

Sequences, Integer Functions

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1 Comments

Even though sequence problems and integer function problems look cosmetically different, a sequence is just a function from \mathbb{N} to (some subset of) \mathbb{R} .

For divisibility problems, try to make one side of the divisibility prime.

If you can find infinitely many values, often you can sub a small unknown number and a large known one, and this gives you the small number.

2 Problems

1. Find all functions $f : \mathbb{N} \rightarrow \mathbb{N}$ such that $f(n+1) > f(f(n))$ for all positive integers n .
2. Let $P(n)$ be the product of the digits of a positive integer n . Let n_1 be a positive integer, and define $n_{i+1} = n_i + P(n_i)$ for each $i \geq 1$. Prove that this sequence is eventually constant.
3. Prove that if a sequence $\{a_n\}_0^\infty$ of nonnegative integers satisfies $a_0 = 0$ and $a_n = n - a_{a_n}$ for all n , then $a_{n+2} > a_n$ for all n .
4. Does there exist a strictly increasing function $f : \mathbb{N} \rightarrow \mathbb{N}$ such that $f(1) = 2$ and $f(f(n)) = f(n) + n$ for all positive integers n ?
5. Find all functions $f : \mathbb{N} \rightarrow \mathbb{N}$ such that $f(f(m)f(n)) = mn$ for all positive integers m and n .
6. Find all functions $f : \mathbb{N} \rightarrow \mathbb{N}$ such that for each positive integer n , $f(n) \mid n^3$ and $\sum_{i=1}^n f(i)$ is a perfect square.
7. Given an integer $k \geq 2$, determine all functions $f : \mathbb{N} \rightarrow \mathbb{N}$ such that $f(x_1)! + f(x_2)! + \cdots + f(x_k)!$ is divisible by $x_1! + x_2! + \cdots + x_k!$ for all positive integers x_1, x_2, \dots, x_k .
8. Let n be a fixed positive integer. Let $\frac{a_1}{b_1}, \dots, \frac{a_k}{b_k}$ be the rational numbers between 0 and 1 inclusive with denominators at most n , written in increasing order and lowest terms.
 - Prove that for each i , $a_{i+1}b_i - a_ib_{i+1} = 1$.
 - Prove that the rational number x with smallest denominator such that $\frac{a_i}{b_i} < x < \frac{a_{i+1}}{b_{i+1}}$ is $\frac{a_i + a_{i+1}}{b_i + b_{i+1}}$.

- Which pairs of numbers appear as consecutive b_i s?
9. Find all functions $f : \mathbb{Z}_{\geq 0} \rightarrow \mathbb{Z}_{\geq 0}$ such that $f(f(f(n))) = f(n+1) + 1$ for all $n \in \mathbb{Z}_{\geq 0}$.
 10. Prove that all terms of the sequence $a_1 = a_2 = a_3 = 1$, $a_{n+1} = (1 + a_{n-1}a_n)/a_{n+2}$ are integers.
 11. Find all functions $f : \mathbb{N} \rightarrow \mathbb{N}$ satisfying

$$f(n) + f(n+1) = f(n+2)f(n+3) - 2023$$

for all $n \in \mathbb{N}$.

12. Find all functions $f : \mathbb{Z} \rightarrow \mathbb{Z}$ such that, for all integers a, b, c satisfying $a + b + c = 0$, we have

$$f(a)^2 + f(b)^2 + f(c)^2 = 2f(a)f(b) + 2f(a)f(c) + 2f(b)f(c).$$

3 Homework

1. Is there an infinite sequence a_1, \dots of primes such that for each i we have $10a_i \leq a_{i+1} < 10a_i + 9$?
2. Find all functions $f : \mathbb{N} \rightarrow \mathbb{N}$ such that $m^2 + f(n) \mid mf(m) + n$ for all positive integers m and n .
3. Is there a function $f : \mathbb{N} \rightarrow \mathbb{N}$ of positive integers such that $\gcd(a_m, a_n) = 1 \iff |m - n| = 1$?