SOME MADMAN'S RAVINGS

GISPISQUARED

Contents

Introduction

I was recently[1] asked what advice I would give to a beginner Olympiad student. Partly for your sake if you're reading this, and partly to inform how I write this, I'll copy (and punctuate) my response here:

- Don't believe people (like whoever writes hdigh) when they say methods of proof is all you should learn before spamming problems. Maybe that works if you're as smart as them, but most of us don't wanna spend most of our training reinventing the wheel.
- On the other hand, don't waste time looking up the latest crazy theorem aopsers start mentioning; those are unlikely to ever be necessary and are probably only rarely useful because problem setters and pscs prefer problems where the easiest solution involves only canonical Olympiad knowledge.
- Most of the problems you're trying right now are gonna be pretty trivial for more advanced students, so whenever you solve or don't solve a problem figure out why it should have been trivial and hopefully this hindsight will start turning into foresight.

If I am to be consistent, then, this book's contents should be of three main types: canonical Olympiad knowledge and techniques, problems on which to practice these techniques, and solutions with motivation that try to make the problems seem as trivial as possible. I've also chosen both to include proofs to the known results (since their proofs are within Olympiad students' grasp and are often the source of canonical techniques), and to separate these proofs from the theorem statements in the same way as with problems (so that students can try to prove the theorems themselves).

There will be two main types of questions:

- *Results*, which are considered important and well-known, and come up sporadically (or in some cases consistently) as steps in the harder problems. Every result comes with an implied "Prove that" attached, unless otherwise noted.
- *Problems*, which will be questions taken from contests (mostly AMC and AIMO in Part 2, and AMO, EGMO and IMO in Part 3).

Since I refuse to rehash stuff that others have done better, I'll refer you to a couple of resources about how to write proofs properly:

- How to Write a Maths Solution
- Notes on English

Cool, hopefully now you know how to write proofs. Guess that means every time you solve a problem you'll get a 7, right?

Of course, to learn you'll need to put the required effort into maths — you're unlikely to learn too much just from reading problems and solutions. I've tried to make sure that all problems are accessible using only prior knowledge (something like high school maths up to y9 or so) and prior exposition; therefore, you should at least try to solve problems before reading the solutions.

[1] February 2022

Part 1. General Recommendations

This part contains some general ideas about how to approach problems, or what to try when stuck.

(With apologies to Pólya)

- How are the objects referred to in the problem defined? What properties do you know them to have?
- Try small or special cases. Can you spot patterns in their structure? In how you solve them? Can you prove any of these patterns in general? Do any of these patterns help?
- Look at stuff that is extremal in some way: biggest, smallest, most connected, most disconnected, most composite, prime, whatever
- Think about what happens if the problem, or the conclusion, is wrong.
- Can you reduce any instance of the problem to a smaller instance? Can you reduce a counterexample to a smaller counterexample?
- Have you seen something similar before? Can you use the result or the method? Can you introduce some auxiliary element to make its use possible?
- Can you draw a diagram to help you understand the problem?

Part 2. **Theory**

0. Notation

Notation	Meaning
$\mathbb{R}$	The real numbers

1. Basics

1.1. Methods of Proof.

1.1.1. *Direct Proof.* This is perhaps the simplest type of proof. The idea is to start with the stuff you're given, do some logical deduction, and finish with what you want to prove.

Result 1.1.1.1. *The remainder when a perfect square is divided by 4 is either 0 or 1.* Solution

1.1.2. *Contradiction.* This is where you assume that what you're trying to prove is wrong and try to derive some kind of logical impossibility. Then the only place where the logic could have gone wrong was in the assumption so the statement you were trying to prove must be true.

Result 1.1.2.1. *There are infinitely many primes.* Solution

1.1.3. *Contrapositive.* It turns out that the statement $A \implies B$ is logically equivalent to the statement $\neg B \implies \neg A$. This is probably easiest to see intuitively with an example: "If x is an integer, then x is rational" is logically equivalent to "If x is not rational, then x is not an integer". Therefore, if we're asked to prove $A \implies B$, it's enough to prove $\neg B \implies \neg A$, which is sometimes easier.

Problem 1.1.3.1. *Let $a, b \in \mathbb{R}$ such that $a + b$ is irrational. Prove that at least one of a and b is irrational.* Solution

1.1.4. *Induction.* Perhaps the hardest to understand of the basic proof techniques, this can be used to prove properties of positive integers where the property for each integer can be related to those of previous integers.

Here is the Principle of Mathematical Induction (PMI):

> Let S be a set of positive integers such that $1 \in S$ and for each $k \in S$, $k+1 \in S$. Then S contains all positive integers.

To prove a statement for all positive integers, we let S be the set of all positive integers for which the statement is true. Then it's enough to prove:

- $1 \in S$. This is called the *base case.*
- If $k \in S$ (the *inductive hypothesis*), then $k + 1 \in S$. This is called the *inductive case.*

Then by PMI, S will contain all positive integers.

There are two ways to make induction superficially more powerful, though they're both equivalent to the usual form of induction:

- Say we want to prove a statement for all integers larger than n, for some n. Then it's enough to prove:
  - The statement is true for $n + 1$.
  - If the statement is true for some integer $k > n$, then it's true for $k + 1$.

  This is equivalent to the normal PMI: to see this, let S be the set of all integers m for which the statement is true for all $m + n$.
- Say we want to use not just the inductive assumption not just for k, but for smaller integers as well. Intuitively this should be fine, since we've in some sense "proved this already" by the time we get to $k + 1$. Formally, to prove a statement $P(n)$ for all positive integers n, it's enough to prove:

– $P(1)$.
– If $P(1), \dots, P(k)$ are all true, then $P(k+1)$ is also true.

Once again this is equivalent to the normal PMI: let S be the set of all integers m for which $P(a)$ is true for all $a \leq m$.

This form of proof by induction is called *strong induction*, and although most proofs by induction only explicitly use $P(k)$, there's no reason to try to make your proof inductive over strong inductive since strong induction gives you more assumptions to work with "for free".

The key idea in both of these reductions to PMI is to somehow encapsulate the extra information you're trying to assume into the framework of standard PMI.

Result 1.1.4.1. *For all positive integers n,*

$$1 + 2 + \cdots + n = \frac{n(n+1)}{2}.$$

Solution

There's also an equivalent statement to PMI — the Well-Ordering Principle.

> Let S be a nonempty set of postiive integers. Then there exists some $x \in S$ such that for all $y \in S$ we have $y \geq x$.

The Well-Ordering Principle is often used in conjunction with contradiction or contrapositive, since it is mainly "about" the existence of some element rather than the properties of the rest of the elements.

We can think of PMI as enabling us to prove something is true by building up larger cases from smaller cases, while well-ordering allows us to prove something is true by assuming there's a counterexample and finding a smaller one. It's nontrivial and instructive to prove they're actually equivalent.

First, we'll need a lemma (auxiliary result) so basic this is probably the only time we'll ever need to justify it:

Result 1.1.4.2. *Both PMI and well-ordering imply that if n is a positive integer then $n \geq 1$.* *Solution*

Result 1.1.4.3. *PMI and well-ordering imply each other.* *Solution*

I find it intriguing that induction and minimality are really just two sides of the same coin. Often you will find that a solution is much more natural to think about and write up in terms of one than the other.

1.2. **Algebra.**

1.2.1. *Factorisations.* I won't have any problems attached to these, but they tend to pop up everywhere so keep an eye out. Here are some common factorisations:

- $x^2 - a^2 = (x+a)(x-a)$
- $x^2 - 2ax + a^2 = (x-a)^2$
- $x^2 + 2ax + a^2 = (x+a)^2$
- $x^3 - a^3 = (x-a)(x^2 + ax + a^2)$
- $x^3 + a^3 = (x+a)(x^2 - ax + a^2)$
- $x^3 + 3x^2a + 3xa^2 + a^3 = (x+a)^3$
- $x^3 - 3x^2a + 3xa^2 - a^3 = (x-a)^3$

Many of these are special cases of the formula

$$x^n - a^n = (x-a)(x^{n-1} + ax^{n-2} + a^2x^{n-3} + \cdots + a^{n-1}).$$

The cases $a = 1$ are especially common.

1.2.2. *Systems of equations.* There are a couple of ways of solving these systems — either you can isolate one variable, substitute into the rest of the equations, and repeat, or you can try and combine the equations in such a way that stuff cancels. The first method is usually fine in school maths and the AMC, but the second is more likely to be useful in harder Olympiad questions.

Sometimes these techniques won't be enough — see Section 2.1.1.

Problem 1.2.2.1. *The difference between two numbers is 20. When 4 is added to each number the larger is three times the smaller. What is the larger of the two original numbers?* Solution

Problem 1.2.2.2. *Find all triples (x, y, z) of real numbers that simultaneously satisfy the equations*

$$\begin{aligned} xy + 1 &= 2z \\ yz + 1 &= 2x \\ zx + 1 &= 2y \end{aligned}$$

Solution

1.2.3. *Quadratics.*

Result 1.2.3.1. *Let m and p be given real numbers. All real numbers x such that*

$$x^2 - 2mx + p = 0$$

are given by $x = m \pm \sqrt{m^2 - p}$. Solution

Result 1.2.3.2. *Let a, b, c be given real numbers. All real numbers x such that*

$$ax^2 + bx + c = 0$$

are given by

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}.$$

In particular, if we let $\Delta = b^2 - 4ac$, then the equation has no real roots if $\Delta < 0$, exactly one real root if $\Delta = 0$, and two real roots if $\Delta > 0$. Solution

This number Δ is called the *discriminant* of the quadratic.

Now, a couple of problems which show how useful both the results and the method are.

Problem 1.2.3.1. *[See Section 1.5.4 if you don't know what number bases are.]*

The number x is 111 when written in base b, but it is 212 when written in base $b-2$. What is x is base 10? Solution

Problem 1.2.3.2. *For each pair of real numbers (r, s), prove that there exists a real numer x that satisfies at least one of the following two equations.*

$$\begin{aligned} x^2 + (r+1)x + s &= 0 \\ rx^2 + 2sx + s &= 0 \end{aligned}$$

Solution

Problem 1.2.3.3. *Find all real numbers x for which $x^3+3x^2+3x+5=0$.* *Solution*

1.2.4. *Inequalities.* At this level, inequalities are mostly about making stuff into squares or, well, "mostly squares". The guiding principle is to try and find an expression which you want to be always nonnegative, figure out where it's 0, and write it in terms of stuff that's 0 there and obviously nonnegative elsewhere.

Result 1.2.4.1. *If a and b are real numbers, then*

$$\frac{a+b}{2} \geq \sqrt{ab}.$$

Solution

Problem 1.2.4.1. *The set S consists of distinct integers such that the smallest is 0 and the largest is 2015. What is the minimum possible average value of the numbers in S?* *Solution*

1.2.5. *Sums of sequences.*

Result 1.2.5.1. *If n is a positive integer and a and b are real numbers, then*

$$\sum_{i=0}^{n}(a+bi) = \frac{(n+1)(2a+bn)}{2}.$$

Solution

Result 1.2.5.2. *If n is a positive integer and r is a real number distinct from 1, then*

$$\sum_{i=0}^{n} r^i = \frac{1-r^{n+1}}{1-r}.$$

Solution

Problem 1.2.5.1. *Prove that for any positive integers n and a, the sum*

$$n+(n+1)+(n+2)+\cdots+(n+a)$$

is never a power of 2. *Solution*

1.3. **Combinatorics.** Combinatorics (the branch of maths that deals with, among other things, finding smart ways to count stuff) is a field with relatively few standard techniques. Often, these are the problems that require the least technical skill and the most ingenuity. With every field, but especially with combinatorics, there is no substitute for practice.

1.3.1. *Addition and Multiplication Principles.* This is the most basic idea in combinatorics. If you can make one from a choices and then one from b choices, the total number of ways you can do this is ab. If you can make either one from a choices or one from b choices, the total number of ways you can do this is $a+b$.

Many combinatorics problems boil down to splitting them into cases, and then applying addition and multiplication principles to count each case.

Result 1.3.1.1. *The number of ways of choosing k things from n, where order matters, is*

$$n(n-1)\cdots(n-k+1) = \frac{n!}{k!}.$$

Solution

Problem 1.3.1.1. *A hockey game between two teams is 'relatively close' if the numbers of goals scored by the two teams never differ by more than two. In how many ways can the first 12 goals of a game be scored if the game is 'relatively close'?* Solution

Problem 1.3.1.2. *An ant's walk starts at the apex of a regular octahedron. It walks along five edges, never retracing its path. It visits each of the other five vertices exactly once. In how many ways can it do this?*

1.3.2. *Combinations.*

Result 1.3.2.1. *The number of ways of choosing k things from n, where order doesn't matter, is*

$$\frac{n!}{(n-k)!k!}.$$

Solution

Problem 1.3.2.1. *How many pairs (a, b) of 3-digit palindromes are there with $a > b$ and $a - b$ also a 3-digit palindrome?* Solution

1.3.3. *Inclusion-Exclusion.* So you try to count something using the addition principle, but you realise you've counted some things twice. Obviously, the way you fix it is you subtract the stuff you doubled up. For more tedious problems this could also be overcounted, but this can be continued and hopefully the stuff you've overcounted gets easier to count each time.

Problem 1.3.3.1. *How many integers between 1 and 1000 inclusive are divisible by at least one of 4, 6 and 10?*

1.3.4. *Bijections.* A bijection is simply a way of counting one set of stuff by identifying each element in the set with an element from another set that's easier to count. We've already used a bijection in our solution to Problem 1.3.2.1.

To come up with a bijection, it's often useful to list out the elements of each set for some small cases, and pair them up in what looks like a natural way. Often that natural way will generalise.

Result 1.3.4.1. *The number of ways of putting n identical things into k boxes is*

$$\binom{n+k-1}{k-1}.$$

Problem 1.3.4.1. *How many paths are there from the bottom left to the top right of a 4×7 grid, if you can only go up or right?*

1.3.5. *Recurrences.* Recurrences are a useful way to solve problems where you're asked for the number of ways of doing something that depends on n. To set up a recurrence, we make a sequence where the ith term is the answer for $n = i$. Then, we find a way of relating each terms to previous terms.

Perhaps an example or two will serve to clarify things:

Problem 1.3.5.1. *How many ways are there of tiling a $2 \times n$ grid with dominoes?*

Problem 1.3.5.2. *How many sequences of 10 traffic lights (each green, yellow or red) are there such that a green light is always followed by a yellow light, while a red light is never followed by a red light?*

1.4. **Geometry.** Whenever you see a geometrical statement or problem (including everywhere in the geometry sections of this book), your first instinct should always be to draw a diagram to understand it. Diagrams should be

- Accurate — use ruler and compass
- Large — they should take up the whole page. Go landscape if it gives you more space.

This is important because your chances of solving, or even understanding, a problem are proportional to how easily you can see things in your diagram.

1.4.1. *Parallels.* You should know what parallel lines are: they point in the same direction and never meet.

Let ABC and DEF be points on two parallel lines (in that order, and in the same direction along each line), and let X and Y be points on BE such that $XBEY$ is in that order. Then

$$\angle ABX = \angle CBE = \angle DEX = \angle FEY,$$

the other four angles are equal, and the angles in these two sets are supplementary (add up to 180°).

Result 1.4.1.1. *For any three points A, B, C,*

$$\angle ABC + \angle BCA + \angle CAB = 180^\circ.$$

1.4.2. *Congruence.* Two triangles ABC and XYZ are called *congruent* if

$$AB = XY,\ AC = XZ,\ BC = YZ,$$

$$\angle BAC = \angle YXZ,\ \angle ABC = \angle XYZ,\ \angle ACB = \angle XZY.$$

We write "ABC is congruent to XYZ" as $\triangle ABC \cong \triangle XYZ$.

There are congruence tests which enable us to determine when two triangles are congruent, so that knowing some of these equalities we can deduce the others.

Triangles ABC and XYZ are congruent if any of the following hold:

- $AB = XY$, $AC = XZ$ and $BC = YZ$. (SSS)
- $AB = XY$, $AC = XZ$ and $\angle BAC = \angle YXZ$. (SAS)
- $AB = XY$ and $\angle BAC = \angle YXZ$ and $\angle ABC = \angle XYZ$. (AAS)

Note: SSA (two sides and an unincluded angle) on its own isn't enough, but we can fix it:

- $AB = XY$ and $AC = XZ$ and $\angle ABC = \angle XYZ$ and $AB < AC$. (Fixed SSA)

In the special case where $\angle ABC = \angle XYZ = 90^\circ$, this is known as RHS.

Result 1.4.2.1. *Let ABC be a triangle, and let M be the midpoint of BC. The following statements are equivalent:*

- $AB = AC$
- $\angle ABC = \angle ACB$
- $AM \perp BC$.

If any of those statements are true the triangle is called *isosceles*.

Problem 1.4.2.1. *Let ABC be an isosceles triangle with $AB = BC$. Let D be a point on BC such that $\angle DBC = 20^\circ$. Let E be a point on AB with $AE = AD$. What is $\angle BDE$?*

Result 1.4.2.2. *Let $ABCD$ be a parallelogram (that is, $AB||CD$ and $BC||DA$). If AC intersects BD at P, then P is the midpoint of AC and of BD.*

1.4.3. *Circles.* A *circle* is the set of points a distance r from a central point O. Thus, any two points on the circle form an isosceles triangle with O.

The circumference (perimeter) of the circle is $2\pi r$, while the area is πr^2.

Result 1.4.3.1. *Let A, B, C, M be points in the plane such that M is the midpoint of AB. Then C lies on the circle centred at M passing through A and B if and only if $\angle ACB = 90°$.*

The next few results are generalisations of this. The proofs are linked in Section 2.3.1.

Result 1.4.3.2. *If A, B, C are points on a circle centred at O with A on the same side of BC as O, then $\angle BOC = 2\angle BAC$.*

Result 1.4.3.3. *If A, B, C are points on a circle centred at O with A and O on opposite sides of BC, then $\angle BOC = 360° - 2\angle BAC$.*

Result 1.4.3.4. *If B and C lie on the same side of AD then there is a circle passing through all of A, B, C, D if and only if $\angle ABD = \angle ACD$.*

Result 1.4.3.5. *If B and D lie on opposite sides of AC, then there is a circle passing through all of A, B, C, D if and only if $\angle ABC + \angle ADC = 180°$.*

Problem 1.4.3.1. *For $n \geq 3$, a pattern can be made by overlapping n circles, each of circumference 1 unit, so that each circle passes through a central point and the resulting pattern has order-n rotational symmetry. For instance, the diagram shows the pattern where $n = 7$.*

If the total length of visible arcs is 60 units, what is n?

1.4.4. *Similarity.* We say that triangles ABC and XYZ are *similar* if for some real r (called the ratio of similitude),

$$\frac{BC}{YZ} = \frac{CA}{ZX} = \frac{AB}{XY} = r,$$
$$\angle ABC = \angle XYZ,\ \angle BCA = \angle YZX,\ \angle CAB = \angle ZXY.$$

We write "ABC is similar to XYZ" as $\triangle ABC \sim \triangle XYZ$.

Triangles ABC and XYZ are similar if any of the following hold:

- $\frac{BC}{YZ} = \frac{CA}{ZX} = \frac{AB}{XY}$ (PPP)
- $\frac{AB}{XY} = \frac{BC}{YZ}$, $\angle ABC = \angle XYZ$ (PAP)
- $\angle ABC = \angle XYZ$, $\angle BCA = \angle YZX$ (AA)

Once again, PPA (two sides and an unincluded angle) doesn't work. However, we can fix it in the same way:

- $\frac{AB}{XY} = \frac{AC}{XZ}$, $\angle ABC = \angle XYZ$ and $AB < AC$. (Fixed PPA)

In the special case where $\angle ABC = \angle XYZ = 90°$, this is known as RHS.

Result 1.4.4.1. *Let ABC and ADE be similar triangles with the same orientation (that is, $\triangle ABC \sim \triangle ADE$ and both triangles are labelled either clockwise or anticlockwise). Then ABD and ACE are also similar and similarly oriented.*

Problem 1.4.4.1. *Let A, B, C, D be points on a circle such that $\triangle ABC$ is equilateral, and D lies on minor arc BC. Prove that $AD = BD + CD$.*

1.4.5. *Areas.* We assign a positive real number, known as an *area*, to each polygon in the plane such that

- the area of a rectangle with sidelengths a and b is ab;
- the areas of two congruent triangles are equal; and
- if two polygons have disjoint interiors then the area of their union equals the sum of their areas.

The area of a polygon $P_1, P_2, \ldots, P_n$ is denoted $|P_1 P_2 \cdots P_n|$. We may use these properties to deduce some well-known facts about areas.

Result 1.4.5.1. *Let ABC be a triangle such that $\angle ABC = 90°$. Then,*

$$|ABC| = \frac{1}{2} \times AB \times BC.$$

Result 1.4.5.2. *Let ABC be a triangle, and let D be a point on line BC such that $AD \perp BC$. Then,*

$$|ABC| = \frac{1}{2} \times AD \times BC.$$

An important special case of this is that two triangles with the same height have areas in the same ratio as their bases, and two triangles with the same base have areas in the same ratio as their heights.

Result 1.4.5.3. *Let ABC and XYZ be similar triangles with ratio of similitude r. Then,*

$$\frac{|ABC|}{|XYZ|} = r^2.$$

Problem 1.4.5.1. *A triangle ABC is divided into four regions by three lines parallel to BC. The lines divide AB into four equal segments. If the second largest region has area 225, what is the area of ABC?*

Problem 1.4.5.2. *Let $ABCD$ be a parallelogram. Point P is on AB produced such that DP bisects BC at X. Point Q is on BA produced such that CQ bisects AD at M. Lines DP and CQ meet at O. If the area of parallelogram $ABCD$ is 192, find the area of triangle POQ.*

Problem 1.4.5.3. *The area of triangle ABC is 300. In triangle ABC, Q is the midpoint of BC, P is a point on AC between C and A such that $CP = 3PA$, R is a point on side AB such that the area of $\triangle PQR$ is twice the area of $\triangle RBQ$. Find the area of $\triangle PQR$.*

Result 1.4.5.4. *In triangle ABC, point P is on AB such that AP bisects $\angle BPC$. Then, $\frac{BP}{PC} = \frac{BA}{AC}$.*

1.4.6. *Pythagoras.* Pythagoras' Theorem is arguably the most famous theorem in mathematics. You should aim to find multiple proofs of it, to cement your understanding of the techniques developed in this section on geometry.

Four proofs are given in the linked solutions, with a fifth in Section 2.3.2.

Result 1.4.6.1. *If $\angle ABC = 90°$, then $AB^2 + BC^2 = AC^2$.*

Problem 1.4.6.1. *Let $ABCD$ be a square and let E and F be points on BC and CD, respectively, such that AEF is an equilateral triangle.*

Find the length BE.

1.4.7. *Trigonometry.* These concepts are easiest to define working on the coordinate plane. Let O be the origin, and let A be the point $(1,0)$. See the diagram. Let $P=(x,y)$ be a point on the unit circle (that is, $x^2+y^2=1$) such that the counterclockwise angle $\angle AOP$ is θ. We define

$$\cos(\theta)=x,\ \sin(\theta)=y,\ \tan(\theta)=\frac{y}{x}.$$

Result 1.4.7.1. *For all θ we have* $\cos(-\theta)=\cos(\theta)=\sin(90-\theta)=-\cos(180-\theta)$.

Result 1.4.7.2. *Let ABC be a triangle with $\angle ABC=90^\circ$. Then,*

$$\sin(\angle ACB)=\frac{AB}{AC},\ \cos(\angle ACB)=\frac{BC}{AC},\ \tan(\angle ACB)=\frac{AB}{BC}.$$

Result 1.4.7.3. *We have*

$$\sin(0)=0,\ \sin(30)=\frac{1}{2},\ \sin(45)=\frac{\sqrt{2}}{2},\ \sin(60)=\frac{\sqrt{3}}{2},\ \sin(90)=1.$$

For the next two results we define ABC to be a triangle with sidelengths $a=BC,\ b=CA,\ c=AB$. We use the shorthand $\angle A=\angle BAC$ and similarly for $\angle B$ and $\angle C$.

Result 1.4.7.4.

$$\frac{a}{\sin\angle A}=\frac{b}{\sin\angle B}=\frac{c}{\sin\angle C}.$$

Further, if A,B,C lie on a circle with radius R then these quantities are all equal to $2R$.

Result 1.4.7.5.

$$\cos\angle A=\frac{b^2+c^2-a^2}{2bc}.$$

Problem 1.4.7.1. *In quadrilateral $PQRS$ we have $PS=5,\ SR=6,\ RQ=4$, and $\angle P=\angle Q=60^\circ$. Find the length of PQ.*

Problem 1.4.7.2. *Let $ABCD$ be a trapezium with $AB\|CD$ such that its vertices A,B,C,D lie on a circle with centre O. Let the diagonals AC and BD intersect at a point M. Assume that $\angle AMD=60^\circ$ and $MO=1$.*

What is the difference between the lengths AB and CD?

1.5. **Number Theory.**

1.5.1. *Divisibility.* For integers a and b, we say $a\mid b$ (read "a divides b") if there is some integer c with $b=a\times c$.

Result 1.5.1.1. *If a and b are positive integers with $a\mid b$ then $a\le b$.*

Result 1.5.1.2. *If $a\mid b$ and $a\mid c$ then $a\mid bx+cy$ for all integers x and y.*

Problem 1.5.1.1. *Find all integers n such that $n^2+1\mid n^3+n^2-n-15$.*

1.5.2. *Primes.* We define a *prime* as a positive integer larger than 1 which is not divisible by any positive integer other than 1 and itself.

Result 1.5.2.1. *Every positive integer larger than 1 has a prime factor.*

This was what allowed us to prove Result 1.1.2.1.

The Fundamental Theorem of Arithmetic (proved in Section 2.4.1) states that each positive integer has a unique prime factorisation; that is, we can write a positive integer n uniquely as

$$n = p_1^{e_1} p_2^{e_2} \cdots p_k^{e_k},$$

where p_i are all prime and e_i are all positive integers.

Prime factorisations allow us to view statements about divisibility and multiplication in terms of the exponents e_i.

In what follows, let

$$a = p_1^{e_1} p_2^{e_2} \cdots p_m^{e_m}, \ b = q_1^{f_1} q_2^{f_2} \cdots q_k^{e_k}.$$

Result 1.5.2.2. *$a \mid b$ if and only if for each i we have that $p_i = q_j$ for some j, and that $e_i \leq f_j$.*

Result 1.5.2.3. *a is a perfect kth power if and only if $k \mid e_i$ for all i.*

Result 1.5.2.4. *The lcm is found by taking the maximum power of each prime that divides either a or b; the gcd is found by taking the minimum power of each prime that divides both a and b. Therefore, $\gcd(a, b) \times lcm(a, b) = ab$.*

Result 1.5.2.5. *If $a \mid c$ and $b \mid c$ then $lcm(a, b) \mid c$.*

1.5.3. *Factorisations.* Factorisations allow us to make equations nicer to work with. For example, solving the equation $xy + x + y = 3$ over the integers becomes much easier when we express it as $(x + 1)(y + 1) = 4$.

Here are the most useful factorisations:

$$axy + bx + cy = d \iff (ax + c)(ay + b) = ad + bc$$
$$a^k - b^k = (a - b)\left(a^{k-1} + a^{k-2}b + \cdots + b^{k-1}\right)$$

Problem 1.5.3.1. *Find all right-angled triangles with positive integer sides such that their area and perimeter are equal.*

Problem 1.5.3.2. *Prove that $1^k + 2^k + \cdots + n^k$ is divisible by $1 + 2 + \cdots + n$ for all positive integers n and odd positive integers k.*

Problem 1.5.3.3. *Prove that if $2^n + 1$ is prime for a positive integer n, then n is a power of 2.*

1.5.4. *Number bases.*

Result 1.5.4.1. *Given positive integers $n > 1$ and k, prove that there are unique nonnegative integers $m, a_0, a_1, \ldots, a_m$ such that $a_m > 0$, $0 \leq a_i < n$ for all i, and*

$$k = a_0 n^0 + a_1 n^1 + \cdots + a_m n^m.$$

This is called the *base-n representation* of k, and is often written as

$$(\overline{a_m a_{m-1} \cdots a_0})_n.$$

When the a_is are specific digits, the parentheses and the bar over the a_is are often dropped. So for example, the decimal number 16 can be expressed as 10000_2.

Problem 1.5.4.1. *If $234_{b+1} - 234_{b-1} = 70_{10}$, what is 234_b in base 10?*

Problem 1.5.4.2. *A sequence $\{a_i\}$ begins with $a_1 = 0$, and for each i the number a_{i+1} is the smallest integer larger than a_i which is not equal to $2a_k - a_j$ for any j, k with $1 \leq j < k \leq i$. So the sequence begins 0,1,3,4,9,...*

Find the 2000th term of this sequence.

2. Not-so-basics

2.1. Algebra.

2.1.1. *Systems of equations.*

2.1.2. *Polynomials.*

A polynomial is just an expression of the form

$$P(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_0 x^0,$$

where each of the a_is is a constant and $a_n \neq 0$. The number n is called the *degree* of the polynomial. The term $a_n x^n$ is called the *leading term*. The expression $P(x) = 0$ is also a polynomial, and it's defined to have degree $-\infty$.

Result 2.1.2.1. *Let $A(x)$ and $B(x)$ be polynomials. Then,*

- $\deg(A \pm B) \leq \max(\deg A, \deg B)$. *Equality occurs unless the leading terms of A and B cancel.*
- $\deg(A \times B) = \deg(A) + \deg(B)$.

Many proofs in polynomial questions proceed by (strong) induction on the degree. The following few examples illustrate a few of the ways in which we can reduce a polynomial to a smaller-degree polynomial.

Result 2.1.2.2. *Let $A(x)$ and $B(x)$ be polynomials with $B(x) \neq 0$. There exist unique polynomials $Q(x)$ and $R(x)$ such that $\deg R < \deg B$ and*

$$A(x) = Q(x)B(x) + R(x).$$

Solution

We say that a polynomial $P(x)$ divides another polynomial $Q(x)$ if there is a polynomial $R(x)$ such that $P(x) = Q(x)R(x)$. With this terminology, an important corollary is that for any real number r and any polynomial $A(x)$, the polynomial $x - r$ divides the polynomial $A(x) - A(r)$.

Result 2.1.2.3. *If a polynomial*

$$P(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_0 x^0$$

has roots $r_1, \ldots, r_n$ then for each i,

$$a_{n-i} = (-1)^i a_n \sum_{j_1 < \cdots < j_i} \prod_{k=1}^{i} r_{j_k}.$$

Solution

2.2. Combinatorics.

2.3. Geometry.

2.3.1. *Directed Angles.*

2.3.2. *The Incircle.*

2.4. Number Theory.

2.4.1. *Fundamental Theorem of Arithmetic.*

2.4.2. *Modular Arithmetic.*

Part 3. Practice

Part 4. Solutions

1. Basics

1.1. Methods of Proof.

Result 1.1.1.1. If you start trying some small cases, what you'll eventually find is that if n is an even integer, then n^2 leaves a remainder of 0 when divided by 4, and if n is an odd integer, then n^2 leaves a remainder of 1 when divided by 4. Once you've conjectured this, all that's left is to recall what it means for a number to be even or odd, and then the proof falls out quite naturally:

Proof. Let the perfect square be n^2. We split into cases depending on the parity of n.

- If n is even, let $n = 2m$ for some integer m. Then
$$n^2 = (2m)^2 = 4m^2,$$
which leaves a remainder of 0 when divided by 4.
- If n is odd, let $n = 2m+1$ for some integer m. Then
$$n^2 = (2m+1)^2 = 4m^2 + 4m + 1 = 4(m^2+m) + 1,$$
which leaves a remainder of 1 when divided by 4.

In either case, the remainder left when dividing n^2 by 4 is either 0 or 1, which is what we wanted to prove. □

Result 1.1.2.1. The key here is to assume, for contradiction, that there are only finitely many primes. Then we want to prove a suitable contradiction — a nice way of doing this is to find a number that isn't 1 but isn't divisible by any of our finitely many primes. The idea of constructing such a number by multiplying everything and adding 1 is surprisingly common in Olympiad maths.

Proof. Assume that there are only finitely many primes $p_1, p_2, \ldots, p_n$. Then, consider the number $A = p_1 p_2 \cdots p_n + 1$. Clearly A is a positive integer larger than 1, so it must have a prime factor, which means that for some i, $p_i \mid A$. But $p_i \mid A - 1$, so $p_i \mid A - (A-1) = 1$, which is our contradiction. □

Problem 1.1.3.1. This is more an exercise in logic than in maths. You could be stuck for ages trying to prove the problem directly, but as soon as you try to use some indirect approach (like contrapositive, contradiction, or assuming one part of the conclusion is false and proving the other is true) the problem pretty much solves itself. I think the cleanest solution in this particular case uses the contrapositive.

Proof. We prove the contrapositive: that if a and b are both rational, then so is $a+b$.

Let $a = \frac{w}{x}$, $b = \frac{y}{z}$. Then
$$a + b = \frac{w}{x} + \frac{y}{z} = \frac{wz + xy}{xz},$$
which is clearly rational. □

Result 1.1.4.1. This is a classic induction problem. Apart from being instructive because it isolates the idea of induction, it does highlight a minor point. In the inductive step, it's just as acceptable to assume the problem is true for k and prove it for $k+1$ as to assume the problem is true for $k-1$ and prove it for k. In this particular case, the latter is somewhat easier.

Proof. We prove this by induction on n.

Base case $n = 1$: We have LHS $= 1 = \frac{1\times 2}{2} =$ RHS.

Inductive step: Assume the problem is true for $n = k - 1$. Then,

$$\begin{aligned} 1 + 2 + \cdots + k &= (1 + 2 + \cdots + k - 1) + k \\ &= \frac{k(k-1)}{2} + k \\ &= \frac{k(k-1) + 2k}{2} \\ &= \frac{k(k+1)}{2}, \end{aligned}$$

so the problem is true for $n = k$. □

Result 1.1.4.2. This is a good example of how solutions to the same problem can look quite different when written up with PMI or with well-ordering.

First assume PMI. A direct induction will prove the statement quite easily.

Now we can try assuming well-ordering. Looking for a contradiction, assume there is some positive integer $a < 1$. A contradiction is easiest to find by applying well-ordering to the set of all positive integers.

Let's write this up.

Proof. First, assume PMI.

Base case $n = 1$: clearly $1 \geq 1$.

Inductive step: assume that $k \geq 1$. Then $k + 1 > k \geq 1$, completing the induction.

Now, assume well-ordering. Assume there's some $a \in \mathbb{N}$ with $a < 1$. Since $\mathbb{N}$ is nonempty, we may let b be its smallest element. Since ab is a positive integer, we have $b \leq ab$. On the other hand, since $a < 1$ we get that $ab < b$. This contradiction concludes the proof. □

Result 1.1.4.3. Since this is an "if and only if" problem, we will probably need to find separate proofs in each direction.

First, let's use induction to prove well-ordering. Our desired conclusion is that any nonempty set of positive integers has a smallest element. Intuitively, what we would like to do is to check if 1 is in it, then if 2 is in it, and so on until we find an element that's in it. Once we've found that element, we wish to prove that it's the smallest element. However, this is hard to write up because we'd like our process to "finish" after a finite number of steps, while induction only talks about things that are true for all positive integers. To make our lives easier, we instead try to prove the contrapositive of well-ordering: that if S is a set of positive integers with no smallest element then S is empty.

Now, our argument looks similar: we claim that 1 is not in S, that 2 is not in S, and so on forever. To make sure that we can actually keep repeating this argument, we need to show that if we found an element in S it must be the smallest element. Proving this requires tweaking our inductive assumption.

Now, let's use well-ordering to prove PMI. You'll probably get nowhere if you aren't completely clear about what you're trying to prove (you may as well replace PMI by gibberish), so let's write out PMI in full:

Let S be a set of positive integers. If $1 \in S$, and if the statement $\forall a \in S,\ a+1 \in S$ is true, then S contains all positive integers.

Once again we use indirect proof — this time it's proof by contradiction (contrapositive also works). Let's assume that there is a set S that satisfies both conditions but doesn't contain all positive integers. We hope to use well-ordering to find a contradiction.

The key idea is to consider the smallest integer that isn't in S, which is possible by well-ordering. Then the condition implies that either it's not the smallest, or $1 \notin S$ — a contradiction either way.

Let's write it up.

Proof. First we prove that if PMI is true, then so is well-ordering. Assume PMI, and we'll prove the contrapositive of well-ordering: that if S is a set of positive integers with no smallest element, then it is empty.

I prove by strong induction that for each positive integer n, if $m \in S$ then $m \geq n+1$.

Base case $n = 1$: if $1 \in S$, then 1 would be the smallest element in S (since if $m \in S$ then m is a positive integer so $m \geq 1$). So since S has no smallest element, 1 is not the smallest element in S so taking the contrapositive we get that 1 is not in S.

Inductive step: Assume that for any $m \in S$ we know that $m \geq k+1$. I claim that $k+1 \notin S$. Indeed, if $k+1$ were in S, then it would be the smallest element of S. But since S has no smallest element, $k+1$ can't be in S. Therefore, if $m \in S$ we know $m-(k+1)$ is a positive integer so $m-(k+1) \geq 1$, so $m \geq k+2$, as required for the inductive step.

This completes the induction. Now if $n \in S$ we've proven that $n \geq n+1$, a contradiction so S must be empty.

Now I prove that if well-ordering is true, then so is PMI. Assume for contradiction that well-ordering is true but PMI is not. Then, there is a set S of positive integers that contains 1 and such that for each $a \in S,\ a+1 \in S$ but that does not contain all positive integers. Then, the set $\mathbb{N} \setminus S$ is nonempty so by well-ordering it contains a smallest element a.

Since we know that $1 \in S$, we know that $a \neq 1$. So since $a \geq 1$, $a-1$ is a positive integer. Since $a-1 < a$ and a is the smallest member of $\mathbb{N} \setminus S$, $a-1$ is not in $\mathbb{N} \setminus S$ so $a-1 \in S$. So $(a-1)+1 = a \in S$, which contradicts the assumption that $a \notin S$. $\square$

1.2. **Algebra.**

Problem 1.2.2.1. Not much to say here — interpret as a system of linear equations and solve however you like.

Answer: 26.

Proof. Let the two numbers be a and b, with $a > b$. Then, $b = a - 20$ so

$$\begin{aligned} a + 4 &= 3(a - 20 + 4) \\ &= 3a - 48 \\ 2a &= 52 \\ a &= 26, \end{aligned}$$

so the larger of the two numbers is 26.

To prove that this actually works, note that if $a = 26$ and $b = 6$, then $a - b = 20$ and $a + 4 = 30 = 3 \times 10 = 3(b + 4)$ as needed. □

Problem 1.2.2.2. Since we don't like the 1s in our equations, we subtract two equations to get rid of them. Alternatively, we subtract two equations because that's one of the most obvious things to do with a system of equations. Either way, once we've done that the rest of the problem is pretty routine.

Answer: $(x, y, z) = (1, 1, 1), \left(-2, -2, \frac{5}{2}\right), \left(-2, \frac{5}{2}, -2\right), \left(\frac{5}{2}, -2, -2\right)$.

Proof. Subtract the third equation from the first:

$$\begin{aligned} xy - xz &= 2z - 2y \\ x(y - z) + 2(y - z) &= 0 \\ (x + 2)(y - z) &= 0 \end{aligned}$$

So either $x = -2$ or $y = z$. Similarly we can deduce that either $z = -2$ or $x = y$. Now we split into four cases:

- $x = -2,\ z = -2$. Then $2y = zx + 1 = 5 \implies y = \frac{5}{2}$.
- $x = -2,\ x = y$. Then similar to the above we get $z = \frac{5}{2}$.
- $y = z\ z = -2$. In the same way we get $x = \frac{5}{2}$.
- $y = z,\ x = y$. Then $x^2 + 1 = 2x \implies (x - 1)^2 = 0 \implies x = 1$, so $x = y = z = 1$.

So the only solutions are what we claim they are. It is easy to check that these solutions all satisfy the original equations. □

Result 1.2.3.1. First, there are a few ways of seeing that this is the answer.

One way is to try to factorise $x^2 - 2mx + p$ as $(x - a)(x - b)$. Then $a + b = 2m$ and $ab = p$, and a and b are the values of x we want. Then the key idea is that the way of using the $a + b = 2m$ condition is to let $a = m + c,\ b = m - c$ so that

$$p = ab = (m + c)(m - c) = m^2 - c^2,$$

so that $c = \sqrt{m^2 - p}$.

Another way is to notice that $x^2 - 2mx + p$ looks a lot like $x^2 - 2mx + m^2 = (x - m)^2$. I'll do the rest in the proof.

Proof. We have

$$\begin{aligned} x^2 - 2mx + p &= x^2 - 2mx + m^2 + p - m^2 \\ &= (x - m)^2 - \left(m^2 - p\right) \end{aligned}$$

If $m^2 - p < 0$ then clearly there are no solutions. Otherwise, we have

$$\begin{aligned} x^2 - 2mx + p &= (x-m)^2 - \left(\sqrt{m^2-p}\right)^2 \\ &= \left(x - m - \sqrt{m^2-p}\right)\left(x - m + \sqrt{m^2-p}\right), \end{aligned}$$

so it equals 0 if and only if $x = m \pm \sqrt{m^2 - p}$. ☐

Result 1.2.3.2. The key here is to get this equation into a form such that we can apply the previous result.

Proof. We have

$$\begin{aligned} 0 &= ax^2 + bx + c \\ 0 &= x^2 + \frac{b}{a}x + \frac{c}{a} \\ &= x^2 - 2\frac{-b}{2a}x + \frac{c}{a} \\ x &= \frac{-b}{2a} \pm \sqrt{\frac{b^2}{4a^2} - \frac{c}{a}} \\ &= \frac{-b \pm \sqrt{4a^2\left(\frac{b^2}{4a^2} - \frac{c}{a}\right)}}{2a} \\ &= \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \end{aligned}$$

as needed.

If $\Delta < 0$ there are clearly no solutions. If $\Delta = 0$ the unique solution is $x = \frac{-b}{2a}$. If $\Delta > 0$ there are two solutions given by our equation. ☐

Problem 1.2.3.1. Once again not much to say here — interpret the number bases in the usual way and solve the resulting quadratic.

Answer: 57.

Proof. We have

$$\begin{aligned} 111_b &= 212_{b-2} \\ b^2 + b + 1 &= 2(b-2)^2 + (b-2) + 2 \\ &= 2b^2 - 7b + 8 \\ 0 &= b^2 - 8b + 7 \\ b &= 4 \pm \sqrt{4^2 - 7} \\ &= 4 \pm 3 \end{aligned}$$

Therefore, since $b > 2$ we get $b = 7 \implies x = 7^2 + 7 + 1 = 57$.

Finally, 57 is indeed 111 in base 7 and 212 in base 5. ☐

Problem 1.2.3.2. The key here is to use the discriminant (Δ in Result 1.2.3.2). In particular, it's enough to prove that at least one of the two Δs is nonnegative. The easiest way of doing this is to assume that the first is negative and prove that the second isn't.

Proof. Assume that there is no real number x such that $x^2+(r+1)x+s=0$. Then the discriminant $(r+1)^2-4s$ is negative, so $4s>(r+1)^2$.

Since $(r+1)^2 \geq 0$, we know that $s>0$. Also, $4(s-r)>(r+1)^2-4r=(r-1)^2 \geq 0$. So since $s>0$ and $4(s-r)>0$, their product $4s^2-4sr$ is also positive so the discriminant of the second quadratic is positive, meaning that it has at least one real solution. □

Problem 1.2.3.3. At first glance, this looks like our methods can't help since we have a cubic not a quadratic. However, the same trick used in Result 1.2.3.1 of recognising a common factorisation does in fact work.

Answer: $x=-1-\sqrt[3]{4}$.

Proof. We subtract 4 from both sides to make the LHS into something we recognise:

$$\begin{aligned} x^3+3x^2+3x+1 &= -4 \\ (x+1)^3 &= -4 \\ x+1 &= -\sqrt[3]{4} \\ x &= -1-\sqrt[3]{4}. \end{aligned}$$

To show that this number works, we can either substitute it in and do the algebra, or notice that each step above was actually an equivalence so the implications run backwards as well. □

Result 1.2.4.1. Since we want to use the fact that squares are nonnegative, we collect all the terms on one side. The rest is recognition, which can be helped by noticing that we want equality to occur when $a=b$.

Proof.

$$\text{RHS} - \text{LHS} = \frac{a+b}{2} - \sqrt{ab} = \frac{a+b-2\sqrt{ab}}{2} = \frac{(\sqrt{a}-\sqrt{b})^2}{2} \geq 0,$$

as needed. □

Problem 1.2.4.1. We have to have a 0 and a 2015 in the set, but apart from them the rest of the terms should be as small as possible. This means that we can apply Result 3 to get a function we want to minimise. A little algebraic trickery means it's enough to minimise

$$n + \frac{4032}{n}.$$

Then, by Result 7, the minimum of this over $\mathbb{R}$ is $2\sqrt{4032}$ at $n=\sqrt{4032} \approx 63.5$, which means either 63 or 64 should minimise the expression over $\mathbb{N}$. In fact both do, so we should try to force the expression into something that looks like $(n-63)(n-64)$, and indeed doing that solves the problem.

Answer: 62.

Proof. Let n be the number of elements in S, and let $S=\{s_1,s_2,\ldots,s_n\}$, where the s_is are in increasing order. Then

$$s_i \geq i-1 \,\forall\, i<n,$$

and $s_n = 2015$, so the average is at least

$$\begin{aligned}\frac{0+1+\cdots+n-2+2015}{n} &= \frac{\frac{(n-2)(n-1)}{2}+2015}{n}\\ &= \frac{n^2-3n+4032}{2n}\\ &= \frac{n^2-127n+4032}{2n}+62\\ &= \frac{(n-63)(n-64)}{2n}+62.\end{aligned}$$

Since n is an integer, the first term is 0 if n is either 63 or 64 and positive otherwise, which means that the minimum value is 62, achieved when S is either $\{0, 1, \ldots, 61, 2015\}$ or $\{0, 1, \ldots, 61, 62, 2015\}$. □

Result 1.2.5.1. There are three ways I know of doing this. One of them is a standard induction, but the other two are more interesting.

For the first way, we notice that we already know the special case (see Result 1.1.4.1) where $a = 0$ and $b = 1$. A little algebra allows us to reduce the whole problem to this particular case.

Proof. We have

$$\begin{aligned}\sum_{i=0}^{n}(a+bi) &= \sum_{i=0}^{n} a + \sum_{i=0}^{n} bi\\ &= a(n+1) + b\sum_{i=0}^{n} i\\ &= a(n+1) + b\frac{n(n+1)}{2}\\ &= \frac{(2a+bn)(n+1)}{2},\end{aligned}$$

as needed. □

For the second way, we use a trick called *Gaussian pairing* — we pair the first term with the last term and so on — so that each pair has the same sum.

Proof. We have

$$\begin{aligned}\sum_{i=0}^{n}(a+bi) &= \sum_{i=0}^{n}(a+b(n-i))\\ &= \frac{1}{2}\left(\sum_{i=0}^{n}(a+bi) + \sum_{i=0}^{n}(a+b(n-i))\right)\\ &= \frac{1}{2}\left(\sum_{i=0}^{n}(2a+bn)\right)\\ &= \frac{(n+1)(2a+bn)}{2},\end{aligned}$$

as needed. □

Result 1.2.5.2. The main idea here comes from extending a couple of the common factorisations:

- $1 - r^2 = (1 - r)(1 + r)$
- $1 - r^3 = (1 - r)(1 + r + r^2)$

We get the $a = 1$ case of the general formula given in 1.2.1. Let's write it up.

Proof. We have
$$1 - r^{n+1} = (1 - r)(1 + r + r^2 + \cdots + r^n),$$
since all the middle terms cancel. Dividing both sides by $1 - r$ yields the desired result. □

Problem 1.2.5.1. So you sum your sequence using Result 1.2.5.1, which gets you a closed form.

Then you're left having to prove that $a+1$ and $2n+a$ cannot both be powers of 2. Playing around with special cases tells you at least one is odd, and from there it's easy to finish.

Proof. From Result 1.2.5.1, we have that the given sum is equal to
$$\frac{(a + 1)(2n + a)}{2}.$$
So assume, for contradiction, that this is a power of 2. Then we have that twice the sum is also a power of 2, so $(a + 1)(2n + a)$ is a power of 2. Since $a + 1$ and $2n + a$ both divide powers of 2, they must each be a power of 2. Since a and n are positive integers, both $a + 1$ and $2n + a$ are at least 2 so they're even. So their difference, $2n - 1$, must also be even, which is a contradiction. □

1.3. **Combinatorics.**

Result 1.3.1.1. Not much to say here: just apply the multiplication principle.

Proof. There are n ways to choose the first thing, $n - 1$ ways to choose the second (since the first has been chosen), $n-2$ to choose the third, and so on up to $n-k+1$ for the kth. Thus the total number of ways is
$$n(n - 1) \cdots (n - k + 1),$$
as claimed. □

Problem 1.3.1.1. So you try some small cases because of course you do, and you notice that every 2 goals apart from the first 2, the number of ways is multiplied by 3. With the help of like, a tree diagram, it's easy to see that after an odd number of goals there are 3 ways of scoring the next 2. The rest is easy:

Answer: 972.

Proof. There are 2 ways for the first goal to be scored.

Assume an odd number of goals has been scored. Then the difference between the two teams' goals is 1, so we can WLOG team A has one more goal than team B. Then the next two goals can be AB, BA, BB so there are 3 ways for the next two goals to be scored.

Finally, there are 2 ways for the last 2 goals to be scored.

Thus, the total number of ways for 12 goals to be scored is $2 \times 3^5 \times 2 = 972$. □

Result 1.3.2.1. The key to proving this is to notice that we have already proven Result 1.3.1.1 where order does matter. To go from order mattering to order not mattering, we need to find out how much we've overcounted by: that is, how many permutations, where order matters, go to the same combination, where order doesn't.

Proof. By Result 1.3.1.1, there are $\frac{n!}{k!}$ ways of choosing k things from n, where order matters.

Let there be x ways of choosing k things from n, where order doesn't matter. Then for each such way, each permutation of those k things is counted in our $\frac{n!}{(n-k)!}$ from above. Applying Result 1.3.1.1 again, we have that $k! \times x = \frac{n!}{(n-k)!}$, which rearranges to the claimed formula. □

1.3.1. *Problem 1.3.2.1.* So upon reading the problem we can let our palindromes be $\overline{xyx}$ and $\overline{ztz}$. Clearly, $x > z$, after which it's clear why $y \geq t$.

So by the multiplication principle, it suffices to count two things:

- The number of ways of choosing digits x and z with $x > z > 0$
- The number of ways of choosing digits y and t with $y \geq t$.

Both these things are quite easy to count using this section's formula.

Answer: 1980.

Proof. Let the palindromes be $\overline{xyx}$ and $\overline{ztz}$. Since their difference is 3 digits and $a > b$, we have $x > z$.

Since the second digit of our subtraction can't carry, we must have $y \geq t$. So it's enough to count the number of ways we can choose x and z, and separately the number of ways we can choose y and t.

Since x and z are distinct digits from 1 to 9, the number of choosing them is $\binom{9}{2} = 36$. Since y and t are not-necessarily-distinct digits from 0 to 9, the number of ways of choosing them is $\binom{10}{2} + 10 = 55$. Thus, the answer is $36 \times 55 = 1980$. □

2. Not-so-basics

2.1. **Algebra.**

Result 2.1.2.2. It is natural to split the proof into two parts: existence and uniqueness. Uniqueness is the easier part: it can be proven by the usual method of assuming two representations exist, and proving they are in fact the same.

The proof of existence proceeds by induction on the degree. The key is to reduce $A(x)$ to a polynomial with smaller degree by cancelling the leading term. If we do this in a way that lets us control $Q(x)$ and $R(x)$, the induction will work.

Proof. First we prove such a representation exists, by strong induction on the degree of A. Let $d = \deg B$.

Base cases $\deg A < d$: clearly $Q(x) = 0,\ R(x) = B(x)$ satisfies all the conditions.

Inductive step: let $a_n x^n$ be the leading term of A, and let $b_d x^d$ be the leading term of B. We may write

$$A(x) = \frac{a_n}{b_d} x^{n-d} B(x) + A_1(x)$$

for some polynomial A_1. Since the first term on the RHS cancels the $a_n x^n$, this polynomial $A_1(x)$ can be represented as $Q_1(x)B(x) + R_1(x)$. But then choosing $Q(x) = Q_1(x) + \frac{a_n}{b_3} x^{n-d} B(x),\ R(x) = R_1(x)$ provides a representation for A.

Now we prove the representation is unique. Assume there are polynomials Q_1, R_1, Q_2, R_2 such that

$$A(x) = B(x)Q_1(x) + R_1(x) = B(x)Q_2(x) + R_2(x).$$

Then, $B(x)(Q_1(x) - Q_2(x)) = R_2(x) - R_1(x)$. The degree of the RHS is less than d; therefore, so is the degree of the LHS. But since the degree of the LHS is at least D unless $Q_1 = Q_2$, we get $Q_1 = Q_2$ and therefore $R_1 = R_2$. $\square$

Result 2.1.2.3. First we need to understand what the problem is saying. Try special cases: $i = 0,\ i = n,\ n = 1,\ n = 2$ and so on, until you know what it's saying.

The idea here is to use the corollary from the previous result multiple times to factorise our polynomial fully, then expand it again. Then when we extract the x^{n-i} coefficient, each term that contributes to it is a product where x appears $n - i$ times, and the rest is a product of i (r_i)s and a constant term. Since each combination of i $(-r_i)$s appears exactly once in the expansion, we get the claimed formula.

The neatest way of writing this up is to use induction.

Proof. By induction on the degree.

Base case $n = 0$: there are no r_is so all that we have to prove is $a_0 = a_0$, which is obvious.

Inductive step: Since r_n is a root of $P(x)$, we can write $P(x) = (x - r_n)Q(x)$ for some polynomial $Q(x)$ of degree $n - 1$. Then we know that $Q(x)$ has roots $r_1, \dots, r_{n-1}$ so by the inductive hypothesis, we know that in $Q(x)$:

- The coefficient of x^{n-i} is
$$(-1)^{i-1} a_n \sum_{j_1 < \cdots < j_{i-1}} \prod_{k=1}^{i-1} r_{j_k}.$$
- The coefficient of x^{n-i-1} is
$$(-1)^{i} a_n \sum_{j_1 < \cdots < j_{i}} \prod_{k=1}^{i} r_{j_k}.$$

Since the coefficient of x^{n-i} in $P(x)$ is the coefficient of x^{n-i-1} in $Q(x)$ minus r_n times the coefficient of x^{n-i} in $Q(x)$, it's what we claim it is. Since this argument works for each i, the induction is complete. $\square$

Part 5. Other resources

1. Basics

- AOPS Resources
- Methods of Proof
- How to write a solution
- OTIS Excerpts, chapter 1

2. NT

- Intermediate Number Theory
- Olympiad Number Theory through Challenging Problems
- Modern Olympiad Number Theory

3. Algebra

- OTIS Excerpts, chapters 2 to 5
- Polynomials
- Sequences

4. Combi

- Combinatorics
- Olympiad Combinatorics
- Combinatorics and Combinatorial Geometry

5. Geo

- EGMO Chapters 1 to 3
- EGMO Chapter 8
- A Beautiful Journey through Olympiad Geometry
- Lemmas in Olympiad Geometry

6. Problems

- Melbourne Uni Maths Comp past papers
- BMO past papers
- International contest collection (see especially EGMO, IMO, IMO Shortlist, RMM)

7. Assorted

- OTIS Excerpts
- HDIGH Olympiad page
- HDIGH Olympiad resources
- Evan Chen's page
- Po-Shen Loh's page
- CJ Quines' page
- Yufei Zhao's page
- Alexander Remonov's page
- Konrad Pilch's page
- Problem Solving Strategies

Todos

- Master listing of results
- Observations to add:
  - Angles to similarity to lengths to similarity to conclusion
- Diagram for problem 1.4.3.1 and for trigonometry definition