Modular Arithmetic 3

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1 Choosing good mods

Prove that:

- Squares are 0, 1 or 4 mod each of $\{5,8\}$, and 0 or 1 mod 3.
- Cubes are 0, 1 or -1 mod each of $\{7,9\}$.

Often a problem will be solved by considering it under an appropriate mod. In general, for nth powers, try looking mod m where $\varphi(m)$ is a small multiple of n. You can also try choosing a mod which divides a bunch of terms in an equation.

However, remember that if you find a single solution to an equation, then that solution is still a solution in every mod so you won't be able to find a contradiction.

2 Problems

1. Find all positive integers a, b such that $a^4 + b^4 = 10a^2b^2 - 2022$.
2. Find all positive integers n such that $2^n + 7^n$ is a perfect square.
3. Find all pairs of positive integers x, y such that $x! + 5 = y^3$.
4. Find all primes p, q for which $pq \mid (5^p - 2^p)(5^q - 2^q)$.
5. Prove that if p is prime, then $2^p + 3^p$ is not a nontrivial perfect power.
6. Find all positive integers x, y, z such that $3^x + 4^y = 5^z$.
7. Find all integers a, b such that $a^3 + (a+1)^3 + \cdots + (a+6)^3 = b^4 + 1$.
8. What is the least residue mod n of the product of the elements of $\mathbb{Z}_n^*$?

3 Homework

1. Find all positive integers a for which $1! + 2! + \cdots + a!$ is a perfect cube.
2. Prove that if m and n are natural numbers, then $3^m + 3^n + 1$ is not a perfect square.
3. Find all primes p and q such that $p + q = (p - q)^3$.