Diophantine Equations

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1 Techniques

- Factorisations
- Mods
- Quadratic discriminant trick
- Take out GCD

2 Problems

1. Find all positive integers x, y such that $x^2 - 3xy + 2y^2 = 2023$.
2. Let a, b, c, d be positive integers with $ab = cd$. Prove that there exist positive integers p, q, r, s such that $a = pq, b = rs, c = pr, d = qs$. Hence prove that $a^2 + b^2 + c^2 + d^2$ is not prime.
3. Find all right-angled triangles with positive integer sides such that their area and perimeter are equal.
4. Let a, b, c be positive integers with $a^2 + b^2 = c^2$, such that no positive integer larger than 1 divides all of them. Prove that there exist positive integers x, y, z such that a, b, c equal $x^2 - y^2, 2xy, x^2 + y^2$ in some order.
5. Prove that there are infinitely many positive integers which are not the sum of a square and a prime.
6. Find all pairs of integers x, y such that $x^4 + 2x^2y + y^3 = 0$.
7. Prove that the equation $y^2 = x^3 + 7$ has no integer solutions.

3 Homework

1. Prove that the equation $x^3 + 3 = 4y(y + 1)$ has no integer solutions.
2. Find all triples of positive integers x, y, z such that $x^3 + y^3 + z^3 - 3xyz$ is prime.
3. Do there exist primes x, y, z such that $x^2 + y^3 = z^4$?