Sequences, Integer Functions

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1 Comments

Even though sequence problems and integer function problems look cosmetically different, a sequence is just a function from $\mathbb{N}$ to (some subset of) $\mathbb{R}$.

For divisibility problems, try to make one side of the divisibility prime.

If you can find infinitely many values, often you can sub a small unknown number and a large known one, and this gives you the small number.

2 Problems

1. Find all completely multiplicative functions $f : \mathbb{N} \to \mathbb{R}$ such that for all $a, b \in \mathbb{N}$, at least two of $f(a), f(b), f(a+b)$ are equal.
2. Find all functions $f : \mathbb{N} \to \mathbb{N}$ such that $f(n+1) > f(f(n))$ for all positive integers n.
3. Does there exist a strictly increasing function $f : \mathbb{N} \to \mathbb{N}$ such that $f(1) = 2$ and $f(f(n)) = f(n) + n$ for all positive integers n?
4. Prove that all terms of the sequence $a_1 = a_2 = a_3 = 1$, $a_{n+1} = (1 + a_{n-1}a_n)/a_{n+2}$ are integers.
5. Does there exist a strictly increasing function $f : \mathbb{N} \to \mathbb{N}$ such that $f(mn) = f(m) + f(n)$ for all positive integers m and n?
6. Let $f : \mathbb{Z} \to \mathbb{N}$ be a function such that for any integers m and n, $f(m-n) \mid f(m) - f(n)$. Prove that for all integers m and n, if $f(m) \leq f(n)$ then $f(m) \mid f(n)$.
7. Let $P(n)$ be the product of the digits of a positive integer n. Let n_1 be a positive integer, and define $n_{i+1} = n_i + P(n_i)$ for each $i \geq 1$. Prove that this sequence is eventually constant.
8. Find all functions $f : \mathbb{Z}_{\geq 0} \to \mathbb{Z}_{\geq 0}$ such that $f(f(f(n))) = f(n+1) + 1$ for all $n \in \mathbb{Z}_{\geq 0}$.
9. Let $a_1, a_2, \ldots$ be a sequence of integers such that the average of every consecutive group of a_is is a perfect square. Prove that the sequence is constant.

3 Homework

1. Is there a sequence $a_1, \ldots, a_n$ of primes such that for each i we have $10a_i \leq a_{i+1} < 10a_i + 9$?
2. Find all functions $f : \mathbb{N} \to \mathbb{N}$ such that $m^2 + f(n) \mid mf(m) + n$ for all positive integers m and n.
3. Is there a function $f : \mathbb{N} \to \mathbb{N}$ of positive integers such that $\gcd(a_m, a_n) = 1 \iff |m-n| = 1$?