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How to Interpret the Coefficients of Spatial Models: Spillovers, Direct and Indirect Effects

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Abstract This paper briefly reviews how to derive and interpret coefficients of spatial regression models, including topics of direct and indirect (spatial spillover) effects. These topics have been addressed in the spatial econometric literature over the past 5–6 years, but often at a level sometimes difficult for students new to the field. Our goal is to overcome this handicap by carefully presenting the mathematics behind these spatial effects and clearly illustrating how they work using two small fictive datasets and one large dataset with real data. The motivation for the paper is primarily pedagogical. Theoretical and conceptual impediments associated with the application of procedures are discussed.

Keywords Spatial effects · Spatial spillovers · Spatial models · Spatial econometrics

1 Introduction

This paper briefly reviews how to derive and interpret coefficients of spatial regression models, including topics of direct and indirect (spatial spillover) effects. LeSage and Pace (2009) and Elhorst (2010) address these topics in greater detail, but at a level sometimes difficult for students new to the field of spatial econometrics. Our goal is to overcome this handicap by carefully presenting the

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mathematics behind these spatial effects and clearly illustrating how they work using two small fictive datasets and one large dataset with real data.

As emphasized by LeSage and Dominguez (2012), in many empirical applications the coefficients of spatial models often are interpreted incorrectly as if they were simple partial derivatives. Although this interpretation is correct for properly specified standard linear models (SLM), or even models with spatially lagged errors, it does not hold for models with spatial dependence in the dependent variable or in the explanatory variables. Such models include the spatial autoregression (SAR) model, the mixed spatial autoregression spatial error (SAC, or, sometimes called, SARAR) model, the spatially-lagged X model (SLX), the spatial Durbin error model (SDEM), and the variety of spatial Durbin models (SDM). Hence, the regression coefficients of these latter models must be interpreted differently and with caution.

Given the potential variety of mathematical expositions and interpretations, depending on the model, we discuss seven common spatial models. We limit our focus to models which are widely used and discussed in the literature. We do not extend our remarks to some of the more recently introduced models—e.g., spatial models with two or more weights matrices as discussed by LeSage and Pace (2011). We also exclude other spatial model specifications discussed by LeSage and Pace (2009), Manski (1993) and Elhorst (2010)—although the principles discussed here apply by extension.

Figure 1, which is similar to part of a figure presented in Elhorst (2010), shows the relationships among these seven models. First, we discuss the standard linear

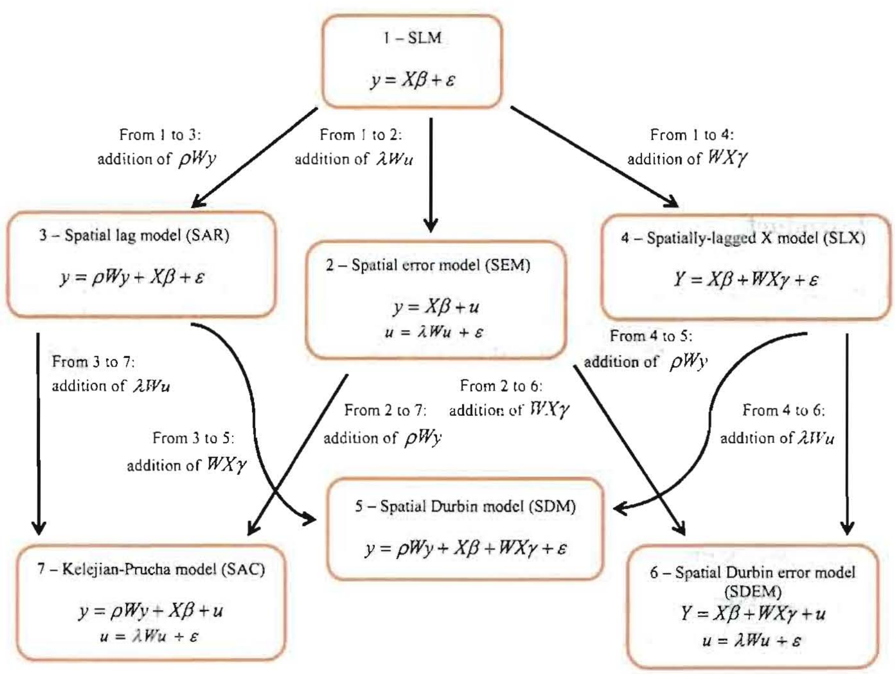

Fig. 1 Some spatial models

regression model (SLM) estimated by OLS. We begin with this model because it is the simplest and most familiar. While it is a non-spatial model, it is commonly used as a diagnostic tool for model specification evaluation and as a benchmark for comparisons with spatial models. As part of this section, we also present the spatial error model (SEM) since the interpretation of the coefficients is similar to interpreting a SLM model. The spatial lag model (SAR) is introduced in Sect. 3. Due to endogenous spatial dependence in this model, it is more challenging to interpret the coefficients, and we build the presentation here with additional details. This section also discusses the SAC model (introduced by Kelejian and Prucha 1998), which is similar in many aspects to the SAR model. We then present in Sect. 4 two regression models with spatial lags only in the independent variables—the spatially-lagged X model (SLX) as well as the spatial Durbin error model (SDEM), both of which contain exogenous spatial dependence. Section 5 discusses the interpretation of coefficients for a spatial Durbin model (SDM) which contains both exogenous and endogenous spatial dependencies, thus making the interpretation of the coefficients more complicated than for the preceding models.

For ease of presentation, each of the models in Sects. 2 through 5 is introduced using an artificially small study domain (three geographic regions). The illustrated spatial effects are then expanded by introducing in Sect. 6 a slightly more complex study domain (eight irregular geographically regions). Section 7 of the paper further expands the illustration of spatial effects using a fairly large dataset for counties in the southeastern U.S.

We conclude the paper with a discussion section which lays out a rationale for why this type of coefficient interpretation is useful for spatial analysts when seeking to understand the extent and strength of spatial spillovers in ecological regression models. We also include some important precautions regarding the limitations of such parameter explorations and why econometricians often prefer summary measures of spatial effects over detailed unit-specific effects.

1.1 SLM and SEM Models

The interpretation of coefficients for the non-spatial standard linear regression model (SLM) is straightforward and can be used as a benchmark for comparison with the spatial models. In particular, residual diagnostics from a SLM are useful for understanding the nature of spatial patterning in the data and, consequently, possible model misspecification. Following the notational conventions used by LeSage and Pace (2009), this familiar model can be expressed as:

$$y = \alpha i_n + X\beta + \varepsilon, \quad \varepsilon \sim N(0, \sigma^2 I_n), \tag{1}$$

where y is the n × 1 dependent variable vector; X is the n × p matrix of explanatory variables; having removed the usual first column of ones from the design matrix, X, i_n is an n-dimensional column vector of ones; α is the intercept coefficient; β represents the regression slope coefficients in a p × 1 vector; and ε is white noise error. The model and assumptions underlying it are richly detailed in most beginning-level statistics textbooks.

Provided that X is exogenous, we have the following expected value for the dependent variable:

$$E[y|X] = E[\alpha i_n + X\beta + \varepsilon|X] = \alpha i_n + X\beta.$$

To interpret statistically the coefficients of this model, we calculate the partial derivatives of Eq. (1) for each specific exogenous variable k. For a dataset with n observations, we obtain the following $n \times n$ matrix:

$$\begin{pmatrix} \frac{\partial y_1}{\partial x_{1k}} & \cdots & \frac{\partial y_1}{\partial x_{nk}} \\ \cdots & \cdots & \cdots \\ \frac{\partial y_n}{\partial x_{1k}} & \cdots & \frac{\partial y_n}{\partial x_{nk}} \end{pmatrix} = \begin{pmatrix} \beta_k & \cdots & 0 \\ \cdots & \cdots & \cdots \\ 0 & \cdots & \beta_k \end{pmatrix} = \beta_k \begin{pmatrix} 1 & \cdots & 0 \\ \cdots & \cdots & \cdots \\ 0 & \cdots & 1 \end{pmatrix} = \beta_k In . \quad (2)$$

As all cross-partial derivatives in this matrix are zeros, the matrix is easily understood as representing the scalar coefficient of the kth variable multiplied by an identity matrix. With p regressors, there are p such matrices. The coefficient interpretation is straightforward:

$$\frac{\partial y_j}{\partial x_{ik}} = \beta_k \quad \text{for} \quad i = j, \quad \text{and} \quad \frac{\partial y_j}{\partial x_{ik}} = 0 \quad \text{for} \quad i \neq j.$$

Only the terms in the major diagonal are non-zero and the model does not present spatial spillovers. That is, the model parameters are estimated under the explicit assumption that the observations are independent; changes in values for one observation do not "spill over" to affect values of other observations.

In order to make the implications of this discussion clearer, we present a simple illustrative example with three neighboring geographic regions as shown in Fig. 2.

Assume the dependent variable, y, is regional per capita income and there are two explanatory variables: x_1, a measure of regional human capital level, and x_2, an index indicating the quality of local infrastructure. Now, arbitrarily set the slope coefficients of the SLM model as $\beta_1 = 2$ and $\beta_2 = 3$, respectively.

The matrix of partial derivatives, relative to human capital, x_{i1}, is:

$$\begin{pmatrix} \frac{\partial y_1}{\partial x_{11}} & \frac{\partial y_1}{\partial x_{21}} & \frac{\partial y_1}{\partial x_{31}} \\ \frac{\partial y_2}{\partial x_{11}} & \frac{\partial y_2}{\partial x_{21}} & \frac{\partial y_2}{\partial x_{31}} \\ \frac{\partial y_3}{\partial x_{11}} & \frac{\partial y_3}{\partial x_{21}} & \frac{\partial y_3}{\partial x_{31}} \end{pmatrix} = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{pmatrix}.$$

Fig. 2 Three fictitious regions

Based on this matrix, we can estimate the approximate impact on each region's per capita income of an increase of 10 units in the human capital level of region 3:

$$\begin{pmatrix} \Delta y_1 \\ \Delta y_2 \\ \Delta y_3 \end{pmatrix}_{SLM} \approx \begin{pmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{pmatrix} \begin{pmatrix} \Delta x_{11} \\ \Delta x_{21} \\ \Delta x_{31} \end{pmatrix} = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ 10 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 20 \end{pmatrix}$$

Only the third region is affected by the increased human capital in region 3, resulting in an increase of 20 units on region 3's per capita income. We observe no spillovers between the regions, as the OLS estimator assumes independence among the observations. Said another way, all cross-partial derivatives are zero.

We also introduce here the spatially-lagged errors model (SEM) because the results are similar to those for the SLM. The structural form of the SEM is specified by the following equations:

$$y = \alpha i_n + X\beta + u, \quad u = \lambda Wu + \varepsilon, \quad \varepsilon \sim N\left(0, \sigma^2 I_n\right),$$

where λ is the coefficient expressing the average strength of spatial correlation among the errors (conditional on W) and W is the weight matrix representing the spatial structure of neighbor influences among the residuals.

Algebraically manipulating these equations, we obtain the reduced form data generation process (DGP) for the SEM model:

$$y = \alpha i_n + X\beta + (I - \lambda W)^{-1}\varepsilon.$$

The expected value for the dependent variable in the SEM is the same as that for the SLM:

$$E[y|X] = \alpha i_n + X\beta.$$

The partial derivatives matrices are also the same:

$$\begin{pmatrix} \frac{\partial y_1}{\partial x_{1k}} & \cdots & \frac{\partial y_1}{\partial x_{nk}} \\ \cdots & \cdots & \cdots \\ \frac{\partial y_n}{\partial x_{1k}} & \cdots & \frac{\partial y_n}{\partial x_{nk}} \end{pmatrix} = \begin{pmatrix} \beta k & \cdots & 0 \\ \cdots & \cdots & \cdots \\ 0 & \cdots & \beta_k \end{pmatrix}.$$

Note that we can define the structural relationship between the SLM and SEM models as follows. Let $y = \alpha i_n + X\beta + u$, and $u = F(W)u + \varepsilon$, where $F(W)$ is a function of the spatial weights matrix. In pairs of models that differ only in the $F(W)$ function, for instance $F(W) = 0$ for SLM, and $F(W) = \lambda W$ for SEM, the expected values and the interpretation of the coefficients are the same.

2 SAR and SAC Models

The structural form for the SAR model is described by the following equations:

$$y = \alpha i_n + \rho Wy + X\beta + \varepsilon, \quad \varepsilon \sim N\left(0, \sigma^2 I_n\right),$$

where ρ is the coefficient for the endogenous variable Wy, a variable representing a function of neighboring values of the dependent variable.

The SAC model, when compared to the spatial lag model, presents a specification setup parallel to that described above for the SLM and SEM models. The SAC model[1] is given as:

$$y = \alpha i_n + \rho W y + X\beta + u, \quad u = \lambda W u + \varepsilon, \quad \varepsilon \sim N(0, \sigma^2 I_n).$$

Hence, the expected value and the coefficient interpretation are the same for the SAR and SAC models, and we continue the discussion only for the first of the two. Due to spatially structured endogeneity present in the model, the beta coefficient interpretation is more complicated than for the SLM or the SEM models, as we must now acknowledge and account for likely spatial spillovers in the model.

Manipulating the SAR equation, we obtain the model's reduced form DGP:

$$y = (I - \rho W)^{-1}(\alpha i_n + X\beta + \varepsilon). \tag{3}$$

This expression has the following expected value, which differs from the first two models (SLM and SEM) discussed above:

$$\begin{aligned} E[y|X] &= E[(I - \rho W)^{-1}(\alpha i_n + X\beta + \varepsilon)] = (I - \rho W)^{-1} E[(\alpha i_n + X\beta + \varepsilon)] \\ &= (I - \rho W)^{-1}(\alpha i_n + X\beta). \end{aligned}$$

Again we obtain the partial derivatives of expression (3) for a specific exogenous variable k:

$$\begin{pmatrix} \frac{\partial y_1}{\partial x_{1k}} & \cdots & \frac{\partial y_1}{\partial x_{nk}} \\ \cdots & \cdots & \cdots \\ \frac{\partial y_n}{\partial x_{1k}} & \cdots & \frac{\partial y_n}{\partial x_{nk}} \end{pmatrix} = (I - \rho W)^{-1} \begin{pmatrix} \beta_k & \cdots & 0 \\ \cdots & \cdots & \cdots \\ 0 & \cdots & \beta_k \end{pmatrix} = \beta_k (I - \rho W)^{-1}. \tag{4}$$

The $n \times n$ matrix $(I - \rho W)^{-1}$ has non-zero elements off the major diagonal, and these non-zero cross-partial derivatives imply the existence of spillovers. This can be seen by expressing the power series expansion of the matrix $(I - \rho W)^{-1}$:

$$(I - \rho W)^{-1} = I + \rho W + \rho 2W2 + \rho 3W3 + \ldots$$

Using this power series, we show below that the SAR model generates a process of "global spillover" indicating that changes in an independent variable anywhere in the study domain will affect the value of the dependent variable everywhere, even when the declaration of neighborhood influences implicit in the matrix W represents simple 1st-order contiguity. The elements of the W matrix are constrained to lie in the interval $\{0, 1\}$ through the process of row standardizing, and the allowable range of the parameter ρ is $\{1/\lambda_{\min}, 1\}$, where $\lambda_{\min}$ is the smallest eigenvalue of the

[1] The two parts of the SAC model are represented here as using the same weights matrix, W. This is a matter of notational convenience and not a requirement of the model. Different W matrices can be used without altering the point of the example.

W matrix. Thus, spatial spillovers in the model that come through the inverse matrix extend throughout the study domain but dampen out as higher powers of both ρ and W are taken. In order to facilitate an understanding of the consequences of this matrix, we again present an example based on Fig. 2. We specify a simple row normalized contiguity matrix of spatial weights:

$$W = \begin{pmatrix} 0 & 1 & 0 \\ 1/2 & 0 & 1/2 \\ 0 & 1 & 0 \end{pmatrix}.$$

For this weight matrix we obtain by elementary matrix arithmetic:

$$(I - \rho W) = \begin{pmatrix} 1 & -\rho & 0 \\ -\rho/2 & 1 & -\rho/2 \\ 0 & -\rho & 1 \end{pmatrix}.$$

We then determine the inverse of this matrix:

$$(I - \rho W)^{-1} = \left(\frac{1}{1-\rho^2}\right) \begin{pmatrix} 1 - \rho^2/2 & \rho & \rho^2/2 \\ \rho/2 & 1 & \rho/2 \\ \rho^2/2 & \rho & 1 - \rho^2/2 \end{pmatrix}.$$

Finally, based on expression (4), we obtain the matrix of derivatives for this illustration:

$$\begin{pmatrix} \frac{\partial y_1}{\partial x_{1k}} & \frac{\partial y_1}{\partial x_{2k}} & \frac{\partial y_1}{\partial x_{3k}} \\ \frac{\partial y_2}{\partial x_{1k}} & \frac{\partial y_2}{\partial x_{2k}} & \frac{\partial y_2}{\partial x_{3k}} \\ \frac{\partial y_3}{\partial x_{1k}} & \frac{\partial y_3}{\partial x_{2k}} & \frac{\partial y_3}{\partial x_{3k}} \end{pmatrix} = \left(\frac{\beta_k}{1-\rho^2}\right) \begin{pmatrix} \left(1 - \rho^2/2\right) & \rho & \rho^2/2 \\ \rho/2 & 1 & \rho/2 \\ \rho^2/2 & \rho & 1 - \rho^2/2 \end{pmatrix}. \quad (5)$$

Clearly the results depend on the exogenously chosen weight matrix. However, as discussed in LeSage and Pace (2011), the sensitivity to W is less strong than commonly believed. Note also that the model's beta coefficients are not represented simply by partial derivatives along the matrix diagonal, as observed before for the SLM and SEM models. The beta coefficients now are only part of these derivatives, and the spatial coefficient rho also enters the calculation. In addition, both the beta and rho coefficients are present in the cross-partial, off-diagonal derivatives.

Setting $\rho = 0$, we obtain the SLM model described in Sect. 2, Eq. (2). If $\rho \neq 0$, the off-diagonal elements are different from zero, implying spatial spillovers. Note that the diagonal derivatives have the following important properties:

$$\frac{\partial y_1}{\partial x_{1k}} = \frac{\partial y_3}{\partial x_{3k}} = \left(\frac{\beta_j}{1-\rho^2}\right)\left(1 - \rho^2/2\right) = \beta_k\left(\frac{2-\rho^2}{2-2\rho^2}\right) > \beta_k,$$

and

$$\frac{\partial y_2}{\partial x_{2k}} = \left(\frac{\beta_j}{1-\rho^2}\right) > \beta_k .$$

That is, the direct effect on a region's dependent variable resulting from a change in an explanatory variable for that region, represented by these partial derivatives, includes not only the β_j, but also an additional spillover effect via feedback controlled by the strength of the spatial coefficient, ρ. The non-diagonal cross-partial derivatives represent the indirect spillovers (LeSage and Dominguez 2012), that is, the influences on the dependent variable y in a region rendered by change in x in some other region.

We present an example. As above, the dependent variable is per capita income, and we allow for a small increase of 10 units in x_1 (human capital level) in region 3, with declared $\beta_1 = 2$. We obtain the following approximation:

$$\begin{pmatrix} \Delta y_1 \\ \Delta y_2 \\ \Delta y_3 \end{pmatrix} \approx \left(\frac{2}{1-\rho^2}\right) \begin{pmatrix} \left(1-\rho^2/2\right) & \rho & \rho^2/2 \\ \rho/2 & 1 & \rho/2 \\ \rho^2/2 & \rho & 1-\rho^2/2 \end{pmatrix} \begin{pmatrix} \Delta x_{11} = 0 \\ \Delta x_{21} = 0 \\ \Delta x_{31} = 10 \end{pmatrix}$$
$$= \left(\frac{20}{1-\rho^2}\right) \begin{pmatrix} \rho^2/2 \\ \rho/2 \\ 1-\rho^2/2 \end{pmatrix} .$$

The results clearly depend on the ρ coefficient. If $\rho = 0$, we obtain the estimates found above for the SLM model:

$$\begin{pmatrix} \Delta y_1 \\ \Delta y_2 \\ \Delta y_3 \end{pmatrix}_{SLM} \approx \left(\frac{20}{1}\right) \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 20 \end{pmatrix} ,$$

and an increase in human capital level in region 3 has an impact only in this region; there are no spatial spillovers.

However, if $\rho = 0.2$, a reasonably small positive spatial coefficient, the results are:

$$\begin{pmatrix} \Delta y_1 \\ \Delta y_2 \\ \Delta y_3 \end{pmatrix}_{SAR(0.2)} \approx \left(\frac{20}{1-(0.2)^2}\right) \begin{pmatrix} (0.2)^2/2 \\ (0.2)/2 \\ 1-(0.2)^2/2 \end{pmatrix} = \begin{pmatrix} 0.42 \\ 2.08 \\ 20.42 \end{pmatrix} .$$

Subtracting the SLM results from the SAR results, we obtain the degree of spatial spillover for $\rho = 0.2$:

$$\begin{pmatrix} \Delta y_1 \\ \Delta y_2 \\ \Delta y_3 \end{pmatrix}_{SAR(0.2)} - \begin{pmatrix} \Delta y_1 \\ \Delta y_2 \\ \Delta y_3 \end{pmatrix}_{SLM} = \begin{pmatrix} 0.42 \\ 2.08 \\ 0.42 \end{pmatrix} .$$

Notice that the model presents spillover even in region 3 due to positive feedback, a larger spillover in region 2 (neighbor to region 3), and a smaller one for region 1 (a more distant neighbor). If $\rho = 0.8$, a large positive spatial correlation, the spillovers are much larger:

$$\begin{pmatrix} \Delta y_1 \\ \Delta y_2 \\ \Delta y_3 \end{pmatrix}_{SAR(0.8)} \approx \left(\frac{20}{1-(0.8)^2}\right)\begin{pmatrix} (0.8)^2/2 \\ (0.8)/2 \\ 1-(0.8)^2/2 \end{pmatrix} = \begin{pmatrix} 17.8 \\ 22.2 \\ 37.8 \end{pmatrix},$$

$$\begin{pmatrix} \Delta y_1 \\ \Delta y_2 \\ \Delta y_3 \end{pmatrix}_{SAR(0.8)} - \begin{pmatrix} \Delta y_1 \\ \Delta y_2 \\ \Delta y_3 \end{pmatrix}_{SLM} = \begin{pmatrix} 17.8 \\ 22.2 \\ 17.8 \end{pmatrix}.$$

It is simple and straightforward to present the matrix for the partial derivatives, and the numerical results in these examples, given only three observations. However, for most empirical studies with many more observations, such as in Kirby and LeSage (2009), who analyzed 63,000 census tracts in the U.S., to present the results performed above might not be feasible (Elhorst 2010). In order to overcome some of these difficulties, and for additional reasons commented upon in our final section, LeSage and Pace (2009) present a means of summarizing the direct, indirect and total effects in such models through an averaging process. The first is represented by the mean of the diagonal terms of the partial derivatives matrix. The indirect effect is defined as the mean of the off-diagonal elements in each row (or column). The total effect is represented as the sum of the direct and indirect effects. Kelejian et al. (2013), and Ward and Gleditsch (2007) discuss similar attributes, which they call, respectively, emanating effects and equilibrium effects.

In order to further clarify the use of these concepts, we illustrate the direct, indirect and total effects using the matrix in Eq. (5). Following the convention of LeSage and Pace (2009), we use the expression $S_k(W)$to represent the partial derivative matrix for variable k, with $k = \{1,2,3\}$. The expression indicates clearly that the results are a function of the exogenously chosen spatial weights matrix:

$$S_k(W) = \begin{pmatrix} \frac{\partial y_1}{\partial x_{1k}} & \frac{\partial y_1}{\partial x_{2k}} & \frac{\partial y_1}{\partial x_{3k}} \\ \frac{\partial y_2}{\partial x_{1k}} & \frac{\partial y_2}{\partial x_{2k}} & \frac{\partial y_2}{\partial x_{3k}} \\ \frac{\partial y_3}{\partial x_{1k}} & \frac{\partial y_3}{\partial x_{2k}} & \frac{\partial y_3}{\partial x_{3k}} \end{pmatrix} = \left(\frac{\beta_k}{1-\rho^2}\right)\begin{pmatrix} \left(1-\rho^2/2\right) & \rho & \rho^2/2 \\ \rho/2 & 1 & \rho/2 \\ \rho^2/2 & \rho & 1-\rho^2/2 \end{pmatrix}. \tag{6}$$

The mean of the diagonal terms represents a summary measure of the individual direct effects (DE):

$$\text{DE} = \left(\frac{1}{3}\right)\left(\frac{\beta_k}{1-\rho^2}\right)\left[\left(1-\rho^2/2\right)+1+\left(1-\rho^2/2\right)\right] = \left(\frac{\beta_k}{3(1-\rho^2)}\right)(3-\rho^2).$$

The individual indirect effects are found in the off-diagonal elements of the partial derivatives matrix. The summary measure (IE) of these individual indirect effects, introduced by LeSage and Pace (2009), is a cumulative measure that sums the off-diagonal elements across rows (or columns) and divides by the number of rows (columns) to obtain an average spillover effect across the study area:

$$\begin{aligned}\text{IE} &= \left(\frac{1}{3}\right)\left(\frac{\beta_j}{1-\rho^2}\right)\left[\rho/2 + \rho^2/2 + \rho + \rho + \rho^2/2 + \rho/2\right] \\ &= \left(\frac{\beta_k}{3(1-\rho^2)}\right)\left(3\rho + \rho^2\right).\end{aligned}$$

The total effect (TE) is the sum of these two:

$$\text{TE} = \text{DE} + \text{IE} = \left(\frac{\beta_k}{(1-\rho^2)}\right)(1+\rho).$$

Given that the trace of a matrix is the sum of the diagonal terms, $tr(A) = \sum a_{ii}$, and that the sum of all the elements in a matrix can be obtained by the expression, $i_n' A\ i_n = \sum a_{ij}$, the expressions for DE, IE and TE can be written for a general problem as:

(a) DE $= \left(\frac{1}{n}\right) tr(S_k(W))$,
(b) IE = TE − DE, and
(c) TE $= \left(\frac{1}{n}\right) i_n'(S_k(W))\ i_n$.

These expressions are now applied to a numerical example. If $\rho = \frac{1}{2}$ and $\beta_k = 2$, we obtain:

$$S_k(W) = \left(\frac{2}{1-\left(\frac{1}{4}\right)}\right)\begin{pmatrix}\left(1-\frac{1}{8}\right) & \frac{1}{2} & \frac{1}{8} \\ \frac{1}{4} & 1 & \frac{1}{4} \\ \frac{1}{8} & \frac{1}{2} & 1-\frac{1}{8}\end{pmatrix} = \begin{pmatrix}\frac{7}{3} & \frac{4}{3} & \frac{1}{3} \\ \frac{2}{3} & \frac{8}{3} & \frac{2}{3} \\ \frac{1}{3} & \frac{4}{3} & \frac{7}{3}\end{pmatrix}.$$

$$DE = \left(\frac{1}{n}\right) tr(S_k(W)) = \left(\frac{1}{3}\right)\left(\frac{7}{3}+\frac{8}{3}+\frac{7}{3}\right) = \frac{22}{9}.$$

$$TE = \left(\frac{1}{n}\right) i_n'(S_k(W)) i_n = \left(\frac{1}{3}\right)\begin{pmatrix}1 & 1 & 1\end{pmatrix}\begin{pmatrix}\frac{7}{3} & \frac{4}{3} & \frac{1}{3} \\ \frac{2}{3} & \frac{8}{3} & \frac{2}{3} \\ \frac{1}{3} & \frac{4}{3} & \frac{7}{3}\end{pmatrix}\begin{pmatrix}1 \\ 1 \\ 1\end{pmatrix} = 4.$$

$$IE = 4 - \frac{22}{9} = \frac{14}{9}.$$

In words, the direct effect represents the expected average change across all observations for the dependent variable of in a particular region due to an increase of one unit for a specific explanatory variable in this region. Under the above conditions ($\rho = ½$ and $\beta_k = 2$) for the three regions in Fig. 2, the direct effect for region 2 (8/3) is slightly higher than the direct effect for regions 1 and 3 (each 7/3) because of spillover feedback from two immediately adjacent regions rather than one. The average direct effect is the mean of these three numbers, 22/9.

Based on the columns of the partial derivative matrix, we can discern the spillover effects of a change in an independent variable of one region on the dependent variable in the other regions. For example, a one-unit change in the first explanatory variable in region 1, x_{11}, affects the value of the dependent variable in region 2,y_2, by 2/3 and also in region 3,y_3, by 1/3, due to spatial spillover effects. The sum of these two spillovers, $\frac{2}{3} + \frac{1}{3} = \frac{3}{3}$, is total indirect effect of this unit change in x_{11}. Similarly, sums are obtained for a unit change in this same variable in the other regions: $x_{21}$and x_{31}, respectively, with the indirect values $\frac{4}{3} + \frac{4}{3} = \frac{8}{3}$ and $\frac{1}{3} + \frac{2}{3} = \frac{3}{3}$. For the system, the average indirect effect is 1/3 of the total of these region-specific indirect effects: (1/3)(3/3 + 8/3 + 3/3) = 14/9.

Based on the rows of the matrix, the indirect effects (cross-partials) represent the changes in the dependent variable of a particular region arising from a one unit increase in an explanatory variable in another region. For example, due to spatial spillovers, a one-unit change in the first explanatory variable in region 2, x_{21}, affects the dependent variable of the first region, y_1, by 4/3. Spillovers to region 1 also occur from a one-unit change in the first explanatory variable in region 3, x_{31}, by 1/3. The sum of these two spillovers, $\frac{4}{3} + \frac{1}{3} = \frac{5}{3}$, is total spillover (or indirect) effect on the dependent variable of region 1 of this unit change in the independent variable in all regions other than region 1. Similarly, total effects in regions 2 and 3 are obtained for a unit change in this same variable in all regions but the reference region, with the values $\frac{2}{3} + \frac{2}{3} = \frac{4}{3}$ and $\frac{1}{3} + \frac{4}{3} = \frac{5}{3}$, respectively. For the system, as above for the columns, the average indirect effect is 1/3 of the total of these region-specific indirect effects: (1/3)(5/3 + 4/3 + 5/3) = 14/ 9. And the total effect for the study area is the sum of the DE and IE: 22/9 + 14/ 9 = 36/9 = 4. As discussed above, these same results apply for the SAC model.

MCMC Bayesian sampling methods are an efficient approach for determining the statistical significance of the direct, indirect and total effects. The spatial parameters are estimated in each sampling step and a different matrix is obtained in each one. The DE, IE and TE are them estimated for each of these matrices, and a distribution for each of these effects is obtained; see Kirby and LeSage (2009), LeSage and Dominguez (2012), LeSage and Pace (2009).

In the next section we elaborate the SDEM and SLX models, both of which incorporate spatially lagged exogenous variables.

3 SDEM and SLX Models

The SDEM and SLX models differ only in the error term, similar to the relationship between the pairs SEM and SLM, and SAC and SAR, and hence also have a similar interpretation. The following equation presents the Spatially-lagged *X* model (SLX), which is similar in notation to the SLM model, save for the addition of the term $WX\gamma$:

$$y = \alpha i_n + X\beta + WX\gamma + \varepsilon, \quad \varepsilon \sim N\left(0, \sigma^2 I_n\right),$$

where γ is the $k \times 1$ coefficient vector for the exogenous spatially lagged independent variables. In the equation it is necessary to keep the intercept term (column of ones) separate from the design matrix, X, to avoid perfect multicollinearity when the X-terms are spatially lagged:

The Spatial Durbin error model (SDEM) is similar in specification, differing only in the specification of the disturbance term:

$$Y = \alpha i_n + X\beta + WX\gamma + u, \quad u = \lambda Wu + \varepsilon, \quad \varepsilon \sim N\left(0, \sigma^2 I_n\right).$$

Due to spatially structured exogeneity present in the models, the gamma coefficients' interpretation is more complicated than for the SLM or the SEM models. However, in some aspects the interpretation is simpler than for the SAR or the SAC models. The models' DGPs both have the following expected value, which differs from models discussed in Sects. 2 and 3:

$$Y = \alpha i_n + X\beta + WX\gamma$$

Again we obtain the partial derivatives of this expression for a specific exogenous variable k:

$$\begin{pmatrix} \dfrac{\partial y_1}{\partial x_{1k}} & \cdots & \dfrac{\partial y_1}{\partial x_{nk}} \\ \cdots & \cdots & \cdots \\ \dfrac{\partial y_n}{\partial x_{1k}} & \cdots & \dfrac{\partial y_n}{\partial x_{nk}} \end{pmatrix} = \begin{pmatrix} \beta_k & \cdots & w_{1n}\gamma_k \\ \cdots & \cdots & \cdots \\ w_{n1}\gamma_k & \cdots & \beta_k \end{pmatrix} = \beta_k I + W\gamma_k.$$

For the pair of models SLM/SEM the matrix of partial derivatives is diagonal. For the SAR/SAC pair, the matrix is non-diagonal due to the spatial dependence represented by $(I - \rho W)^{-1}$. Here, for the pair SLX/SDEM, the matrix is also non-diagonal because of spatially-lagged independent variables in X.

As estimated for the SAR/SAC models, we present the direct, indirect and total effects for the SDEM/SLX model. The direct effect is the mean value of the elements of the main diagonal of the partial derivatives matrix. For these models each diagonal element is equal to β_k. Therefore, DE $= \beta_k$.

The expression for the indirect effect with the mean value of the partial derivatives in each row (or column) is:

$$\text{IE} = \gamma_k\left(\frac{1}{n}\right)\left[\left(\sum_{i=1}^{n} w_{1i}\right) + \left(\sum_{i=1}^{n} w_{2i}\right) + \cdots + \left(\sum_{i=1}^{n} w_{ni}\right)\right]$$

If the weight matrix, W, is row normalized, as it is for most common spatial analytic applications, this expression reduces to:

$$\mathrm{IE} = \gamma_k\left(\frac{1}{n}\right)[(1) + (1) + \cdots + (1)] = \gamma_k.$$

The TE is given by:

$$\mathrm{TE} = \mathrm{DE} + \mathrm{IE} = \beta_k + \gamma_k.$$

That is, the result can be estimated irrespective of calculating the full partial derivative matrix. By way of illustration, we again use the row normalized contiguity weight matrix for Fig. 2:

$$\begin{pmatrix} \frac{\partial y_1}{\partial x_{1k}} & \frac{\partial y_1}{\partial x_{2k}} & \frac{\partial y_1}{\partial x_{3k}} \\ \frac{\partial y_2}{\partial x_{1k}} & \frac{\partial y_2}{\partial x_{2k}} & \frac{\partial y_2}{\partial x_{3k}} \\ \frac{\partial y_3}{\partial x_{1k}} & \frac{\partial y_3}{\partial x_{2k}} & \frac{\partial y_3}{\partial x_{3k}} \end{pmatrix} = \begin{pmatrix} \beta_k & \gamma_k & 0 \\ \frac{\gamma_k}{2} & \beta_k & \frac{\gamma_k}{2} \\ 0 & \gamma_k & \beta_k \end{pmatrix}. \tag{7}$$

Again, for illustrative purposes, using the value $\beta_k = 2$, as above, and including an exogenous spatial lag with $\gamma_k = 1$, we obtain the following partial derivative matrix:

$$\begin{pmatrix} \frac{\partial y_1}{\partial x_{1k}} & \frac{\partial y_1}{\partial x_{2k}} & \frac{\partial y_1}{\partial x_{3k}} \\ \frac{\partial y_2}{\partial x_{1k}} & \frac{\partial y_2}{\partial x_{2k}} & \frac{\partial y_2}{\partial x_{3k}} \\ \frac{\partial y_3}{\partial x_{1k}} & \frac{\partial y_3}{\partial x_{2k}} & \frac{\partial y_3}{\partial x_{3k}} \end{pmatrix} = \begin{pmatrix} 2 & 1 & 0 \\ \frac{1}{2} & 2 & \frac{1}{2} \\ 0 & 1 & 2 \end{pmatrix}.$$

And the values for the DE, IE and TE are:

$$\begin{aligned} DE &= \beta_k = 2, \\ IE &= \gamma_k = 1, \\ TE &= DE + IE = 3. \end{aligned}$$

Again we present a simple approximation. As above, the dependent variable is per capita income, and we allow for a small increase of 10 units in x_1 (human capital level) in region 3:

$$\begin{pmatrix} \Delta y_1 \\ \Delta y_2 \\ \Delta y_3 \end{pmatrix}_{SDEM} \approx \begin{pmatrix} 2 & 1 & 0 \\ \frac{1}{2} & 2 & \frac{1}{2} \\ 0 & 1 & 2 \end{pmatrix} \begin{pmatrix} \Delta x_{11} = 0 \\ \Delta x_{21} = 0 \\ \Delta x_{31} = 10 \end{pmatrix} = \begin{pmatrix} 0 \\ 5 \\ 20 \end{pmatrix}.$$

Unlike what was observed for the SAC model, note here that the spillover effect is strictly local. An increase in human capital in the third region affects only the second region, because the two are contiguous immediate neighbors. Region 1 is unaffected as it is not defined as contiguous in the 1st—order adjacency matrix, W.

In the next section, we discuss the spatial Durbin model (SDM), which includes in a single specification the features present in the SAC and in the SDEM models. This model features prominently in the LeSage and Pace (2009) textbook, as it is a general specification within which other common spatial models are nested.

4 SDM Model

The spatial Durbin model (SDM) is given as:

$$y = \rho Wy + \alpha i_n + X\beta + WX\gamma + \varepsilon, \quad \varepsilon \sim N(0, \sigma^2 I_n).$$

Rearranging terms yields:

$$y = (I - \rho W)^{-1}(\alpha i_n + X\beta + WX\gamma + \varepsilon).$$

The matrix of partial derivatives for this model is:

$$\begin{pmatrix} \frac{\partial y_1}{\partial x_{1k}} & \cdots & \frac{\partial y_1}{\partial x_{nk}} \\ \cdots & \cdots & \cdots \\ \frac{\partial y_n}{\partial x_{1k}} & \cdots & \frac{\partial y_n}{\partial x_{nk}} \end{pmatrix} = [I - \rho W]^{-1} \begin{pmatrix} \beta_k & \cdots & w_{1n}\gamma_k \\ \cdots & \cdots & \cdots \\ w_{n1}\gamma_k & \cdots & \beta_k \end{pmatrix}$$
$$= [I - \rho W]^{-1}[\beta_k I + W\gamma_k].$$

Note that for the SDM model there are non-diagonal terms due to the exogenous parameter β_k and the endogenous spatial lag parameter, ρ, (as with the SAR and SAC models), but now also due to the exogenous spatial lag parameter, γ_k (as with the SDEM and SLX models).

By way of illustration, we again use the row normalized contiguity weight matrix for Fig. 2 to derive the partial derivatives matrix:

$$\begin{pmatrix} \frac{\partial y_1}{\partial x_{1k}} & \frac{\partial y_1}{\partial x_{2k}} & \frac{\partial y_1}{\partial x_{3k}} \\ \frac{\partial y_2}{\partial x_{1k}} & \frac{\partial y_2}{\partial x_{2k}} & \frac{\partial y_2}{\partial x_{3k}} \\ \frac{\partial y_3}{\partial x_{1k}} & \frac{\partial y_3}{\partial x_{2k}} & \frac{\partial y_3}{\partial x_{3k}} \end{pmatrix} = \left(\frac{1}{1-\rho^2}\right) \begin{pmatrix} \left(1 - {\rho^2}/{2}\right) & \rho & {\rho^2}/{2} \\ {\rho}/{2} & 1 & {\rho}/{2} \\ {\rho^2}/{2} & \rho & 1 - {\rho^2}/{2} \end{pmatrix} \begin{pmatrix} \beta_k & \gamma_k & 0 \\ \frac{\gamma_k}{2} & \beta_k & \frac{\gamma_k}{2} \\ 0 & \gamma_k & \beta_k \end{pmatrix}$$

$$= \left(\frac{1}{1-\rho^2}\right) \begin{pmatrix} \left(1 - {\rho^2}/{2}\right)\beta_k + \frac{\rho\gamma_k}{2} & \rho\beta_k + \gamma_k & \frac{\rho^2\beta_k}{2} + \frac{\rho\gamma_k}{2} \\ \frac{\rho\beta_k}{2} + \frac{\gamma_k}{2} & \beta_k + \rho\gamma_k & \frac{\rho\beta_k}{2} + \frac{\gamma_k}{2} \\ \frac{\rho^2\beta_k}{2} + \frac{\rho\gamma_k}{2} & \rho\beta_k + \gamma_k & \left(1 - {\rho^2}/{2}\right)\beta_k + \frac{\rho\gamma_k}{2} \end{pmatrix}. \tag{8}$$

Using expressions (i), (ii) and (iii), from Sect. 3, we derive the DE, IE and TE summary measures of spatial effects:

$$\begin{aligned}
\mathrm{DE} &= \frac{3-\rho^2}{3(1-\rho^2)}\beta_k + \frac{2\rho}{3(1-\rho^2)}\gamma_k,\\
\mathrm{TE} &= \frac{3+3\rho}{3(1-\rho^2)}(\beta_k+\gamma_k),\\
\mathrm{IE} &= \frac{3\rho+\rho^2}{3(1-\rho^2)}\beta_k + \frac{3+\rho}{3(1-\rho^2)}\gamma_k.
\end{aligned}$$

For the spatial Durbin model, the results for DE, TE and IE depend on two scalar parameter vectors β_k and γ_k in addition to the scalar spatial regression parameter ρ. In the spatial lag (SAR) and SAC models, these effects depend only on the first of these parameter vectors. Hence, in the SAR and SAC models, the ratio between the DE and IE depends only on the spatial coefficient ρ: $\frac{DE}{IE} = \frac{3-\rho^2}{3\rho+\rho^2}$. This limitation does not carry over to the spatial Durbin model, which is one of the interesting features of this model (Elhorst 2010).

Again, using the values $\rho = \frac{1}{2}$ and $\beta_1 = 2$, as above, and including an exogenous spatial lag with $\gamma_1 = 1$, we obtain the following values for DE, IE and TE:

$$\begin{aligned}
\mathrm{DE} &= \frac{26}{9},\\
\mathrm{TE} &= 6,\\
\mathrm{IE} &= \frac{28}{9}.
\end{aligned}$$

When comparing these SDM results with those obtained for the spatial lag (SAR) model, we find that they are all larger due to the exogenous interaction and when comparing with the SDEM model, they are greater because of the endogenous dependence.

Moreover, for the spatial Durbin model, the indirect effect can be divided into two parts: the local effects due to the γ_1 coefficient, and the global effects arising from the inverse matrix involving ρ (Elhorst 2010). The first are local because their impact is only on immediate neighbors (under an assumed first-order contiguity weights matrix), while the global influence affects all regions through the matrix $(I-\rho W)^{-1}$.

In Sects. 2, 3, 4 and 5 we have described how to interpret the coefficients of the SLM, SEM, SAR, SAC, SLX, SDEM and SDM models. The following section now compares these models in an illustrative simulation involving eight geographic regions.

5 Illustrative Simulation

We describe in this section a simple simulation using the expressions presented in the previous sections. We define as observational units the eight regions of Fig. 3.

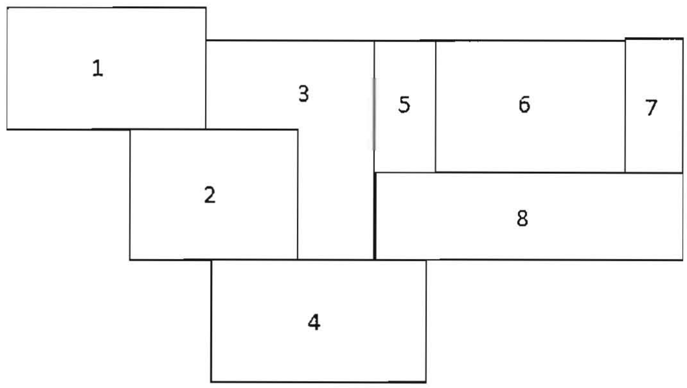

Fig. 3 Eight fictitious regions

This illustration has three objectives: (1) to show a more realistic problem with several irregular regions; (2) to compare the five models discussed in the above sections; and (3) to present and compare the consequences of shocks in central versus peripheral regions on the distribution of spatial spillovers among the regions.

In this example, as above, the dependent variable is per capita income. Only one explanatory variable, x_1 (human capital level), varies. The coefficient for this variable again is given by $\beta_1 = 2$. We assume two values for ρ, $\rho = 0.2$ and $\rho = 0.5$; and one for γ: $\gamma 1 = {}^1\!/_2$. The spatial weights matrix is a row normalized 1st-order queen contiguity matrix. For a discussion about weights matrices, see Corrado and Fingleton (2012), Harris et al. (2011), Leenders (2002), and Stakhovych and Bijmolt (2009).

We compare six models: SLM (and consequently the spatial error model, SEM), two spatial lag models (SAR) with different spatial coefficients, $\rho = 0.2$ or $\rho = 0.5$, (and so also the Kelejian–Prucha model, SAC), a SDEM model (and so also the SLX model), and two spatial Durbin models (SDM), each with one of the two endogenous spatial parameter values.

Table 1 shows the results for DE, IE and TE for these models. These effects were estimated using the matrices shown in Eqs. (2), (5), (7) and (8), respectively, for the SLM, SAR, SDEM and SDM models. For the SLM, the DE is smaller and the IE is zero, as spillovers are absent from the model. As seen in the spatial lag models, the larger the spatial correlation the greater are the DE, IE and TE. The spatial Durbin error model (SDEM) has a similar value as the SLM for the DE, as there is no spillover feedback, only local indirect effects. The values for the IE depend on the magnitude of the gamma coefficient. Due both to local and global indirect effects, the spatial Durbin model (SDM) shows larger spillovers than the two models with a spatially lagged dependent variable (SAR and SAC). The SDM also has larger spillovers than the two models with only spatially lagged independent variables (SLX and SDEM) .

Table 2 shows the results of two approximations using these same models. In the first, there is an increase of ten units on human capital level in region 7, which is

Table 1 Direct effect, indirect effect and total effect for different models

Model	Effects		
	DE	IE	TE
SLM (OLS estimation) (& SEM)	2.00	0.00	2.00
Lag ($\rho = 0.2$) (SAR/SAC)	2.03	0.47	2.50
Lag ($\rho = 0.5$) (SAR/SAC)	2.21	1.79	4.00
SDEM (& SLX)	2.00	1.00	3.00
Spatial Durbin ($\rho = 0.2$) (SDM)	2.09	1.66	3.75
Spatial Durbin ($\rho = 0.5$) (SDM)	2.43	3.57	6.00

located in the periphery of the study area. In the second, this same increase is applied to region 3, more centrally located.

We notice first that total spillovers in the two examples are the same, ten times the TE (the latter shown in the final column of Table 1). Intuitively, it might be expected that spillovers in a central region would be greater than in a peripheral one. However, this is not the result because, for the chosen weight matrix (row normalized 1st-order queen), total spillovers are the same. Other weights matrices will give different results, indicating the importance of choosing a matrix in full consideration of the specific empirical problem being addressed (Corrado and Fingleton 2012; Harris et al. 2011; Piras and Lozano-Gracia 2012). But see the cautions in this regard raised by LeSage and Pace (2011).

Nevertheless, notice how the spillovers are distributed among the regions. For the first simulation, they are mostly localized in regions 6 and 8, which are the 1st-order neighbors of region 7. For the SDEM (& SLX) models, the spillovers are distributed only locally to these regions. In the second simulation, spillovers are more evenly distributed among the regions, because region 3 is located in the central part of the study domain.

6 Illustration with Real Data

This section reviews the same concepts and models discussed above, but now with real data and a reasonably large study domain. This example has two main objectives: (1) to illustrate the effects with real data and a greater number of observations, and (2) to introduce the use of maps as a tool to diagnose and discuss the results.

We apply the different non-spatial and spatial models in this section purely for purposes of spatial effect illustration. In empirical practice such explorations would normally follow standard procedures of model diagnostic examinations and goodness-of-fit comparisons. Moreover, we assume that excessive reliance on such routine specification tests is initially overcome by introducing and motivating the form of spatial dependence based on theoretical considerations (Gibbons and Overman 2012). For instance, in some settings spatial dependence and/or spatial heterogeneity might confidently be theoretically motivated, thus justifying the use

Table 2 Approximations for the regional impact on per capital income due to increase in human capital level in two specific regions

Region	Model					
	SLM (& SEM)	Lag (0.2) (SAR/SAC)	Lag (0.5) (SAR/SAC)	Spatial Durbin error (SDEM/SLX)	Spatial Durbin (0.2)	Spatial Durbin (0.5)
Increase of ten units of human capital in region 7						
1	0.00	0.01	0.18	0.00	0.02	0.36
2	0.00	0.01	0.32	0.00	0.04	0.65
3	0.00	0.11	1.28	0.00	0.39	2.56
4	0.00	0.09	0.89	0.00	0.32	1.79
5	0.00	0.23	1.92	0.00	0.82	3.84
6	0.00	2.13	6.48	5.00	7.44	12.96
7	20.00	20.23	21.79	20.00	20.80	23.59
8	0.00	2.19	7.13	5.00	7.67	14.25
Total	20	25	40	30	37.5	60
Increase of ten units of human capital in region 3						
1	0.00	0.88	2.85	2.00	3.07	5.70
2	0.00	0.96	3.51	2.00	3.36	7.02
3	20.00	20.31	22.65	20.00	21.08	25.30
4	0.00	0.91	3.21	2.00	3.20	6.41
5	0.00	0.86	2.78	2.00	3.00	5.55
6	0.00	0.10	0.95	0.00	0.35	1.89
7	0.00	0.04	0.51	0.00	0.15	1.02
8	0.00	0.94	3.55	2.00	3.29	7.10
Total	20	25	40	30	37.5	60

of a particular spatial model (LeSage and Dominguez 2012). A well-defined theoretical model should optimally be the basis for a specific empirical spatial model. In addition, sometimes the use of a specific spatial model can be justified if there is a large probability of important omitted variables with spatial correlation.

We now turn to our empirical illustrative analysis, which is similar in perspective to the dynamic emanating index, presented in Kelejian and Mukerji (2011), and as the emanating elasticities, discussed in Hondroyiannis et al. (2009). The data refer to 1387 counties in the southeast of the U.S. The dependent variable is the rate of child poverty from the U.S. Census of 2000, defined as the proportion of children under the age of 18 living in a household with income below the official poverty threshold. Figure 4 maps the values and geographic distribution for this variable, previously analyzed in a larger, nationwide, study by Voss et al. (2006). The data display large numerical variation among the counties and strong spatial clustering. Clusters of high poverty levels occur in south Texas, parts of the Mississippi delta and old "cotton belt", as well as large portions of the Appalachian Region. Large regions of low or relatively low poverty separate these high poverty regions.

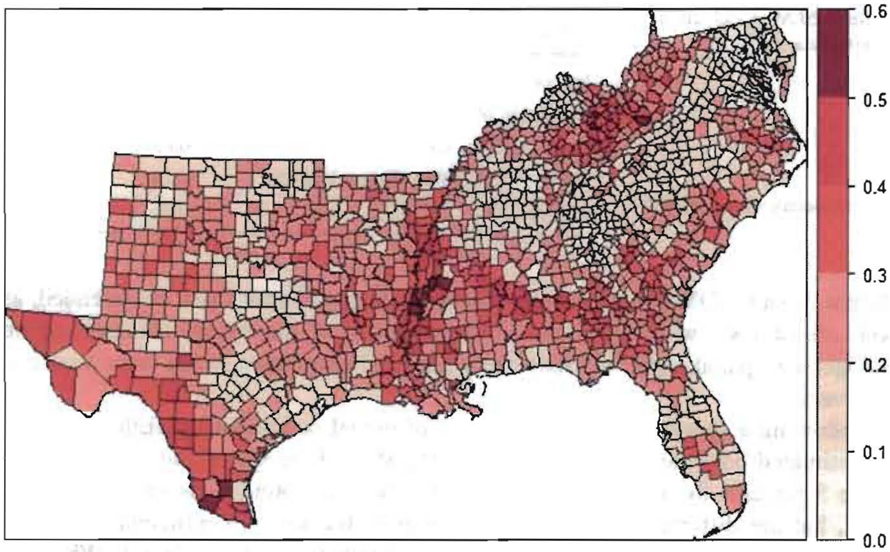

Fig. 4 Child poverty for counties in the southeastern U.S.

For purposes of illustration, we use three explanatory variables: the proportion of female-headed families with children but no spouse, the county unemployment rate (proportion), and the proportion of persons age 18+ with at least a high school education. All variables, including the dependent variable, are transformed to ensure a relatively normal dependent variable and reasonable linearities between the dependent variable and each independent variable.

Table 3 shows the results for the SLM estimated by OLS. In this non-spatial model, all coefficients have the anticipated sign and show strong statistical significance. Higher proportions of female headed families and county unemployment are associated with high poverty rates, as are lower the proportions of persons age 18 + with at least a HS education.

Upon first examination, the simple linear regression model estimated by OLS shows reasonable results. Residual diagnostics, however, suggest the need for a spatial model; they are highly spatially structured, strongly implying non-independent errors—a violation of the OLS model assumption of independence. Figure 5 shows the residuals of the OLS model. Clustering is clearly indicated, with contiguous areas of positive residuals and other contiguous areas with clusters with negative residuals, suggesting strong spatial autocorrelation of the errors. The Moran test on the residuals ($I = 0.451$ for a row standardized 1st-order queen matrix) is highly significant (z score = 28).

Lagrange multiplier diagnostics for spatial dependence in the OLS residuals are summarized in Table 4. All four tests are significant, suggesting the need for a spatial model, but it remains unclear exactly which model to prefer. There is some preference in the Lagrange multiplier diagnostics for a spatial error model (SEM), but the appropriateness of a spatial lag model (SAR) or, more probably, a Spatial

Table 3 SLM model, child poverty data

Variables	Coefficient	SE
Intercept	−0.06	0.011
Proportion of female headed families	0.39	0.024
Unemployment rate	0.83	0.038
Proportion of highly educated people	−0.14	0.006
Adjusted R^2	0.68	

All parameters are significant at 5 %

Durbin model (SDM) or a Kelejien–Prucha SAC model could also be defended, as both robust tests were significant, indicating that if the model incorporates an endogenous spatial lag it still may show a spatial correlation in the error term and vice versa.

Following a specific-to-general strategy of model determination (Elhorst 2010), we estimated both the spatial error model (SEM) and the spatial lag model (SAR). Table 5 presents the results. The parameters show the same signs observed in the SLM, but are different in magnitude. Moreover, the spatial coefficients (λ in the SEM model andρin the SAR model) are both positive and significant. Which of these models should we prefer? Or should we estimate another spatial model?

Figure 6 shows the residuals from the spatial lag model. We observe that this map has a spatial pattern similar to that generated by plotting the OLS residuals; however, we do observe some differences, as positive and negative residuals seems more scattered. Moreover, the residuals for the spatial lag model are somewhat closer to zero, which we intuit by comparing the scales in Figs. 5 and 6.

The next two figures analyze outliers from the SAR (spatial lag) model. Counties with negative residuals are scattered in all parts of the study area, indicating a non-systematic spatial distribution (Fig. 7). However, the outliers with positive residuals are mostly located in Texas (Fig. 8). These results suggest that a SAR model with additional explanatory variables might be considered, for instance a spatial model with a Texas dummy. But there are other alternatives as well. The difference in log likelihoods between the SEM and SAR models (Table 5) suggests a mild preference for the spatial error model, and the map of the SAR residuals implies strongly the inadequacy of a simple spatial lag model. This suggests further investigation of the SDM or SAC alternatives.

Before proceeding to examine these other potential spatial models, we examine for implications of spatial effects in the SAR model. Table 6 presents the results for the direct, indirect and total effects for each of the variables using Eq. (5) and expressions (ai, (b) and (c) from Sect. 3, above. All variables show a considerable IE due to spatial spillovers.

After estimating these effects, a natural question to be answered is: What would be the consequences if we changed the value of an explanatory variable in just one specific county? To illustrate an answer we arbitrarily increased by 20 percentage points (from 19 % to 39 %), the proportion of households with children headed by females in Autauga County, Alabama, and held all other variables constant. Table 7 shows the results for the twenty counties that changed most in their poverty levels due only to the increase in Autauga, all of them not far from Autauga County. In the

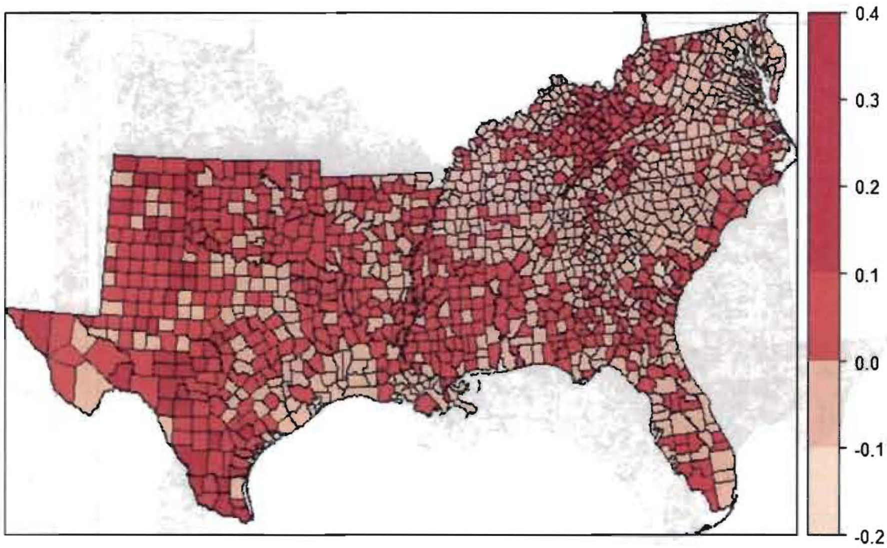

Fig. 5 Residuals of the SLM model

Table 4 Lagrange multiplier tests for the OLS model residuals, child poverty data

Test	Result
LM error	Significant
LM lag	Significant
LM error robust	Significant
LM lag robust	Significant

Table 5 Spatial error model (SEM) and spatial lag model (SAR), child poverty data

Variables	Parameter estimates	
	SEM	SAR
Intercept	−0.05	−0.14
Proportion of female headed families	0.62	0.34
Unemployment rate	0.39	0.52
Proportion of highly educated people	−0.13	−0.11
Rho spatial parameter	–	0.45
Lambda spatial parameter	0.76	–
Log likelihood	2397	2270

All variables were significant at 5 %

first three columns, the table shows the adjustment in the explanatory variables, as described. The table also shows the original values for poverty levels for these counties in the fourth column. For instance, Autauga's original poverty level was 13.7 %. Due to the direct and indirect spatial effects of a change in level only in one independent variable in a single county, there are increases in poverty levels in *all*

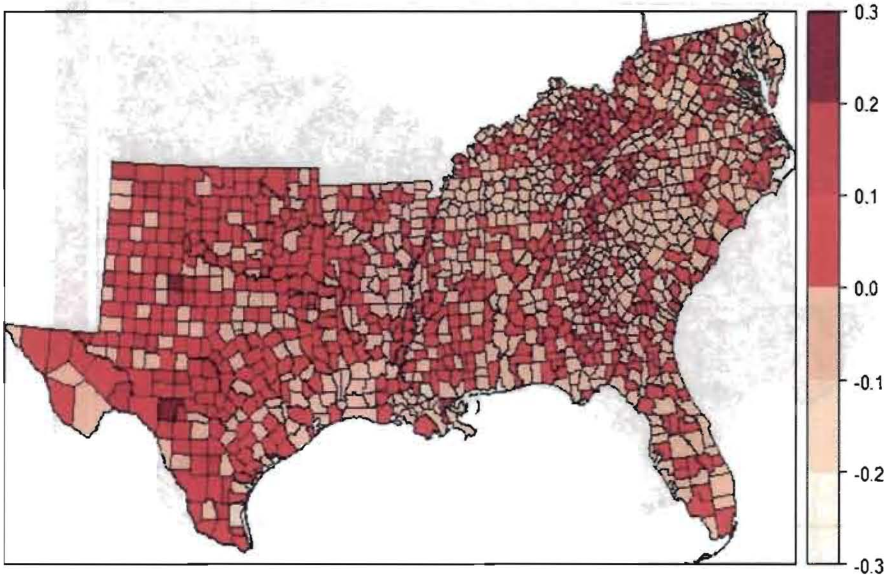

Fig. 6 Residuals for the spatial lag model (SAR), child poverty data

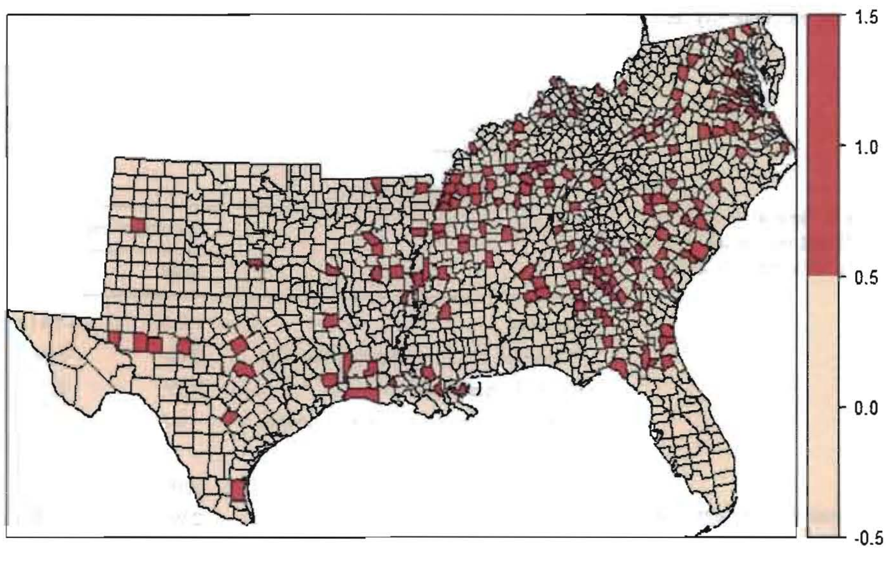

Fig. 7 Outliers with negative residuals of the spatial lag model, child poverty data

counties arising from this change in Autauga County. These changes result from spatial spillovers and, in the case of Autauga, spillover feedbacks. The table shows the updated values for poverty indicating that, in Autauga, poverty would increase to 16.2 %, an increase of 2.50 percentage points. The increase in this county is much

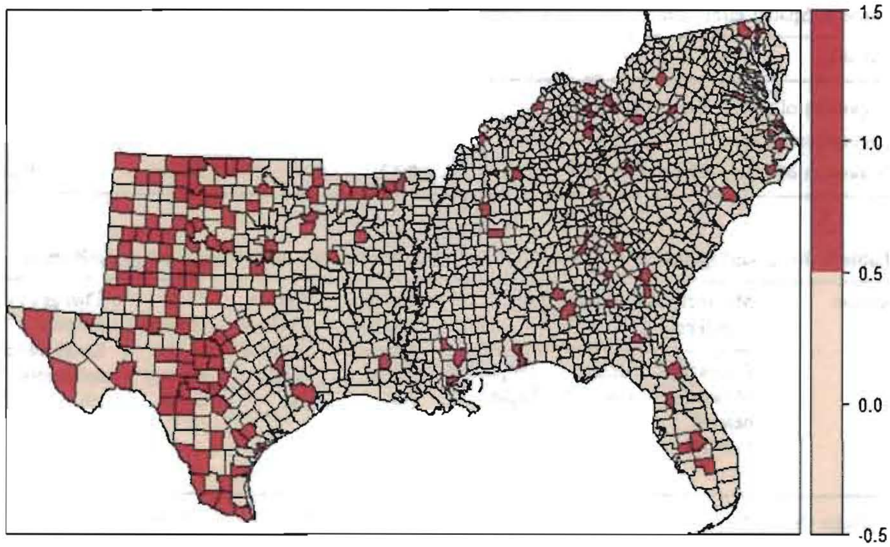

Fig. 8 Outliers with positive residuals of the spatial lag model, child poverty data

larger than in others because of the strong direct effect. For the other 19 counties, changes are all positive, however with much smaller magnitudes, ranging from 0.0208 % for Dallas County (a 1st-order neighbor) to less than 0.0001 % for Coffee County (a 3rd-order neighbor). The final column in Table 7 shows the increase in poverty resulting from spillovers, and Fig. 9 graphs these results. For Autauga we observe an increase of 0.0038 % due to feedback spillovers. Notice that this value is not the largest in this column of the table. Five other counties had values above this. These five are the immediate 1st-order neighbors of Autauga County. The next twelve regions are the 2nd-order neighbors (i.e., neighbors of the neighbors) and two of the counties (Hale and Coffee) are 3rd-order neighbors.

To show the complete results for the 1387 counties in a table like Table 7 would be tiresome and not particularly illuminating. However, a useful way to study results such as these is using maps. Figure 9 shows the results for total changes in poverty levels for all counties in the southeast U.S. arising from the simulated change in one independent variable in Autauga County. The highest value, as discussed above, is for Autauga County due to the combined direct and indirect (feedback) effects of 2.50 %. In the figure below, Autauga County is located in the middle of the darker area, the only county with a value above 0.03 %. For the five immediate neighbors to Autauga County (shared boundaries) we observe increases in poverty values between 0.015 and 0.021 %, a result of spatial spillover from the simulated increase in female headed households in Autauga County. The 2nd- and 3rd-order neighbors show smaller increases in poverty. For counties further away and not shown in Table 7 or Fig. 9, the values are even smaller, and tend toward zero with increase in distance from Autauga County (Fig. 10). As anticipated, spillovers are clustered

Table 6 Spatial direct effects, indirect effects and total effects: SAR model

Variable	DE	IE	TE
Proportion of female headed families	0.36	0.26	0.62
Unemployment rate	0.54	0.40	0.94
Proportion of highly educated people	−0.12	−0.09	−0.20

Table 7 Estimated spillovers due to increase in household structure in Autauga County: SAR model

Region	Modifications in explanatory variables			Poverty levels—Original values (%)	Poverty levels—Updated values (%)	Total changes in poverty levels (%)	Changes in poverty levels due to spillovers (%)
	Proportion of female-headed families (%)	Unemp. rate (%)	Proportion higher education (%)				
Autauga Co.	20	0	0	13.720	16.216	2.4962	0.0038
Dallas Co.	0	0	0	40.977	40.998	0.0208	0.0208
Lowndes Co.	0	0	0	41.801	41.822	0.0208	0.0208
Elmore Co.	0	0	0	14.352	14.372	0.0203	0.0203
Chilton Co.	0	0	0	19.867	19.883	0.0158	0.0158
Montgomery Co.	0	0	0	25.269	25.284	0.0157	0.0157
Perry Co.	0	0	0	49.236	49.237	0.0007	0.0007
Wilcox Co.	0	0	0	48.503	48.504	0.0006	0.0006
Crenshaw Co.	0	0	0	28.737	28.738	0.0005	0.0005
Macon Co.	0	0	0	44.130	44.130	0.0005	0.0005
Coosa Co.	0	0	0	19.548	19.548	0.0005	0.0005
Butler Co.	0	0	0	31.607	31.608	0.0002	0.0002
Bullock Co.	0	0	0	44.998	44.999	0.0002	0.0002
Bibb Co.	0	0	0	28.109	28.109	0.0002	0.0002
Pike Co.	0	0	0	30.016	30.016	0.0002	0.0002
Tallapoosa Co.	0	0	0	24.610	24.610	0.0002	0.0002
Shelby Co.	0	0	0	7.384	7.384	0.0002	0.0002
Marengo Co.	0	0	0	33.861	33.861	0.0001	0.0001
Hale Co.	0	0	0	34.142	34.142	0.0000	0.0000
Coffee erte.	0	0	0	22.535	22.535	0.0000	0.0000
…	…	…	…	…	…	…	…

around Autauga, and the impact diminishes rather quickly with distance due to the effect of higher powers of ρ and the elements of W (both less than 1).

Figure 11 shows only spatial spillovers, the final column of Table 7. For Autauga County, the direct effect has been removed, leaving only the feedback effect

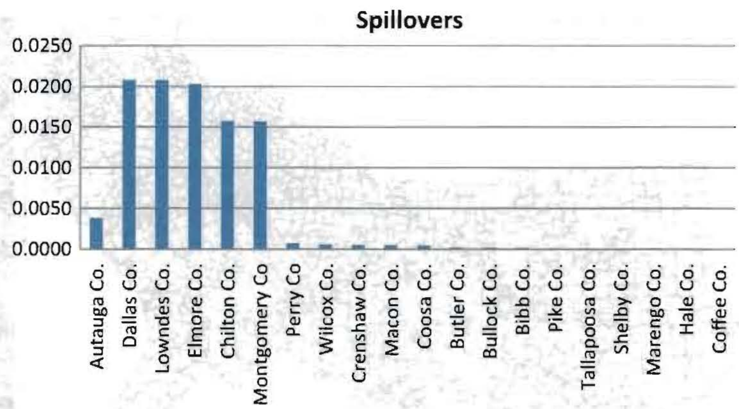

Fig. 9 Poverty spillover effects from an arbitrary increase of 20 percentage points in household structure in Autauga County

resulting from spatial spillovers in the county system. The largest spillovers are for the immediate neighbors of Autauga.

After discussing the results for the spatial lag (SAR) model in Sect. 3, we proceeded in Sects. 4 and 5 to elaborate the SAC, SDEM and SDM models. For the southeastern counties dataset, and the selected variables, the LM robust tests indicate that possibly a preferred model is one of these three, so we examine these models here.

Table 8 shows the results. The Kelejian–Prucha (SAC) model produces parameter estimates for the covariates with the same sign as the OLS, SEM and SAR models. However, they are closer in numerical value to the parameter estimates from the SEM model. Regarding the spatial coefficients, only one, λ, is positive, while ρ is negative (after controlling for error dependence). This, once again, may indicate that the spatial error model (SEM) is the preferred model for these data and this particular specification.

Negative values for spatial parameters occasionally are observed in empirical studies (e.g., in studies of competition between firms in a specific urban center, or in the competition for natural resources between species). An interesting illustration from the social sciences appears in Tolnay et al. (1996). However, they are much less commonly found than positive correlation (Griffith and Arbia 2010), and therefore merit careful scrutiny from a theoretical perspective.

We interpret the negative sign for the estimated ρ parameter in the SAC model to represent residual spatial dependence *after accounting for* the overwhelmingly dominant spatial patterning in these data (positive spatial dependence) reflected in the estimated λ parameter. There are several small regions in the dataset where the central county of a metropolitan area has a very different level of poverty (higher) than neighboring suburban counties (lower). Washington DC (here treated as a county) is a recognizable example, but the pattern is repeated in metropolitan areas throughout the south. The λ parameter, having accounted for the dominant positive

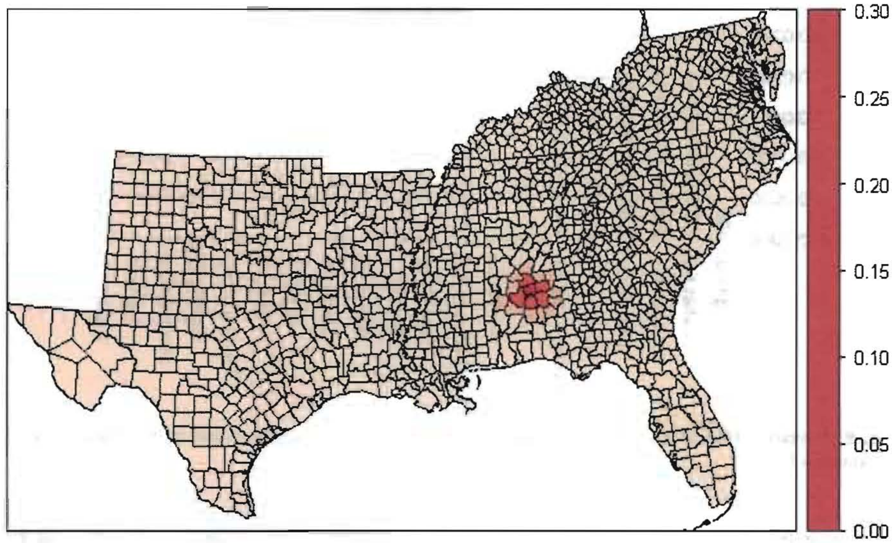

Fig. 10 Total changes in poverty levels for the spatial lag model, child poverty data

spatial dependence in the data permits the ρ parameter to reflect the remaining, less common, negative spatial dependence found predominantly in metropolitan areas.

The SDEM and SEM models present quite similar results for the main explanatory variables and also for the estimated lambda parameter (compare Tables 5, 8). The similarity results from the fact that the two models differ in specification only by the inclusion of the lagged independent variables in the SDEM. For these lagged exogenous variables in the SDEM model, we observe that two are negative (spatially lagged proportion of female headed families and lagged proportion of highly educated people) and one positive (lagged unemployment rate). A likelihood ratio test reveals the SDEM to be significantly superior to the SEM, suggesting against the more simple SEM model.

The SEM, SDM and SAC models all show similar values for the parameter estimates of the main explanatory variables. In the SDM, however, parameter estimates for the three lagged independent variables are either non-significant (lagged unemployment rate), or have signs contrary to the respective non-lagged explanatory variable. This is the expected outcome for the SDM and permits us to carry out a common factors test for the suitability of the spatial error (SEM) model (Anselin 1988). The estimated γ parameters (for the lagged exogenous variables) are sufficiently different from the (negative) product of ρ and β, again suggesting against the simpler SEM. A likelihood ratio test shows the SDM to be significantly superior to the SEM and to the SAR (Table 5). This same result is observed for the SAC model, although this latter model showed a lower likelihood than the SDM (Table 8).

Given these findings, we continue our discussion with the direct, indirect and total effects of the spatial Durbin model (SDM). Table 9 presents the results. Here

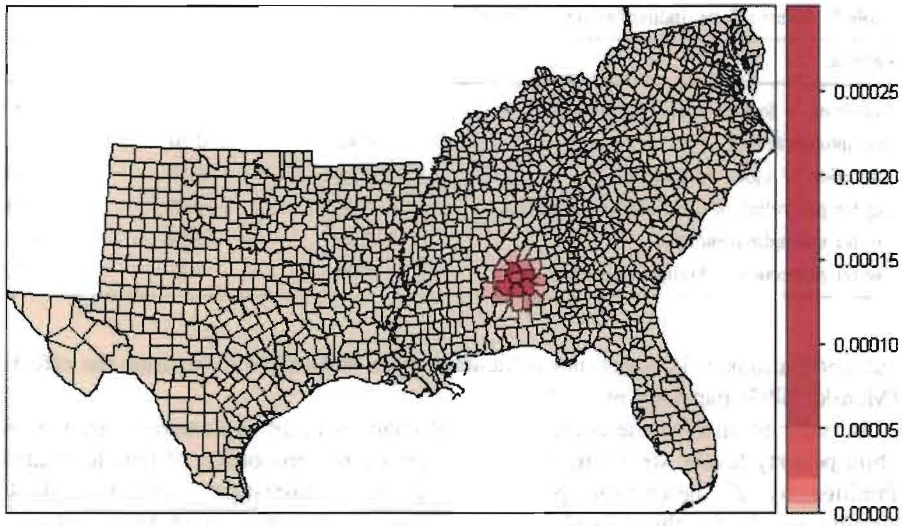

Fig. 11 Changes in poverty levels due to spillovers for the spatial lag model, child poverty data

Table 8 Kelejian–Prucha (SAC), Spatial Durbin Error (SDEM) and Spatial Durbin (SDM) models, child poverty data

Variables	Coefficients		
	Kelejian–Prucha model (SAC)	Spatial Durbin error model (SDEM)	Spatial Durbin model (SDM)
Intercept	0.04*	−0.08	−0.03
Proportion of female headed families	0.61	0.58	0.62
Unemployment rate	0.36	0.46	0.37
Proportion of highly educated people	−0.13	−0.13	−0.13
Lag for proportion of female headed families	–	−0.29	−0.59
Lag for unemployment rate	–	0.56	0.10*
Lag for proportion of highly educated people	–	−0.03	0.07
Rho	−0.17	–	0.67
Lambda	0.83	0.68	–
Log likelihood	2401	2428	2439

All variables are significant at 5 when not specified

* Not significant

the signs for the main effects for the explanatory variables and for the lagged variables are mostly different, as anticipated. This may result from an observation in the econometric literature that when a model includes endogenous and exogenous

Table 9 Direct effects, indirect effects and total effects for the SDM model, child poverty data

Variable	DE	IE	TE
Proportion of female headed families	0.70	1.17	1.87
Unemployment rate	0.42	0.70	1.12
Proportion of highly educated people	−0.14	−0.23	−0.38
Lag for proportion of female headed families	−0.66	−1.09	−1.76
Lag for unemployment rate	0.11	0.19	0.31
Lag for proportion of highly educated people	0.08	0.13	0.22

autocorrelations, it is sometimes difficult or even impossible to separate the effects (Manski 1993; Partridge et al. 2012).

In order to analyze the consequences of changes in an explanatory variable on child poverty levels, we again arbitrarily increase the proportion of female headed families by 20 percentage points but, in this illustration using the SDM, simultaneously for three focal counties: Putnam (FL), Bossier (LA) and Brazos (TX). Figure 12 shows the total impact and geographic extent on county poverty levels. The three clusters of increased poverty are easily seen in these subregions, with the central county being most affected overwhelmingly by the direct effect (as discussed above for Autauga County). Poverty also increases in nearby counties due to spatial spillover of the changed household structure variable in the focal counties. Figure 13, showing only the spatial spillovers, clearly reveals these effects, however, using a different scale to make the map more insightful. Notice that for the three counties that are given a simulated increase in the family structure variable, feedback spillovers are highly negative. This arises from the negative influence of the coefficient estimated for the spatially lagged proportion of female headed families (Table 8). The spatial spillover effects in surrounding counties remain positive (darker red in Fig. 13) but the negative spatial feedback to the focal counties makes the spatial Durbin model a particularly interesting model in terms of understanding and properly interpreting indirect spatial effects.

7 Discussion and Conclusion

In this paper we provide numerical and geographical illustrations of spatial spillover effects as richly elaborated in the pathbreaking book by LeSage and Pace (2009) and elsewhere. For students coming to this material with limited background, a clear understanding of these important spatial concepts is important, but often difficult to grasp. Our illustrations begin with fictive numbers on a simplistic study area consisting of three spatial regions. Spatial effects are shown for seven commonly discussed spatial models in the spatial econometric literature: (1) the standard linear model (no spatial effects), (2) the spatial error model (SEM) with spatially lagged errors, (3) the spatial autoregression (SAR) model with endogenous spatial lags, (4) the Kelejian–Prucha (SAC) model with both endogenous and error lags, (5) the spatially-lagged X model (SLX), (6) the spatial Durbin error model (SDEM), and (7)

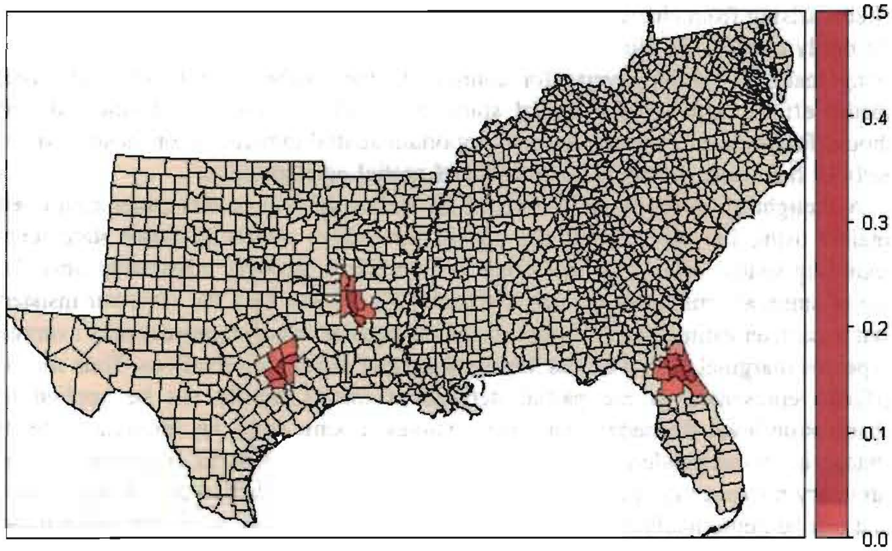

Fig. 12 Total changes in poverty levels for the spatial Durbin model (SDM), child poverty data

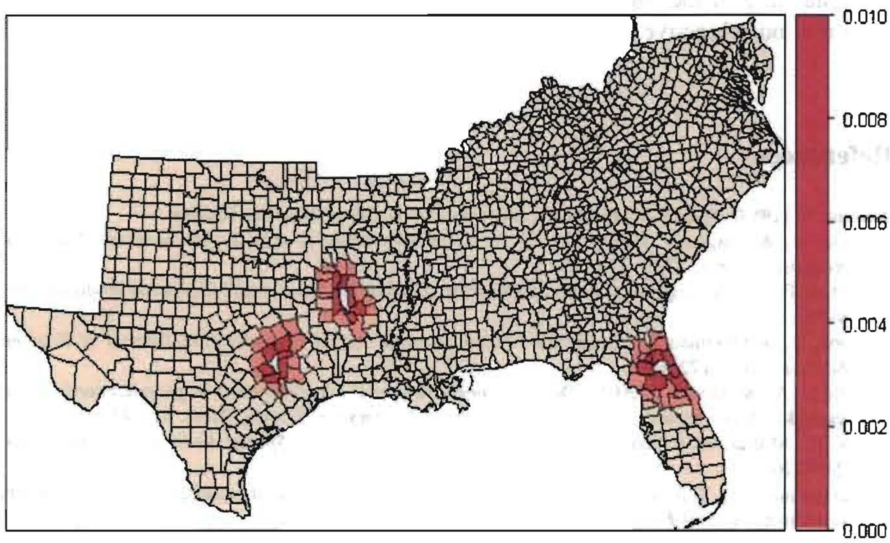

Fig. 13 Changes in poverty levels due only to spatial spillovers for the spatial Durbin mode (SDM), child poverty data. (Color figure online)

the spatial Durbin model (SDM) with both substantive endogenous and exogenous spatial lags. For each of these, we show the mathematics behind the spatial effects and illustrate both the direct and indirect (spillover and feedback) effects. The simple example is expanded to a study area with eight regions to illustrate spatial effects arising from changes in boundary and centrally-located spatial units. Finally, we richly expand the illustrations and apply the methodology to a large study area using real data (child poverty for counties in the southeastern U.S.). The direct spatial effects and indirect spatial spillover effects are clearly documented and should facilitate interpreting of these important spatial influences for those who are early in their understanding of the field of spatial econometrics.

A thoughtful, anonymous, review of an earlier version of this paper cautioned against using the elements of the partial derivatives matrix to make statements regarding spatial effects for individual observations. Strongly advocating only the use of summary measures of spatial effects (DE, IE and TE), the reviewer insisted that parameter estimates in regression models (spatial or not) represent only average expected marginal effects across the entire sample of data observations. Thus spatial effects represented in the partial derivatives matrix should not be applied to observation-level dependent variable changes arising from an observation-level change in an independent. This is a valid critique. We would add, however, that the summary measures of spatial effects represent averages of actual partial derivatives that can be numerically estimated. Our purpose here has been to explain what those partial derivatives are and how they arise. The reviewer conceded that material such as that presented here should be limited to classroom demonstrations of spatial spillovers. If our explanation and derivation of these effects can assist student understanding of the mathematics behind spatial spillovers and spatial effects, we have met our objective.

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