ν_p notation and LTE

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1 Definition

Let p be a prime and let r be a rational number. We let $\nu_p(r)$ be the exponent of p in the prime factorisation of r.

2 Basic properties

- $\nu_p(ab) = \nu_p(a) + \nu_p(b)$.
- $\nu_p(a+b) \geq \min(\nu_p(a), \nu_p(b))$.
- $\nu_p(a^n) = n\nu_p(a)$.
- If a is a positive integer, then $\nu_p(a) \leq \log_p(a)$.
- If $\nu_p(a+b) > \min(\nu_p(a), \nu_p(b))$ then $\nu_p(a) = \nu_p(b)$.
- $\frac{a}{b}$ is an integer iff for all p, $\nu_p(a) \geq \nu_p(b)$.
- a is a perfect kth power iff for all p, $k \mid \nu_p(a)$.
- $\nu_p(\gcd(a,b)) = \min(\nu_p(a), \nu_p(b))$.
- $\nu_p(\operatorname{lcm}(a,b)) = \max(\nu_p(a), \nu_p(b))$.

3 Less basic properties

- Legendre's Formula:

$$\nu_p(n!) = \sum_{i=1}^{\infty} \left\lfloor \frac{n}{p^i} \right\rfloor = \frac{n - s_p(n)}{p-1} < \frac{n}{p-1}.$$

- Lifting the Exponent (LTE):

  Let p be an odd prime, and let a and b be integers such that $p \mid a - b$ but $p \nmid a$. Let k be a positive integer.

  Then, $\nu_p\left(a^k - b^k\right) = \nu_p(a-b) + \nu_p(k)$.

- LTE for $p = 2$:

  Let a and b be odd integers, and let k be a positive integer.

  If k is even, we have $\nu_2(a^k - b^k) = \nu_2(a-b) + \nu_2(a+b) + \nu_2(k) - 1$.

  If k is odd, we have $\nu_2(a^k - b^k) = \nu_2(a-b)$.

4 Problems

1. Let x, y, z be rational numbers such that $xy,\ yz,\ zx,\ x+y+z$ are all integers.

   Prove that $x,\ y,\ z$ are all integers.

2. Let p be an odd prime. Prove that

$$p^2 \mid 1^p + 2^p + \cdots + p^p.$$

3. Let a, b, c be positive integers such that $c \mid a^c - b^c$. Prove that $c(a-b) \mid a^c - b^c$.

4. Prove that for all positive integers n,

$$\binom{2n}{n} \mid \operatorname{lcm}(1, 2, \ldots, 2n).$$

5. Find all pairs of positive integers x, p such that p is prime, $x \leq 2p$, and $x^{p-1} \mid (p-1)^x + 1$.

6. Let a, b, n be positive integers such that $a > b > 1$ and b is odd. If $b^n \mid a^n - 1$, prove that $na^b > 3^n$.

7. Find all natural numbers n such that $n^2 \mid 2^n + 1$.

5 Homework

1. Let a be a positive integer such that $4(a^n + 1)$ is a perfect cube for all positive integers n. Prove that $a = 1$.
2. Let n and k be positive integers. Assume that for each positive integer m, there exists a positive integer a such that $a^k \equiv n \pmod{m}$. Prove that n is a perfect kth power.
3. Find all positive integers n and k such that

$$k! = (2^n - 1)(2^n - 2)(2^n - 4) \cdots (2^n - 2^{n-1}).$$