# Density Arguments, Floor Functions

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## 1 Techniques

Density arguments: these involve taking a "global" view that involves estimating how many numbers have a specific property in a specific range. The aim is usually to apply pigeonhole.

Floor functions: recall $\lfloor x \rfloor$ is the largest integer which is at most $x$. Most often, problems which involve $\lfloor x \rfloor$ will be solved by considering $x - \lfloor x \rfloor$ (also denoted $\{x\}$), or $\lceil x \rceil - x$.

## 2 Problems

1. Is there a positive integer which can be written as the sum of 2023 distinct 2022nd powers in at least 2021 different ways?
2. Prove that the sequence $a_i = \left\lfloor (\sqrt{2}+1)^i \right\rfloor$ alternates between even and odd integers.
3. Let $r$ be an irrational root of a polynomial $P(x)$ of degree $d$ with integer coefficients. Prove that there is a real number $C$ such that for any integer $q$ we have $\{qr\} \geq \frac{C}{q^{d-1}}$.
4. Define
$$q(n) = \left\lfloor \frac{n}{\lfloor \sqrt{n} \rfloor} \right\rfloor.$$
Determine all positive integers $n$ for which $q(n) > q(n+1)$.
5. Let $P$ be a polynomial of degree larger than 1 with integer coefficients. Prove that there are infinitely many positive integers which cannot be written in the form $P(x+1) + P(x+2) + \cdots + P(x+k)$ for positive integers $x$ and $k$.
6. Let $x$ be an irrational number. Prove that there are infinitely many positive integers $n$ such that $\{nx\} < \frac{1}{n}$.
7. Prove that for some constant $C > 0$, the following statement holds:

   Let $m \geq 2$ be an integer, $A$ a finite set of integers (not necessarily positive), and $B_1, B_2, \ldots, B_m$ subsets of $A$. Suppose that for every $k = 1, 2, \ldots, m$, the sum of the elements of $B_k$ is $2^k$. Then $A$ contains at least $\frac{Cm}{\log_2 m}$ elements.

## 3 Homework

1. Determine all positive integers $M$ such that the sequence $a_0, a_1, a_2, \dots$ defined by

$$a_0 = M + \frac{1}{2} \qquad \text{and} \qquad a_{k+1} = a_k \lfloor a_k \rfloor \quad \text{for } k = 0, 1, 2, \dots$$

   contains at least one integer term.

2. Let $n > 1$ be an integer, and let $a$ be an integer coprime to $n$. Prove that there exist integers $x, y$ with $0 < |x| < \sqrt{n}$, $0 < |y| < \sqrt{n}$ and $ay \equiv x \pmod{n}$.

3. Prove that every positive integer is the root of a polynomial all of whose coefficients are of the form $2^a - 2^b$ for positive integers $a$ and $b$.