Bounding

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1 Techniques

- WLOG arguments: break symmetry to bound one of the variables.
- If a and b are integers with $a > b$ then $a \geq b + 1$.
- If a and b are integers with $b \neq 0$ and $a \mid b$ then $|a| \leq |b|$. Thus, try to get big dividing small. Polynomial long division is helpful here.
- Look for "dominant terms" that grow larger than everything else. For sufficiently large n and any $c > 1$, we have $c < \log n < n^c < c^n < n! < n^n$.
- If you need to prove a specfic bound of the form "$f(x) > g(x)$ for $x > k$," try induction.
- If you have a factorial, $n! \equiv 0 \pmod{c}$ for $n \geq c$ so mods are often nice.

2 Problems

1. Find all positive integers k such that there is a positive integer n for which
$$k = \frac{n^2 - 29}{3n + 11}.$$
2. Find all positive integers a, b, c such that $ab + bc + ca = abc + 2$.
3. Find all pairs of positive integers x, y such that $x^3 + y^3 = (x + y)^2$.
4. Find all pairs of positive integers x, y such that $1 + 2^x + 2^{2x} = y^2$.
5. Find all pairs of positive integers x, y such that $x! + 5 = y^3$.
6. Find all pairs a, b of positive integers such that $2017^a = b^6 - 32b + 1$.
7. Find all pairs x, y of positive integers such that $3^x - 8^y = 2xy + 1$.

3 Homework

1. Find all pairs of positive integers x, y such that $x^2 - y! = 2001$.
2. Find all pairs of positive integers x, y such that $y^2(x-1) = x^5 - 1$.
3. Find all triples a, b, c of positive integers such that $a \mid bc - 1,\ b \mid ca - 1$ and $c \mid ab - 1$.