

# Descent and Floor Functions

Andres Buritica Monroy

## 1 Techniques

Descent works like this: assign some positive integer quantity (the “size” of a solution, e.g. the sum of the absolute values of the variables) to each solution, take the solution with lowest “size” that doesn’t fit your claimed pattern and derive a contradiction.

Floor functions: recall  $\lfloor x \rfloor$  is the largest integer which is at most  $x$ . Most often, problems which involve  $\lfloor x \rfloor$  will be solved by considering  $x - \lfloor x \rfloor$  (also denoted  $\{x\}$ ), or  $\lceil x \rceil - x$ , and doing inequalities.

## 2 Problems

1. Prove that  $\lfloor \sqrt{n} + \sqrt{n+1} \rfloor = \lfloor \sqrt{4n+1} \rfloor$  for all positive integers  $n$ .
2. Prove that if  $x, y, z$  are integers such that  $x^2 + y^2 + z^2 = (xy)^2$ , then  $x = y = z = 0$ .
3. Let  $a$  and  $b$  be positive irrational numbers such that  $\frac{1}{a} + \frac{1}{b} = 1$ . Let  $A = \{\lfloor na \rfloor : n \in \mathbb{N}\}$ , and  $B = \{\lfloor nb \rfloor : n \in \mathbb{N}\}$ . Prove that the sets  $A$  and  $B$  together contain each positive integer exactly once.
4. Let  $f$  be a function defined on the nonnegative integers such that  $f(2x) = 2f(x)$ ,  $f(4x+1) = 4f(x) + 3$ , and  $f(4x-1) = 2f(2x-1) - 1$ . Prove that  $f$  is injective.
5. Find all positive integers  $n$  such that  $1 + \lfloor \sqrt{n} \rfloor$  divides  $n$ .
6. Solve over integers:  $6(6a^2 + 3b^2 + c^2) = 5n^2$ .
7. Prove that for any positive integer  $n$  which is not a perfect square, there is a positive integer  $k$  such that

$$n = \left\lfloor k + \sqrt{k} + \frac{1}{2} \right\rfloor.$$

### 3 Homework

1. Let's say you have a set  $S$  of positive rational numbers such that  $1 \in S$ , and if  $x \in S$  then both  $x + 1$  and  $\frac{1}{x}$  are in  $S$ . Prove that  $S$  contains all positive rationals.
2. Let  $p$  and  $q$  be coprime. Prove that

$$\sum_{i=1}^{q-1} \left\lfloor \frac{ip}{q} \right\rfloor = \frac{(p-1)(q-1)}{2}.$$

3. Prove that there does not exist a list of 2022 positive integers such that if you remove any one of them, the rest can be split into two groups of equal sum.