Review and Extension — Size Arguments

Andres Buritica

1   Exam review

2 Key concepts for this term

- Bounding: WLOG, dominant terms, induction, mods.
- Descent
- How to deal with floor functions
- Diophantine equations: factorisations, mods, discriminant, taking out GCD.

3 Problems

1. Find all pairs of positive integers x, y such that $x^3 + y^3 = (x + y)^2$.

2. Find all pairs of positive integers x, y such that $1 + 2^x + 2^{2x} = y^2$.

3. Find all pairs of positive integers x, y such that $x! + 5 = y^3$.

4. Find all positive integers n such that $1 + \lfloor \sqrt{n} \rfloor$ divides n.

5. Prove that for any positive integer n which is not a perfect square, there is a positive integer k such that
$$n = \left\lfloor n + \sqrt{n} + \frac{1}{2} \right\rfloor.$$

6. Let d be a positive integer which is not a perfect square. Prove that there exist positive integers x_0, y_0 such that for any pair x, y of positive integers satisfying $x^2 - dy^2 = 1$, there is a positive integer k satisfying
$$x + y\sqrt{d} = (x_0 + y_0\sqrt{d})^k.$$

7. Prove that there are infinitely many positive integers which are not the sum of a square and a prime.

8. Find all pairs of integers x, y such that $x^4 + 2x^2y + y^3 = 0$.

9. Prove that the equation $y^2 = x^3 + 7$ has no integer solutions.

4 Homework

Instructions: solve and submit any three problems from Section 3.