

# Sequences and Integer Functions

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## 1 Problems

1. Prove that there does not exist a function  $f : \mathbb{N} \rightarrow \mathbb{N}$  such that for any distinct positive integers  $i$  and  $j$ ,  $\gcd(f(i) + j, f(j) + i) = 1$ .
2. Find all functions  $f : \mathbb{Z} \rightarrow \mathbb{Z}$  such that  $f(m + f(n)) = f(m) - n$  for all integers  $m$  and  $n$ .
3. Consider a sequence of positive integers  $a_1, a_2, \dots$  which satisfies  $a_n = a_{n-1}^2 + a_{n-2}^2 + a_{n-3}^2$  for all  $n \geq 3$ . Prove that if  $a_k = 1997$  then  $k \leq 4$ .
4. Prove that any infinite sequence of integers in arithmetic progression has an infinite subsequence in geometric progression.
5. Prove that there exists a strictly increasing sequence  $\{a_n\}_1^\infty$  of positive integers such that for any  $k \geq 0$ , the sequence  $\{k + a_n\}$  contains only finitely many primes.
6. The sequence  $\{a_i\}_1^\infty$  is defined by  $a_1 = 1$  and  $a_{n+1} = a_n^2 + 1$  for  $n \geq 1$ . Prove that there are infinitely many primes which divide some  $a_i$ .
7. Let  $n$  be a positive integer. Define a sequence by letting  $a_1 = n$ , and for each  $i > 1$  choosing  $a_i$  such that  $0 \leq a_i < i$  and  $\frac{a_1 + \dots + a_i}{i}$  is an integer. Prove that this sequence is eventually constant.

## 2 Homework

1. Find all functions  $f : \mathbb{N} \rightarrow \mathbb{N}$  satisfying  $f(n + f(n)) = f(n)$  for all  $n$  such that 1 is in the range of  $f$ .
2. Find all monotonically increasing functions  $f : \mathbb{N} \rightarrow \mathbb{Z}_{\geq 0}$  such that  $f(mn) = f(m) + f(n)$  for all nonnegative integers  $m$  and  $n$ .
3. Let  $a, b$  be odd positive integers. Define the sequence  $c_n$  by choosing  $c_1 = a, c_2 = b$  and for each  $i > 2$  letting  $c_i$  be the largest odd divisor of  $c_{i-1} + c_{i-2}$ . Prove that this sequence is eventually constant (that is, there is an  $m$  such that for any  $i, j > m$ ,  $a_i = a_j$ ).