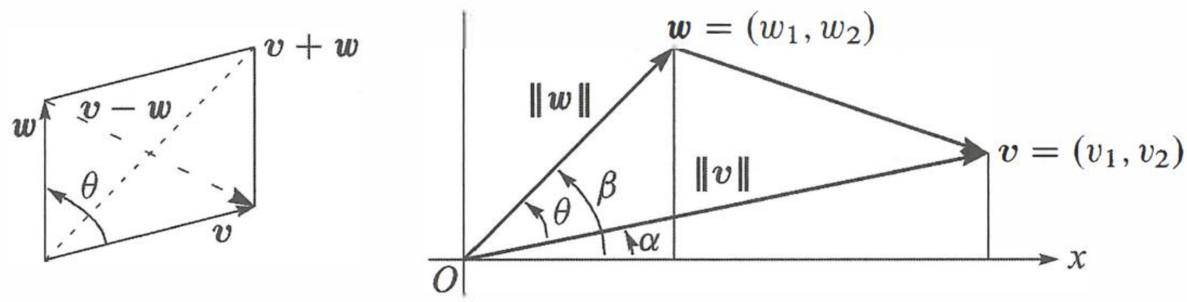
23 and 28 from section 1.2

Triangle inequality $\|\boldsymbol{v}+\boldsymbol{w}\|^2 \leq (\|\boldsymbol{v}\|+\|\boldsymbol{w}\|)^2$ **or** $\|\boldsymbol{v}+\boldsymbol{w}\| \leq \|\boldsymbol{v}\|+\|\boldsymbol{w}\|$.



23 The figure shows that $\cos\alpha = v_1/\|\boldsymbol{v}\|$ and $\sin\alpha = v_2/\|\boldsymbol{v}\|$. Similarly $\cos\beta$ is _____ and $\sin\beta$ is _____. The angle θ is $\beta-\alpha$. Substitute into the trigonometry formula $\cos\beta\cos\alpha+\sin\beta\sin\alpha$ for $\cos(\beta-\alpha)$ to find $\cos\theta = \boldsymbol{v}\cdot\boldsymbol{w}/\|\boldsymbol{v}\|\,\|\boldsymbol{w}\|$.

$$\cos\beta = \frac{\omega_1}{|\omega|}$$

$$\sin\beta = \frac{\omega_2}{|\omega|}$$

$$\boldsymbol{\cos(\beta-\alpha) = \cos\beta\cos\alpha + \sin\beta\sin\alpha = \frac{\omega_1}{|\omega|}\frac{v_1}{|v|} + \frac{\omega_2}{|\omega|}\frac{v_2}{|v|} = \frac{\omega_1 v_1 + \omega_2 v_2}{|\omega||v|} = \frac{v*\omega}{|v||\omega|}}$$

Section 1.2. Problem 28: Can three vectors in the xy plane have $u\cdot v<0$ and $v\cdot w<0$ and $u\cdot w<0$?

$uv<0 \to \cos(u,v)<0 \to 90° < angle(u,v) < 270°$, same to the rest. It's possible.

4 and 13 from section 1.3

Section 1.3. Problem 4: Find a combination $x_1w_1+x_2w_2+x_3w_3$ that gives the zero vector:

$$w_1 = \begin{bmatrix}1\\2\\3\end{bmatrix};\ w_2 = \begin{bmatrix}4\\5\\6\end{bmatrix};\ w_3 = \begin{bmatrix}7\\8\\9\end{bmatrix}.$$

Those vectors are (independent)(dependent). The three vectors lie in a ________. The matrix W with those columns is *not invertible*.

$x = \begin{bmatrix}1\\-2\\1\end{bmatrix}$, Those vectors are dependent. The three vectors lie in a plane.

Section 1.3. Problem 13: The very last words say that the 5 by 5 centered difference matrix *is not* invertible. Write down the 5 equations $Cx = b$. Find a combination of left sides that gives zero. What combination of b_1, b_2, b_3, b_4, b_5 must be zero?

$$[x_2, x_3 - x_1, x_4 - x_2, x_5 - x_3, -x_4] = [b_1, \dots, b_5]$$

$$b_1 + b_3 + b_5 = 0$$

29 and 30 from section 2.1

Section 2.1. Problem 29: Start with the vector $u_0 = (1, 0)$. Multiply again and again by the same "Markov matrix" $A = [.8\ .3; .2\ .7]$. The next three vectors are u_1, u_2, u_3:

$$u_1 = \begin{bmatrix} .8 & .3 \\ .2 & .7 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} .8 \\ .2 \end{bmatrix} \qquad u_2 = Au_1 = \underline{\qquad} \qquad u_3 = Au_2 = \underline{\qquad\quad}.$$

What property do you notice for all four vectors u_0, u_1, u_2, u_3.

$$u_2 = \begin{bmatrix} 0.7 \\ 0.3 \end{bmatrix}, u_3 = \begin{bmatrix} 0.65 \\ 0.35 \end{bmatrix}$$

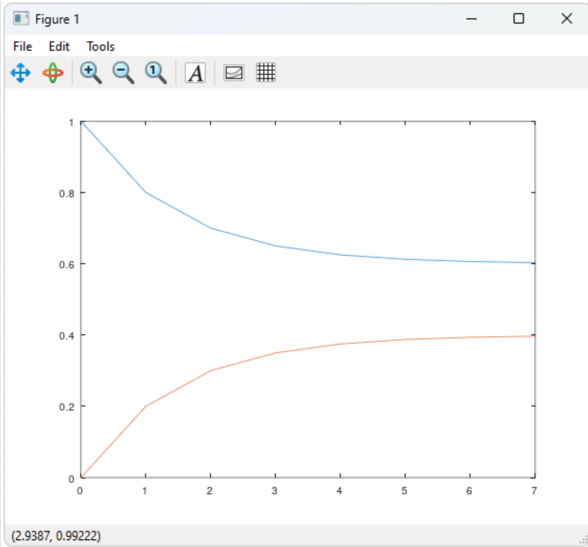
All four vectors have components that sum to one.

Section 2.1. Problem 30: Continue Problem 29 from $u_0 = (1, 0)$ to u_7, and also from $v_0 = (0, 1)$ to v_7. What do you notice about u_7 and v_7? Here are two MATLAB codes, with while and for. They plot u_0 to u_7 and v_0 to v_7.
The u's and the v's are approaching a steady state vector s. Guess that vector and check that $As = s$. If you start with s, then you stay with s.

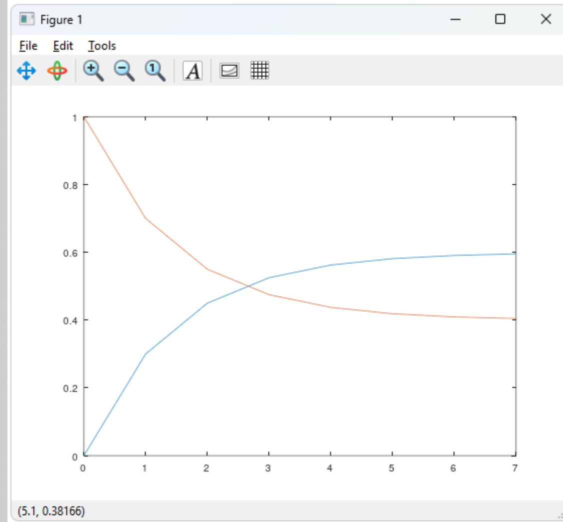
$$s = [0.6, 0.4]$$

$$As = \begin{bmatrix} 0.8 & 0.3 \\ 0.2 & 0.7 \end{bmatrix} \begin{bmatrix} 0.6 \\ 0.4 \end{bmatrix} = \begin{bmatrix} 0.6 \\ 0.4 \end{bmatrix}$$

```
u = [1 ; 0]; A= [.8 .3 ; .2 .7]
x = u; k = [0 : 7];
while size(x,2) <= 7
u = A*u; x = [x u];
end
plot(k, x)
```




```
v = [0 ; 1]; A= [.8 .3 ; .2 .7]
x = v; k = [0 : 7];
for j = 1 : 7
  v = A*v; x = [x v];
end
plot(k, x)
```



20 and 32 from section 2.2

Section 2.2. Problem 20: Three planes can fail to have an intersection point, even if no planes are parallel. The system is singular if row 3 of A is a __________ of the first two rows. Find a third equation that can't be solved together with $x + y + z = 0$ and $x - 2y - z = 1$.

The system is singular if row 3 of A is a linear combination of the first two rows.

For example, 2x-y=1 cannot be solved by given $x + y + z = 0$ and $x - 2y - z = 1$

Section 2.2. Problem 32: Start with 100 equations $Ax = 0$ for 100 unknowns $x = (x_1, \ldots, x_{100})$. Suppose elimination reduces the 100th equation to $0 = 0$, so the system is "singular".
(a) Elimination takes linear combinations of the rows. So this singular system has the singular property: Some linear combination of the 100 **rows** is ________.
(b) Singular systems $Ax = 0$ have infinitely many solutions. This means that some linear combination of the 100 **columns** is _________.
(c) Invent a 100 by 100 singular matrix with no zero entries.
(d) For your matrix, describe in words the row picture and the column picture of $Ax = 0$. Not necessary to draw 100-dimensional space.

$\boxed{\text{Solution}}$ (12 points)
(a) Zero. (b) Zero. (c) There are many possible answers. For instance, the matrix for which every row is $(1\ 2\ 3\ \cdots\ 100)$. (d) The row picture is 100 copies of the hyperplane in 100-space defined by the equation

$$x_1 + 2x_2 + 3x_3 + \cdots + 100x_{100} = 0.$$

The column picture is the 100 vectors proportional to $(1\ 1\ 1\ \cdots\ 1)$ of lengths $10, 20, \ldots, 1000$.

22 and 29 from section 2.3

Section 2.3. Problem 22: The entries of A and x are a_{ij} and x_j. So the first component of Ax is $\sum a_{1j}x_j = a_{11}x_1 + \cdots + a_{1n}x_n$. If E_{21} subtracts row 1 from row 2, write a formula for
(a) the third component of Ax
(b) the $(2,1)$ entry of $E_{21}A$
(c) the $(2,1)$ entry of $E_{21}(E_{21}A)$
(d) the first component of $E_{21}Ax$.

a. $\sum a_{3j}x_j = a_{31}x_1 + \cdots + a_{3n}x_n$
b. $(a_{21} - a_{11})x_1$
c. $(a_{21} - 2a_{11})x_1$
d. $\sum (a_{2j} - a_{1j})x_j$ (This is the second component)

$\boxed{\text{Solution}}$ (4 points)
(a) $\sum a_{3j}x_j$. (b) $a_{21} - a_{11}$. (c) $a_{21} - 2a_{11}$. (d) $\sum a_{1j}x_j$.

Section 2.3. Problem 29: Find the triangular matrix E that reduces "*Pascal's matrix*" to a smaller Pascal:

$$E\begin{bmatrix}1&0&0&0\\1&1&0&0\\1&2&1&0\\1&3&3&1\end{bmatrix} = \begin{bmatrix}1&0&0&0\\0&1&0&0\\0&1&1&0\\0&1&2&1\end{bmatrix}.$$

Which matrix M (multiplying several E's) reduces Pascal all the way to I?

$$E_{21} = \begin{bmatrix}1&0&0&0\\-1&1&0&0\\0&-1&1&0\\0&0&-1&1\end{bmatrix}, E_{32} = \begin{bmatrix}1&0&0&0\\0&1&0&0\\0&-1&1&0\\0&0&-1&1\end{bmatrix}, E_{43} = \begin{bmatrix}1&0&0&0\\0&1&0&0\\0&0&1&0\\0&0&-1&1\end{bmatrix}, M = E_{43}E_{32}E_{21} = \begin{bmatrix}1&0&0&0\\-1&1&0&0\\1&-2&1&0\\-1&3&-3&1\end{bmatrix}$$

32 and 36 from section 2.4

Section 2.4. Problem 32: Suppose you solve $Ax = b$ for three special right sides b:

$$Ax_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}; \quad Ax_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}; \quad Ax_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}.$$

If the three solutions x_1, x_2, x_3 are the columns of a matrix X, what is A times X?

$$AX = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Section 2.4. Problem 36: Suppose A is m by n, B is n by p, and C is p by q. Then the multiplication count for $(AB)C$ is $mnp + mpq$. The multiplication count from A times BC with $mnq + npq$ separate multiplications.
(a) If A is 2 by 4, B is 4 by 7, and C is 7 by 10, do you prefer $(AB)C$ or $A(BC)$?
(b) With N-component vectors, would you choose $(u^T v)w^T$ or $u^T(vw^T)$?
(c) Divide by $mnpq$ to show that $(AB)C$ is faster when $n^{-1} + q^{-1} < m^{-1} + p^{-1}$.

a. $(AB)C = 2 * 4 * 7 + 2 * 7 * 10 = 14^2 = 196, A(BC) = 4 * 7 * 10 + 2 * 4 * 10 = 360.$ Therefore, we prefer $(AB)C$
b. $(u^T v)w^T = 2N, u^T(vw^T) = 2N^2$, therefore, we prefer $(u^T v)w^T$
c. $\frac{(AB)C}{mnpq} = \frac{1}{q} + \frac{1}{n}, \frac{A(BC)}{mnpq} = \frac{1}{m} + \frac{1}{p} \rightarrow (AB)C$ is faster when $\frac{1}{q} + \frac{1}{n} < \frac{1}{m} + \frac{1}{p}$

7 from section 2.5

Section 2.5. Problem 7: If A has row 1 + row 2 = row 3, show that A is not invertible:
(a) Explain why $Ax = (1, 0, 0)$ cannot have a solution.
(b) Which right sides (b_1, b_2, b_3) might allow a solution to $Ax = b$?
(c) What happens to row 3 in elimination?

a. $row_1 x = 1, row_2 x = 0, row_3 x = 0 \rightarrow row_1 x + row_2 x = x(row_1 + row_2) \neq x(row_3)$
b. $row_1 x = b_1, row_2 x = b_2, row_3 x = b_3 \rightarrow b_1 + b_2 = b_3$
c. The row_3 will be 0