

Lecture Notes on the ARV Algorithm for Sparsest Cut

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Abstract

One of the landmarks in approximation algorithms is the $O(\sqrt{\log n})$ -approximation algorithm for the *Uniform Sparsest Cut problem* by Arora, Rao and Vazirani from 2004. The algorithm is based on a *semidefinite program* that finds an embedding of the nodes respecting the triangle inequality. Their core argument shows that a random hyperplane approach will find two large sets of $\Theta(n)$ many nodes each that have a distance of $\Theta(1/\sqrt{\log n})$ to each other if measured in terms of $\|\cdot\|_2^2$.

Here we give a detailed set of lecture notes describing the algorithm. For the proof of the *Structure Theorem* we use a cleaner argument based on expected maxima over k -neighborhoods that significantly simplifies the analysis.

1 Introduction

Let $G = (V, E)$ be a complete, undirected graph on $|V| = n$ nodes and let $c : E \rightarrow \mathbb{R}_{\geq 0}$ be a cost function on the edges. For a subset $S \subseteq V$ of nodes, let $\delta(S) := \{\{i, j\} \in E \mid |\{i, j\} \cap S| = 1\}$ be the induced *cut*. We abbreviate $c(\delta(S)) := \sum_{e \in \delta(S)} c(e)$ as the cost of the cut. The (*Uniform*) *Sparsest Cut* problem is then to find the cut that minimizes the *cost-over-separated-pairs* ratio:

$$\min \left\{ \frac{c(\delta(U))}{|U| \cdot |V \setminus U|} \mid \emptyset \subset U \subset V \right\}.$$

There is also a non-uniform version of the problem where each pair $i, j \in V$ has an associated *demand* $d(i, j) \geq 0$ and one aims for the cut minimizing the ratio $c(\delta(S))/d(\delta(S))$. We will now see the celebrated algorithm by Arora, Rao and Vazirani [ARV04] that finds a $O(\sqrt{\log n})$ -approximation in polynomial time.

For the algorithm we will not try to optimize any constant. To fix some notation, we will denote any vector in bold font, like $\mathbf{v}_i \in \mathbb{R}^m$. If we write $i \sim V$, then we mean that i is a uniform random node from V . We denote $N(0, 1)$ as the *1-dimensional Gaussian distribution* with mean 0 and variance 1. In particular, a random variable $g \sim N(0, 1)$ has *density* $\frac{1}{\sqrt{2\pi}} e^{-x^2/2}$. If we write $\mathbf{g} \sim N^m(0, 1)$, then we mean that \mathbf{g} is an *m -dimensional Gaussian*. Recall that the vector $\mathbf{g} = (g_1, \dots, g_m)$ can be generated by sampling each coordinate independently with $g_i \sim N(0, 1)$. In reverse, for any pair of orthonormal vectors $\mathbf{u}, \mathbf{v} \in \mathbb{R}^m$, the inner products $\langle \mathbf{g}, \mathbf{u} \rangle, \langle \mathbf{g}, \mathbf{v} \rangle$ are independently distributed from $N(0, 1)$.

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2 A semidefinite program

Sparsest Cut is an unusual problem in the sense that it minimizes the ratio of two functions. Let us assume for the sake of simplicity that we guessed the cost C^* and the size S^* of an optimum cut, say with $S^* \leq \frac{n}{2}$. Then we define a *semidefinite programming relaxation*

$$\begin{aligned} \sum_{i \in V} \|\mathbf{v}_i\|_2^2 &= S^* & (I) \\ \sum_{\{i,j\} \in \binom{V}{2}} \|\mathbf{v}_i - \mathbf{v}_j\|_2^2 &= S^* \cdot (n - S^*) & (II) \\ \sum_{\{i,j\} \in E} c_{ij} \|\mathbf{v}_i - \mathbf{v}_j\|_2^2 &= C^* & (III) \\ \|\mathbf{v}_i - \mathbf{v}_j\|_2^2 &\leq \|\mathbf{v}_i - \mathbf{v}_k\|_2^2 + \|\mathbf{v}_k - \mathbf{v}_j\|_2^2 \quad \forall i, j, k \in V \cup \{0\} & (IV) \\ \|\mathbf{v}_i\|_2^2 &\leq 1 \quad \forall i \in V & (V) \\ \|\mathbf{v}_0\|_2^2 &= 0 & (VI) \end{aligned}$$

where we use an artificial index 0 with $\mathbf{v}_0 := \mathbf{0}$, so that the triangle inequality also holds for the origin.

Lemma 1. *If there is cut $U^* \subseteq V$ of cost C^* and size S^* , then the above SDP has a solution.*

Proof. One could choose 1-dimensional vectors by defining

$$\mathbf{v}_i := \begin{cases} 1 & \text{if } i \in U^* \\ 0 & \text{if } i \notin U^*. \end{cases}$$

Then the only non-trivial case is verifying the *triangle inequalities* in (IV). These are satisfied by our choice of $\mathbf{v}_i \in \{0, 1\}$ since if $\|\mathbf{v}_i - \mathbf{v}_j\|_2^2 = 1$, then i and j have to be on different sides of the cut U^* and any node k has to be either not on the side of i or not on the side of j . \square

We can solve the semi-definite program (SDP) in polynomial time [GLS93]; let $\{\mathbf{v}_i\}_{i \in V} \subseteq \mathbb{R}^m$ be the solution. Due to the triangle inequalities (IV) we can define a *metric* $d : V \times V \rightarrow \mathbb{R}_{\geq 0}$ by setting $d(i, j) := \|\mathbf{v}_i - \mathbf{v}_j\|_2^2$. Note that while $\|\cdot\|_2$ is always a metric, $\|\cdot\|_2^2$ is not a metric on all points sets. For sets of nodes $A, B \subseteq V$ we define $d(i, A) := \min\{d(i, j) : j \in A\}$ and $d(A, B) := \min\{d(i, j) : i \in A, j \in B\}$.

3 A ball rounding scheme

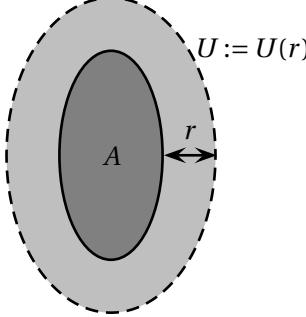
Given that family $\{\mathbf{v}_i\}_{i \in V}$ of SDP vectors, there are several natural rounding procedures that would come to mind. For example one could try the *hyperplane rounding* that Goemans and Williamson [GW94] have used for MaxCut. The natural algorithm for Sparsest Cut would be to take a random Gaussian $\mathbf{g} \sim N^m(0, 1)$ and set $U := \{i \in V \mid \langle \mathbf{g}, \mathbf{v}_i \rangle \geq 0\}$. Assume for the sake of simplicity that we are in the *balanced* case of Sparsest Cut with $S^* = \Theta(n)$. Then an edge $(i, j) \in E$ has a contribution to the objective function of $\Theta(d(i, j)/n^2)$. On the other hand, the probability that (i, j) is separated is roughly proportional to the Euclidean distance $\|\mathbf{v}_i - \mathbf{v}_j\|_2$ and even if the hyperplane generates perfectly balanced cuts, the expected contribution of an edge (i, j) to the hyperplane cut would be $\Theta(\sqrt{d(i, j)}/n^2)$. In other words, short edges would be separated far too likely.

The second best idea that one might have, would be to select a node $i \in V$ and take a random cut $U := \{j \in V \mid d(i, j) \leq r\}$ where $r \sim [0, 1]$. Now every edge is cut with a probability not exceeding $d(i, j)$. On the other hand, this argument seems to not give any guarantee on the size of U and $V \setminus U$, hence the

objective function can be arbitrarily bad again. But a slight fix of this rounding argument can work. We only need a large “core” of nodes so that the remaining nodes still have a decently large distance to it.

Lemma 2. Suppose we have a set of nodes $A \subseteq V$ with $|A| \geq \alpha n$ and $\sum_{i \in V} d(i, A) \geq \beta \cdot S^*$. Then the best cut of the form $\{i \in V \mid d(i, A) \leq r\}$ is a $\frac{2}{\alpha\beta}$ -approximation.

Proof. Recall that the optimum value of the objective function is $\frac{C^*}{S^*(n-S^*)}$. Suppose we sample $r \sim [0, 1]$ and take $U := U(r) := \{i \in V \mid d(i, A) \leq r\}$ as a random cut.



Then for an edge $(i, j) \in E$, say with $d(i, A) \leq d(j, A)$, we have

$$\Pr_{r \sim [0,1]}[(i, j) \in \delta(U)] = \Pr_{r \sim [0,1]}[d(i, A) \leq r \leq d(j, A)] \stackrel{\text{triangle inequality}}{\leq} d(i, j).$$

Hence the expected cost of the cut U is

$$\mathbb{E}_{r \sim [0,1]}[c(\delta(U))] \leq \sum_{(i,j) \in E} d(i, j) = C^*.$$

Note that in any case $|U| \geq |A| \geq \alpha n$. We know that $d(i, A) \leq 2$ for all $i \in V$ and hence $\Pr_{r \sim [0,1]}[i \notin U] = \Pr_{r \sim [0,1]}[r < d(i, A)] \geq \frac{1}{2}d(i, A)$. Hence

$$\mathbb{E}_{r \sim [0,1]}[|V \setminus U|] \geq \frac{1}{2} \sum_{i \in V} d(i, A) \geq \frac{\beta}{2} \cdot S^*.$$

Then $\mathbb{E}_{r \sim [0,1]}[|U| \cdot |V \setminus U|] \geq \frac{\alpha\beta}{2} \cdot S^* \cdot n$. In other words, the random cut seems to have the right expected nominator and denominator to satisfy the claim. But this is *not* enough to argue that their ratio satisfies $\mathbb{E}_{r \sim [0,1]}[\frac{c(\delta(U))}{|U| \cdot |V \setminus U|}] \leq \frac{2}{\alpha\beta} \cdot \frac{C^*}{S^*(n-S^*)}$. The following insight comes to rescue:

Fact. Let $\mathbf{a}, \mathbf{b} \in \mathbb{R}_{\geq 0}^m$ be non-negative numbers and \mathcal{D} be a distribution over indices in $[m]$.

Then

$$\min_{i \in \{1, \dots, m\}} \left\{ \frac{\mathbf{a}_i}{\mathbf{b}_i} \right\} \leq \frac{\mathbb{E}_{i \sim \mathcal{D}}[\mathbf{a}_i]}{\mathbb{E}_{i \sim \mathcal{D}}[\mathbf{b}_i]}.$$

Now, this fact implies that *best* choice of U (over all $r \in [0, 1]$) will indeed satisfy the claim and the lemma is proven. \square

We should also remark that if A is given, we can find the cut U in polynomial time as we only need to try out at most n^2 many values of r .

4 The case of heavy clusters

Let $B(i, r) := \{j \in V \mid d(i, j) \leq r\}$ be the “ball” of radius r around node i . A slight annoyance of the ARV algorithm is that it requires a case split. If we can find a *cluster center* $i^* \in V$, then we can use Lemma 2 to get a constant factor approximation by just taking a ball around the center i^* .

Lemma 3. *Suppose there is a node $i^* \in V \cup \{0\}$ with $|B(i^*, \frac{1}{8} \cdot \frac{S^*}{n})| \geq \frac{n}{4}$. Then in polynomial time one can find a cut that gives a $O(1)$ -approximation.*

Proof. We set $A := B(i^*, \frac{1}{8} \cdot \frac{S^*}{n})$. Then by assumption $|A| \geq \frac{n}{4}$. Moreover, bounding the average distance of pairs of nodes from above and from below gives

$$\frac{1}{2} \cdot \frac{S^*}{n} \leq \frac{S^*}{n} \cdot \underbrace{\frac{n - S^*}{n}}_{\geq 1/2} \stackrel{(II)}{\leq} \mathbb{E}_{i,j \sim V} [d(i, j)] \stackrel{\text{triangle ineq.}}{\leq} 2 \cdot \mathbb{E}_{i \sim V} \left(d(i, A) + \frac{1}{8} \cdot \frac{S^*}{n} \right).$$

This can be rearranged to $\mathbb{E}_{i \sim V} [d(i, A)] \geq \frac{1}{8} \cdot \frac{S^*}{n}$. We obtain a 64-approximation by applying Lemma 2. \square

5 An algorithm for the main case

From now on we make the assumption that no cluster exists:

$$\left| B\left(i, \frac{1}{8} \cdot \frac{S^*}{n}\right) \right| < \frac{n}{4} \quad \forall i \in V \cup \{0\}.$$

We will prove that in this case, there are sets $L, R \subseteq V$ of size $|L|, |R| \geq \Omega(n)$ with $d(L, R) \geq \Delta \cdot \frac{S^*}{n}$ for $\Delta := \Theta(1/\sqrt{\log n})$. Then choosing Lemma 2 with $A := L$ will give a $O(\frac{1}{\Delta})$ -approximation. Before we start proving this, we want to further simplify the situation. Note that by (I) we have $\mathbb{E}_{i \sim V} [d(i, \mathbf{0})] = \frac{S^*}{n}$, and hence at most half the nodes can have a distance of more than $2 \cdot \frac{S^*}{n}$ to $\mathbf{0}$. Moreover we have $|B(\mathbf{0}, \frac{1}{8} \cdot \frac{S^*}{n})| \leq \frac{n}{4}$. Then we only loose a constant factor if we delete those nodes and assume that $\frac{1}{8} \cdot \frac{S^*}{n} \leq d(i, \mathbf{0}) \leq 2 \cdot \frac{S^*}{n}$ for all remaining nodes. Next we scale the vectors \mathbf{v}_i by a factor of $\sqrt{n/S^*}$, which scales the distances $d(i, j)$ by a factor of $\frac{n}{S^*}$. After this transformation it suffices to prove the following structure theorem:

Theorem 4 (ARV Structure Theorem). *Given any set of $|V| = n$ vectors $\{\mathbf{v}_i\}_{i \in V} \subseteq \mathbb{R}^m$ with $\frac{1}{8} \leq \|\mathbf{v}_i\|_2^2 \leq 2$ and $|B(i, \frac{1}{8})| \leq \frac{3}{4}n$ for all $i \in V$ that satisfy the triangle inequalities*

$$\|\mathbf{v}_i - \mathbf{v}_j\|_2^2 \leq \|\mathbf{v}_i - \mathbf{v}_k\|_2^2 + \|\mathbf{v}_k - \mathbf{v}_j\|_2^2 \quad \forall i, j, k \in V.$$

Then there is a polynomial time algorithm that with constant probability finds sets $L, R \subseteq V$ of size $|L|, |R| \geq \Omega(n)$ with $d(L, R) \geq \Delta$ for $\Delta := \Theta(1/\sqrt{\log n})$.

From now on, we complete ignore the cost function and only use the properties given in the Structure Theorem. Such sets L and R with $d(L, R) \geq \Delta$ are called Δ -separated. Let $c' > 0$ be a small enough constants. The algorithm to produce the Δ -separated sets is as follows:

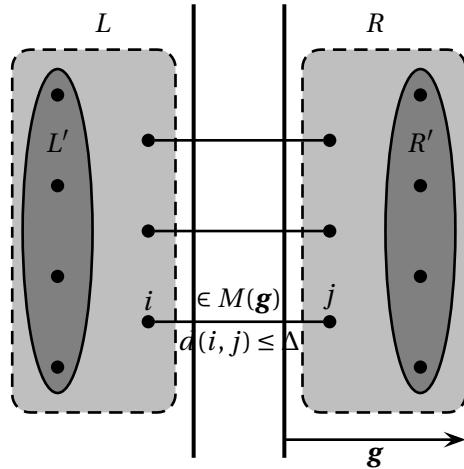
Well-separated sets algorithm

Input: Vectors $\{\mathbf{v}_i\}_{i \in V} \subseteq \mathbb{R}^m$ satisfying the triangle inequality with $\frac{1}{8} \leq \|\mathbf{v}_i\|_2^2 \leq 2$ and $|B(i, \frac{1}{8})| \leq \frac{3}{4}n$ for all $i \in V \cup \{0\}$.

Output: Either Δ -separated sets L', R' of size $|L'|, |R'| \geq \frac{c'}{2}n$ or FAIL

- (1) Pick a random Gaussian $\mathbf{g} \sim N^m(0, 1)$
- (2) Set $L := \{i \in V \mid \langle \mathbf{v}_i, \mathbf{g} \rangle \leq -1\}$ and $R := \{i \in V \mid \langle \mathbf{v}_i, \mathbf{g} \rangle \geq 1\}$
- (3) If either $|L| \leq c'n$ or $|R| \leq c'n$ then FAIL
- (4) Compute any inclusion-wise maximal matching $M(\mathbf{g}) \subseteq \{(i, j) \in L \times R \mid d(i, j) \leq \Delta\}$
- (5) If $|M(\mathbf{g})| > \frac{c'}{2}n$ then FAIL
- (6) Return $L' := \{i \in L \mid i \text{ not covered by } M(\mathbf{g})\}$ and $R' := \{i \in V \mid i \text{ not covered by } M(\mathbf{g})\}$

Observe that any pair $i \in L'$ and $j \in R'$ that remains will have $d(i, j) > \Delta$ as otherwise the matching $M(\mathbf{g})$ would not have been maximal. Also, if the algorithm reaches (6), then $\min\{|L'|, |R'|\} \geq \frac{c'}{2}n$.



The first step is to argue that with constant probability both L and R have size at least $\Omega(n)$.

Lemma 5. There is an absolute constant $c' > 0$ so that $\Pr_{\mathbf{g} \sim N^m(0, 1)}[\min\{|L|, |R|\} \geq c'n] \geq c'$.

Proof. We will prove that $\mathbb{E}[|L| \cdot |R|] \geq \Omega(n^2)$ which then implies the claim. Fix any $i \in V$ and select one of the at least $\frac{1}{4}n$ nodes j with $d(i, j) \geq \frac{1}{8}$. Let \mathbf{w} be the orthogonal projection of \mathbf{v}_j on \mathbf{v}_i , see figure below. Let $\alpha \in [0, \frac{\pi}{2}]$ be the angle between $\mathbf{v}_i - \mathbf{v}_j$ and \mathbf{w} and let $\beta \in [0, \frac{\pi}{2}]$ be the angle between \mathbf{v}_j and \mathbf{w} . Due to the triangle inequalities, the angle spanned by the points $\mathbf{0}$, \mathbf{v}_i and \mathbf{v}_j is non-obtuse and $\alpha + \beta \leq \frac{\pi}{2}$. Then we have either $\alpha \leq \frac{\pi}{4}$ and

$$\|\mathbf{w}\|_2 = \underbrace{\cos(\alpha)}_{\geq 1/\sqrt{2}} \cdot \underbrace{\|\mathbf{v}_i - \mathbf{v}_j\|_2}_{\geq 1/\sqrt{8}} \geq \frac{1}{4}$$

or otherwise we have $\beta \leq \frac{\pi}{4}$ and

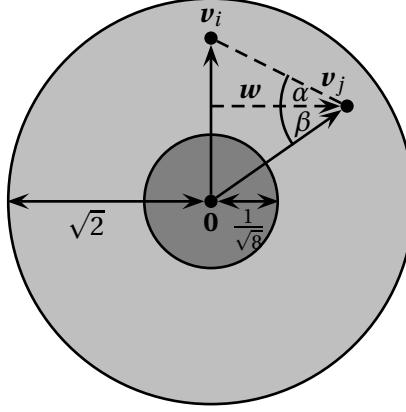
$$\|\mathbf{w}\|_2 = \underbrace{\cos(\beta)}_{\geq 1/\sqrt{2}} \cdot \underbrace{\|\mathbf{v}_j\|_2}_{\geq 1/\sqrt{8}} \geq \frac{1}{4}$$

Either way, $\|\mathbf{w}\|_2 \geq \frac{1}{4}$. Since $\mathbf{v}_i \perp \mathbf{w}$, the inner products $\langle \mathbf{g}, \mathbf{v}_i \rangle$ and $\langle \mathbf{g}, \mathbf{w} \rangle$ are independent random variables and we can estimate

$$\begin{aligned} \Pr[i \in L \text{ and } j \in R] &\geq \Pr[-2 \leq \langle \mathbf{g}, \mathbf{v}_i \rangle \leq -1 \text{ and } \langle \mathbf{g}, \mathbf{w} \rangle \geq 3] \\ &= \Pr\left[-\frac{1}{\underbrace{\|\mathbf{v}_i\|_2}_{\in [\frac{1}{\sqrt{8}}, \frac{1}{\sqrt{2}}]}} \leq \langle \mathbf{g}, \frac{\mathbf{v}_i}{\|\mathbf{v}_i\|_2} \rangle \leq -\frac{2}{\|\mathbf{v}_i\|_2}\right] \cdot \Pr\left[\langle \mathbf{g}, \frac{\mathbf{w}}{\|\mathbf{w}\|_2} \rangle \geq \underbrace{\frac{3}{\|\mathbf{w}\|_2}}_{\leq 12}\right] > 0 \end{aligned}$$

which is some tiny, yet absolute constant. Note that in case the latter event happens, then indeed

$$\langle \mathbf{g}, \mathbf{v}_j \rangle = \underbrace{\langle \mathbf{g}, \mathbf{v}_i \rangle}_{\geq -2} + \underbrace{\frac{\langle \mathbf{v}_i, \mathbf{v}_j \rangle}{\|\mathbf{v}_i\|_2^2}}_{\in [0, 1]} + \underbrace{\langle \mathbf{g}, \mathbf{w} \rangle}_{\geq 3} \geq 1.$$



□

This implies that with a constant probability, the algorithm does not fail in (3). The main technical part lies in proving that $\mathbb{E}_{\mathbf{g} \sim N^m(0,1)}[|M(\mathbf{g})|] \leq cn$ where we can make the constant c as small as we want, at the expense of a smaller value of Δ . If we choose $c := (\frac{c'}{2})^2$, then $\Pr[|M(\mathbf{g})| > \frac{c'}{2}] \leq \frac{c'}{2}$ and the success probability of the algorithm is at least $\frac{c'}{2}$.

6 The proof of the Structure Theorem

The following geometric theorem by Arora, Rao and Vazirani is the heart of their $O(\sqrt{\log n})$ -approximation for Sparsest Cut. To be precise, the original ARV result [ARV04] only showed this theorem for $\Delta = \Theta((\log n)^{-2/3})$ and needed a lot of extra work to get the $O(\sqrt{\log n})$ -approximation. The claim as it is stated here was first proven by Lee [Lee05]. For an edge set E' , let $\beta(E')$ be the size of the maximum matching.

Theorem 6 ([ARV04, Lee05]). *For any constant $c > 0$ there is a choice of $\Delta := \Theta_c(1/\sqrt{\log n})$ so that the following holds: Let $\{\mathbf{v}_i\}_{i \in V} \subseteq \mathbb{R}^m$ be a set of $|V| = n$ vectors satisfying the triangle inequality*

$$\|\mathbf{v}_i - \mathbf{v}_j\|_2^2 \leq \|\mathbf{v}_i - \mathbf{v}_k\|_2^2 + \|\mathbf{v}_k - \mathbf{v}_j\|_2^2 \quad \forall i, j, k \in V.$$

For a vector $\mathbf{g} \in \mathbb{R}^m$ define

$$E(\mathbf{g}) := \{(i, j) \in V \times V \mid \langle \mathbf{v}_j - \mathbf{v}_i, \mathbf{g} \rangle \geq 2 \text{ and } \|\mathbf{v}_i - \mathbf{v}_j\|_2^2 \leq \Delta\}.$$

Then $\mathbb{E}_{\mathbf{g} \sim N^m(0,1)}[\beta(E(\mathbf{g}))] \leq cn$.

Here we think of $E(\mathbf{g})$ as directed edges. Let $M(\mathbf{g})$ be a maximum matching attaining $\beta(E(\mathbf{g}))$. We will assume the existence of such a matching $M(\mathbf{g})$ and lead this to a contradiction. By inducing on a subgraph and reducing the constant c one can even assume that for *every* node the probability of having an outgoing edge is at least c and the same is true for ingoing edges. First, there is no harm in assuming that $M(\mathbf{g})$ has the reverse edges of $M(-\mathbf{g})$, which implies that each node has an outgoing edge with the same probability as it has an incoming edge.

Lemma 7. *Assume that Theorem 6 is false for vectors $\{\mathbf{v}_i\}_{i \in V} \subseteq \mathbb{R}^m$. Then there is a subset $V' \subseteq V$ of size $|V'| \geq cn$ and a matching $M'(\mathbf{g}) \subseteq M(\mathbf{g})$ on V' so that every node $i \in V'$ has an outgoing edge in $M'(\mathbf{g})$ with probability at least $\frac{c}{8}$ and an ingoing edge with probability at least $\frac{c}{8}$.*

Proof. For each node $i \in V$ define $p(i) := \Pr_{\mathbf{g} \sim N^m(0,1)}[i \text{ has outgoing edge in } M(\mathbf{g})]$. If there is a node i with $p(i) \leq \frac{c}{8}$, then we imagine to delete the node from the graph and remove from $M(\mathbf{g})$ any edge containing node i . Note that this decreases the expected size of the matching by at most $2 \cdot \frac{c}{8}$. We continue this procedure until no such node exists anymore. Let V' be the remaining set of nodes with $M'(\mathbf{g}) := M(\mathbf{g}) \cap (V' \times V')$. Then $\mathbb{E}_{\mathbf{g} \sim N^m(0,1)}[|M'(\mathbf{g})|] \geq \mathbb{E}_{\mathbf{g} \sim N^m(0,1)}[|M(\mathbf{g})|] - n \cdot \frac{c}{4} \geq \frac{c}{2}n$. Then there must be at least $|V'| \geq cn$ many nodes left. \square

After changing the constants and adapting the value of n , we assume to have n nodes and for every node $i \in V$, the matching $M(\mathbf{g})$ has an outgoing and an incoming edge with probability at least c .

We call an edge (i, j) Δ -short if $d(i, j) \leq \Delta$. For a node $i \in V$, let $\Gamma(i) := \{j \in V \mid d(i, j) \leq \Delta\}$ be the neighborhood of i with respect to the graph of Δ -short edges. Moreover, let $\Gamma_k(i) := \Gamma_{k-1}(\Gamma(i))$ be the nodes that can be reached from i via at most k many Δ -short edges.

Lemma 8. *For any $k \in \mathbb{Z}_{\geq 0}$ and $i' \in \Gamma_k(i)$ one has $\|\mathbf{v}_i - \mathbf{v}_{i'}\|_2 \leq \sqrt{k\Delta}$.*

Proof. We have $\|\mathbf{v}_i - \mathbf{v}_{i'}\|_2^2 = d(i, i') \leq k \cdot \Delta$ by the SDP triangle inequality. Taking square roots gives the claim. \square

At the heart of the arguments lies the fact that the value of Lipschitz functions is well concentrated. Recall that a function $F : \mathbb{R}^m \rightarrow \mathbb{R}$ is called L -Lipschitz, if $|F(\mathbf{x}) - F(\mathbf{y})| \leq L \cdot \|\mathbf{x} - \mathbf{y}\|_2$ for all $\mathbf{x}, \mathbf{y} \in \mathbb{R}^m$.

Lemma 9 (Concentration for Lipschitz Functions (Sudakov-Tsirelson, Borrell)). *Let $F : \mathbb{R}^m \rightarrow \mathbb{R}$ be an L -Lipschitz function with Gaussian mean $\mu := \mathbb{E}_{\mathbf{g} \sim N^m(0,1)}[F(\mathbf{g})]$. Then $\Pr_{\mathbf{g} \sim N^m(0,1)}[|F(\mathbf{g}) - \mu| \geq \alpha] \leq 2e^{-\frac{1}{2} \cdot (\alpha/L)^2}$ for any $\alpha \geq 0$.*

We define a function

$$F_{i,k}(\mathbf{g}) := \max\{\langle \mathbf{g}, \mathbf{v}_i - \mathbf{v}_j \rangle \mid j \in \Gamma_k(i)\}.$$

In other words, $F_{i,k}(\mathbf{g})$ gives the maximum inner product $\langle \mathbf{g}, \mathbf{v}_i - \mathbf{v}_j \rangle$ over all nodes $j \in V$ that are within k many Δ -short edges of node i . Note that $F_{i,k}(\mathbf{g}) \geq \langle \mathbf{g}, \mathbf{v}_i - \mathbf{v}_{i'} \rangle = 0$ for all $\mathbf{g} \in \mathbb{R}^m$ as $i \in \Gamma_k(i)$.

Lemma 10. *The function $F_{i,k} : \mathbb{R}^m \rightarrow \mathbb{R}$ is $\sqrt{k \cdot \Delta}$ -Lipschitz.*

Proof. Fix $\mathbf{g}, \mathbf{g}' \in \mathbb{R}^m$ and assume for the sake of symmetry that $F(\mathbf{g}) \geq F(\mathbf{g}')$. Let $j, j' \in \Gamma_k(i)$ be the nodes

attaining $F(\mathbf{g})$ and $F(\mathbf{g}')$, resp. Then

$$|F(\mathbf{g}) - F(\mathbf{g}')| = \langle \mathbf{g}, \mathbf{v}_i - \mathbf{v}_j \rangle - \langle \mathbf{g}', \mathbf{v}_i - \mathbf{v}_{j'} \rangle \leq \langle \mathbf{g} - \mathbf{g}', \mathbf{v}_i - \mathbf{v}_j \rangle \stackrel{\text{Cauchy-Schwarz}}{\leq} \|\mathbf{g} - \mathbf{g}'\|_2 \cdot \underbrace{\|\mathbf{v}_i - \mathbf{v}_j\|_2}_{\leq \sqrt{k \cdot \Delta} \text{ by Lem. 8}}.$$

Here we used in the first inequality that $j' \in \Gamma_k(i)$ maximizes $\langle \mathbf{g}', \mathbf{v}_i - \mathbf{v}_{j'} \rangle$. \square

Now, let $\mu_{i,k} := \mathbb{E}_{\mathbf{g} \sim N^m(0,1)}[F_{i,k}(\mathbf{g})]$ be the expected maximum inner product over k -neighbors of node $i \in V$. One useful argument will be a nice relation between expectations of neighbors:

Lemma 11. For any node $i \in V$ and $i' \in \Gamma(i)$ and $k \in \mathbb{Z}_{\geq 0}$ one has $\mu_{i',k+1} \geq \mu_{i,k}$.

Proof. We have

$$\begin{aligned} \mu_{i',k+1} &= \mathbb{E}_{\mathbf{g} \sim N^m(0,1)} \left[\max \left\{ \langle \mathbf{v}_{i'} - \mathbf{v}_i, \mathbf{g} \rangle + \langle \mathbf{v}_i - \mathbf{v}_j, \mathbf{g} \rangle \mid j \in \Gamma_{k+1}(i') \right\} \right] \\ &\stackrel{\Gamma_{k+1}(i') \supseteq \Gamma_k(i)}{\geq} \mathbb{E}_{\mathbf{g} \sim N^m(0,1)} \left[\max \left\{ \langle \mathbf{v}_i - \mathbf{v}_j, \mathbf{g} \rangle \mid j \in \Gamma_k(i) \right\} \right] + \underbrace{\mathbb{E}_{\mathbf{g} \sim N^m(0,1)} [\langle \mathbf{v}_{i'} - \mathbf{v}_i, \mathbf{g} \rangle]}_{=0} = \mu_{i,k}. \end{aligned}$$

\square

On the other hand, we can get the following upper bound:

Lemma 12. For any $k \in \mathbb{Z}_{\geq 0}$ and any $i \in V$ one has $\mu_{i,k} \leq 10\sqrt{\log n} \cdot \sqrt{k\Delta}$.

Proof. For any $j \in \Gamma_k(i)$ we have $\|\mathbf{v}_i - \mathbf{v}_j\|_2 \leq \sqrt{k\Delta}$ and generously

$$\Pr_{\mathbf{g} \sim N^m(0,1)} [\langle \mathbf{v}_i - \mathbf{v}_j, \mathbf{g} \rangle \geq 8\sqrt{\log n} \cdot \sqrt{k\Delta}] \leq \int_{8\sqrt{\log n}}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-x^2/2} dx \leq \frac{1}{2n}.$$

That means $\Pr_{\mathbf{g} \sim N^m(0,1)}[F_{i,k}(\mathbf{g}) \leq 8\sqrt{\log n} \cdot \sqrt{k\Delta}] \geq 1/2$. Again, $F_{i,k}$ is $\sqrt{k\Delta}$ -Lipschitz, hence $\Pr_{\mathbf{g} \sim N^m(0,1)}[|F_{i,k}(\mathbf{g}) - \mu_{i,k}| \geq 2\sqrt{k\Delta}] \leq \frac{1}{2}$ and $2 \leq 2\sqrt{\log(n)}$ for $n \geq 2$. \square

6.1 Extend or expand

The core argument to get to a contradiction is the following:

Lemma 13. Let $\Delta > 0$, $\delta \in \mathbb{R}$ and $k \in \{0, \dots, \frac{1}{100} \log(1/c) \cdot \frac{1}{\Delta}\}$ be parameters and $U \subseteq \{i \in V \mid \mu_{i,k} \geq \delta\}$ be a set of nodes. Then

- (A) either there is a subset $U' \subseteq \Gamma(U)$ so that $|U'| \geq \frac{c}{4} \cdot |U|$ and $\mu_{i,k+1} \geq \delta + 1$ for all $i \in U'$.
- (B) or the neighborhood $U' := \Gamma(U)$ satisfies $|U'| \geq \frac{4}{c} \cdot |U|$ and $\mu_{i,k+1} \geq \delta$ for all $i \in U'$.

Proof. If $|\Gamma(U)| \geq \frac{4}{c} \cdot |U|$, then every node in $i \in \Gamma(U)$ has $\mu_{i,k+1} \geq \delta$ by Lemma 11 and we are in case (B). So suppose that $|\Gamma(U)| < \frac{4}{c} \cdot |U|$. Consider the random matching

$$\tilde{M}(\mathbf{g}) := \left\{ (i, j) \in M(\mathbf{g}) \mid i \in U \text{ and } F_{i,k}(\mathbf{g}) \geq \delta - \frac{1}{2} \right\}$$

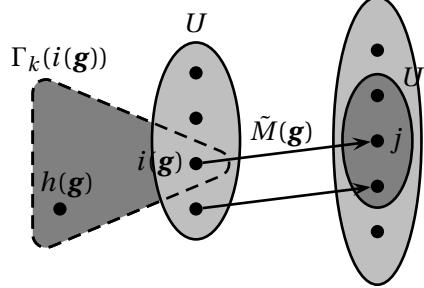
that is the restriction of $M(\mathbf{g})$ to edges that are going out of U and where $F_{i,k}(\mathbf{g})$ is large enough. Note that $\mu_{i,k} = \mathbb{E}_{\mathbf{g} \sim N^m(0,1)}[F_{i,k}(\mathbf{g})] \geq \delta$ for all $i \in U$ and $F_{i,k}$ is $\sqrt{k\Delta}$ -Lipschitz. Hence by Lemma 10 we have

$$\Pr_{\mathbf{g} \sim N^m(0,1)} \left[F_{i,k}(\mathbf{g}) < \delta - \frac{1}{2} \right] \leq 2 \exp \left(-\frac{1}{8k\Delta} \right) < \frac{c}{2} \quad \forall i \in U.$$

This implies that each node in U will have an outgoing edge in $\tilde{M}(\mathbf{g})$ with probability at least $\frac{c}{2}$. Now, define $U' := \{j \in \Gamma(U) \mid \Pr[j \text{ has incoming edge from } \tilde{M}(\mathbf{g})] \geq \frac{c^2}{16}\}$. Since $\tilde{M}(\mathbf{g})$ is a matching we have

$$\frac{c}{2} \cdot |U| \leq \mathbb{E}[|\tilde{M}(\mathbf{g})|] \leq \frac{c^2}{16} \cdot \underbrace{|\Gamma(U) \setminus U'|}_{\leq (4/c) \cdot |U|} + |U'| \leq \frac{c}{4} \cdot |U| + |U'|$$

which implies that $|U'| \geq \frac{c}{4} \cdot |U|$. Now fix a node $j \in U'$. It remains to argue that $\mu_{j,k+1} \geq \delta + 1$ for all $j \in U'$. First, condition on the event that $\tilde{M}(\mathbf{g})$ has an edge incoming to j . We denote that edge by $(i(\mathbf{g}), j) \in \tilde{M}(\mathbf{g})$ with $i(\mathbf{g}) \in U^1$. We know by the definition of $\tilde{M}(\mathbf{g})$ that $\langle \mathbf{v}_j - \mathbf{v}_{i(\mathbf{g})}, \mathbf{g} \rangle \geq 2$. Moreover, we know that there is a node $h(\mathbf{g}) \in \Gamma_k(i(\mathbf{g}))$ so that $\langle \mathbf{v}_{i(\mathbf{g})} - \mathbf{v}_{h(\mathbf{g})}, \mathbf{g} \rangle \geq \frac{\delta - \frac{1}{2}}{\Gamma(U)}$.



Then $\langle \mathbf{v}_j - \mathbf{v}_{h(\mathbf{g})}, \mathbf{g} \rangle = \langle \mathbf{v}_j - \mathbf{v}_{i(\mathbf{g})}, \mathbf{g} \rangle + \langle \mathbf{v}_{i(\mathbf{g})} - \mathbf{v}_{h(\mathbf{g})}, \mathbf{g} \rangle \geq \delta + \frac{3}{2}$. In other words, $\Pr_{\mathbf{g} \sim N^m(0,1)}[F_{j,k+1}(\mathbf{g}) \geq \delta + \frac{3}{2}] \geq \frac{c^2}{16}$. Again, the function $F_{j,k+1}$ is $\sqrt{(k+1)\Delta}$ -Lipschitz and

$$\Pr_{\mathbf{g} \sim N^m(0,1)} \left[|F_{j,k+1}(\mathbf{g}) - \mu_{j,k+1}| \geq \frac{1}{2} \right] \stackrel{\text{Lem. 9}}{\leq} 2 \exp \left(-\frac{1}{8(k+1)\Delta} \right) < \frac{c^2}{16}$$

and hence $\mu_{j,k+1} \geq \delta + 1$. This shows the claim. \square

Now, suppose we run Lemma 13 iteratively, starting with $U := V$ and in each iteration we replace the current U by the set U' . We iterate this until the upper bound on k is reached. Note that case (B) cannot happen more often than case (A) as always $|\Gamma(U)| \leq n$. Then after being $k = \frac{1}{100\Delta} \log(1/c)$ times in Case (A) and $\ell \in \{0, \dots, k\}$ times in Case (B), we end up with a set $U \subseteq V$ with $|U| \geq n \cdot (\frac{c}{4})^{k-\ell} \geq n \cdot (\frac{c}{4})^k$ and $\mu_{i,2k} \geq \mu_{i,k+\ell} \geq k$ for all $i \in U$. On the other hand, $\mu_{i,2k} \leq 10\sqrt{\log n} \cdot \sqrt{2k\Delta}$ by Lemma 12. Choosing $\Delta := \Theta_c(\frac{1}{\sqrt{\log n}})$ and $k := \Theta_c(\sqrt{\log n})$ with proper choice of constants, then gives a contradiction.

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¹ Here we write $i(\mathbf{g})$ to indicate that this node will depend on the choice of \mathbf{g} .

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