Review and Extension: Constructions, Polynomials, Sequences

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1 Exam review

2 Problems

1. What is the maximum possible value of n such that there exists an integer polynomial P such that the equation $P(x) = 5$ has n distinct integer solutions, and $P(x) = 8$ has an integer solution?

2. Find all functions $f : \mathbb{Z} \to \mathbb{Z}$ such that, for all integers a, b, c satsfying $a + b + c = 0$, we have
$$f(a)^2 + f(b)^2 + f(c)^2 = 2f(a)f(b) + 2f(a)f(c) + 2f(b)f(c).$$

3. Let $P(x)$ and $Q(x)$ be polynomials whose coefficients are all equal to 1 or 7. If $P(x)$ divides $Q(x)$, prove that $1 + \deg P(x)$ divides $1 + \deg Q(x)$.

4. Dragos, the early ruler of Moldavia, and Maria the Oracle play the following game. Firstly, Maria chooses a set S of prime numbers. Then Dragos gives an infinite sequence $x_1, x_2, \ldots$ of distinct positive integers. Then Maria picks a positive integer M and a prime number p from her set S. Finally, Dragos picks a positive integer N and the game ends. Dragos wins if and only if for all integers $n \geq N$ the number x_n is divisible by p^M; otherwise, Maria wins. Who has a winning strategy?

5. Let $P(x)$ be a polynomial of degree $n > 1$ with integer coefficients and let k be a positive integer. Prove that there are at most n integers t such that $P^{(k)}(t) = t$, were $P^{(k)}$ is P repeated k times.

6. Does there exist a prime p such that every power of p is a base-10 palindrome?

7. For a positive integer n, let $\varphi(n)$ and $d(n)$ denote the value of the Euler phi function at n and the number of positive divisors of n, respectively. Prove that there are infinitely many positive integers n such that $\varphi(n)$ and $d(n)$ are both perfect squares.

8. For each positive integer k, let $S(k)$ the sum of digits of k in decimal system. Show that there is an integer k, with no 9 in it's decimal representation, such that
$$S(2^{24^{2017}}k) = S(k).$$

3 Problems from previous handouts

1. Find all integers n such that $n^2 + 1$ divides $n^3 + n^2 - n - 15$.

2. Find all right-angled triangles with positive integer sides such that their area and perimeter are equal.

3. Let a, m, n be positive integers with $a > 1$. Prove that $\gcd(a^m - 1, a^n - 1) = a^{\gcd(m,n)} - 1$.

4. Prove that $1^k + 2^k + \cdots + n^k$ is divisible by $1 + 2 + \cdots + n$ for all positive integers n and odd positive integers k.

5. Let a, b, c, d be positive integers with $ab = cd$. Prove that there exist positive integers p, q, r, s such that $a = pq, b = rs, a = pr, d = qs$.

6. Let p be a prime with $p > 3$. Prove that there are positive integers $a < b < \sqrt{p}$ such that $p - b^2 \mid p - a^2$.

7. Find all pairs of positive integers a, b such that

$$b^2 - a \mid a^2 + b \quad \text{and} \quad a^2 - b \mid b^2 + a.$$

8. Prove that for any nonnegative integer n, the number $7^{7^n} + 1$ is the product of at least $2n+3$ (not necessarily distinct) primes.

9. For any positive integer n, prove that $\sum_{d|n} \varphi(d) = n$.

10. Let x and y be positive integers and let p be prime. Assume there are coprime positive integers m and n such that $x^m \equiv y^n \pmod{p}$. Prove that there is a unique positive integer z with $0 \le z < p$ such that
$$x \equiv z^n \pmod{p}, \qquad y \equiv z^m \pmod{p}.$$

11. Let a and b be positive integers such that $a^n + n \mid b^n + n$ for all positive integers n. Prove that $a = b$.

12. Let n and k be positive integers such that $\varphi^k(n) = 1$ (that is, φ iterated k times). Prove that $n \le 3^k$.

13. Prove that for each positive integer n there exist n consecutive positive integers, none of which is a prime power.

14. Let m and n be positive integers. Show that $4mn - m - n$ can never be a perfect square.

15. Find all positive integer solutions to $3^x + 4^y = 5^z$.

16. Let $n > 1$ be a positive integer and let p be a prime. Given that $n \mid p - 1$ and $p \mid n^3 - 1$, prove that $4p - 3$ is a perfect square.

17. Let n be a fixed positive integer. Let $\frac{a_1}{b_1}, \dots, \frac{a_k}{b_k}$ be the rational numbers between 0 and 1 inclusive with denominators at most n, written in increasing order and lowest terms.
    - Prove that for each i, $a_{i+1}b_i - a_i b_{i+1} = 1$.
    - Prove that the rational number x with smallest denominator such that $\frac{a_i}{b_i} < x < \frac{a_{i+1}}{b_{i+1}}$ is $\frac{a_i + a_{i+1}}{b_i + b_{i+1}}$.
    - Which pairs of numbers appear as consecutive b_is?

18. Suppose that $(a_1, b_1), (a_2, b_2), \dots, (a_{100}, b_{100})$ are distinct ordered pairs of nonnegative integers. Let N denote the number of pairs of integers (i, j) satisfying $1 \le i < j \le 100$ and $|a_i b_j - a_j b_i| = 1$. Determine the largest possible value of N over all possible choices of the 100 ordered pairs.

19. For a positive integer n, let $f(n)$ be the number of binary strings of length n that can't be expressed as an m-fold repetition of another binary string for any $m > 1$.

    For example, $f(6) = 54$ since the only strings of length 6 that can be expressed as an m-fold repetition of another binary string for some $m > 1$ are 000000, 001001, 010010, 010101, 011011, 100100, 101010, 101101, 110110, 111111.

(a) Find two functions g and h, in closed form, such that $f = g * h$.

(b) Prove that $n \mid f(n)$.

(c) Find all n for which $n \mid \displaystyle\sum_{i=1}^{n} f(i) \left\lfloor \frac{n}{i} \right\rfloor$.

20. Find all pairs of positive integers x, y such that $1 + 2^x + 2^{2x+1} = y^2$.

21. Find all pairs of positive integers x, y such that $x^3 - y^3 = xy + 61$.

22. Find all triples of positive integers a, b, c such that $a^2 + b + c = abc$.

23. Find all positive integers x, y, n such that $\gcd(x, n+1) = 1$ and $x^n + 1 = y^{n+1}$.

24. Four positive integers x, y, z, t satisfy $xy - zt = x + y = z + t$. Is it possible that xy and zt are both perfect squares?

25. Prove that there are infinitely many triples (a, b, c) of positive integers in arithmetic progression such that $ab + 1$, $bc + 1$ and $ca + 1$ are all perfect squares.

26. Find all solutions in integers to $x^2 + y^2 + z^2 = 2xyz$.

27. Let a and b be two positive integers. Prove that

$$a^2 + \left\lceil \frac{4a^2}{b} \right\rceil$$

is not a square.

28. Let p and q be coprime. Prove that

$$\sum_{i=1}^{q-1} \left\lfloor \frac{ip}{q} \right\rfloor = \frac{(p-1)(q-1)}{2}.$$

29. Find all positive integers n such that $1 + \lfloor \sqrt{n} \rfloor$ divides n.

30. Let r be an irrational root of a polynomial $P(x)$ of degree d with integer coefficients. Prove that there is a real number C such that for any integer q we have $\{qr\} \geq \frac{C}{q^{d-1}}$.

31. Let x be an irrational number.

(a) Prove that for each positive integer n there is a positive integer m such that $\{mx\} < \frac{1}{n}$.

(b) Prove that there are infinitely many positive integers n such that $\{nx\} < \frac{1}{n}$.

32. Prove that the sequence $a_i = \left\lfloor (\sqrt{2} + 1)^i \right\rfloor$ alternates between even and odd integers.

33. Find all positive integers x and y such that if $z = \gcd(x, y)$, then $x + y^2 + z^3 = xyz$.

34. For each positive integer n, let

$$f(n) = \frac{1}{n} \sum_{i=1}^{n} \left\lfloor \frac{n}{i} \right\rfloor.$$

(a) Prove that $f(n+1) > f(n)$ for infinitely many n.

(b) Prove that $f(n+1) < f(n)$ for infinitely many n.

35. Is there a positive integer m for which the equation

$$\frac{1}{a} + \frac{1}{b} + \frac{1}{c} + \frac{1}{abc} = \frac{m}{a+b+c}$$

has infinitely many solutions in positive integers a, b, c?

36. Let p be an odd prime. Prove that

$$p^2 \mid 1^p + 2^p + \cdots + p^p.$$

37. Let a, b, c be positive integers such that $c \mid a^c - b^c$. Prove that $c(a-b) \mid a^c - b^c$.

38. Prove that for all positive integers n,

$$\binom{2n}{n} \mid \operatorname{lcm}(1, 2, \ldots, 2n).$$

39. Find all pairs of positive integers x, p such that p is prime, $x \leq 2p$, and $x^{p-1} \mid (p-1)^x + 1$.

40. Let a, b, n be positive integers such that $a > b > 1$ and b is odd. If $b^n \mid a^n - 1$, prove that $na^b > 3^n$.

41. Find all natural numbers n such that $n^2 \mid 2^n + 1$.

42. Find all positive integers n such that $n \mid 2^n - 1$.

43. Prove that if $\sigma(n) = 2n + 1$, then n is a perfect square.

44. Find all positive integers n such that $n \mid 2^{n-1} + 1$.

45. Find all primes p, q, r such that $p \mid q^r + 1,\ q \mid r^p + 1,\ r \mid p^q + 1$.

46. Let n and m be nonnegative integers, and let p be prime. Prove that

$$\binom{n}{m} \equiv \prod_{i=0}^{k} \binom{n_i}{m_i} \pmod{p},$$

where $n = \sum n_i p^i$ and $m = \sum m_i p^i$.

47. Find all positive integers n for which there exists a function $g : \mathbb{Z}_n \to \mathbb{Z}_n$ such that all the functions

$$g(x), g(x) + x, \ldots, g(x) + 100x$$

are bijections $\mathbb{Z}_n \to \mathbb{Z}_n$.

48. Let k and n be positive integers such that n is odd. Prove that there is an integer a such that $a^{32} \equiv (n+1)^3 \pmod{n^k}$.

49. Prove that for any prime p and positive integer a with $p \nmid a$ there are at least $p-1$ solutions in $\mathbb{Z}_p$ to $x^2 + y^2 \equiv a \pmod{p}$.

50. Let p be prime and let a and b be positive integers such that $p \nmid b$. Prove that there exists a positive integer n such that $p^a \mid n^n - b$.

51. If $p > 3$ is a prime such that $\varphi(p-1) > \frac{p-1}{3}$, prove that there are two consecutive generators mod p.

52. Let p be a prime and let n be a positive integer with $p \nmid n$. Find

$$\sum_{i=1}^{p} \left(\frac{i^2 + n}{p} \right).$$

53. Let $p > 3$ be a prime and let a, b, c be integers with $a \neq 0$. Suppose that $ax^2 + bx + c$ is a perfect square for p consecutive integers x. Prove that $p \mid b^2 - 4ac$.

54. Let P be a nonconstant polynomial with integer coefficients. Prove that for any integer m there exist an integer n and a prime p such that $p^m \mid P(n)$.

55. Let $a_1, a_2, a_3, \ldots$ be a sequence of integers, such that for any positive integers n and k, the quantity

$$\frac{a_n + a_{n+1} + \cdots + a_{n+k-1}}{k}$$

is always the square of an integer. Prove that all a_is are equal.

56. Let a and b be positive integers. Prove that there are infinitely many positive integers n such that $n^b - 1 \nmid a^n + 1$.

57. Prove that there are infinitely many pairs a and b of perfect squares such that they have the same number of digits in decimal, and their concatenation is also a square.

58. Prove that there exist infinitely many positive integers n such that $n^2 + 1 \mid n!$.

59. For which positive integers r and s does there exist a positive integer n such that nr and ns have the same number of divisors?

60. Let p be an integer polynomial and let a be an integer such that $p(p(\cdots(p(a))\cdots)) = a$. Prove that $p(p(a)) = a$.

61. Let a, b, c be integers such that $a/b + b/c + c/a$ and $a/c + c/b + b/a$ are both integers. Prove that $|a| = |b| = |c|$.

62. Prove that if p is prime, then $1 + x + x^2 + \cdots + x^{p-1}$ is irreducible.

63. Let f be a nonconstant integer polynomial and let n and k be positive integers. Prove that there exists a positive integer a such that each of the numbers $f(a), f(a+1), \ldots, f(a+n-1)$ has at least k distinct prime divisors.

64. Prove that if $5 \nmid a$, then $x^5 - x + a$ is irreducible.

65. Find all integer polynomials p such that

   - $p(n) > n$ for all positive integers n, and
   - for each positive integer n there is a positive integer k such that $p^{(k)}(1)$ (p repeated k times) is divisible by n.

66. Find all functions $f : \mathbb{N} \to \mathbb{N}$ such that $f(n+1) > f(f(n))$ for all positive integers n.

67. Does there exist a strictly increasing function $f : \mathbb{N} \to \mathbb{N}$ such that $f(1) = 2$ and $f(f(n)) = f(n) + n$ for all positive integers n?

68. Prove that all terms of the sequence $a_1 = a_2 = a_3 = 1$, $a_{n+1} = (1 + a_{n-1}a_n)/a_{n+2}$ are integers.

69. Does there exist a strictly increasing function $f : \mathbb{N} \to \mathbb{N}$ such that $f(mn) = f(m) + f(n)$ for all positive integers m and n?

70. Let $f : \mathbb{Z} \to \mathbb{N}$ be a function such that for any integers m and n, $f(m-n) \mid f(m) - f(n)$. Prove that for all integers m and n, if $f(m) \leq f(n)$ then $f(m) \mid f(n)$.

71. Let $P(n)$ be the product of the digits of a positive integer n. Let n_1 be a positive integer, and define $n_{i+1} = n_i + P(n_i)$ for each $i \geq 1$. Prove that this sequence is eventually constant.

72. Find all functions $f : \mathbb{Z}_{\geq 0} \to \mathbb{Z}_{\geq 0}$ such that $f(f(f(n))) = f(n+1) + 1$ for all $n \in \mathbb{Z}_{\geq 0}$.

73. Let $a_1, a_2, \ldots$ be a sequence of integers such that the average of every consecutive group of a_is is a perfect square. Prove that the sequence is constant.

74. Find all functions $f : \mathbb{Z}_{>0} \to \mathbb{Z}_{>0}$ such that $a + f(b)$ divides $a^2 + bf(a)$ for all positive integers a and b with $a + b > 2019$.