Sequences and Integer Functions

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1 Problems

1. Prove that there does not exist a function $f : \mathbb{N} \to \mathbb{N}$ such that for any positive integers i and j, $\gcd(f(i) + j, f(j) + i) = 1$.
2. Does one of the first $10^8 + 1$ Fibonacci numbers end with 4 0s?
3. Find all functions $f : \mathbb{Z} \to \mathbb{Z}$ such that $f(m + f(n)) = f(m) - n$ for all integers m and n.
4. Prove that any infinite sequence of integers in arithmetic progression has an infinite subsequence in geometric progression.
5. Prove that the function $f(n) = \lfloor (1 + \sqrt{2})^n \rfloor$ alternates between even and odd integers.
6. Let n be a positive integer. Define a sequence by letting $a_1 = n$, and for each $i > 1$ choosing a_i such that $0 \leq a_i < i$ and $\frac{a_1 + \cdots + a_i}{i}$ is an integer. Prove that this sequence is eventually constant.

2 Homework

1. Find all functions $f : \mathbb{N} \to \mathbb{N}$ such that $f(f(m)f(n)) = mn$ for all positive integers m and n.
2. Find all monotonically increasing functions $f : \mathbb{N} \to \mathbb{Z}_{\geq 0}$ such that $f(mn) = f(m) + f(n)$ for all nonnegative integers m and n.
3. Let a, b be odd positive integers. Define the sequence c_n by choosing $c_1 = a, c_2 = b$ and for each $i > 2$ letting c_i be the largest odd divisor of $c_{i-1} + c_{i-2}$. Prove that this sequence is eventually constant (that is, there is an m such that for any $i, j > m$, $a_i = a_j$).