Density Arguments, Floor Functions

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1 Techniques

Density arguments: these involve taking a "global" view that involves estimating how many numbers have a specific property in a specific range. The aim is usually to apply pigeonhole.

Floor functions: recall $\lfloor x \rfloor$ is the largest integer which is at most x. Most often, problems which involve $\lfloor x \rfloor$ will be solved by considering $x - \lfloor x \rfloor$ (also denoted $\{x\}$), or $\lceil x \rceil - x$.

2 Problems

1. Is there a positive integer which can be written as the sum of 2023 distinct 2022nd powers in at least 2021 different ways?
2. Let a and b be irrational numbers such that $\frac{1}{a} + \frac{1}{b} = 1$. Let $A = \{\lfloor na \rfloor : n \in \mathbb{N}\}$, and $B = \{\lfloor nb \rfloor : n \in \mathbb{N}\}$. Prove that $A \cap B = \emptyset$ and $A \cup B = \mathbb{N}$.
3. Prove that every positive integer is the root of a polynomial all of whose coefficients are of the form $2^a - 2^b$ for positive integers a and b.
4. Let p and q be coprime. Prove that
$$\sum_{i=1}^{q-1} \left\lfloor \frac{ip}{q} \right\rfloor = \frac{(p-1)(q-1)}{2}.$$
5. Find all positive integers n such that $1 + \lfloor \sqrt{n} \rfloor$ divides n.
6. Let r be an irrational root of a polynomial $P(x)$ of degree d with integer coefficients. Prove that there is a real number C such that for any integer q we have $\{qr\} \geq \frac{C}{q^{d-1}}$.
7. Let x be an irrational number. Prove that there are infinitely many positive integers n such that $\{nx\} < \frac{1}{n}$.
8. Prove that the sequence $a_i = \left\lfloor (\sqrt{2}+1)^i \right\rfloor$ alternates between even and odd integers.

3 Homework

1. Define the sequence $a_0 = i,\ a_{n+1} = a_n \lfloor a_n \rfloor$ for each nonnegative integer n. For which positive integers i does there exist a positive integer k such that a_k is a positive integer?

2. Let $n > 1$ be an integer, and let a be an integer coprime to n. Prove that there exist integers x, y with $0 < |x| < \sqrt{n}$, $0 < |y| < \sqrt{n}$ and $ay \equiv x \pmod{n}$.

3. Prove that for some constant $C > 0$, the following statement holds:

   Let $m \geq 2$ be an integer, A a finite set of integers (not necessarily positive), and $B_1, B_2, \dots, B_m$ subsets of A. Suppose that for every $k = 1, 2, \dots, m$, the sum of the elements of B_k is 2^k. Then A contains at least $\frac{Cm}{\log_2 m}$ elements.