

Division Algorithm, Euclid and Bezout

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1 Division Algorithm

- Prove that if a is an integer and b is a positive integer, there is a unique pair (q, r) of integers such that $0 \leq r < b$ and $a = qb + r$.
- Find all integers n such that $2n^2 + 1 \mid n^3 + n^2 - n - 1$.

2 Euclid's Algorithm

We define the *greatest common divisor* of two integers a and b , not both of which are 0, as the largest positive integer d such that $d \mid a$ and $d \mid b$. We notate it by $\gcd(a, b)$.

- Let a and b be integers. Prove that if $a = qb + r$ then $\gcd(a, b) = \gcd(b, r)$.
- Find $\gcd(72, 30)$ by applying the previous result repeatedly. (This is Euclid's Algorithm.)
- Prove that for all positive integers a, m, n we have

$$\gcd(a^m - 1, a^n - 1) = a^{\gcd(m, n)} - 1.$$

3 Bezout's Identity

- Let a and b be integers. Prove that there are integers c and d such that $ac + bd = \gcd(a, b)$.
- Prove that if n, a, b are positive integers such that $n \mid ab$ and $\gcd(n, a) = 1$ then $n \mid b$.
- Let a and b be integers with $\gcd(a, b) = 1$, and let c be an integer. Prove that if $a \mid c$ and $b \mid c$ then $ab \mid c$.
- Let a, b, c, d be positive integers with $\gcd(a, b) = 1$. Prove that there is an integer e such that $a \mid e - c$ and $b \mid e - d$.

4 Problems

1. Let a, b, c, d be positive integers with $ab = cd$. Prove that there exist positive integers p, q, r, s such that $a = pq, b = rs, c = pr, d = qs$.

2. Prove that for positive integers $m, n > 2$ we cannot have

$$2^m - 1 \mid 2^n + 1.$$

3. Find all positive integers n such that $3^{n-1} + 5^{n-1} \mid 3^n + 5^n$.

4. Let n be a composite positive integer. Calculate

$$\gcd((n-1)! + 1, n!).$$

5. Assume that $p_1, \dots, p_m, q_1, \dots, q_n$ are primes such that

$$p_1 p_2 \cdots p_m = q_1 q_2 \cdots q_n.$$

Prove that the q_i s are a permutation of the p_i s.

6. Find all pairs of positive integers a, b such that

$$b^2 - a \mid a^2 + b \quad \text{and} \quad a^2 - b \mid b^2 + a.$$

7. Let m and n be positive integers. Prove that

$$m \mid \gcd(m, n) \binom{m}{n}.$$

8. Find all pairs of positive integers x, y such that $xy^2 + y + 7 \mid x^2y + x + y$.

5 Homework

- (a) Find all integers x, y such that $6x + 2y = 8$.
(b) Find all integers x, y such that $6x + 4y = 8$.
- Let a_1, a_2, \dots, a_n be positive integers, and let d be the largest positive integer such that $d \mid a_i$ for all i .

Prove that there are integers b_1, b_2, \dots, b_n such that

$$d = a_1b_1 + a_2b_2 + \cdots + a_nb_n.$$

- The Fibonacci sequence is defined by $F_1 = F_2 = 1$ and

$$F_{n+1} = F_n + F_{n-1}.$$

Prove that $\gcd(F_n, F_{n+2}) = 1$ for all n .