Review and Extension — Divisibility and Congruences

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1 Exam Review

2 Key concepts for this term

- Induction, strong induction, well-ordering
- If $a \mid b$ and $a \mid c$ then for any integers x and y, $a \mid bx + cy$
- Division algorithm, Euclid's algorithm, Bezout's identity
- Fundamental Theorem of Arithmetic
- GCD, LCM in terms of prime factorisations
- Factorise expressions
- Take out the GCD
- Basic properties of $\mathbb{Z}_p$: operations, inverses, Wilson, Fermat, GCD trick
- Multiplicative and completely multiplicative functions
- Formulas for d, σ and φ
- Prove a problem for prime powers first
- Basic properties of $\mathbb{Z}_n$: operations, inverses, Euler, GCD trick, Chinese Remainder Theorem, generalised Wilson
- How to choose a mod
- Modular contradictions
- Quadratic discriminant trick

3 Problems

4 Homework

Solve and submit any three problems from Section 3 and/or Section 7.

5 Extension: Farey sequences

Let n be a fixed positive integer. Let $\frac{a_1}{b_1}, \ldots, \frac{a_k}{b_k}$ be the rational numbers between 0 and 1 inclusive with denominators at most n, written in increasing order and lowest terms.

- Prove that for each i, $a_{i+1}b_i - a_i b_{i+1} = 1$.
- Prove that the rational number x with smallest denominator such that $\frac{a_i}{b_i} < x < \frac{a_{i+1}}{b_{i+1}}$ is $\frac{a_i + a_{i+1}}{b_i + b_{i+1}}$.
- Which pairs of numbers appear as consecutive b_is?

6 Extension: Dirichlet Convolution and Mobius Inversion

Let $f : \mathbb{N} \to \mathbb{R}$ and $g : \mathbb{N} \to \mathbb{R}$ be two functions. We define the *Dirichlet convolution* $f * g$ as

$$(f * g)(n) = \sum_{d|n} f(d) g\left(\frac{n}{d}\right).$$

We define the functions $d,\ \sigma,\ \varphi$ as before and also define the functions

$$\zeta(n) = 1,\ \psi(n) = n.$$

- Prove that $*$ is associative: that is, $(a * b) * c = a * (b * c)$.
- Prove that if a and b are multiplicative then so is $a * b$.
- Find a function δ such that $\delta * a = a$ for all functions a.
- Find a function μ such that $\mu * \zeta = \delta$.
- Prove that $g = f * \zeta \iff f = g * \mu$.
- Find $\zeta * \zeta,\ \psi * \zeta$ and $\varphi * \zeta$.
- Prove that
$$\sum_{i=1}^{n} f(i) \left\lfloor \frac{n}{i} \right\rfloor = \sum_{j=1}^{n} (f * \zeta)(j).$$

7 Extension: Problems

1. Suppose that $(a_1, b_1), (a_2, b_2), \ldots, (a_{100}, b_{100})$ are distinct ordered pairs of nonnegative integers. Let N denote the number of pairs of integers (i, j) satisfying $1 \leq i < j \leq 100$ and $|a_i b_j - a_j b_i| = 1$. Determine the largest possible value of N over all possible choices of the 100 ordered pairs.

2. For a positive integer n, let $f(n)$ be the number of binary strings of length n that can't be expressed as an m-fold repetition of another binary string for any $m > 1$.

   For example, $f(6) = 54$ since the only strings of length 6 that can be expressed as an m-fold repetition of another binary string for some $m > 1$ are 000000, 001001, 010010, 010101, 011011, 100100, 101010, 101101, 110110, 111111.

   (a) Find two functions g and h, in closed form, such that $f = g * h$.

   (b) Prove that $n \mid f(n)$.

   (c) Find all n for which $n \mid \displaystyle\sum_{i=1}^{n} f(i) \left\lfloor \frac{n}{i} \right\rfloor$.