Division Algorithm, Euclid and Bezout

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1 Division Algorithm

- Prove that if a is an integer and b is a positive integer, there is a unique pair (q, r) of integers such that $0 \leq r < b$ and $a = qb + r$.
- Find all integers n such that $2n^2 + 1 \mid n^3 + n^2 - n - 1$.

2 Euclid's Algorithm

We define the *greatest common divisor* of two integers a and b, not both of which are 0, as the largest positive integer d such that $d \mid a$ and $d \mid b$. We notate it by $\gcd(a, b)$.

- Let a and b be integers. Prove that if $a = qb + r$ then $\gcd(a, b) = \gcd(b, r)$.
- Find $\gcd(72, 30)$ by applying the previous result repeatedly. (This is Euclid's Algorithm.)
- Prove that for all positive integers a, m, n we have

$$\gcd(a^m - 1, a^n - 1) = a^{\gcd(m,n)} - 1.$$

3 Bezout's Identity

- Let a and b be integers. Prove that there are integers c and d such that $ac + bd = \gcd(a, b)$.
- Prove that if n, a, b are positive integers such that $n \mid ab$ and $\gcd(n, a) = 1$ then $n \mid b$.
- Let a and b be integers with $\gcd(a, b) = 1$, and let c be an integer. Prove that if $a \mid c$ and $b \mid c$ then $ab \mid c$.
- Let a, b, c, d be positive integers with $\gcd(a, b) = 1$. Prove that there is an integer e such that $a \mid e - c$ and $b \mid e - d$.

4 Problems

1. The denominators of two irreducible fractions are 600 and 700. Find the minimum possible value of the denominator of their sum (written as an irreducible fraction).

2. Let a, b, c, d be positive integers with $ab = cd$. Prove that there exist positive integers p, q, r, s such that $a = pq, b = rs, c = pr, d = qs$.

3. Prove that for positive integers $m, n > 2$ we cannot have

$$2^m - 1 \mid 2^n + 1.$$

4. Find all positive integers n such that $3^{n-1} + 5^{n-1} \mid 3^n + 5^n$.

5. Let S be a nonempty set of integers such that if a and b are in S, then so is $2a - b$.

   Prove that S is an arithmetic progression. (That is, there are integers a and d such that $n \in S$ if and only if $n = a + kd$ for some integer k.)

6. Let n be a composite positive integer. Calculate

$$\gcd((n-1)! + 1, n!).$$

7. Suppose $a_1, a_2, \dots$ is an infinite strictly increasing sequence of positive integers and $p_1, p_2, \dots$ is a sequence of distinct primes such that $p_n \mid a_n$ for all $n \geq 1$. It turns out that $a_{n+1} - a_n = p_{n+1} - p_n$ for all $n \geq 1$. Prove that the sequence $\{a_n\}$ consists only of prime numbers.

8. Assume that $p_1, \dots, p_m,\ q_1, \dots, q_n$ are primes such that

$$p_1 p_2 \cdots p_m = q_1 q_2 \cdots q_n.$$

   Prove that the q_is are a permutation of the p_is.

9. Let m and n be positive integers. Prove that

$$m \mid \gcd(m, n) \binom{m}{n}.$$

5 Homework

1. (a) Find all integers x, y such that $6x + 2y = 8$.

   (b) Find all integers x, y such that $6x + 4y = 8$.

2. Let $a_1, a_2, \ldots, a_n$ be positive integers, and let d be the largest positive integer such that $d \mid a_i$ for all i.

   Prove that there are integers $b_1, b_2, \ldots, b_n$ such that

   $$d = a_1 b_1 + a_2 b_2 + \cdots + a_n b_n.$$

3. The Fibonacci sequence is defined by $F_1 = F_2 = 1$ and

   $$F_{n+1} = F_n + F_{n-1}.$$

   Prove that $\gcd(F_n, F_{n+2}) = 1$ for all n.