Sequences and Integer Functions

Andres Buritica Monroy

1 Problems

1. Prove that there does not exist a function $f : \mathbb{N} \to \mathbb{N}$ such that for any distinct positive integers i and j, $\gcd(f(i) + j, f(j) + i) = 1$.

2. Consider a sequence of positive integers $a_1, a_2, \ldots$ which satisfies $a_n = a_{n-1}^2 + a_{n-2}^2 + a_{n-3}^2$ for all $n \geq 3$. Prove that if $a_k = 1997$ then $k \leq 4$.

3. Prove that any infinite sequence of integers in arithmetic progression has an infinite subsequence in geometric progression.

4. Prove that there exists a strictly increasing sequence $\{a_n\}_1^\infty$ of positive integers such that for any $k \geq 0$, the sequence $\{k + a_n\}$ contains only finitely many primes.

5. The sequence $\{a_i\}_1^\infty$ is defined by $a_1 = 1$ and $a_{n+1} = a_n^2 + 1$ for $n \geq 1$. Prove that there are infinitely many primes which divide some a_i.

6. Let n be a positive integer. Define a sequence by letting $a_1 = n$, and for each $i > 1$ choosing a_i such that $0 \leq a_i < i$ and $\frac{a_1 + \cdots + a_i}{i}$ is an integer. Prove that this sequence is eventually constant.

2 Homework

1. Find all functions $f : \mathbb{N} \to \mathbb{N}$ satisfying $f(n + f(n)) = f(n)$ for all n such that 1 is in the range of f.

2. Find all monotonically increasing functions $f : \mathbb{N} \to \mathbb{Z}_{\geq 0}$ such that $f(mn) = f(m) + f(n)$ for all nonnegative integers m and n.

3. Let a, b be odd positive integers. Define the sequence c_n by choosing $c_1 = a, c_2 = b$ and for each $i > 2$ letting c_i be the largest odd divisor of $c_{i-1} + c_{i-2}$. Prove that this sequence is eventually constant (that is, there is an m such that for any $i, j > m$, $c_i = c_j$).