Bounding

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1 Techniques

- WLOG arguments: break symmetry to bound one of the variables.
- If a and b are integers with $a > b$ then $a \geq b + 1$.
- If a and b are integers with $b \neq 0$ and $a \mid b$ then $|a| \leq |b|$. Thus, try to get big dividing small.
- Look for "dominant terms" that grow larger than everything else.
- A concave down function on a closed interval achieves its minimum at one of the endpoints.
- Hierarchy of size: let $P(n)$ and $Q(n)$ be polynomials such that $\deg P > \deg Q$, and let $c > 1$ be a constant. For sufficiently large n, $\ln n \ll |Q(n)| \ll |P(n)| \ll c^n \ll n! \ll n^n$.
- Stirling's Approximation: $\left(\frac{n}{e}\right)^n < n! < n\left(\frac{n}{e}\right)^n$.
- Bernoulli's Inequality: for $r \geq 1$ and $x \geq -1$ we have $(1+x)^r \geq 1 + rx$.

2 Problems

1. Prove that for any real polynomial P such that the leading coefficient of P is positive, there is an N such that for any $n > N$, $P(n) > 0$.
2. Find all pairs x, y of positive integers such that $3^x - 8^y = 2xy + 1$.
3. Find all pairs of positive integers x, y such that $1 + 2^x + 2^{2x} = y^2$.
4. Prove that for any real numbers $c > 1$ and $x > 0$, and any polynomial P, there is an N such that for any $n > N$, $c^n > P(n)$.
5. Find all pairs of positive integers x, y such that $1 + 2^x + 2^{2x+1} = y^2$.
6. Find all pairs of positive integers x, y such that $x^3 - y^3 = xy + 61$.
7. Find all triples of positive integers a, b, c such that $a^2 + b + c = abc$.
8. Find all positive integers x, y, n such that $\gcd(x, n+1) = 1$ and $x^n + 1 = y^{n+1}$.
9. Four positive integers x, y, z, t satisfy $xy - zt = x + y = z + t$. Is it possible that xy and zt are both perfect squares?

3 Homework

1. Let n be a nonnegative integer. The powers 2^n and 5^n start with the same digit. What are all possible values of this digit?
2. Find all pairs of positive integers x, y such that $xy^2 + y + 7 \mid x^2y + x + y$.
3. Find all triples a, b, c of positive integers such that $a^3 + b^3 + c^3 = (abc)^2$.