Geometry Problem Session (I)

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Charles' Lemma: In triangle ABC, D is the second intersection of the angle bisector of $\angle BAC$ with the circumcircle of $\triangle ABC$.

Then, $DB = DC$.

Proof: $\angle BAD = \angle BCD$, $\angle BAD = \angle DAC = \angle DBC$. Therefore $\triangle DBC$ is isosceles.

Other stuff:

- $DI = DB = DC$:

  $$\angle DIB = \angle DAB + \angle ABI = \angle CAD + \angle IBC = \angle CBD + \angle IBC = \angle IBD.$$

- It works in directed angles!

  For D to be on internal bisector, we must have $\angle BAD = \angle DAC$.

  If we make our angles directed, $\angle BAD = \angle DAC \iff D$ is on the internal or external bisector.

- Therefore, the same argument using directed angles yields $EB = EC$.
- The same proof shows that $DB = DI_A$.
- D is the midpoint of II_A.
- E is the midpoint of I_BI_C.
- This is the nine-point circle diagram!
- We can also reverse reconstruct: if D is on two of {internal or external angle bisector of $\angle BAC$, circumcircle of ABC, perpendicular bisector of BC} then D is on the third (unless $ABCD$ is a non-cyclic kite).

Why directed angles are useful:

Let's say the problem was: "Let ABC be a triangle and let $D \neq A$ be a point on the circumcircle of $\triangle ABC$ such that the reflection of B over line AD lies on line AC. Prove that $DB = DC$."

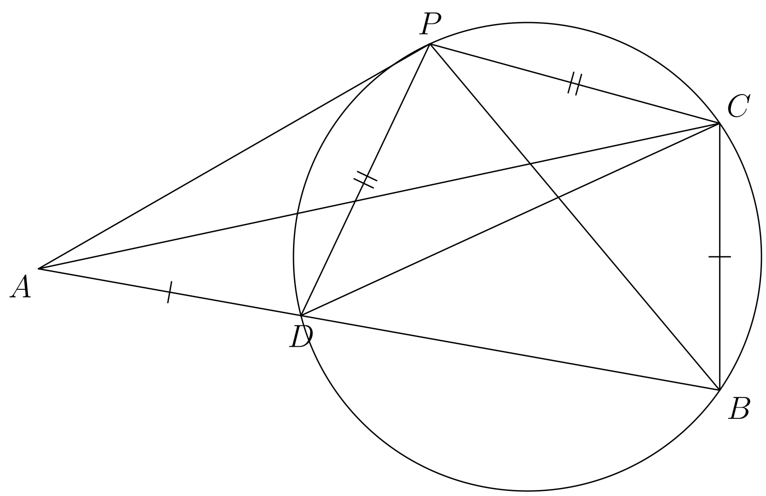
We did the angle chase for D above. The angle chase for E looks like

$$\angle EBC = \angle EAC = 180^\circ - \angle EAB = \angle ECB.$$

With directed angles those two angle chases look exactly the same; in particular, you can write the proof up in half the time.

G3. Triangle ABC satisfies $AB > 2BC$. Let D be the point on side AB such that $AD = BC$. Point P lies on both the bisector of $\angle ABC$ and the perpendicular bisector of CD.

Prove that P lies on the perpendicular bisector of AB.



Lemma: $HA = HG \iff H$ on perp bisector of AG.

Proof: if H lies on perp bisector, then $\triangle HAG \cong \triangle HFG$ (SAS) so $HA = HF$. If $HA = HF$ then $\triangle HAG \cong \triangle HFG$ (SSS) so $\angle HGA = \angle HGF$ so they're both $90°$ and H lies on the perpendicular bisector.

P lies on the angle bisector of $\angle CBD$, and $PC = PD$.

Our diagram suggests that P should be on the circumcircle of $\triangle BDC$.

To prove this, we can reverse reconstruct.

Make P' be the second intersection of the angle bisector of $\angle DBC$ and the circumcircle of $\triangle BCD$.

Charles' Lemma tells us that P' is on the perpendicular bisector of CD.

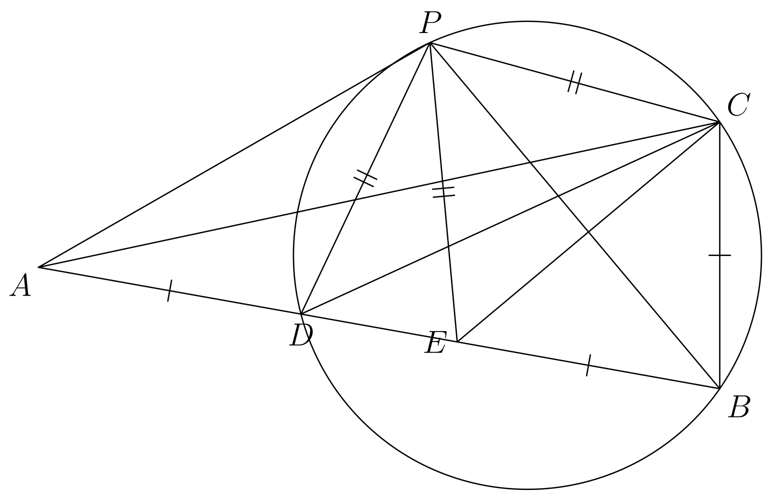
Since P is on the angle bisector of $\angle CBD$ and the perpendicular bisector of CD, we would like to conclude $P = P'$.

Since $AB > 2BC$ we get that $BC < BD$. Since the angle bisector is not perpendicular to CD, we get that $P = P'$ (unique point of intersection).

Solution:

Since $AB > 2BC$ we get that $BC < BD$. Since $BC \neq BD$, P lies on the angle bisector of $\angle CBD$ and the perpendicular bisector of CD, it's well known that P is on the circumcircle of $\triangle BCD$.

So $\angle PCB = \angle PDA$. Since $PC = PD$ and $CB = DA$, we get $\triangle PCB \cong \triangle PDA$ (SAS). This gets us $PB = PA$, which means that P lies on the PB of AB.



Another way to prove P is on the circumcircle:

Construct point E such that $BE = BC$. Important to note that D and E are distinct since $AD + EB = 2BC < AB$.

P is on the bisector of $\angle EBC$ which is the perpendicular bisector of EC.

$PE = PC = PD$. $\triangle PEB \cong \triangle PCB$ (SSS).

To finish the problem from here, note that P is on the PB of DE, which is also the perpendicular bisector of AB.

To show that $PBCD$ is cyclic, we can angle chase

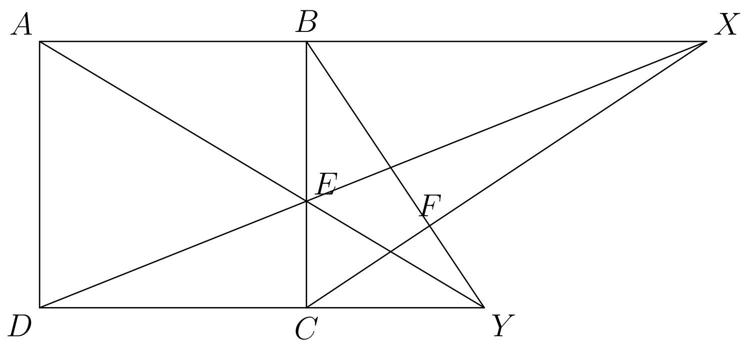
$$\angle PDE = \angle PED = 180^\circ - \angle PEB = 180^\circ - \angle PCB.$$

Or in directed angles:

$$\angle PDE = \angle DEP = \angle BEP = \angle PCB.$$

G4. Let $ABCD$ be a square and let E be a point on the side BC. Let Y be the point where the line AE meets line CD and let X be the point where line DE meets line AB. Let F be the intersection of lines BY and CX.

Show that point E lies on the bisector of angle BFC.



$\triangle AEB \sim \triangle YEC$ (parallel lines) gives $\frac{AB}{CY} = \frac{BE}{EC}$.

But $AB = BC$ so $\frac{AB}{CY} = \frac{BC}{CY} = \frac{BE}{EC}$.

Similarly, we have $\frac{BC}{BX} = \frac{EC}{BE}$.

Flip that to get $\frac{BX}{BC} = \frac{BE}{EC} = \frac{BC}{CY}$.

Since $\angle XBC = \angle BCY = 90°$, we get that $\triangle BXC \sim \triangle CBY$.

Also, $\triangle BFX \sim \triangle YFC$ (AA)

We length chase:

$$\frac{BF}{FC} = \frac{BF}{FY} \times \frac{FY}{FC}$$

$$\begin{aligned}\frac{BF}{FY} &= \frac{BX}{CY} \\ &= \frac{BX}{BC} \times \frac{BC}{CY} \\ &= \left(\frac{BE}{EC}\right)^2 \\ \frac{FY}{FC} &= \frac{FB}{FX} \\ \frac{FY}{FC} &= \frac{FY + FB}{FC + FX} && \text{(addendo)} \\ &= \frac{BY}{CX} \\ &= \frac{CY}{BC} \\ &= \frac{EC}{BE}\end{aligned}$$

So

$$\frac{BF}{FC} = \left(\frac{BE}{EC}\right)^2 \times \frac{EC}{BE}$$
$$= \frac{BE}{EC}.$$

Done by converse of angle bisector theorem.

Addendo:

If we have that $\frac{a}{b} = \frac{c}{d}$, then they're also equal to

$$\frac{a+c}{b+d}.$$

Proof: let $a = kb,\ c = kd$ and so $a + c = k(b + d)$.

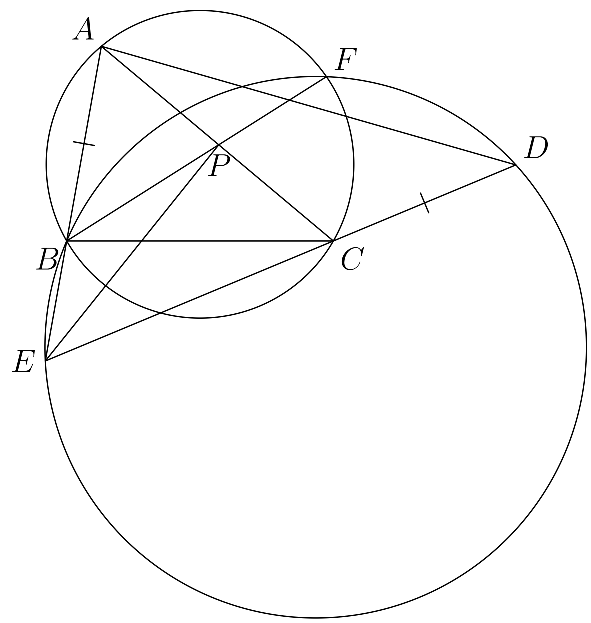
General remark: this problem had many solutions, none of which were particularly easy to find — it did mean, however, that if you persevered with any reasonable approach you were likely to solve the problem.

Additional solution approaches:

- $\triangle BXC \sim \triangle CBY$ implies $\angle BFC = 90°$, so $\triangle BFC \sim \triangle BCY$ so $\frac{BF}{FC} = \frac{BC}{CY} = \frac{BE}{EC}$.
- Construct parallels to BY and CX through E and intersect with AB and CD, then use congruent triangles to prove perpendicularity.
- Construct the centre of the square; prove cyclic (perpendiculars), collinear (Pappus) and angle bisector (Charles).
- Use Pythagoras to find $\frac{BY}{CX}$ instead of similar triangles.
- Prove perpendicularity using $BC^2 + XY^2 = BX^2 + CY^2$.
- Coordinates.

G5. Let $ABCD$ be a convex quadrilateral such that $AB = CD$. The lines AB and CD intersect at the point E, and the circumcircles of the triangles ABC and BDE intersect at B and F. Let P be the intersection of the lines AC and BF.

Prove that EP is the bisector of $\angle AED$.



Motivation:

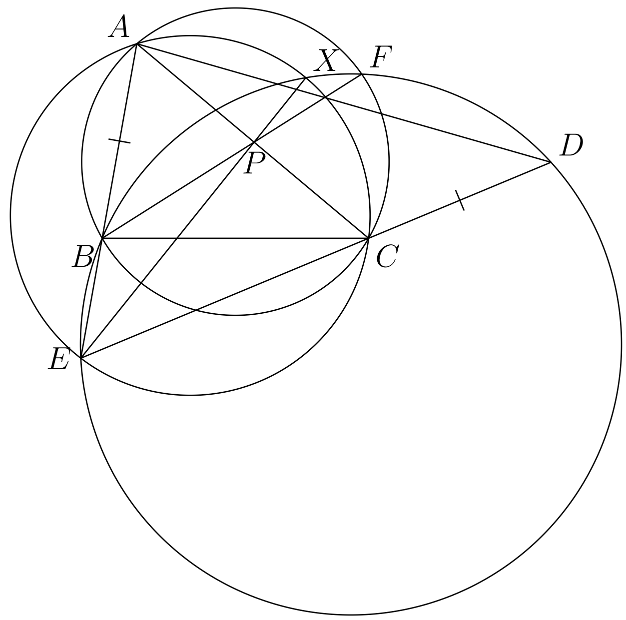
- Miquel point diagram: four circles BDE, ACE, ABZ, CDZ intersect at a point X. X is the centre of the spiral symmetry sending AB to CD, and AC to BD.

  This gives you similar triangles:

  $$\triangle XAB \sim \triangle XCD, \; \triangle XAC \sim \triangle XBD.$$

  We have equal lengths $AB = CD$. Triangles XAB and XCD are now congruent. This means that $XA = XC$, $XB = XD$ and so X lies on the angle bisector of $\angle AEC$.

- We want angles using line EP. We get angles from cyclic quads, so if we extend EP to the circumcircle of BED then we get cyclic quads from which angles follow.



Solution:

Construct X as the intersection of the circumcircles of AEC and BED.

We need to prove:

- P lies on EX. This is true from the radical axis theorem: $AXCE,\ BXFE,\ ABCF$ are cyclic so $AC,\ BF,\ EX$ are concurrent (at P).
- X lies on the angle bisector of $\angle AED$.

$$\angle XAB = 180^\circ - \angle XCE = \angle XCD,$$
$$\angle XBA = 180^\circ - \angle XBE = \angle XDE.$$

$\angle XAB = \angle XCD,\ \angle XBA = \angle XDC,\ AB = CD$ so by AAS,

$$\triangle XAB \cong \triangle XCD.$$

Therefore $XA = XC$ so

$$\angle XEA = \angle XCA = \angle CAX = \angle CEX$$

so we get X lies on the angle bisector. (Or quote Charles' Lemma)

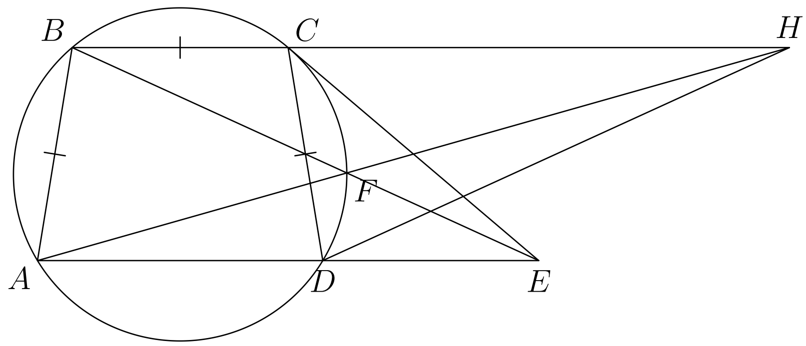
A minor point: people reverse reconstructed X as the intersection of EP with the circumcircle of BED. Then to prove that $AXCE$ is cyclic,

$$PX \times PE = PF \times PB = PA \times PC.$$

However, the converse of POP only holds with directed lengths. Example: $PA \times PC = PB \times PD$ so $ABCD$ is cyclic. But with undirected lengths, we have that $PA \times PC = PB \times PD'$ [where D' is the reflection of D over P] but $ABCD'$ is not cyclic.

G6. Convex cyclic quadrilateral $ABCD$ satisfies $AB = BC = CD$. Let Γ be the circumcircle of $ABCD$. The tangent to Γ at C intersects AD at E. BE intersects Γ again at F. DF and AF intersect BC at G and H respectively.

Prove that the circumcircle of DGH is tangent to CD at D.



STP: $\angle CDG = \angle DHG$.

But $\angle CDG = \angle CDF = \angle CBF = \angle CBE$.

STP: $\angle CBE = \angle DHC$.

We were given $AB = BC = CD$, and we were given $ABCD$ cyclic. That means that $ABCD$ is an isosceles trapezium, so $BC \| AD$.

Since $BC \| AD$, $\angle CBE = \angle DEB$. STP:

$$\angle DEB = \angle DHB \iff BHED \text{ cyclic.}$$

STP: $\angle BDA = \angle BHE$.

We have $\angle BDA = \angle CDB = \angle CFB$.

STP: $\angle CFB = \angle CHE \iff CHEF$ cyclic.

STP: $\angle EFH = \angle ECH$.

But we have $\angle EFH = \angle BFA = \angle CFB = \angle CDB = \angle ECH$.