Descent and Floor Functions

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1 Techniques

Descent works like this: assign some positive integer quantity (the "size" of a solution, e.g. the sum of the absolute values of the variables) to each solution, take the solution with lowest "size" that doesn't fit your claimed pattern and derive a contradiction.

Floor functions: recall $\lfloor x \rfloor$ is the largest integer which is at most x. Most often, problems which involve $\lfloor x \rfloor$ will be solved by considering $x - \lfloor x \rfloor$ (also denoted $\{x\}$), or $\lceil x \rceil - x$, and doing inequalities.

2 Problems

1. Prove that $\lfloor \sqrt{n} + \sqrt{n+1} \rfloor = \lfloor \sqrt{4n+1} \rfloor$ for all positive integers n.
2. Prove that if x, y, z are integers such that $x^2 + y^2 + z^2 = (xy)^2$, then $x = y = z = 0$.
3. Let a and b be positive irrational numbers such that $\frac{1}{a} + \frac{1}{b} = 1$. Let $A = \{\lfloor na \rfloor : n \in \mathbb{N}\}$, and $B = \{\lfloor nb \rfloor : n \in \mathbb{N}\}$. Prove that the sets A and B together contain each posiive integer exactly once.
4. Let f be a function defined on the nonnegative integers such that $f(2x) = 2f(x)$, $f(4x+1) = 4f(x) + 3$, and $f(4x-1) = 2f(2x-1) - 1$. Prove that f is injective.
5. Find all positive integers n such that $1 + \lfloor \sqrt{n} \rfloor$ divides n.
6. Solve over integers: $6(6a^2 + 3b^2 + c^2) = 5n^2$.
7. Prove that for any positive integer n which is not a perfect square, there is a positive integer k such that
$$n = \left\lfloor k + \sqrt{k} + \frac{1}{2} \right\rfloor.$$

3 Homework

1. Let's say you have a set S of positive rational numbers such that $1 \in S$, and if $x \in S$ then both $x+1$ and $\frac{1}{x}$ are in S. Prove that S contains all positive rationals.

2. Let p and q be coprime. Prove that

$$\sum_{i=1}^{q-1} \left\lfloor \frac{ip}{q} \right\rfloor = \frac{(p-1)(q-1)}{2}.$$

3. Prove that there does not exist a list of 2022 positive integers such that if you remove any one of them, the rest can be split into two groups of equal sum.