# Team Level Lectures

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## 1 Farey sequences

Let $n$ be a fixed positive integer. Let $\frac{a_1}{b_1}, \ldots, \frac{a_k}{b_k}$ be the rational numbers between 0 and 1 inclusive with denominators at most $n$, written in increasing order and lowest terms.

- Prove that for each $i$, $a_{i+1}b_i - a_ib_{i+1} = 1$.
- Prove that the rational number $x$ with smallest denominator such that $\frac{a_i}{b_i} < x < \frac{a_{i+1}}{b_{i+1}}$ is $\frac{a_i+a_{i+1}}{b_i+b_{i+1}}$.
- Which pairs of numbers appear as consecutive $b_i$s?

Example problems:

- Suppose that $(a_1, b_1), (a_2, b_2), \ldots, (a_{100}, b_{100})$ are distinct ordered pairs of nonnegative integers. Let $N$ denote the number of pairs of integers $(i, j)$ satisfying $1 \le i < j \le 100$ and $|a_ib_j - a_jb_i| = 1$. Determine the largest possible value of $N$ over all possible choices of the 100 ordered pairs.
- A lattice point in the Cartesian plane is a point whose coordinates are both integers. A lattice polygon is a polygon all of whose vertices are lattice points.

  Let $\Gamma$ be a convex lattice polygon. Prove that $\Gamma$ is contained in a convex lattice polygon $\Omega$ such that the vertices of $\Gamma$ all lie on the boundary of $\Omega$, and exactly one vertex of $\Omega$ is not a vertex of $\Gamma$.

## 2 Dirichlet Convolution and Mobius Inversion

Let $f : \mathbb{N} \to \mathbb{R}$ and $g : \mathbb{N} \to \mathbb{R}$ be two functions. We define the *Dirichlet convolution* $f * g$ as

$$(f * g)(n) = \sum_{d|n} f(d)g\left(\frac{n}{d}\right).$$

We define the functions $d,\ \sigma,\ \varphi$ as before and also define the functions

$$\zeta(n) = 1,\ \psi(n) = n.$$

- Prove that $*$ is associative: that is, $(a * b) * c = a * (b * c)$.

- Prove that if $a$ and $b$ are multiplicative then so is $a * b$.
- Find a function $\delta$ such that $\delta * a = a$ for all functions $a$.
- Find a function $\mu$ such that $\mu * \zeta = \delta$.
- Prove that $g = f * \zeta \iff f = g * \mu$.
- Find $\zeta * \zeta,\ \psi * \zeta$ and $\varphi * \zeta$.
- Prove that
$$\sum_{i=1}^{n} f(i) \left\lfloor \frac{n}{i} \right\rfloor = \sum_{j=1}^{n} (f * \zeta)(j).$$

Example problems:

For a positive integer $n$, let $f(n)$ be the number of binary strings of length $n$ that can't be expressed as an $m$-fold repetition of another binary string for any $m > 1$.

For example, $f(6) = 54$ since the only strings of length 6 that can be expressed as an $m$-fold repetition of another binary string for some $m > 1$ are 000000, 001001, 010010, 010101, 011011, 100100, 101010, 101101, 110110, 111111.

- Find two functions $g$ and $h$, in closed form, such that $f = g * h$.
- Prove that $n \mid f(n)$.
- Find all $n$ for which $n \mid \displaystyle\sum_{i=1}^{n} f(i) \left\lfloor \frac{n}{i} \right\rfloor$.

# 3 Polynomials mod $p$

Let $p$ be prime.

- Prove that unique factorisation holds for polynomials mod $p$. (This is not true for all integers — for instance, $(x-1)^2 \equiv (x-3)^2 \pmod 4$.)
- Prove that for every function $f : \mathbb{Z}_p \to \mathbb{Z}_p$ there is a unique polynomial $P$ in $\mathbb{Z}_p$ of degree less than $p-1$ such that $f(x) = P(x)$ for each $x \in \mathbb{Z}_p$.
- Let $g$ be a generator mod $p$, and let $ab = p - 1$. Prove that
$$\prod_{i=1}^{a} (x - g^{bi}) \equiv x^a - 1 \pmod p.$$

  What does this tell us about the roots of the cyclotomic polynomials in mod $p$?
- Consider all $\binom{p-1}{k}$ products of $k$ elements of $\mathbb{Z}_p$. Prove that their sum is divisible by $p$.
- For any positive integer $n < p - 1$, prove that
$$\sum_{i=1}^{p-1} i^n \equiv 0 \pmod p.$$

Example problems:

- Let $p$ be an odd prime. We compute the product of $(4-x)$, where $x$ varies over all residues mod $p$ except the quadratic residues. Find the least residue of this product mod $p$.
- Find the least residue of the sum of all generators mod $p$.
- Let $\mathbb{Z}/n\mathbb{Z}$ denote the set of integers considered modulo $n$ (hence $\mathbb{Z}/n\mathbb{Z}$ has $n$ elements). Find all positive integers $n$ for which there exists a bijective function $g : \mathbb{Z}/n\mathbb{Z} \to \mathbb{Z}/n\mathbb{Z}$, such that the 101 functions

  $$g(x), \quad g(x)+x, \quad g(x)+2x, \quad \dots, \quad g(x)+100x$$

  are all bijections on $\mathbb{Z}/n\mathbb{Z}$.
- Let $p$ be an odd prime. An integer $x$ is called a quadratic non-residue if $p$ does not divide $x-t^2$ for any integer $t$.

  Denote by $A$ the set of all integers $a$ such that $1 \le a < p$, and both $a$ and $4-a$ are quadratic non-residues. Calculate the remainder when the product of the elements of $A$ is divided by $p$.

# 4 Binomial coefficients mod $p$

- Wolstenholme's Theorem: let $a$ and $b$ be positive integers, and let $p$ be a prime greater than 3. Prove that

  $$\binom{ap}{bp} \equiv \binom{a}{b} \pmod{p^3}.$$

- Lucas' Theorem: let $m = \sum m_i p^i$ and $n = \sum n_i p^i$ be the base-$p$ expansions of $m$ and $n$, where $p$ is prime. Prove that

  $$\binom{m}{n} \equiv \prod \binom{m_i}{n_i} \pmod{p}.$$

# 5 Weak Prime Number Theorem

1. Prove that the sum of the reciprocals of the primes diverges.
2. Let $n$ be a positive integer larger than 1.
   (a) Prove that the product of all primes between $\lceil \frac{n}{2} \rceil$ and $n$ (including $n$, not including $\lceil \frac{n}{2} \rceil$) is less than $2^n$.
   (b) Prove that the product of all primes between 1 and $n$ is at most $4^{n-1}$.
   (c) Find some real number $c$ independent of $n$ such that there are at most $\frac{cn}{\log_2 n}$ primes that are at most $n$.
3. Let $n$ be a positive integer larger than $2^{2^{2^2}}$.
   (a) Let $p$ be a prime.

- Prove that if $p^k \mid \binom{2n}{n}$ then $p^k < 2n$.
- Prove that if $2p \leq 2n < 3p$ then $p \nmid \binom{2n}{n}$.

(b) Prove that

$$\prod_{\substack{p^k \| \binom{2n}{n} \\ p \leq n}} p^k < \binom{2n}{n}.$$

(c) Find some real number $c$ independent of $n$ such that there are at least $\frac{cn}{\log_2 n}$ primes that are at most $n$.

# 6 Build a graph

1. Fifty numbers are chosen from the set $\{1, 2, \ldots, 99\}$, no two of which sum to 99 or 100. Prove that the numbers must be $50, 51, \ldots, 99$.
2. Let $p$ be a prime, and let $a_1, \ldots, a_p$ be integers. Show that there exists an integer $k$ such that the numbers
$$a_1 + k, a_2 + 2k, \ldots, a_p + pk$$
produce at least $\frac{1}{2}p$ distinct remainders upon division by $p$.
3. An international society has its members from six different countries. The list of members has 1978 names, numbered $1, 2, \ldots, 1978$. Prove that there is at least one member whose number is the sum of the numbers of two (not necessarily distinct) members from his own country.
4. A set of positive integers is called *fragrant* if it contains at least two elements and each of its elements has a prime factor in common with at least one of the other elements. Let $P(n) = n^2 + n + 1$. What is the least possible positive integer value of $b$ such that there exists a non-negative integer $a$ for which the set $\{P(a+1), P(a+2), \ldots, P(a+b)\}$ is fragrant?
5. The Fibonacci numbers $F_0, F_1, F_2, \ldots$ are defined inductively by $F_0 = 0, F_1 = 1$, and $F_{n+1} = F_n + F_{n-1}$ for $n \geq 1$. Given an integer $n \geq 2$, determine the smallest size of a set $S$ of integers such that for every $k = 2, 3, \ldots, n$ there exist some $x, y \in S$ such that $x - y = F_k$.
6. There are $4n$ pebbles of weights $1, 2, 3, \ldots, 4n$. Each pebble is coloured in one of $n$ colours and there are four pebbles of each colour. Show that we can arrange the pebbles into two piles so that the following two conditions are both satisfied:
   - The total weights of both piles are the same.
   - Each pile contains two pebbles of each colour.
7. Let $k, m, n$ be integers satisfying $1 < n \leq m - 1 \leq k$. Determine the maximum size of a subset $S$ of the set $\{1, 2, \ldots, k\}$ such that no $n$ distinct elements of $S$ add up to $m$.
8. Let n be an even positive integer. Show that there is a permutation $(x_1, x_2, \ldots, x_n)$ of $(1, 2, \ldots, n)$ such that for every $i \in (1, 2, \ldots, n)$, the number $x_{i+1}$ is one of the numbers $2x_i, 2x_i - 1, 2x_i - n, 2x_i - n - 1$. Here we use the cyclic subscript convention, so that $x_{n+1}$ means $x_1$.

# 7 Other additive number theory

1. Let $n$ be a positive integer and $\{A, B, C\}$ a partition of $(1, 2, 3, \ldots, 3n)$ such that $|A| = |B| = |C| = n$. Prove that there exist $x \in A, y \in B, z \in C$ such that one of $x, y, z$ is the sum of the other two.
2. Suppose that every integer has been given one of the colors red, blue, green or yellow. Let $x$ and $y$ be odd integers so that $|x| \neq |y|$. Show that there are two integers of the same color whose difference has one of the following values: $x$, $y$, $(x + y)$ or $(x - y)$.
3. Let $k$, $m$, $n$ be integers satisfying $1 < n \leq m - 1 \leq k$. Determine the maximum size of a subset $S$ of the set $\{1, 2, \ldots, k\}$ such that no $n$ distinct elements of $S$ add up to $m$.
4. A set $S$ of distinct integers is called sum-free if there does not exist a triple $\{x, y, z\}$ of integers in $S$ such that $x + y = z$. Show that for any set $X$ of distinct integers, $X$ has a sum-free subset $Y$ such that $|Y| > |X|/3$.
5. Prove that there exists a four-coloring of the set $M = \{1, 2, \ldots, 1987\}$ such that any arithmetic progression with 10 terms in the set M is not monochromatic.