I — Congruence and Similarity

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1 Intro

1.1 Congruence

- We say that triangles ABC and XYZ are *congruent* if
$$BC = YZ,\ CA = ZX,\ AB = XY,\ \angle BAC = \angle YXZ,$$
$$\angle ABC = \angle XYZ,\ \angle ACB = \angle XZY.$$
- If all of these conditions are true then we can write
$$\triangle ABC \cong \triangle XYZ.$$
We also know that $|ABC| = |XYZ|$.
- (SSS) If $BC = YZ,\ CA = ZX,\ AB = XY$ then $\triangle ABC \cong \triangle XYZ$.
- (SAS) If $AB = XY,\ \angle ABC = \angle XYZ,\ BC = YZ$ then $\triangle ABC \cong \triangle XYZ$.
- (AAS) If $\angle ABC = \angle XYZ,\ \angle BCA = \angle YZX,\ BC = YZ$ then $\triangle ABC \cong \triangle XYZ$.
- (SSA doesn't work) If $AB = XY,\ BC = YZ,\ \angle BCA = \angle YZX$ then you don't necessarily know $\triangle ABC \cong \triangle XYZ$.
- (fixed SSA) If $AB = XY,\ BC = YZ,\ \angle BCA = \angle YZX$ **and** $AB > BC$ then $\triangle ABC \cong \triangle XYZ$.
- (RHS) If $AB = XY,\ BC = YZ,\ \angle BCA = \angle YZX = 90^\circ$ then $\triangle ABC \cong \triangle XYZ$.

1.2 Similarity

- We say that triangles ABC and XYZ are *similar* if
$$\frac{BC}{YZ} = \frac{CA}{ZX} = \frac{AB}{XY} = r,$$
$$\angle ABC = \angle XYZ,\ \angle BCA = \angle YZX,\ \angle CAB = \angle ZXY.$$
- If all of these conditions are true then we can write
$$\triangle ABC \sim \triangle XYZ.$$
We also know that $\frac{|ABC|}{|XYZ|} = r^2$.
- Triangles can be directly similar ($\overset{+}{\sim}$) or oppositely similar ($\overset{-}{\sim}$).
- (PPP) If $\frac{BC}{YZ} = \frac{CA}{ZX} = \frac{AB}{XY}$ then $\triangle ABC \sim \triangle XYZ$.
- (PAP) If $\frac{AB}{XY} = \frac{BC}{YZ},\ \angle ABC = \angle XYZ$ then $\triangle ABC \sim \triangle XYZ$.
- (AA) If $\angle ABC = \angle XYZ,\ \angle BCA = \angle YZX$ then $\triangle ABC \sim \triangle XYZ$.
- (PPA doesn't work) If $\frac{AB}{XY} = \frac{BC}{YZ},\ \angle BCA = \angle YZX$ then you don't necessarily know $\triangle ABC \sim \triangle XYZ$.
- (fixed PPA) If $\frac{AB}{XY} = \frac{BC}{YZ},\ \angle BCA = \angle YZX$ **and** $AB > BC$ then $\triangle ABC \sim \triangle XYZ$.
- (RHS) If $\frac{AB}{XY} = \frac{BC}{YZ},\ \angle BCA = \angle YZX = 90^\circ$ then $\triangle ABC \sim \triangle XYZ$.

2 Lemmas

All of these are "well-known" diagrams that you should learn to recognise as subconfigurations of problems. However, the ideas used in their proofs are at least as important as the configurations themselves.

1. (Isosceles triangle) Let ABC be a triangle. Prove that $AB = AC$ if and only if $\angle ABC = \angle ACB$.

2. (Power of a point) Let $ABCD$ be a cyclic quadrilateral, and let AC and BD meet at P. Prove that $PA \times PC = PB \times PD$.

3. (Similar switch) Let ABC and ADE be triangles that are directly similar. Prove that $\triangle ABD$ and $\triangle ACE$ are also directly similar.

4. (Ice cream cone theorem) Let PA and PB be tangents to a circle. Prove that $PA = PB$.

5. (Similar figures) Let points $ABCDWXYZ$ be such that $\triangle ABC \overset{+}{\sim} \triangle WXY$ and $\triangle BCD \overset{+}{\sim} \triangle XYZ$. Prove that $\triangle ABD \sim \triangle WXZ$ and $\triangle ACD \sim \triangle WYZ$.

   In this case, we can say that $ABCD \overset{+}{\sim} WXYZ$.

6. (Menelaus' Theorem) Let ABC be a triangle. Let X, Y, Z be collinear points on sides BC, CA and AB respectively. Prove that
$$\frac{AZ}{ZB} \times \frac{BX}{XC} \times \frac{CY}{YX} = -1.$$

7. (Homothetic triangles) Let ABC and XYZ be two triangles such that $BC||YZ$, $CA||ZX$ and $AB||XY$. Prove that AX, BY and CZ are concurrent.

8. (Pythagoras' Theorem) Let ABC be a triangle such that $\angle BAC = 90°$. Prove that $AB^2 + AC^2 = BC^2$.

9. (Diagonals of a parallelogram) Let $ABCD$ be a parallelogram (that is, $AB||CD$ and $BC||DA$). Let AC intersect BD at P. Prove that P is the midpoint of AC and of BD.

10. (Alternate segment switch) Let A, B, C be collinear points, and let D, E be points such that $AD = AE$. Prove that if $\angle ADB = \angle ACD$ then $\angle AEB = \angle ACE$.

11. (Ptolemy's Theorem) Let $ABCD$ be points in that order around a circle. Prove that $AB \times CD + BC \times AD = AC \times BD$.

12. (Diameter of the incircle) Let ABC be a triangle with incentre I and A-excentre I_A. Let the incircle touch BC at D and the A-excircle touch BC at E. Let P be the reflection of D over I. Prove that P is on AE.

13. (Symmedian) Let ABC be a triangle with circumcircle Γ. Let the tangents to Γ at B and C intersect at P, and let the midpoint of BC be M. Prove that $\angle MAB = \angle PAC$.

14. (Harmonic quadrilateral) Let ABC be a triangle. Let Q be a point such that the circumcircle of AQB is tangent to AC and the circumcircle of AQC is tangent to AB. Let D be the reflection of A over Q. Prove that $ABCD$ is cyclic and $AB \times CD = BD \times AC$.

15. (Symmedian) Let ABC be a triangle. Let P be the intersection of the tangents from B and C to the circumcircle of ABC. Let Q be a point such that the circumcircle of AQB is tangent to AC and the circumcircle of AQC is tangent to AB. Prove that A, P, Q are collinear.

16. (Symmedian) Let ABC be a triangle. Let Q be a point such that the circumcircle of AQB is tangent to AC and the circumcircle of AQC is tangent to AB. Let M be the midpoint of BC. Prove that $\angle BAQ = \angle MAC$.

17. (Butterfly Theorem) Let $ABCD$ be a cyclic quadrilateral with circumcentre O, and let AC and BD intersect at P. A line through P intersects AB and CD at E and F, such that OP is perpendicular to EF. Prove that P is the midpoint of EF.

18. (Generalised Movie Theorem) Let ABC and PQR be directly similar triangles. If X, Y, Z are points such that triangles APX, BQY and CRZ are directly similar, then prove that $\triangle XYZ$ is also directly similar to triangles ABC and PQR.

3 Problems

Many of these are stolen from PST, but if you're inters you can't have done that much of PST so it's fine right?

1. Let ABC be a triangle with circumcentre O, orthocentre H, centroid G and nine-point centre N. Prove that H, G, N, O are collinear with $OG : GN : NH = 2 : 1 : 3$.

2. Let ABC be a triangle with $AB < AC$, and let P be a point on segment AC such that $AB = CP$. Prove that the perpendicular bisectors of BC and AP meet on the circumcircle of ABC.

3. Let $ABCDE$ be a convex pentagon such that $\angle BAC = \angle CAD = \angle DAE$ and $\angle ABC = \angle ACD = \angle ADE$. The diagonals BD and CE meet at P. Prove that the line AP passes through the midpoint of CD.

4. Let ABC be a triangle with orthocentre H. Prove that the circumcentres of triangles BCH, CAH and ABH form a triangle congruent to $\triangle ABC$.

5. Let ABC be a triangle and let P be a point on line BC. Let Q and R be points on lines CA and AB respectively, such that $QP = QC$ and $RP = RB$. Prove that the circumcircle of $\triangle AQR$ passes through the circumcentre of $\triangle ABC$.

6. Let ABC be a triangle with circumcentre O and incentre I. Let R and r be the circumradius and inradius of $\triangle ABC$ respectively. Prove that $R^2 - OI^2 = 2Rr$.

7. Let P be a point inside triangle ABC such that $\angle PBA = \angle PCA$. Let Q and R be the feet of the perpendiculars from P to sides AB and AC, and let M be the midpoint of AB. Prove that $MQ = MR$.

8. Let P and Q be on segment BC of an acute triangle ABC such that $\angle PAB = \angle BCA$ and $\angle CAQ = \angle ABC$. Let M an N be the points on AP and AQ, respectively, such that P is the midpoint of AN and Q is the midpoint of AN. Prove that the intersection of BM and CN is on the circumcircle of triangle ABC.

9. Let ABC be a triangle, and let points M and N be on rays AB and AC respectively such that $AM = AN = BC$. Let O be the point such that $AMON$ is a parallelogram. Prove that if O is the A-excentre of triangle ABC, then $\triangle ABC$ is isosceles.