Number Bases

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1 Definitions

Let $b > 1$ be an integer. For any integer $n \geq 1$ there are unique nonnegative integers $k, a_0, \ldots, a_k$ such that each a_i is less than b, $a_k \neq 0$, and

$$n = \sum a_i b^i.$$

This is called the *base-b representation* of n, and we write

$$n = \overline{a_k a_{k-1} \ldots a_0}_{(b)}.$$

The usual decimal representation takes $b = 10$.

The sum of the digits of n in base b is denoted $s_b(n)$.

2 Divisibility tests

Since powers of b have nice properties in modular arithmetic, base-b expansions naturally lead to simple divisibility tests. The most familiar of these are in base 10:

- A number is divisible by 2^n iff the number produced by taking its last n digits is divisible by 2^n.
- The same statement, replacing 2^n by 5^n.
- A number is divisible by 3 iff the sum of its digits is divisible by 3.
- The same statement, replacing 3 by 9.
- A number is divisible by 11 iff the alternating sum of its digits (that is, the sum of $(-1)^i a_i$) is divisble by 11.

Using the language of modular arithmetic, we can generalise these statements to arbitrary bases.

- Let $l < k$. Then, $\overline{a_k a_{k-1} \ldots a_0}_{(b)} \equiv \overline{a_{l-1} \ldots a_0}_{(b)} \pmod{b^l}$. In particular, this congruence is still true $\pmod{d^l}$ for any $d \mid b$.
- $n \equiv s_b(n) \pmod{b-1}$.
- $n \equiv a_0 - a_1 + a_2 - \cdots + (-1)^k a_k \pmod{b+1}$.

For primes p which do not divide any of $b-1, b, b+1$, we can create divisibility tests as follows. Let m be such that $p \mid bm-1$. Then $p \mid bx+y \iff p \mid bmx+my \iff p \mid x+my$. For example, $7 \mid 10a+b \iff 7 \mid a-2b$.

3 Problems

1. Find all b such that $\overline{111}_{(b)} = \overline{212}_{(b-2)}$.
2. A faulty car odometer always skips the digit 4. If the odometer reads 2005, how many kilometres has the car actually travelled?
3. Prove that $s_b(x+y) \leq s_b(x) + s_b(y)$ and $s_b(xy) \leq s_b(x)s_b(y)$.
4. Prove that $\overline{11\ldots1}_{(9)}$ is always a triangular number.
5. Can a number consisting of 5 distinct even digits be a perfect square?
6. Prove that for any base b, there is a perfect square which ends with b distinct digits when written in base b.
7. (a) Prove that for every positive integer n and any base b there is a number divisible by n consisting of only 1s and 0s in base b.

   (b) Prove that if b is even and n is odd, there is a number divisible by n consisting of only odd digits in base b.
8. A sequence $\{x_n\}$ begins with $x_0 = 0$, and for each $i \geq 1$, x_{i+1} is the smallest positive integer greater than x_i such that the set $\{x_0, \ldots, x_{i+1}\}$ does not contain any 3-term arithmetic progressions. Find x_{2000}.

4 Homework

1. Find all b such that $\overline{234}_{(b+1)} - \overline{234}_{(b-1)} = 70$.
2. Find $s_{10}(s_{10}(s_{10}(4444^{4444})))$.
3. Find all b such that $\overline{11111}_{(b)}$ is a perfect square.