Sequences, Integer Functions

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1 Comments

Even though sequence problems and integer function problems look cosmetically different, a sequence is just a function from $\mathbb{N}$ to (some subset of) $\mathbb{R}$.

For divisibility problems, try to make one side of the divisibility prime.

If you can find infinitely many values, often you can sub a small unknown number and a large known one, and this gives you the small number.

2 Problems

1. Find all functions $f : \mathbb{N} \to \mathbb{N}$ such that $f(n+1) > f(f(n))$ for all positive integers n.
2. Prove that if a sequence $\{a_n\}_0^\infty$ of integers satisfies $a_0 = 0$ and $a_n = n - a_{a_n}$ for all n, then $a_{n+2} > a_n$ for all n.
3. Given an integer $k \geq 2$, determine all functions f from the positive integers into themselves such that $f(x_1)! + f(x_2)! + \cdots f(x_k)!$ is divisible by $x_1! + x_2! + \cdots x_k!$ for all positive integers $x_1, x_2, \cdots x_k$.
4. Does there exist a strictly increasing function $f : \mathbb{N} \to \mathbb{N}$ such that $f(1) = 2$ and $f(f(n)) = f(n) + n$ for all positive integers n?
5. Prove that all terms of the sequence $a_1 = a_2 = a_3 = 1,\ a_{n+1} = (1 + a_{n-1}a_n)/a_{n+2}$ are integers.
6. Find all functions $f : \mathbb{N} \to \mathbb{N}$ such that $f(f(m)f(n)) = mn$ for all positive integers m and n.
7. Let $f : \mathbb{Z} \to \mathbb{N}$ be a function such that for any integers m and n, $f(m-n) \mid f(m) - f(n)$. Prove that for all integers m and n, if $f(m) \leq f(n)$ then $f(m) \mid f(n)$.
8. Let n be a fixed positive integer. Let $\frac{a_1}{b_1}, \ldots, \frac{a_k}{b_k}$ be the rational numbers between 0 and 1 inclusive with denominators at most n, written in increasing order and lowest terms.
   - Prove that for each i, $a_{i+1}b_i - a_ib_{i+1} = 1$.
   - Prove that the rational number x with smallest denominator such that $\frac{a_i}{b_i} < x < \frac{a_{i+1}}{b_{i+1}}$ is $\frac{a_i+a_{i+1}}{b_i+b_{i+1}}$.
   - Which pairs of numbers appear as consecutive b_is?

9. Find all functions $f : \mathbb{N} \to \mathbb{N}$ such that for all $m, n \in \mathbb{N}$ we have $m^2 + f(n) \mid mf(m) + n$.
10. Let $P(n)$ be the product of the digits of a positive integer n. Let n_1 be a positive integer, and define $n_{i+1} = n_i + P(n_i)$ for each $i \geq 1$. Prove that this sequence is eventually constant.
11. Find all functions $f : \mathbb{Z}_{\geq 0} \to \mathbb{Z}_{\geq 0}$ such that $f(f(f(n))) = f(n+1) + 1$ for all $n \in \mathbb{Z}_{\geq 0}$.
12. Find all functions $f : \mathbb{N} \to \mathbb{N}$ satisfying

$$f(n) + f(n+1) = f(n+2)f(n+3) - 2023$$

for all $n \in \mathbb{N}$.

13. Find all functions $f : \mathbb{Z} \to \mathbb{Z}$ such that, for all integers a, b, c satsfying $a + b + c = 0$, we have

$$f(a)^2 + f(b)^2 + f(c)^2 = 2f(a)f(b) + 2f(a)f(c) + 2f(b)f(c).$$

3 Homework

1. Is there a sequence $a_1, \ldots$ of primes such that for each i we have $10a_i \leq a_{i+1} < 10a_i + 9$?
2. Find all functions $f : \mathbb{N} \to \mathbb{N}$ such that $m^2 + f(n) \mid mf(m) + n$ for all positive integers m and n.
3. Is there a function $f : \mathbb{N} \to \mathbb{N}$ of positive integers such that $\gcd(a_m, a_n) = 1 \iff |m-n| = 1$?