Review and Extension — Divisibility

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1 Key concepts for this term

- Induction, strong induction and well-ordering.
- If $a \mid b$ and $a \mid c$ then for any integers x and y, $a \mid bx + cy$.
- Factorisations:

$$axy + bx + cy = d \iff (ax + c)(ay + b) = ad + bc$$
$$a^k - b^k = (a - b)\left(a^{k-1} + a^{k-2}b + \cdots + b^{k-1}\right)$$

- Division Algorithm: for any integer a and positive integer b there exist unique integers q and r such that $a = qb + r$ and $0 \leq r < b$.
- Euclid's Algorithm: if $a = qb + r$ then $\gcd(a, b) = \gcd(b, r)$.
- Bezout's Identity: for any integers a and b there exist integers x and y such that $\gcd(a, b) = ax + by$.
- Fundamental Theorem of Arithmetic: prime factorisations are unique.
- When dealing with divisibility problems it's often more convenient to think of numbers in terms of prime factorisations.

2 Arithmetic Functions

We define:

- The number of positive divisors function $d(n)$.
- The sum of positive divisors function $\sigma(n)$.
- The totient function $\varphi(n)$: the number of positive integers which are at most n and coprime to n.

Find formulae for $d(n)$ and $\sigma(n)$ where $n = p_1^{e_1} p_2^{e_2} \cdots p_k^{e_k}$.

3 Problems

1. Given positive integers $n > 1$ and k, prove that there are unique nonnegative integers $m, a_0, a_1, \ldots, a_m$ such that $a_m > 0$, $0 \le a_i < n$ for all i, and
$$k = a_0 n^0 + a_1 n^1 + \cdots + a_m n^m.$$
2. Prove that $1^k + 2^k + \cdots + n^k$ is divisible by $1 + 2 + \cdots + n$ for all positive integers n and odd positive integers k.
3. Prove that if $2^n + 1$ is prime for a positive integer n, then n is a power of 2.
4. Let m and n be positive integers. Prove that $\sqrt[m]{n}$ is an integer or irrational.
5. Let a, b, c be positive integers such that $a^3 + b^3 = 2^c$. Prove that $a = b$.
6. Let a and b be positive integers such that $a \mid b^2 \mid a^3 \mid b^4 \mid \cdots$

   Prove that $a = b$.
7. Prove that every positive integer is a sum of one or more numbers of the form $2^r 3^s$, where r and s are nonnegative integers and no summand divides another.
8. Prove that for positive integers $m, n > 2$ we cannot have
$$2^m - 1 \mid 2^n + 1.$$
9. Find all positive integers n such that $3^{n-1} + 5^{n-1} \mid 3^n + 5^n$.
10. Let n be a composite positive integer. Calculate
$$\gcd((n-1)! + 1, n!).$$
11. Let S be a nonempty set of integers such that if a and b are in S, then so is $2a - b$.

    Prove that S is an arithmetic progression. (That is, there are integers a and d such that $n \in S$ if and only if $n = a + kd$ for some integer k.)
12. Find all pairs of positive integers a, b such that
$$b^2 - a \mid a^2 + b \quad \text{and} \quad a^2 - b \mid b^2 + a.$$
13. Let m and n be positive integers. Prove that
$$m \mid \gcd(m, n)\binom{m}{n}.$$
14. Find all pairs of positive integers x, y such that $xy^2 + y + 7 \mid x^2 y + x + y$.
15. Prove that $d(n) \le 2\sqrt{n}$ for all n.
16. Let a, b, p be positive integers such that p is prime and $\operatorname{lcm}(a, a+p) = \operatorname{lcm}(b, b+p)$. Prove that $a = b$.

17. Prove that for all n,

$$\sigma(1) + \sigma(2) + \cdots + \sigma(n) \leq n^2.$$

18. Prove that if ab is a perfect square, then so are $\frac{a}{\gcd(a,b)}$ and $\frac{b}{\gcd(a,b)}$.

19. Prove that for all composite n apart from 6,

$$\sqrt{n} \leq \varphi(n) \leq n - \sqrt{n}.$$

20. Let n be an even positive integer such that $\sigma(n) = 2n$. Prove that $n = 2^{p-1}\left(2^p - 1\right)$, where p is a prime.

21. For any positive integer n, prove that $\displaystyle\sum_{d|n} \varphi(d) = n$.

4 Homework

- Complete the survey: https://forms.gle/RYsmy3SBhQa9y4Yh9
- Solve and submit any three problems from the previous section.