

# Bounding

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## 1 Techniques

- WLOG arguments: break symmetry to bound one of the variables.
- If  $a$  and  $b$  are integers with  $a > b$  then  $a \geq b + 1$ .
- If  $a$  and  $b$  are integers with  $b \neq 0$  and  $a \mid b$  then  $|a| \leq |b|$ . Thus, try to get big dividing small. Polynomial long division is helpful here.
- Look for “dominant terms” that grow larger than everything else. For sufficiently large  $n$  and any  $c > 1$ , we have  $c < \log n < n^c < c^n < n! < n^n$ .
- If you need to prove a specific bound of the form “ $f(x) > g(x)$  for  $x > k$ ,” try induction.
- If you have a factorial,  $n! \equiv 0 \pmod{c}$  for  $n \geq c$  so mods are often nice.

## 2 Problems

1. Find all positive integers  $k$  such that there is a positive integer  $n$  for which

$$k = \frac{n^2 - 29}{3n + 11}.$$

2. Find all positive integers  $a, b, c$  such that  $ab + bc + ca = abc + 2$ .
3. Prove that if  $n > 11$  then  $n^2 - 19n + 89$  is not a square.
4. Find all positive integers  $n$  such that  $2^n \leq (n + 1)^2$ .
5. Find all positive integers  $a, b, c$  such that  $a \mid b + c$ ,  $b \mid c + a$ ,  $c \mid a + b$ .
6. Let  $S(n)$  be the sum of the digits of  $n$ . Find  $S(S(S(4444^{4444})))$ .
7. For a natural number  $N$ , consider all distinct perfect squares that can be obtained from  $N$  by deleting one digit from its decimal representation. Prove that the number of such squares is bounded by some value that doesn't depend on  $N$ .
8. Find all pairs  $a, b$  of positive integers such that  $2017^a = b^6 - 32b + 1$ .

### 3 Homework

1. Find all pairs of positive integers  $x, y$  such that  $x^2 - y! = 2001$ .
2. Find all pairs of positive integers  $x, y$  such that  $y^2(x - 1) = x^5 - 1$ .
3. Find all triples  $a, b, c$  of positive integers such that  $a \mid bc - 1$ ,  $b \mid ca - 1$  and  $c \mid ab - 1$ .