

Constructions and Existence Proofs

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1 Existence Results

Number theory has quite a few results about the existence of a number satisfying certain properties that you should be aware of. We've already seen many of these:

- There are infinitely many primes.
- (Bezout's Identity) For any coprime integers a and b , there are integers x and y such that $ax + by = 1$.
- (Chinese Remainder Theorem) For any positive integers a_1, \dots, a_n , and any pairwise coprime positive integers b_1, \dots, b_n , there is exactly one residue $x \pmod{b_1 \dots b_n}$ such that $x \equiv a_i \pmod{b_i}$ for each i .

The pigeonhole principle isn't strictly a number-theoretic result but is also often useful.

2 Techniques

You will learn much of this by trying problems, but in brief:

- Try small cases. TRY SMALL CASES!
- Build larger constructions from smaller ones; build constructions that satisfy more conditions from constructions that satisfy fewer.
- Try to control properties of your construction — rely on simple properties as opposed to “randomness” (exception: pigeonhole).

3 Problems

1. Prove that for each positive integer n , there are n consecutive positive integers, none of which is a prime power.
2. (a) Prove that for every positive integer n there is a number divisible by n consisting of only 1s and 0s.
(b) Prove that if n is not a multiple of 5, there is a number divisible by n consisting of only 1s and 2s.
3. Prove that for every positive integer n , there is a set S of n distinct positive integers such that every subset of S has a geometric mean which is a positive integer.
4. Prove that there is an infinite set of positive integers such that the sum of any finite subset is not a perfect power.
5. Prove that for every positive integer n , there are infinitely many terms of the Fibonacci sequence which are divisible by n .
6. Prove that for every positive integer n , there is a positive integer X such that

$$X, 2X, 3X, \dots, nX$$

are all nontrivial perfect powers.

7. Does there exist an infinite sequence of integers a_1, a_2, \dots such that $\gcd(a_m, a_n) = 1 \iff |m - n| = 1$?
8. Let m and c be integers. Prove that for any infinite sequence a_1, a_2, \dots of positive integers which contains every positive integer exactly once, there are integers x, y, k such that $x < y$ and $a_x + a_{x+1} + \dots + a_y = mk + c$.

4 Homework

1. Prove that for any positive integer c and any prime p , there is a positive integer x such that $x^x \equiv c \pmod{p}$.
2. Let x be an irrational number, and let a and b be real numbers such that $0 \leq a < b \leq 1$. Prove that there is an integer n such that $a < \{nx\} < b$. Hence prove that there is a power of 2 whose decimal representation starts with 2023.
3. Prove that every arithmetic progression $a, a + b, \dots$ where $\gcd(a, b) = 1$ has infinitely many terms which are not divisible by any perfect square larger than 1.