Constructions and Existence Proofs

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1 Existence Results

Number theory has quite a few results about the existence of a number satisfying certain properties that you should be aware of. We've already seen many of these:

- There are infinitely many primes.
- (Bezout's Identity) For any coprime integers a and b, there are integers x and y such that $ax + by = 1$.
- (Chinese Remainder Theorem) For any positive integers $a_1, \dots, a_n$, and any pairwise coprime positive integers $b_1, \dots, b_n$, there is exactly one residue $x \pmod{b_1 \dots b_n}$ such that $x \equiv a_i \pmod{b_i}$ for each i.

The pigeonhole principle isn't strictly a number-theoretic result but is also often useful.

2 Techniques

You will learn much of this by trying problems, but in brief:

- Try small cases. TRY SMALL CASES!
- Build larger constructions from smaller ones; build constructions that satisfy more conditions from constructions that satisfy fewer.
- Try to control properties of your construction — rely on simple properties as opposed to "randomness" (exception: pigeonhole).

3 Problems

1. Prove that for each positive integer n, there are n consecutive positive integers, none of which is a prime power.
2. (a) Prove that for every positive integer n there is a number divisible by n consisting of only 1s and 0s.

   (b) Prove that if n is not a multiple of 5, there is a number divisible by n consisting of only 1s and 2s.
3. Prove that for every positive integer n, there is a set S of n distinct positive integers such that every subset of S has a geometric mean which is a positive integer.
4. Prove that there is an infinite set of positive integers such that the sum of any finite subset is not a perfect power.
5. Prove that for every positive integer n, there are infinitely many terms of the Fibonacci sequence which are divisible by n.
6. Prove that for every positive integer n, there is a positive integer X such that

$$X,\ 2X,\ 3X,\ \ldots,\ nX$$

   are all nontrivial perfect powers.
7. Does there exist an infinite sequence of integers $a_1, a_2, \ldots$ such that $\gcd(a_m, a_n) = 1 \iff |m - n| = 1$?
8. Let m and c be integers. Prove that for any infinite sequence $a_1, a_2, \ldots$ of positive integers which contains every positive integer exactly once, there are integers x, y, k such that $x < y$ and $a_x + a_{x+1} + \cdots + a_y = mk + c$.

4 Homework

1. Prove that for any positive integer c and any prime p, there is a positive integer x such that $x^x \equiv c \pmod{p}$.

2. Let x be an irrational number, and let a and b be real numbers such that $0 \leq a < b \leq 1$. Prove that there is an integer n such that $a < \{nx\} < b$. Hence prove that there is a power of 2 whose decimal representation starts with 2023.

3. Prove that every arithmetic progression $a, a+b, \ldots$ where $\gcd(a,b) = 1$ has infinitely many terms which are not divisible by any perfect square larger than 1.