Descent, Vieta Jumping, and Pell equations

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1 Techniques

Vieta jumping and Pell equations are two special cases of the technique of *infinite descent*. As we know, this technique involves assigning some notion of *size* to every set of positive integers satisfying a certain property, and then proving that if a certain condition is not satisfied, one can always get from one set to another set of smaller size.

In Vieta jumping, the way we get from one solution to the next is by considering the second root of a quadratic which we know has one integer root; then, Vieta's formulas provide us with information that allows us to prove the new solution has smaller size.

In Pell equations, the key step is provided by *Brahmagupta's Identity*:

$$(a^2 - nb^2)(c^2 - nd^2) = (ac \pm nbd)^2 - n(ad \pm bc)^2.$$

2 Pell Equations

We consider the equation $a^2 - nb^2 = \pm 1$ over positive integers a, b, where n is not a square. In homework question 3, you will prove that a solution always exists.

If we consider a solution (a_0, b_0), then we may find more solutions by repeatedly applying Brahmagupta's Identity. More explicitly, if (a, b) is a solution then we get the bigger solution $(a_0 a + nb_0 b, a_0 b + b_0 a)$.

Note that if $a_0^2 - nb_0^2 = 1$, then all solutions thus generated satisfy $a^2 - nb^2 = 1$. Otherwise, the RHSs alternate between 1 and -1.

I claim that this process yields all solutions, assuming we begin with the smallest solution (a_0, b_0). Consider the descent step

$$(a, b) \to (|a_0 a - nb_0 b|, |a_0 b - b_0 a|).$$

- Prove that this step inverts the previous type of step.
- Prove that this step gets us from a solution to a smaller solution, unless $(a, b) = (a_0, b_0)$.

When we instead have an equation of the form $a^2 - nb^2 = \pm k$, we may use the same descent step. The descent can only finish when $a_0 b - b_0 a = 0$ or $|a_0 b - b_0 a| \geq b$. In the second case we may deduce $b^2 < \frac{b_0^2 |k|}{2(a_0 - 1)}$, which leaves us with a finite set of solutions from which all solutions are generated.

3 Problems

1. Let a and b be positive integers. Prove that if $\frac{a^2+b^2}{ab+1}$ is an integer, then it is a square.
2. Prove that infinitely many triangular numbers are squares.
3. Find all solutions in integers to $x^2 + y^2 + z^2 = 2xyz$.
4. Prove that there are infinitely many triples (a, b, c) of positive integers in arithmetic progression such that $ab + 1$, $bc + 1$ and $ca + 1$ are all perfect squares.
5. Prove that there are infinitely many pairs of positive integers a, b such that $a \mid b^2 + 1$ and $b \mid a^2 + 1$.
6. Prove that if $2 + \sqrt{28n^2 + 1}$ is an integer, then it is a perfect square.
7. Let a and b be two positive integers. Prove that

$$a^2 + \left\lceil \frac{4a^2}{b} \right\rceil$$

is not a square.

4 Homework

1. Find all positive integers n such that
$$\sqrt{\frac{7^n+1}{2}}$$
is prime.

2. Let a and b be positive integers. Show that if $4ab-1$ divides $(4a^2-1)^2$, then $a=b$.

3. Let n be a positive integer.

   (a) Prove that there are infinitely many pairs (a,b) of positive integers such that
   $$\left|b\sqrt{n}-a\right| < \frac{1}{b}.$$

   (b) Prove that there are infinitely many pairs (a,b) of positive integers such that a^2-nb^2 equals the same value for each such pair.

   (c) Prove that there are positive integers a_1, b_1, a_2, b_2 such that $a_1^2-nb_1^2=a_2^2-nb_2^2=d$, $a_1 \equiv a_2 \pmod{d}$, and $b_1 \equiv b_2 \pmod{d}$.

   (d) Prove that there are positive integers a_0, b_0 such that $a_0^2-nb_0^2=1$.