Integer Polynomials

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1 Basic facts

We say that $p(x)$ is *divisible* by a polynomial $q(x)$ if there is a polynomial $r(x)$ (not necessarily with integer coefficients) such that $p(x) = q(x)r(x)$. We also write $q(x) \mid p(x)$.

Examples:

- If $p(x) = 2x + 2$, does $p(x) \mid x^2 - 1$?

  One thing that this shows is that even if $p(x) \mid q(x)$ as polynomials, substituting specific integers for x won't necessarily divisible numbers. For example, $p(2) = 6$, which does not divide $2^2 - 1 = 3$.

- If $p(x) = 3x + 1$, does $p(x) \mid 9x^2 + 1$?

For the following, p is a polynomial with integer coefficients.

- (Division algorithm) For any polynomial q there are unique polynomials f and r such that $p(x) = q(x)f(x) + r(x)$ and $\deg r < \deg q$.

  If q has integer coefficients, then f and r have rational coefficients. If q has a leading coefficient of ± 1, then f and r have integer coefficients.

  Examples:

  - Let $p(x) = 2x^3 + x + 1$, and $q(x) = x^2 + 3x - 5$.
  - Let $p(x) = x^3 + x + 1$, and $q(x) = 2x^2 + 3x - 5$.

- (Factor theorem) If a is a root of p, then $x - a \mid p(x)$.

  "a is a root of p" just means $p(a) = 0$.

- (Remainder theorem) If $p(a) = c$ then $x - a \mid p(x) - c$.
- (*) If a and b are integers, then $a - b \mid p(a) - p(b)$. (This only works if p is an integer polynomial!)
- (Finite differences) If c is a constant and p is a polynomial with leading coefficient ax^n, then $p(x) - p(x - c)$ is a polynomial with leading coefficient nax^{n-1}.
- (Rational Root Theorem) If y and z are integers with $\gcd(y, z) = 1$ such that $p(y/z) = 0$, and if a_0 is the constant term and a_n is the leading coefficient of p, then $y \mid a_0$ and $z \mid a_n$.

2 Problems

1. Prove that every nonconstant integer polynomial has a composite number in its image.
2. Do there exist two quadratics $ax^2 + bx + c$ and $(a+1)x^2 + (b+1)x + (c+1)$ which both have two integer roots?
3. Let P be an integer polynomial such that if $P(x)$ is an integer then x is rational. Prove that P is linear.
4. Find all polynomials $P(x)$ with integer coefficients such that if $m \mid n$ then $P(m) \mid P(n)$.
5. Prove that for any two distinct polynomials P and Q with coefficients in $\{0, 1, \ldots, 9\}$, either $P(-2) \neq Q(-2)$ or $P(-5) \neq Q(-5)$.
6. Let $P(x)$ and $Q(x)$ be polynomials with integer coefficients such that the leading coefficient of $P(x)$ is 1. Suppose that $P(n)^n$ divides $Q(n)^{n+1}$ for infinitely many positive integers n. Prove that $P(n)$ divides $Q(n)$ for infinitely many positive integers n.
7. Let $n \geq 2$ be an integer and $P(x)$ be a polynomial with nonnegative integer coefficients satisfying $P(1) = 1$ and $x^n P(1/x) = P(x)$ for all x. Prove that there exist infinitely many pairs x, y of positive integers such that $x|P(y)$ and $y|P(x)$.
8. Given are positive integers a, b satisfying $a \geq 2b$. Does there exist a polynomial $P(x)$ of degree at least 1 with coefficients from the set $\{0, 1, 2, \ldots, b-1\}$ such that $P(b) \mid P(a)$?

3 Homework

1. Prove that if P is a polynomial with integer coefficients and leading term $a_0 n^k$ such that $m \mid P(n)$ for all n, then $m \mid k! a_0$.
2. Prove that for every polynomial $P(x)$ of degree at least 2 with integer coefficients, there is an infinite arithmetic progression of integers which does not contain $P(k)$ for any integer k.
3. Is there a polynomial f of degree 2023 with integer coefficients such that

$$f(n), f(f(n)), f(f(f(n))), \cdots$$

are pairwise relatively prime for any integer n?