c) Find some real number c independent of n such that there are at least $\frac{cn}{\log_2 n}$ primes that are at most n.