Prime factorisations and Arithmetic Functions

Andres Buritica

June 5, 2022

1 Prime factorisations

From last time:

- Assume that $p_1, \ldots, p_m,\ q_1, \ldots, q_n$ are primes such that

$$p_1 p_2 \cdots p_m = q_1 q_2 \cdots q_n.$$

  Prove that the q_is are a permutation of the p_is.

Therefore, each positive integer has a unique prime factorisation (the Fundamental Theorem of Arithmetic). In particular we can write a positive integer n uniquely as

$$n = p_1^{e_1} p_2^{e_2} \cdots p_k^{e_k},$$

where p_i are all prime and e_i are all positive integers.

Prime factorisations allow us to view statements about divisibility and multiplication in terms of the exponents e_i.

In what follows, let

$$a = p_1^{e_1} p_2^{e_2} \cdots p_m^{e_m},\ b = q_1^{f_1} q_2^{f_2} \cdots q_k^{e_k}.$$

- $a \mid b$ if and only if for each i we have that $p_i = q_j$ for some j, and that $e_i \leq f_j$.
- a is a perfect kth power if and only if $k \mid e_i$ for all i.
- The lcm is found by taking the maximum power of each prime that divides either a or b; the gcd is found by taking the minimum power of each prime that divides both a and b.
- $\gcd(a, b) \times \operatorname{lcm}(a, b) = ab$.

2 Arithmetic functions

We define:

- The number of positive divisors function $d(n)$.
- The sum of positive divisors function $\sigma(n)$.
- The totient function $\varphi(n)$: the number of positive integers which are at most n and coprime to n.

A function $f : \mathbb{N} \to \mathbb{R}$ is called multiplicative if for any coprime positive integers a and b, we have

$$f(a)f(b) = f(ab).$$

It's called completely multiplicative if this equation holds for *any* positive integers a and b

- Prove that the values at the primes of a completely multiplicative function completely define the function (unless these values are all 0, in which case $f(1)$ can be 0 or 1).
- Prove that the values at prime powers of a multiplicative function completely define it (once again, unless these values are all 0).
- Prove that d and σ are multiplicative. (φ is also multiplicative, but we will prove this next term.)
- Find formulae for $d(n), \sigma(n), \varphi(n)$ where $n = p_1^{e_1} p_2^{e_2} \cdots p_k^{e_k}$.

3 Problems

1. Prove that $d(n) \leq 2\sqrt{n}$.
2. Let a, b, p be positive integers such that p is prime and $\operatorname{lcm}(a, a+p) = \operatorname{lcm}(b, b+p)$. Prove that $a = b$.
3. Prove that for all n,
$$\sigma(1) + \sigma(2) + \cdots + \sigma(n) \leq n^2.$$
4. Prove that if ab is a perfect square, then so are $\frac{a}{\gcd(a,b)}$ and $\frac{b}{\gcd(a,b)}$.
5. Prove that for all composite n apart from 6,
$$\sqrt{n} \leq \varphi(n) \leq n - \sqrt{n}.$$
6. Let n be an even positive integer such that $\sigma(n) = 2n$. Prove that $n = 2^{p-1}(2^p - 1)$, where p is a prime.
7. For any positive integer n, prove that $\sum_{d|n} \varphi(d) = n$.

4 Homework

1. The cells in a jail are numbered from 1 to 100, and there are 100 buttons also numbered from 1 to 100. For each i, the ith button opens a closed cell and closes an open cell, affecting only the multiples of i.

   For example, if the 47th cell is open and the 94th cell is closed, then pressing the 47th button will close the 47th cell and open the 94th cell.

   Initially all cells are closed. The warden presses the first button, then the second, and so on, for all 100 buttons. Which cells are open at the end?

2. Find all primes p such that $p^{2022} + p^{2023}$ is a perfect square.

3. Prove that for any positive integer n we have $\sigma(n) \geq d(n)\sqrt{n}$.