Bounding

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1 Techniques

- WLOG arguments: break symmetry to bound one of the variables.
- If a and b are integers with $a > b$ then $a \geq b + 1$.
- If a and b are integers with $b \neq 0$ and $a \mid b$ then $|a| \leq |b|$. Thus, try to get big dividing small. Polynomial long division is helpful here.
- Look for "dominant terms" that grow larger than everything else. For sufficiently large n and any $c > 1$, we have $c < \log n < n^c < c^n < n! < n^n$.
- If you need to prove a specfic bound of the form "$f(x) > g(x)$ for $x > k$," try induction.
- If you have a factorial, $n! \equiv 0 \pmod{c}$ for $n \geq c$ so mods are often nice.

2 Problems

1. Find all positive integers k such that there is a positive integer n for which
$$k = \frac{n^2 - 29}{3n + 11}.$$
2. Find all positive integers a, b, c such that $ab + bc + ca = abc + 2$.
3. Prove that if $n > 11$ then $n^2 - 19n + 89$ is not a square.
4. Find all positive integers n such that $2^n \leq (n + 1)^2$.
5. Find all positive integers a, b, c such that $a \mid b + c,\ b \mid c + a,\ c \mid a + b$.
6. Let $S(n)$ be the sum of the digits of n. Find $S(S(S(4444^{4444})))$.
7. For a natural number N, consider all distinct perfect squares that can be obtained from N by deleting one digit from its decimal representation. Prove that the number of such squares is bounded by some value that doesn't depend on N.
8. Find all pairs a, b of positive integers such that $2017^a = b^6 - 32b + 1$.

3 Homework

1. Find all pairs of positive integers x, y such that $x^2 - y! = 2001$.
2. Find all pairs of positive integers x, y such that $m^2 + (m+1)^2 = n^4 + (n+1)^4$.
3. Find all triples a, b, c of positive integers such that $a \mid bc - 1,\ b \mid ca - 1$ and $c \mid ab - 1$.