Homework Review and Proofwriting

Andres Buritica

1 Lecture 10 Homework

1. Prove that for any positive integer c and any prime p, there is a positive integer x such that $x^x \equiv c \pmod{p}$.
2. Let x be an irrational number, and let a and b be real numbers such that $0 \leq a < b \leq 1$. Prove that there is an integer n such that $a < \{nx\} < b$. Hence prove that there is a power of 2 whose decimal representation starts with 2023.
3. Prove that every arithmetic progression $a, a+b, \ldots$ where $\gcd(a, b) = 1$ has infinitely many terms which are not divisible by any perfect square larger than 1.

2 Lecture 11 Homework

1. Find all functions $f : \mathbb{N} \to \mathbb{N}$ such that $f(f(m)f(n)) = mn$ for all positive integers m and n.
2. Find all monotonically increasing functions $f : \mathbb{N} \to \mathbb{Z}_{\geq 0}$ such that $f(mn) = f(m) + f(n)$ for all nonnegative integers m and n.
3. Let a, b be odd positive integers. Define the sequence c_n by choosing $c_1 = a, c_2 = b$ and for each $i > 2$ letting c_i be the largest odd divisor of $c_{i-1} + c_{i-2}$. Prove that this sequence is eventually constant (that is, there is an m such that for any $i, j > m$, $a_i = a_j$).

3 Homework

1. The denominators of two irreducible fractions are 600 and 700. Find the minimum value of the denominator of their sum (written as an irreducible fraction).
2. We call the *main divisors* of a composite number n the two largest of its divisors other than n. Composite numbers a and b are such that their main divisors coincide. Prove that $a = b$.
3. Suppose $a_1, a_2, \ldots$ is an infinite strictly increasing sequence of positive integers and $p_1, p_2, \ldots$ is a sequence of distinct primes such that $p_n \mid a_n$ for all $n \geq 1$. It turned out that $a_n - a_k = p_n - p_k$ for all $n, k \geq 1$. Prove that the sequence $\{a_n\}$ consists only of prime numbers.