Team Level Lectures

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1 Farey sequences

Let n be a fixed positive integer. Let $\frac{a_1}{b_1}, \ldots, \frac{a_k}{b_k}$ be the rational numbers between 0 and 1 inclusive with denominators at most n, written in increasing order and lowest terms.

- Prove that for each i, $a_{i+1}b_i - a_i b_{i+1} = 1$.
- Prove that the rational number x with smallest denominator such that $\frac{a_i}{b_i} < x < \frac{a_{i+1}}{b_{i+1}}$ is $\frac{a_i + a_{i+1}}{b_i + b_{i+1}}$.
- Which pairs of numbers appear as consecutive b_is?

Suppose that $(a_1, b_1), (a_2, b_2), \ldots, (a_{100}, b_{100})$ are distinct ordered pairs of nonnegative integers. Let N denote the number of pairs of integers (i, j) satisfying $1 \leq i < j \leq 100$ and $|a_i b_j - a_j b_i| = 1$. Determine the largest possible value of N over all possible choices of the 100 ordered pairs.

2 Dirichlet Convolution and Mobius Inversion

Let $f : \mathbb{N} \to \mathbb{R}$ and $g : \mathbb{N} \to \mathbb{R}$ be two functions. We define the *Dirichlet convolution* $f * g$ as

$$(f * g)(n) = \sum_{d|n} f(d) g\left(\frac{n}{d}\right).$$

We define the functions $d,\ \sigma,\ \varphi$ as before and also define the functions

$$\zeta(n) = 1,\ \psi(n) = n.$$

- Prove that $*$ is associative: that is, $(a * b) * c = a * (b * c)$.
- Prove that if a and b are multiplicative then so is $a * b$.
- Find a function δ such that $\delta * a = a$ for all functions a.
- Find a function μ such that $\mu * \zeta = \delta$.
- Prove that $g = f * \zeta \iff f = g * \mu$.
- Find $\zeta * \zeta,\ \psi * \zeta$ and $\varphi * \zeta$.

- Prove that
$$\sum_{i=1}^{n} f(i) \left\lfloor \frac{n}{i} \right\rfloor = \sum_{j=1}^{n} (f * \zeta)(j).$$

For a positive integer n, let $f(n)$ be the number of binary strings of length n that can't be expressed as an m-fold repetition of another binary string for any $m > 1$.

For example, $f(6) = 54$ since the only strings of length 6 that can be expressed as an m-fold repetition of another binary string for some $m > 1$ are 000000, 001001, 010010, 010101, 011011, 100100, 101010, 101101, 110110, 111111.

- Find two functions g and h, in closed form, such that $f = g * h$.
- Prove that $n \mid f(n)$.
- Find all n for which $n \mid \sum_{i=1}^{n} f(i) \left\lfloor \frac{n}{i} \right\rfloor$.

3 More theorems about mod p

- Wolstenholme's Theorem: let a and b be positive integers, and let p be a prime greater than 3. Prove that
$$\binom{ap}{bp} \equiv \binom{a}{b} \pmod{p^3}.$$
- Lucas' Theorem: let $m = \sum m_i p^i$ and $n = \sum n_i p^i$ be the base-p expansions of m and n, where p is prime. Prove that
$$\binom{m}{n} \equiv \prod \binom{m_i}{n_i} \pmod{p}.$$
- Let p be a prime. What is the sum of all the generators mod p?
- Quadratic residues:
  - Let p be an odd prime. Working in mod p, let S be a set such that for any nonzero a, $a \in S \iff -a \notin S$.

    Prove that for each x, $|xS \setminus S|$ is even if and only if x is a quadratic residue mod p.
  - Find $\left(\frac{2}{p}\right)$ and $\left(\frac{-1}{p}\right)$.
  - Let p and q be distinct odd primes, such that p is 1 mod 4.

    Prove that p is a quadratic residue mod q iff q is a quadratic residue mod p.
  - What if instead both p and q are 3 mod 4?

4 Weak Prime Number Theorem

1. Prove that the sum of the reciprocals of the primes diverges.
2. Let n be a positive integer larger than 1.
   (a) Prove that the product of all primes between $\lceil \frac{n}{2} \rceil$ and n (including n, not including $\lceil \frac{n}{2} \rceil$) is less than 2^n.
   (b) Prove that the product of all primes between 1 and n is at most 4^{n-1}.
   (c) Find some real number c independent of n such that there are at most $\frac{cn}{\log_2 n}$ primes that are at most n.
3. Let n be a positive integer larger than $2^{2^{2^2}}$.
   (a) Let p be a prime.
      - Prove that if $p^k \mid \binom{2n}{n}$ then $p^k < 2n$.
      - Prove that if $2p \leq 2n < 3p$ then $p \nmid \binom{2n}{n}$.

   (b) Prove that
   $$\prod_{\substack{p^k \| \binom{2n}{n} \\ p \leq n}} p^k < \binom{2n}{n}.$$
   (c) Find some real number c independent of n such that there are at least $\frac{cn}{\log_2 n}$ primes that are at most n.