Induction and Divisibility

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1 Induction and variants

Arguably, the defining property of the integers is the **Principle of Mathematical Induction**: if we have a set $S \subseteq \mathbb{N}$ such that $\forall a \in S,\ a+1 \in S$ then $S = \mathbb{N}$.

To prove that some sentence $P(n)$ is true for all positive integers n, we follow the following structure.

- Prove that $P(1)$ is true.
- Prove that if $P(n)$ is true, then $P(n+1)$ is true.

If we've done both of those things, why can we conclude that $P(a)$ is true for all $a \in \mathbb{N}$?

We may use induction to prove some foundational results about the integers:

- For all positive integers n, $n \geq 1$.
- If S is a nonempty set of positive integers, there is some $a \in S$ such that for any $b \in S$ we have $a \leq b$. This is known as the **Well-Ordering Principle**.

The final addition to our induction toolkit will be **strong induction**: if S is a set of positive integers such that $1 \in S$ and

$$\forall a \in \mathbb{N},\ (\forall b \in \mathbb{N},\ b \leq a \implies b \in S) \implies a+1 \in S,$$

then $\mathbb{N} = S$.

2 Divisibility

For integers a and b, we say $a \mid b$ (read "a divides b") if there is some integer c with $b = a \times c$.

- Prove that if a and b are positive integers with $a \mid b$ then $a \leq b$.
- Prove that if $a \mid b$ and $a \mid c$ then $a \mid bx + cy$ for all integers x and y.

We define a *prime* as a positive integer larger than 1 which is not divisible by any positive integer other than 1 and itself.

- Prove that every positive integer larger than 1 has a prime factor.
- Prove that there are infinitely many primes.

Also look for factorisations when trying problems: e.g.

$$axy + bx + cy = d \iff (ax + c)(ay + b) = ad + bc$$
$$a^k - b^k = (a - b)\left(a^{k-1} + a^{k-2}b + \cdots + b^{k-1}\right)$$

3 Problems

1. Given positive integers $n > 1$ and k, prove that there are unique nonnegative integers $m, a_0, a_1, \ldots, a_m$ such that $a_m > 0,\ 0 \le a_i < n$ for all i, and

$$k = a_0 n^0 + a_1 n^1 + \cdots + a_m n^m.$$

2. Find all right-angled triangles with positive integer sides such that their area and perimeter are equal.

3. Prove that $1^k + 2^k + \cdots + n^k$ is divisible by $1 + 2 + \cdots + n$ for all positive integers n and odd positive integers k.

4. Prove that if $2^n + 1$ is prime for a positive integer n, then n is a power of 2.

5. Prove that if a is an integer and b is a positive integer, there is a unique pair (q, r) of integers such that $0 \le r < b$ and $a = qb + r$.

6. Let m and n be positive integers. Prove that $\sqrt[m]{n}$ is an integer or irrational.

7. Let a, b, c be positive integers such that $a^3 + b^3 = 2^c$. Prove that $a = b$.

8. Let a and b be positive integers such that $a \mid b^2 \mid a^3 \mid b^4 \mid \cdots$

   Prove that $a = b$.

9. Prove that every positive integer is a sum of one or more numbers of the form $2^r 3^s$, where r and s are nonnegative integers and no summand divides another.

4 Homework

1. Prove that for any natural number n, the fraction

$$\frac{21n+4}{14n+3}$$

is irreducible.

2. Determine all pairs (a, b) of non-negative integers such that

$$\frac{a+b}{2} - \sqrt{ab} = 1.$$

3. Prove that if there are two terms of an arithmetic progression which are coprime integers, then there is an infinite subset of that progression all of whose elements are coprime integers.