Descent, Vieta Jumping, and Pell equations

Andres Buritica

1 Techniques

There is really one technique here: assign some positive integer quantity (the "size" of a solution, e.g. the sum of the absolute valies of the variables) to each solution, take the solution with lowest "size" that doesn't fit your claimed pattern and derive a contradiction.

2 Problems

1. Find all solutions in integers to $a^4 + b^4 = c^2$.
2. Let a and b be positive integers. Prove that if $\frac{a^2+b^2}{ab+1}$ is an integer, then it is a square.
3. Pell Equations:
   - Let d be a positive integer which is not a perfect square. Prove that there exist positive integers x_0, y_0 such that for any pair x, y of positive integers satisfying $x^2 - dy^2 = 1$, there is a positive integer k satisfying
   $$x + y\sqrt{d} = (x_0 + y_0\sqrt{d})^k.$$
   - Find the recurrence relating a solution (x_n, y_n) to the next solution (x_{n+1}, y_{n+1}).
   - Do the same for $x^2 - dy^2 = -1$.

   In the case where the RHS is 1, it is known that there is always a solution. When the RHS is -1, there need not be solutions (eg $x^2 - 3y^2 = -1$ has no solutions).
4. Prove that there are infinitely many triples (a, b, c) of positive integers in arithmetic progression such that $ab + 1$, $bc + 1$ and $ca + 1$ are all perfect squares.
5. Find all solutions in integers to $x^2 + y^2 + z^2 = 2xyz$.
6. Let a and b be two positive integers. Prove that
   $$a^2 + \left\lceil \frac{4a^2}{b} \right\rceil$$
   is not a square.

3 Homework

1. We call a 5-tuple of integers *arrangeable* if its elements can be labelled a, b, c, d, e in some order such that $a - b + c - d + e = 0$. Determine all 2022-tuples of integers such that if we place them in order around a circle, then any 5-tuple of numbers in consecutive positions is arrangeable.

2. Find all positive integers n such that
$$\sqrt{\frac{7^n + 1}{2}}$$
is prime.

3. Let a and b be positive integers. Show that if $4ab - 1$ divides $(4a^2 - 1)^2$, then $a = b$.