Density Arguments, Floor Functions

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1 Techniques

Density arguments: these involve taking a "global" view that involves estimating how many numbers have a specific property in a specific range. The aim is usually to apply pigeonhole.

Floor functions: recall $\lfloor x \rfloor$ is the largest integer which is at most x. Most often, problems which involve $\lfloor x \rfloor$ will be solved by considering $x - \lfloor x \rfloor$ (also denoted $\{x\}$), or $\lceil x \rceil - x$.

2 Problems

1. Is there a positive integer which can be written as the sum of 2023 distinct 2022nd powers in at least 2021 different ways?
2. Prove that the sequence $a_i = \left\lfloor (\sqrt{2}+1)^i \right\rfloor$ alternates between even and odd integers.
3. Define
$$q(n) = \left\lfloor \frac{n}{\lfloor \sqrt{n} \rfloor} \right\rfloor.$$
Determine all positive integers n for which $q(n) > q(n+1)$.
4. Let P be a polynomial of degree larger than 1 with integer coefficients. Prove that there are infinitely many positive integers which cannot be written in the form $P(x+1) + P(x+2) + \cdots + P(x+k)$ for positive integers x and k.
5. Let x be an irrational number. Prove that there are infinitely many positive integers n such that $\{nx\} < \frac{1}{n}$.
6. Prove that for some constant $C > 0$, the following statement holds:

   Let $m \geq 2$ be an integer, A a finite set of integers (not necessarily positive), and $B_1, B_2, \ldots, B_m$ subsets of A. Suppose that for every $k = 1, 2, \ldots, m$, the sum of the elements of B_k is 2^k. Then A contains at least $\frac{Cm}{\log_2 m}$ elements.

3 Homework

1. Determine all positive integers M such that the sequence $a_0, a_1, a_2, \ldots$ defined by

$$a_0 = M + \frac{1}{2} \qquad \text{and} \qquad a_{k+1} = a_k \lfloor a_k \rfloor \quad \text{for } k = 0, 1, 2, \ldots$$

contains at least one integer term.

2. Let $n > 1$ be an integer, and let a be an integer coprime to n. Prove that there exist integers x, y with $0 < |x| < \sqrt{n}$, $0 < |y| < \sqrt{n}$ and $ay \equiv x \pmod{n}$.

3. Prove that every positive integer is the root of a polynomial all of whose coefficients are of the form $2^a - 2^b$ for positive integers a and b.