Descent and Floor Functions

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1 Techniques

Descent works like this: assign some positive integer quantity (the "size" of a solution, e.g. the sum of the absolute values of the variables) to each solution, take the solution with lowest "size" that doesn't fit your claimed pattern and derive a contradiction.

Floor functions: recall $\lfloor x \rfloor$ is the largest integer which is at most x. Most often, problems which involve $\lfloor x \rfloor$ will be solved by considering $x - \lfloor x \rfloor$ (also denoted $\{x\}$), or $\lceil x \rceil - x$, and doing inequalities.

2 Problems

1. Find all solutions in integers to $a^3 + 2b^3 + 4c^3 = 0$.
2. Prove that $\lfloor \sqrt{n} + \sqrt{n+1} \rfloor = \lfloor \sqrt{4n+1} \rfloor$.
3. Prove that if x, y, z are integers such that $x^2 + y^2 + z^2 = (xy)^2$, then $x = y = z = 0$.
4. Let a and b be positive irrational numbers such that $\frac{1}{a} + \frac{1}{b} = 1$. Let $A = \{\lfloor na \rfloor : n \in \mathbb{N}\}$, and $B = \{\lfloor nb \rfloor : n \in \mathbb{N}\}$. Prove that the sets A and B together contain each posiive integer exactly once.
5. Solve over integers: $6(6a^2 + 3b^2 + c^2) = 5n^2$.
6. Find all positive integers n such that $1 + \lfloor \sqrt{n} \rfloor$ divides n.
7. Let f be a function defined on the nonnegative integers such that $f(2x) = 2f(x)$, $f(4x+1) = 4f(x) + 3$, and $f(4x-1) = 2f(2x-1) - 1$. Prove that f is injective.
8. Prove that for any positive integer n which is not a perfect square, there is a positive integer k such that
$$n = \left\lfloor k + \sqrt{k} + \frac{1}{2} \right\rfloor.$$
9. Find all triples (x, y, z) of integers such that
$$x^3 + 2y^3 + 4z^3 - 6xyz = 0.$$

3 Homework

1. Let's say you have a set S of positive rational numbers such that $1 \in S$, and if $x \in S$ then both $x+1$ and $\frac{1}{x}$ are in S. Prove that S contains all positive rationals.

2. Let p and q be coprime. Prove that

$$\sum_{i=1}^{q-1} \left\lfloor \frac{ip}{q} \right\rfloor = \frac{(p-1)(q-1)}{2}.$$

3. A list of 2022 positive integers is given, such that if you remove any one of them, the rest can be split into two groups of equal sum. Prove that all the numbers in the list are equal.