SOME MADMAN'S RAVINGS

GISPISQUARED

Contents

Introduction

If you can't be bothered reading, see the tldr here.

I'm writing this since I think that most of the resources out there that try to teach you olympiad maths have all the theory you need to solve IMO-level problems (and sometimes much more!), but it's much more rare to see people who tell you how to look at problems and figure out approaches that may, or in some cases should, work. True, much of this is an individual learning experience, but I think a lot of this can be written more explicitly.

This may reduce some of the magic of figuring this stuff out for yourself; to mitigate this effect, I have relegated all the proofs and solutions to the back so that you can try prove everything before reading the solutions.

Apart from the methods of proof chapter, the overwhelming majority of this book will be problems and solutions. There will be two main types of problems:

- *Results*, which are considered important and well-known, and come up sporadically (or in some cases consistently) as steps in the harder problems.
- *Problems*, which will be questions taken from contests (mostly AMC and AIMO in Part 2, and AMO, EGMO and IMO in Part 3).

The ideas used to prove results are usually considered pretty basic ideas, though I think that's not because they're simple but because they're used often enough that they become standard. Then again I usually gave up far too early and looked at the proofs, so maybe I just didn't give myself enough time to prove them back when I was learning them.

Since I refuse to rehash stuff that others have done better, I'll refer you to a couple of resources about how to write proofs properly:

- How to Write a Maths Solution
- Notes on English

Cool, hopefully now you know how to write proofs. Guess that means every time you solve a problem you'll get a 7, right?

Oh, and apologies for the bad formatting typesetting hyperlinking etc.

Part 1. Theory

1. Basics

1.1. Methods of Proof.

1.1.1. *Direct Proof.* This is perhaps the simplest type of proof. The idea is to start with the stuff you're given, do some logical deduction, and finish with what you want to prove.

Result 1. *The remainder when a perfect square is divided by 4 is either 0 or 1.* *Solution*

1.1.2. *Contradiction.* This is where you assume that what you're trying to prove is wrong and try to derive some kind of logical impossibility. Then the only place where the logic could have gone wrong was in the assumption so the statement you were trying to prove must be true.

Result 2. *There are infinitely many primes.* *Solution*

1.1.3. *Contrapositive.* It turns out that the statement $A \implies B$ is logically equivalent to the statement $\neg B \implies \neg A$. This is probably easiest to see intuitively with an result: "If x is an integer, then x is rational" is logically equivalent to "If x is not rational, then x is not an integer". Therefore, if we're asked to prove $A \implies B$, it's enough to prove $\neg B \implies \neg A$, which is sometimes easier.

Problem 1. *Let $a, b \in \mathbb{R}$ such that $a + b$ is irrational. Prove that at least one of a and b is irrational.* *Solution*

1.1.4. *Induction.* Perhaps the hardest to understand of the basic proof techniques, this can be used to prove properties of positive integers where the property for each integer can be related to those of previous integers.

Here is the Principle of Mathematical Induction (PMI):

> Let S be a set of positive integers such that $1 \in S$ and for each $k \in S$, $k + 1 \in S$. Then S contains all positive integers.

To prove a statement for all positive integers, we let S be the set of all positive integers for which the statement is true. Then it's enough to prove:

- $1 \in S$. This is called the *base case.*
- If $k \in S$ (the *inductive hypothesis*), then $k + 1 \in S$. This is called the *inductive case.*

Then by PMI, S will contain all positive integers.

There are two ways to make induction superficially more powerful, though they're both equivalent to the usual form of induction:

- Say we want to prove a statement for all integers larger than n, for some n. Then it's enough to prove:
  - The statement is true for n.
  - If the statement is true for some integer $k > n$, then it's true for $k+1$.

  This is equivalent to the normal PMI: to see this, let S be the set of all integers m for which the statement is true for all $m + n$.

- Say we want to use not just the inductive assumption not just for k, but for smaller integers as well. Intuitively this should be fine, since we've in some sense "proved this already" by the time we get to $k+1$. Formally, to prove a statement $P(n)$ for all positive integers n, it's enough to prove:
  - $P(1)$.
  - If $P(1), \ldots, P(k)$ are all true, then $P(k+1)$ is also true.

  This form of proof by induction is called *strong induction*, and although most proofs by induction only explicitly use $P(k)$, there's no reason to try to make your proof inductive over strong inductive since strong induction gives you more assumptions to work with "for free".

The key idea in both of these reductions to PMI is to somehow encapsulate the extra information you're trying to assume into the framework of standard PMI.

Result 3. *For all positive integers n,*

$$1 + 2 + \cdots + n = \frac{n(n+1)}{2}.$$

Solution

To conclude the Methods of Proof section, I'll include one final result that combines most of what we've covered so far.

Result 4. *The* Well-Ordering Principle *states that any set of positive integers has a least element. It's equivalent to PMI — that is, PMI is true if and only if well-ordering is true.* *Solution*

I find it intriguing that induction and minimality are really just two sides of the same coin. Often you will find that a solution is much more natural to think about and write up in terms of one than the other.

1.2. **Algebra.**

1.2.1. *Factorisations.* I won't have any problems attached to these, but they tend to pop up everywhere so keep an eye out. Here are some common factorisations:

- $x^2 - a^2 = (x+a)(x-a)$
- $x^2 - 2ax + a^2 = (x-a)^2$
- $x^2 + 2ax + a^2 = (x+a)^2$
- $x^3 - a^3 = (x-a)(x^2 + ax + a^2)$
- $x^3 + a^3 = (x+a)(x^2 - ax + a^2)$
- $x^3 + 3x^2a + 3xa^2 + a^3 = (x+a)^3$
- $x^3 - 3x^2a + 3xa^2 - a^3 = (x-a)^3$

The cases $a = 1$ are especially common.

1.2.2. *Systems of equations.* There are a couple of ways of solving these systems — either you can isolate one variable, substitute into the rest of the equations, and repeat, or you can try and combine the equations in such a way that stuff cancels. The first method is usually fine in school maths and the AMC, but the second is more likely to be useful in harder Olympiad questions.

Sometimes these techniques won't be enough — see here.

Problem 2. *The difference between two numbers is 20. When 4 is added to each number the larger is three times the smaller. What is the larger of the two original numbers?* *Solution*

Problem 3. *Find all triples (x, y, z) of real numbers that simultaneously satisfy the equations*

$$xy + 1 = 2z$$
$$yz + 1 = 2x$$
$$zx + 1 = 2y$$

Solution

1.2.3. *Quadratics.*

Result 5. *Let m and p be given real numbers. All real numbers x such that*

$$x^2 - 2mx + p = 0$$

are given by $x = m \pm \sqrt{m^2 - p}$. Solution

Result 6. *Let a, b, c be given real numbers. All real numbers x such that*

$$ax^2 + bx + c = 0$$

are given by

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}.$$

In particular, if we let $\Delta = b^2 - 4ac$, then the equation has no real roots if $\Delta < 0$, exactly one real root if $\Delta = 0$, and two real roots if $\Delta > 0$. Solution

This number Δ is called the *discriminant* of the quadratic.

Now, a couple of problems which show how useful both the results and the method are.

Problem 4. *[See here if you don't know what number bases are.]*

The number x is 111 when written in base b, but it is 212 when written in base $b - 2$. What is x is base 10? Solution

Problem 5. *For each pair of real numbers (r, s), prove that there exists a real numer x that satisfies at least one of the following two equations.*

$$x^2 + (r + 1)x + s = 0$$
$$rx^2 + 2sx + s = 0$$

Solution

Problem 6. *Find all real numbers x for which $x^3 + 3x^2 + 3x + 5 = 0$. Solution*

1.2.4. *Inequalities.* At this level, inequalities are mostly about making stuff into squares or, well, "mostly-squares". The guiding principle is to try and find an expression which you want to be always nonnegative, figure out where it's 0, and write it in terms of stuff that's 0 there and obviously nonnegative elsewhere.

Result 7. *If a and b are real numbers, then*

$$\frac{a+b}{2} \geq \sqrt{ab}.$$

Solution

Problem 7. *The set S consists of distinct integers such that the smallest is 0 and the largest is 2015. What is the minimum possible average value of the numbers in S? Solution*

1.2.5. *Sums of sequences.*

Result 8. *If n is a positive integer and a and b are real numbers, then*

$$\sum_{i=0}^{n}(a+bi) = \frac{(n+1)(2a+bn)}{2}.$$

Solution

Result 9. *If n is a positive integer and r is a real number distinct from 1, then*

$$\sum_{i=0}^{n} r^i = \frac{1-r^{n+1}}{1-r}.$$

Solution

1.2.6. *Polynomials.* A polynomial is just an expression of the form

$$P(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_0 x^0,$$

where each of the a_is is a constant and $a_n \neq 0$. The number n is called the *degree* of the polynomial. The expression $P(x) = 0$ is also a polynomial, and it's defined to have degree $-\infty$.

Result 10. *If a polynomial $P(x)$ of degree n has a root r, then there is a polynomial $Q(x)$ of degree $n-1$ such that*

$$P(x) = (x-r)Q(x).$$

Solution

Result 11. *If a polynomial*

$$P(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_0 x^0$$

has roots $r_1, \ldots, r_n$ then for each i,

$$a_{n-i} = (-1)^i a_n \sum_{j_1 < \cdots < j_i} \prod_{k=1}^{i} r_{j_k}.$$

Solution

1.3. **Combinatorics.**

1.3.1. *Addition and Multiplication Principles.*

Problem 8. *A hockey game between two teams is 'relatively close' if the number of goals scored by the two teams never differ by more than two. In how many ways can the first 12 goals of a game be scored if the game is 'relatively close'?* *Solution*

1.3.2. *Permutations and Combinations.*

1.3.3. *Double Counting and Combinatorial Identities.*

1.3.4. *Venn Diagrams and PIE.*

1.3.5. *Supermarket Principle.*

1.3.6. *Recurrences.*

1.4. **Geometry.**

1.4.1. *Congruence.*

1.4.2. *Basic angle chasing.*

1.4.3. *Similarity.*

1.4.4. *Pythagoras.*

1.4.5. *Common right-angled triangles.*

1.5. **Number Theory.**

1.5.1. *Primes and FTOA.*

1.5.2. *SFFT.*

1.5.3. *Divisibility.*

1.5.4. *Number bases.* hi

2. Not-so-basics

2.1. **Algebra.**

2.1.1. *Systems of equations.* hi

2.2. **Combinatorics.**

2.3. **Geometry.**

2.4. **Number Theory.**

Part 2. Practice

Part 3. Solutions

Result 1 If you start trying some small cases, what you'll eventually find is that if n is an even integer, then n^2 leaves a remainder of 0 when divided by 4, and if n is an odd integer, then n^2 leaves a remainder of 1 when divided by 4. Once you've conjectured this, all that's left is to recall what it means for a number to be even or odd, and then the proof falls out quite naturally:

Proof. Let the perfect square be n^2. We split into cases depending on the parity of n.

- If n is even, let $n = 2m$ for some integer m. Then
$$n^2 = (2m)^2 = 4m^2,$$
which leaves a remainder of 0 when divided by 4.
- If n is odd, let $n = 2m + 1$ for some integer m. Then
$$n^2 = (2m + 1)^2 = 4m^2 + 4m + 1 = 4(m^2 + m) + 1,$$
which leaves a remainder of 1 when divided by 4.

In either case, the remainder left when dividing n^2 by 4 is either 0 or 1, which is what we wanted to prove. □

Result 2 The key here is to assume, for contradiction, that there are only finitely many primes. Then we want to prove a suitable contradiction — a nice way of doing this is to find a number that isn't 1 but isn't divisible by any of our finitely many primes. The idea of constructing such a number by multiplying everything and adding 1 is surprisingly common in Olympiad maths.

Proof. Assume that there are only finitely many primes $p_1, p_2, \ldots, p_n$. Then, consider the number $A = p_1 p_2 \cdots p_n + 1$. Clearly A is a positive integer larger than 1, so it must have a prime factor, which means that for some i, $p_i \mid A$. But $p_i \mid A - 1$, so $p_i \mid A - (A - 1) = 1$, which is our contradiction. □

Problem 1 This is more an exercise in logic than in maths. You could be stuck for ages trying to prove the problem directly, but as soon as you try to use some indirect approach (like contrapositive, contradiction, or assuming one part of the conclusion is false and proving the other is true) the problem pretty much solves itself. I think the cleanest solution in this particular case uses the contrapositive.

Proof. We prove the contrapositive: that if a and b are both rational, then so is $a + b$.

Let $a = \frac{w}{x}$, $b = \frac{y}{z}$. Then
$$a + b = \frac{w}{x} + \frac{y}{z} = \frac{wz + xy}{xz},$$
which is clearly rational. □

Result 3 This is a classic induction problem. Apart from being instructive because it isolates the idea of induction, it does highlight a minor point. In the inductive step, it's just as acceptable to assume the problem is true for k and prove it for $k + 1$ as to assume the problem is true for $k - 1$ and prove it for k. In this particular case, the latter is somewhat easier.

Proof. We prove this by induction on n.

Base case $n = 1$: We have LHS $= 1 = \frac{1\times 2}{2} =$ RHS.

Inductive step: Assume the problem is true for $n = k - 1$. Then,

$$\begin{aligned} 1 + 2 + \cdots + k &= (1 + 2 + \cdots + k - 1) + k \\ &= \frac{k(k-1)}{2} + k \\ &= \frac{k(k-1) + 2k}{2} \\ &= \frac{k(k+1)}{2}, \end{aligned}$$

so the problem is true for $n = k$. □

Result 4 Since this is an "if and only if" problem, we will probably need to find separate proofs in each direction.

First, let's use induction to prove well-ordering. Our desired conclusion is that every nonempty set of positive integers has a smallest element. Intuitively, what we would like to do is to check if 1 is in it, then if 2 is in it, and so on until we first find an element that's in it. But this quickly becomes circular and it's difficult to make airtight. The trick is to utilise proof by contrapositive — start with a set of positive integers that has no smallest element, and prove it's empty using our sequential checking process. Make sure you actually use PMI somewhere, otherwise it's probably a fakesolve.

Now, let's use well-ordering to prove PMI. You'll probably get nowhere if you aren't completely clear about what you're trying to prove (you may as well replace PMI by gibberish), so let's write out PMI in full:

Let S be a set of positive integers. If $1 \in S$, and if the statement $\forall a \in S,\ a+1 \in S$ is true, then S contains all positive integers.

Once again we use indirect proof — this time it's proof by contradiction (contrapositive also works). Let's assume that there is a set S that satisfies both conditions but doesn't contain all positive integers. We hope to use well-ordering to find a contradiction.

The key idea is to consider the smallest integer that isn't in S, which is possible by well-ordering. Then the condition implies that either it's not the smallest, or $1 \notin S$ — a contradiction either way.

Let's write it up.

Proof. First we prove that if PMI is true, then so is well-ordering. Assume PMI, and we'll prove the contrapositive of well-ordering: that if S is a set of positive integers with no smallest element, then it is empty.

I prove by strong induction that for each positive integer n, $n \notin S$. Clearly this is sufficient.

Base case $n = 1$: if $1 \in S$, then 1 would be the smallest element in S. So since S has no smallest element, 1 is not the smallest element in S so 1 is not in S. (Notice where I used the contrapositive here?)

Strong inductive step: Assume that for all $i = 1, 2, \ldots, k,\ i \notin S$. I claim that $k + 1 \notin S$. Indeed, if $k + 1$ were in S, then it would be the smallest element of S. But since S has no smallest element, $k + 1$ can't be in S.

This completes the induction, so no integer is in S meaning that S is empty as needed.

Now I prove that if well-ordering is true, then so is PMI. Assume for contradiction that well-ordering is true but PMI is not. Then, there is a set S of positive integers that contains 1 and such that for each $a \in S$, $a+1 \in S$ but that does not contain all positive integers. Then, the set $\mathbb{N} \setminus S$ is nonempty so by well-ordering it contains a smallest element a.

Since we know that $1 \in S$, we know that $a \neq 1$. So $a-1$ is a positive integer. Since $a-1 < a$ and a is the smallest member of $\mathbb{N} \setminus S$, $a-1 \notin \mathbb{N} \setminus S \implies a-1 \in S$. So since $a-1 \in S$, $a \in S$ which contradicts the assumption that $a \notin S$. ☐

Problem 2 Not much to say here — interpret as a system of linear equations and solve however you like.

Answer: 26.

Proof. Let the two numbers be a and b, with $a > b$. Then, $b = a - 20$ so

$$\begin{aligned} a + 4 &= 3(a - 20 + 4) \\ &= 3a - 48 \\ 2a &= 52 \\ a &= 26, \end{aligned}$$

so the larger of the two numbers is 26.

To prove that this actually works, note that if $a = 26$ and $b = 6$, then $a - b = 20$ and $a + 4 = 30 = 3 \times 10 = 3(b + 4)$ as needed. ☐

Problem 3 Since we don't like the 1s in our equations, we subtract two equations to get rid of them. Alternatively, we subtract two equations because that's one of the most obvious things to do with a system of equations. Either way, once we've done that the rest of the problem is pretty routine.

Answer: $(x, y, z) = (1, 1, 1), \left(-2, -2, \frac{5}{2}\right), \left(-2, \frac{5}{2}, -2\right), \left(\frac{5}{2}, -2, -2\right)$.

Proof. Subtract the third equation fron the first:

$$\begin{aligned} xy - xz &= 2z - 2y \\ x(y - z) + 2(y - z) &= 0 \\ (x + 2)(y - z) &= 0 \end{aligned}$$

So either $x = -2$ or $y = z$. Similarly we can deduce that either $z = -2$ or $x = y$. Now we split into four cases:

- $x = -2,\ z = -2$. Then $2y = zx + 1 = 5 \implies y = \frac{5}{2}$.
- $x = -2,\ x = y$. Then similar to the above we get $z = \frac{5}{2}$.
- $y = z\ z = -2$. In the same way we get $x = \frac{5}{2}$.
- $y = z,\ x = y$. Then $x^2 + 1 = 2x \implies (x - 1)^2 = 0 \implies x = 1$, so $x = y = z = 1$.

So the only solutions are what we claim they are. It is easy to check that these solutions all satisfy the original equations. ☐

Result 5 First, there are a few ways of seeing that this is the answer.

One way is to try to factorise $x^2 - 2mx + p$ as $(x-a)(x-b)$. Then $a + b = 2m$ and $ab = p$, and a and b are the values of x we want. Then the key idea is that the way of using the $a + b = 2m$ condition is to let $a = m + c,\ b = m - c$ so that

$$p = ab = (m+c)(m-c) = m^2 - c^2,$$

so that $c = \sqrt{m^2 - p}$.

Another way is to notice that $x^2 - 2mx + p$ looks a lot like $x^2 - 2mx + m^2 = (x-m)^2$. I'll do the rest in the proof.

Proof. We have

$$\begin{aligned} x^2 - 2mx + p &= x^2 - 2mx + m^2 + p - m^2 \\ &= (x-m)^2 - \left(m^2 - p\right) \end{aligned}$$

If $m^2 - p < 0$ then clearly there are no solutions. Otherwise, we have

$$\begin{aligned} x^2 - 2mx + p &= (x-m)^2 - \left(\sqrt{m^2-p}\right)^2 \\ &= \left(x - m - \sqrt{m^2-p}\right)\left(x - m + \sqrt{m^2-p}\right), \end{aligned}$$

so it equals 0 if and only if $x = m \pm \sqrt{m^2 - p}$. $\square$

Result 6 The key here is to get this equation into a form such that we can apply the previous result.

Proof. We have

$$\begin{aligned} 0 &= ax^2 + bx + c \\ 0 &= x^2 + \frac{b}{a}x + \frac{c}{a} \\ &= x^2 - 2\frac{-b}{2a}x + \frac{c}{a} \\ x &= \frac{-b}{2a} \pm \sqrt{\frac{b^2}{4a^2} - \frac{c}{a}} \\ &= \frac{-b \pm \sqrt{4a^2\left(\frac{b^2}{4a^2} - \frac{c}{a}\right)}}{2a} \\ &= \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \end{aligned}$$

as needed.

If $\Delta < 0$ there are clearly no solutions. If $\Delta = 0$ the unique solution is $x = \frac{-b}{2a}$. If $\Delta > 0$ there are two solutions given by our equation. $\square$

Problem 4 Once again not much to say here — interpret the number bases in the usual way and solve the resulting quadratic.

Answer: 57.

Proof. We have

$$\begin{aligned}
111_b &= 212_{b-2} \\
b^2+b+1 &= 2(b-2)^2+(b-2)+2 \\
&= 2b^2-7b+8 \\
0 &= b^2-8b+7 \\
b &= 4 \pm \sqrt{4^2-7} \\
&= 4 \pm 3
\end{aligned}$$

But since $b > 2$ we get $b = 7 \implies x = 7^2+7+1 = 57$.
Finally, 57 is indeed 111 in base 7 and 212 in base 5. □

Problem 5 The key here is to use the discriminant (Δ in Result 6). In particular, it's enough to prove that at least one of the two Δs is nonnegative. The easiest way of doing this is to assume that the first is negative and prove that the second isn't.

Proof. Assume that there is no real number x such that $x^2+(r+1)x+s = 0$. Then the discriminant $(r+1)^2-4s$ is negative, so $4s > (r+1)^2$.

Since $(r+1)^2 \geq 0$, we know that $s > 0$. Also, $4(s-r) > (r+1)^2-4r = (r-1)^2 \geq 0$. So since $s > 0$ and $4(s-r) > 0$, their product $4s^2-4sr$ is also positive so the discriminant of the second quadratic is positive, meaning that it has at least one real solution. □

Problem 6 At first glance, this looks like our methods can't help since we have a cubic not a quadratic. However, the same trick used in Result 5 of recognising a common factorisation does in fact work.

Answer: $x = -1-\sqrt[3]{4}$.

Proof. We subtract 4 from both sides to make the LHS into something we recognise:

$$\begin{aligned}
x^3+3x^2+3x+1 &= -4 \\
(x+1)^3 &= -4 \\
x+1 &= -\sqrt[3]{4} \\
x &= -1-\sqrt[3]{4}.
\end{aligned}$$

To show that this number works, we can either substitute it in and do the algebra, or notice that each step above was actually an equivalence so the implications run backwards as well. □

Result 7 Since we want to use the fact that squares are nonnegative, we collect all the terms on one side. The rest is recognition, which can be helped by noticing that we want equality to occur when $a = b$.

Proof.

$$\text{RHS} - \text{LHS} = \frac{a+b}{2} - \sqrt{ab} = \frac{a+b-2\sqrt{ab}}{2} = \frac{(\sqrt{a}-\sqrt{b})^2}{2} \geq 0,$$

as needed. □

Problem 7 We have to have a 0 and a 2015 in the set, but apart from them the rest of the terms should be as small as possible. This means that we can apply Result 3 to get a function we want to minimise. A little algebraic trickery means it's enough to minimise

$$n + \frac{4032}{n}.$$

Then, by Result 7, the minimum of this over $\mathbb{R}$ is $2\sqrt{4032}$ at $n = \sqrt{4032} \approx 63.5$, which means either 63 or 64 should minimise the expression over $\mathbb{N}$. In fact both do, which means we should try to force the expression into something that looks like $(n-63)(n-64)$, and indeed doing that solves the problem.

Answer: 62.

Proof. Let n be the number of elements in S, and let $S = \{s_1, s_2, \ldots, s_n\}$, where the s_is are in increasing order. Then

$$s_i \geq i - 1 \,\forall\, i < n,$$

and $s_n = 2015$, so the average is at least

$$\begin{aligned}\frac{0+1+\cdots+n-2+2015}{n} &= \frac{\frac{(n-2)(n-1)}{2} + 2015}{n}\\ &= \frac{n^2 - 3n + 4032}{2n}\\ &= \frac{n^2 - 127n + 4032}{2n} + 62\\ &= \frac{(n-63)(n-64)}{2n} + 62.\end{aligned}$$

Since n is an integer, the first term is 0 if n is either 63 or 64 and positive otherwise, which means that the minimum value is 62, achieved when S is either $\{0, 1, \ldots, 61, 2015\}$ or $\{0, 1, \ldots, 61, 62, 2015\}$. □

Result 8 There are three ways I know of doing this. One of them is a standard induction, but the other two are more interesting.

For the first way, we notice that we already know the special case (see Result 3) where $a = 0$ and $b = 1$. A little algebra allows us to reduce the whole problem to this particular case.

Proof. We have

$$\begin{aligned}\sum_{i=0}^{n}(a+bi) &= \sum_{i=0}^{n} a + \sum_{i=0}^{n} bi\\ &= a(n+1) + b\sum_{i=0}^{n} i\\ &= a(n+1) + b\frac{n(n+1)}{2}\\ &= \frac{(2a+bn)(n+1)}{2},\end{aligned}$$

as needed. □

For the second way, we use a trick called *Gaussian pairing* — we pair the first term with the last term and so on — so that each pair has the same sum.

Proof. We have

$$\begin{aligned}\sum_{i=0}^{n}(a+bi) &= \sum_{i=0}^{n}(a+b(n-i)) \\ &= \frac{1}{2}\left(\sum_{i=0}^{n}(a+bi)+\sum_{i=0}^{n}(a+b(n-i))\right) \\ &= \frac{1}{2}\left(\sum_{i=0}^{n}(2a+bn)\right) \\ &= \frac{(n+1)(2a+bn)}{2},\end{aligned}$$

as needed. □

Result 9 The main idea here comes from extending a couple of the common factorisations:

- $1-r^2=(1-r)(1+r)$
- $1-r^3=(1-r)(1+r+r^2)$

Let's write it up.

Proof. We have

$$1-r^{n+1}=(1-r)(1+r+r^2+\cdots+r^n),$$

since all the middle terms cancel. Dividing both sides by $1-r$ yields the desired result. □

Result 10 For those of you who have seen polynomial long division, this should seem very familiar, and indeed the same idea of dividing from highest to lowest degree works. The neatest and most rigorous way to write it up uses induction.

Proof. By strong induction on the degree.

Base case $n=1$: if $ar-b=0$ then $\frac{b}{a}=r$ so $ax-b=a(x-\frac{b}{a})=a(x-r)$, as needed. Clearly $Q(x)=a$ has degree 0.

Inductive step: Write

$$P(x)=a_nx^{n-1}(x-r)+P_1(x),$$

so that $P_1(x)$ is a polynomial of degree less than n and $P_1(r)=P(r)-0=0$. By the strong inductive hypothesis there is a polynomial $Q_1(x)$ of degree less than $n-1$ such that $P_1(x)=(x-r)Q_1(x)$. Then,

$$P(x)=(x-r)\left(a_nx^{n-1}+Q_1(x)\right),$$

so if we let $Q(x)=x^{n-1}+Q_1(x)$ we have $P(x)=(x-r)Q(x)$. Finally, $Q(x)$ has degree $n-1$ since $Q_1(x)$ has degree less than $n-1$ and a_nx^{n-1} has degree $n-1$. □

Result 11 First we need to understand what the problem is saying. Try special cases: $i=0,\ i=n,\ n=1,\ n=2$ and so on, until you know what it's saying.

The idea here is to use the previous result multiple times to factorise our polynomial fully, then expand it again. Then when we extract the x^{n-i} coefficient, each term that contributes to it is a product where x appears $n-i$ times, and the rest is a product of i (r_i)s and a constant term. Since each combination of i $(-r_i)$s appears exactly once in the expansion, we get the claimed formula.

The neatest way of writing this up is to use induction.

Proof. By induction on the degree.

Base case $n = 0$: there are no r_is so all that we have to prove is $a_0 = a_0$, which is obvious.

Inductive step: Since r_n is a root of $P(x)$, we can write $P(x) = (x - r_n)Q(x)$ for some polynomial $Q(x)$ of degree $n - 1$. Then we know that $Q(x)$ has roots $r_1, \dots, r_{n-1}$ so by the inductive hypothesis, we know that in $Q(x)$:

- The coefficient of x^{n-i} is
$$(-1)^{i-1} a_n \sum_{j_1 < \cdots < j_{i-1}} \prod_{k=1}^{i-1} r_{j_k}.$$
- The coefficient of x^{n-i-1} is
$$(-1)^{i} a_n \sum_{j_1 < \cdots < j_i} \prod_{k=1}^{i} r_{j_k}.$$

Since the coefficient of x^{n-i} in $P(x)$ is the coefficient of x^{n-i-1} in $Q(x)$ minus r_n times the coefficient of x^{n-i} in $Q(x)$, it's what we claim it is. Since this argument works for each i, the induction is complete. $\square$

Part 4. TLDR

This part contains some ideas about how to approach problems; basically, I've taken what I think are the most important bits and condensed them together here.

(With apologies to Pólya)

- How are the objects referred to in the problem defined? What properties do you know them to have?
- Try small or special cases. Can you spot patterns in their structure? In how you solve them? Can you prove any of these patterns in general? Do any of these patterns help?
- Look at stuff that is extremal in some way: biggest, smallest, most connected, most disconnected, most composite, prime, whatever
- Think about what happens if the problem, or the conclusion, is wrong.
- Can you reduce any instance of the problem to a smaller instance? Can you reduce a counterexample to a smaller counterexample?
- Have you seen something similar before? Can you use the result or the method? Can you introduce some auxiliary element to make its use possible?
- Can you draw a diagram to help you understand the problem?