① a) Price elasticity of demand:

$$\varepsilon := \frac{P}{Q} \cdot \frac{\partial Q}{\partial P}$$

From $P = a_0 - a_1 Q + v \Rightarrow Q = \dfrac{a_0 - P + v}{a_1}$

Thus, $\dfrac{\partial Q}{\partial P} = \dfrac{-1}{a_1}$. Plugging in ε:

$$\varepsilon = \frac{a_0 - a_1 Q + v}{Q} \cdot \frac{(-1)}{a_1} = 1 - \frac{(a_0 + v)}{a_1 Q}$$

- Q analysis:

Note: Since we know the price elasticity of demand will be negative, we focus on the monotonicity of $-\varepsilon$.

See that $\dfrac{\partial \varepsilon}{\partial Q} = -\dfrac{a_0 + v}{a_1 Q^2}$.

So if $\dfrac{a_0 + v}{a_1} > 0$ (A1) $\Rightarrow$ $-\varepsilon$ is (strictly) decreasing in Q

we assume (A1) ↑ "positive prices in demand"

if $\dfrac{a_0 + v}{a_1} < 0 \Rightarrow -\varepsilon$ is (strictly) increasing in Q

- v analysis: (measure of willingness to pay of consumers)

See that $\dfrac{\partial \varepsilon}{\partial v} = \dfrac{1}{a_1 Q}$, we assume $Q > 0$

So, if $a_1 > 0$ (A2) $\Rightarrow -\varepsilon$ is strictly increasing in v

we assume (A2) ↑ "downward slopping demand curve"

if $a_1 < 0 \Rightarrow -\varepsilon$ is strictly decreasing in v

b) Cournot equilibrium, Fixed N, fixed F

Firm i solves

$$\max_{q_i} \; P \cdot q_i - \overbrace{\{F + (b_0 - b_1 q_i + \eta)\, q_i\}}^{\text{total cost}}$$

$$\max_{q_i} \; (a_0 - a_1 Q + v)\, q_i - \{F + (b_0 - b_1 q_i + \eta) q_i\}$$

FOC: $a_0 + v - a_1 \sum_{j \neq i}^{N} q_j - 2 q_i a_1 - (b_0 + \eta) + 2 b_1 q_i = 0$

$$\Rightarrow \quad q_i \cdot 2(b_1 - a_1) = a_1 \sum_{j \neq i}^{N} q_i + b_0 + \eta - a_0 - v$$

SOC entail $b_1 - a_1 \leq 0$ or $a_1 \geq 0$ when $b_1 = 0$ (A2)

Using symmetry of firms:

$$q \cdot 2(b_1 - a_1) - a_1 N q = b_0 + \eta$$

$$q_i^* = \frac{b_0 + \eta - a_0 - v}{2(b_1 - a_1) - a_1 N}$$

Assuming $b_1 = 0$ $\Rightarrow$ $\boxed{q_i^* = \frac{a_0 + v - b_0 - \eta}{a_1(1+N)}}$ and $Q^* = N q^*$

Hence, the equilibrium price is:

$$P^* = a_0 + a_1 N \cdot \frac{b_0 + \eta - a_0 - v}{a_1(1+N)} + v$$

After some manipulation $\boxed{P^* = \frac{a_0 + v + N(b_0 + \eta)}{N+1}}$

As $N \to \infty$
$P^* \to b_0 + \eta$

Profits in equilibrium are:

$$\pi^* = P^* \cdot q_i - \{F + (b_0 - \underbrace{\cancel{b_1}}_{\overset{\text{by assumption}}{0}} q + \eta) q_i\}$$

Therefore $\boxed{\pi^* = \frac{1}{a_1} \left(\frac{a_0 + v - b_0 - \eta}{N+1} \right)^2 - F}$

c) Entry condition $\iff$ Profits greater or equal than zero.

$$\pi \geq 0 \iff \left(\frac{a_0 + v - b_0 - \eta}{N+1}\right)^2 - a_1 F \geq 0 \qquad (\text{if } a_1 \geq 0)$$

$$\iff \frac{a_0 + v - b_0 - \eta}{N+1} \geq \sqrt{a_1 F}$$

$$\iff N \leq \frac{a_0 + v - b_0 - \eta}{\sqrt{a_1 F}} - 1$$

Hence, the equilibrium number of firms is $\boxed{N^* = \frac{a_0 + v - b_0 - \eta}{\sqrt{a_1 F}} - 1}$

d) The markup or Lerner index is

$$\mathcal{L}(N) = \frac{P^*(N) - C'(q^*(N))}{P^*(N)} \Rightarrow \mathcal{L}(N) = \frac{a_0 + v - b_0 - \eta}{a_0 + v - N(b_0 + \eta)}$$

Plugging in N^* we get

$$\boxed{\mathcal{L}(N^*) = \frac{\sqrt{a_1 F}}{\sqrt{a_1 F} + b_0 + \eta}}$$

Let I_H be Herfindahl index: (degree of concentration of the industry)

$$I_H \equiv \sum_{i=1}^{n} \alpha_i^2 \quad \text{and} \quad \frac{P - \sum_{i=1}^{n} \alpha_i C'(q_i^*)}{P} = \frac{I_H}{\varepsilon}$$

where $\alpha_i \equiv \frac{q_i}{Q}$ is the market share of i.

By symmetry, $I_H = \sum_{i=1}^{N} N^{-2} = \boxed{\frac{1}{N}} = \mathcal{L} \cdot \varepsilon$

Then, pluggin in N^*

$$\boxed{I_H = \frac{\sqrt{a_1 F}}{a_0 + v - b_0 - \eta - \sqrt{a_1 F}}}$$

Plugging in the equilibrium variables in ε:

$$\varepsilon(N) = 1 - \frac{(a_0+v)(1+N)}{N(a_0+v-b_0-\eta)}, \quad \text{and} \quad \varepsilon(N^*) \text{ is}$$

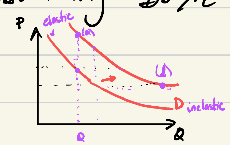
$$\boxed{\varepsilon = -\frac{\sqrt{a_1 F} + b_0 + \eta}{a_0 + v - b_0 - \eta - \sqrt{a_1 F}}}$$

e) First, we analyse $\Delta\varepsilon$:

$\boxed{\Delta v}$ Taking derivatives of ε wrp to v:

$$\frac{\partial \varepsilon}{\partial v} = \frac{\sqrt{a_1 F} + b_0 + \eta}{(a_0 + v - b_0 - \eta - \sqrt{a_1 F})^2} \leq 0$$

, assuming $b_0, \eta > 0$



then $-\varepsilon$ is (strictly) decreasing in v

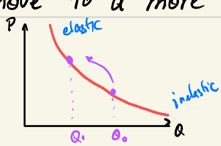
Notice that in a) we get the opposite result. In the previous case we had an exogenous market structure, but now, Q^*, N^*, P^* are determined endogenously. Thus, if we think of v as a measure of the willingness to pay: when N is fixed, the effect of v on the demand (shift to the right) increases elasticity. But, with endogenous market structure N and Q increase as well, and this effect dominates the direct effect of v, since ε is decreasing in Q and N.

To see this clearly, notice $q_i^*(N^*) = \sqrt{\frac{F}{a_1}}$, $Q_i^* = N^* q$ and N^* is strictly increasing in v.

$\boxed{\Delta \eta}$ Taking derivatives wrp to η we obtain

$$\frac{\partial \varepsilon}{\partial \eta} = \frac{a_0+v}{(a_0+v-b_0-\eta-\sqrt{a_1 F})^2} \geq 0 \Rightarrow \varepsilon \text{ is (strictly) increasing in } \eta$$

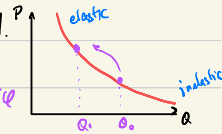
Note that $C'(q) = (b_0+\eta) - 2b_1 q_i$, so η is a measure of marginal cost (MgC). Hence, when MgC increases, total production Q and the number of firms N falls, so we move to a more elastic portion of the demand curve:



$\boxed{\Delta F}$ Taking derivatives wrp to F yields

$$\frac{\partial \varepsilon}{\partial F} = \sqrt{\frac{a_1}{F}} \cdot \frac{a_0+v}{2\left(a_0+v-b_0-\eta-\sqrt{a_1F}\right)^2} \geq 0,$$ $-\varepsilon$ is increasing wrp to F

Again, we compare with (a), in the very short run, fixed costs are sunk and do not affect elasticity. Recall $q_i^* = \sqrt{\frac{a_1}{F}}$, thus when F rises q_i^*, N^* and Q^* drops, and the firm production moves to a more elastic portion of the demand.



$\boxed{\Delta \ln \mathcal{L}}$: $\ln \mathcal{L}$ can be interpreted as the percentag of markup over price

$$\ln \mathcal{L} = \frac{1}{2}\ln a_1 + \frac{1}{2}\ln F - \ln\left(\sqrt{a_1F}+b_0+\eta\right)$$

Therefore:

Δ $$\frac{\partial \ln \mathcal{L}}{\partial v} = 0$$ When N and Q is exogenous (a), an increase in willingness to pay increases prices and thus markups, but when the market structure is endogenous, more firms enter the market and total production rises; the effects cancel and markups remain identical.

Δ $$\frac{\partial \ln \mathcal{L}}{\partial \eta} = -\frac{1}{\sqrt{a_1F}+b_0+\eta} \leq 0$$ An increase in the marginal costs decreases markups directly, and indirectly through an increase in prices due to a decline in total production and number of firms

Δ $$\frac{\partial \ln \mathcal{L}}{\partial F} = \frac{b_0+\eta}{2F\left(\sqrt{a_1F}+b_0+\eta\right)} \geq 0$$: As fixed cost rises, the equilibrium number of firms decreases, so competition declines an the quantity produce (Q), therefore prices and markups increase.

$\boxed{\Delta \ln I_H}$

$$\ln I_H = \frac{1}{2}\ln a_1 + \frac{1}{2}\ln F - \ln\left(a_0+v-b_0-\eta-\sqrt{a_1F}\right)$$

concentration in the industry

$$\frac{\partial \ln I_H}{\partial v} = \frac{1}{a_0+v-b_0-\eta-\sqrt{a_1F}} \leq 0$$ As the maximum willingness to pay rises more firms enter the market and the industry is less concentrated.

$$\frac{\partial \ln I_H}{\partial \eta} = \frac{1}{a_0+v-b_0-\eta-\sqrt{a_1F}} \geq 0$$ As marginal cost rises, N falls and HH Index increases since market shares grow.

$$\frac{\partial \ln I_H}{\partial F} = \frac{1}{2F} \cdot \frac{a_0+v-b_0-\eta}{a_0+v-b_0-\eta-\sqrt{a_1F}} \geq 0$$ Similarly, Fixed costs hamper entry market, N drops as well as market shares.

f) Firms collude perfectly, then, Firms problem is:

$$\max_Q \; P \cdot Q - C\left(\frac{Q}{N}\right) \cdot N \Rightarrow$$

$$\max_Q \; (a_0 - a_1 Q + v) Q - \left\{ NF + \left(b_0 - b_1 \frac{Q}{N} + \eta\right) Q \right\}$$

FOC: $a_0 + v - 2a_1 Q - (b_0 + \eta) + 2b_1 \frac{Q}{N} = 0$

$$Q = \frac{a_0 + v - b_0 - \eta}{2 \cdot \left(a_1 - \frac{b_1}{N}\right)} \quad \text{when } b_1 = 0: \quad \boxed{Q^{**} = \frac{a_0 + v - b_0 - \eta}{2 \cdot a_1}}$$

Firms are symmetric and split equally, thus $\boxed{q^{**} = \frac{1}{N} \cdot \frac{a_0 + v - b_0 - \eta}{2 \cdot a_1}}$

And prices are $\boxed{P^* = \frac{a_0 + v + b_0 + \eta}{2}}$

Notice that equilibrium price does NOT depend on N.

Equilibrium profits for Firm i are $\boxed{\pi_i^* = \frac{1}{4a_1 N} \cdot (a_0 + v - b_0 - \eta)^2 - F}$

Zero profits give us $\pi_i^*(N) = 0$ iff $\boxed{N^* = \frac{(a_0 + v - b_0 - \eta)^2}{4 a_1 F}}$

Hence, Lerner Index is

$$\mathcal{L} = \frac{P^* - C'(N^*)}{P^*} \Rightarrow \mathcal{L} = \frac{a_0 + v - b_0 - \eta}{a_0 + v + b_0 + \eta},$$

the Herfindahl index is

$$I_H = \frac{1}{N^*} = \frac{4 a_1 F}{(a_0 + v - b_0 - \eta)^2}$$

And the price elasticity of demand is $\varepsilon = 1 - \frac{(a_0 + v)}{a_1 Q} = 1 - \frac{2(a_0 + v)}{a_0 + v - b_0 - \eta}$

$$\boxed{\varepsilon = -\frac{a_0 + v + b_0 + \eta}{a_0 + v - b_0 - \eta}}$$

g) $\ln P = c_0 - c_1 \ln Q + \xi$

a) Taking advantage of the log form of the demand:

$$\varepsilon = \frac{d \log Q}{d \log P} = -\frac{1}{\cancel{Q}} \cdot \frac{1}{c_1} \cdot \cancel{Q} \Rightarrow \boxed{\varepsilon = -\frac{1}{c_1}}$$

We have constant price elasticity of demand.

b) Firm i solves

$$\max_{q_i} \frac{e^{c_0+\xi}}{Q^{c_1}} \cdot q_i - F - (b_0+\eta) q_i$$

FOC: $e^{c_0+\xi} \cdot \left(Q^{-c_1} - c_1 Q^{-c_1-1} \cdot q_i \right) = b_0 + \eta$

By symmetry, $q_i = q$ and FOC reduces to

$$\boxed{q^* = \left(\frac{(N-c_1) e^{c_0+\xi}}{(b_0+\eta) N^{c_1+1}} \right)^{1/c_1}} \quad \text{and} \quad Q^* = N q^*$$

Prices are

$$\boxed{P^* = \frac{(b_0+\eta) N}{N - c_1}}$$

d) the Lerner Index is

$$\mathcal{L} = \frac{P - C'(q)}{P} \Rightarrow \mathcal{L} = 1 - \frac{\cancel{b_0+\eta}}{(\cancel{b_0+\eta})} \cdot \frac{N-c_1}{N} \Rightarrow \boxed{\mathcal{L} = \frac{c_1}{N}}$$

And like previous function, $\boxed{I_H = \frac{1}{N}}$

e) $\Delta \eta$, Δv, ΔF when N is fixed, i.e. exogenous market structure, ε, I_H and $\mathcal{L}$ do not depend on the parameters. This is a consequence of the constant elasticity:

$$\mathcal{L} = \frac{\alpha_i}{\varepsilon} = \frac{1}{N} \underset{\text{constant}}{\varepsilon^{-1}} \quad \text{and} \quad \alpha_i = \varepsilon \mathcal{L} = \frac{1}{N} \quad \text{and} \quad I_H = N(\varepsilon \mathcal{L})^2 = \frac{1}{N}$$