Review and Extension — Size Arguments

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1 Exam review

2 Key concepts for this term

- WLOG
- Big divides small
- Dominant terms
- Graphical reasoning
- Relative growth rates of common functions
- Infinite descent
- Vieta jumping
- Pell equations
- Counting numbers in a range
- Dealing with floor functions

3 Problems

1. Find all pairs of positive integers x, y such that $1 + 2^x + 2^{2x+1} = y^2$.
2. Find all pairs of positive integers x, y such that $x^3 - y^3 = xy + 61$.
3. Find all triples of positive integers a, b, c such that $a^2 + b + c = abc$.
4. Find all positive integers x, y, n such that $\gcd(x, n + 1) = 1$ and $x^n + 1 = y^{n+1}$.
5. Four positive integers x, y, z, t satisfy $xy - zt = x + y = z + t$. Is it possible that xy and zt are both perfect squares?
6. Prove that there are infinitely many triples (a, b, c) of positive integers in arithmetic progression such that $ab + 1$, $bc + 1$ and $ca + 1$ are all perfect squares.
7. Find all solutions in integers to $x^2 + y^2 + z^2 = 2xyz$.

8. Let a and b be two positive integers. Prove that

$$a^2 + \left\lceil \frac{4a^2}{b} \right\rceil$$

is not a square.

9. Let p and q be coprime. Prove that

$$\sum_{i=1}^{q-1} \left\lfloor \frac{ip}{q} \right\rfloor = \frac{(p-1)(q-1)}{2}.$$

10. Find all positive integers n such that $1 + \lfloor \sqrt{n} \rfloor$ divides n.

11. Let r be an irrational root of a polynomial $P(x)$ of degree d with integer coefficients. Prove that there is a real number C such that for any integer q we have $\{qr\} \geq \frac{C}{q^{d-1}}$.

12. Let x be an irrational number.

    (a) Prove that for each positive integer n there is a positive integer m such that $\{mx\} < \frac{1}{n}$.

    (b) Prove that there are infinitely many positive integers n such that $\{nx\} < \frac{1}{n}$.

13. Prove that the sequence $a_i = \left\lfloor (\sqrt{2}+1)^i \right\rfloor$ alternates between even and odd integers.

14. Find all positive integers x and y such that if $z = \gcd(x, y)$, then $x + y^2 + z^3 = xyz$.

15. For each positive integer n, let

$$f(n) = \frac{1}{n} \sum_{i=1}^{n} \left\lfloor \frac{n}{i} \right\rfloor.$$

    (a) Prove that $f(n+1) > f(n)$ for infinitely many n.

    (b) Prove that $f(n+1) < f(n)$ for infinitely many n.

16. Is there a positive integer m for which the equation

$$\frac{1}{a} + \frac{1}{b} + \frac{1}{c} + \frac{1}{abc} = \frac{m}{a+b+c}$$

has infinitely many solutions in positive integers a, b, c?

4 Homework

Solve and submit any three problems from Section 3.

5 Extension: density of the primes

1. Let n be a positive integer larger than 1.

   (a) Prove that the product of all primes between $\lceil \frac{n}{2} \rceil$ and n (including n, not including $\lceil \frac{n}{2} \rceil$) is less than 2^n.[1]

   (b) Prove that the product of all primes between 1 and n is at most 4^{n-1}.

   (c) Find some real number c independent of n such that there are at most $\frac{cn}{\log_2 n}$ primes that are at most n.

2. Let n be a positive integer larger than $2^{2^{2^2}}$.

   (a) Let p be a prime.

   - Prove that if $p^k \mid \binom{2n}{n}$ then $p^k < 2n$.
   - Prove that if $2p \leq 2n < 3p$ then $p \nmid \binom{2n}{n}$.

   (b) Prove that

$$\prod_{\substack{p^k \| \binom{2n}{n} \\ p \leq n}} p^k < \binom{2n}{n}.$$

   (c) Find some real number c independent of n such that there are at least $\frac{cn}{\log_2 n}$ primes that are at most n.

[1] Hint: binomial coefficients are integers.