Chapter03: Descriptive Measures

Measures of Central Tendency

- The ***center of a distribution*** can be measured by several statistics: arithmetic mean, median, mode and others.
- The center of a distribution provides us with the ***summary information*** about that location around which the data ***predominately vary***.
  It can be considered as the ***most representative*** data value.
- Each of the central tendency measures has ***advantages*** and ***disadvantages*** depending on the underlying distribution of the data and problem at hand.
- In general, using just a locational summary measure for a full-fledged distribution of the sample is accompanied with substantial ***loss of information*** (e.g., shape of the distribution)

The Mid-Range

- The value halfway between x_{min} and x_{max}: $x_{mid} = \frac{x_{max}+x_{min}}{2}$
- Does not work well for skewed distributions.
- Depends on ***just two*** observations, which are also the most extreme ones. Therefore, it ignores all the other sample data.

The Mode

- The ***most frequently*** observed data value (or histogram bar mid-point) is described by the mode.
  In essence, it is the value that has the ***highest likelihood*** of being observed in our data.
- The mode is a valid statistic for categorical and metric variables.
- For ***multi-modal*** distributions the definition needs to be modified:
  - If several modes are in ***close vicinity*** of each other, then we pick a representative value from this ensemble of modes.
  - For a small number of outstanding modes, we could report locations of all modes as a set of ***representative values***.
  - Recall, multimodality usually hints at the fact that ***distinct underlying mechanisms*** have generated a ***heterogeneous*** set of data.

The Median

- The median is the middle number with 50% of the observations larger than the median and 50% of the observations less than the median.
- The median is only valid for metric data.
- The median is closely related to the order statistic. Data in the order statistic are sorted ascending.

- Odd number of sample observations n:

$$\underbrace{x_{[1]}, x_{[2]}, \dots, x_{\left[\frac{N+1}{2}-1\right]}}_{50\%\ observations \le x_{\left[\frac{N+1}{2}\right]}},\ \overbrace{x_{\left[\frac{N+1}{2}\right]}}^{Median},\ \underbrace{x_{\left[\frac{N+1}{2}+1\right]}, \dots, x_{[N-1]}, x_{[N]}}_{50\%\ observations \ge x_{\left[\frac{N+1}{2}\right]}}$$

- Even number of observations N:

$$\underbrace{x_{[1]}, x_{[2]}, \dots, x_{\left[\frac{N}{2}\right]}}_{50\%\ observations \le x_{\left[\frac{N}{2}\right]}},\ \overbrace{\frac{x_{\left[\frac{N}{2}\right]}+x_{\left[\frac{N}{2}+1\right]}}{2}}^{Artifical\ Median},\ \underbrace{x_{\left[\frac{N}{2}+1\right]}, \dots, x_{[N-1]}, x_{[N]}}_{50\%\ observations \ge x_{\left[\frac{N}{2}+1\right]}}$$

- Some software packages offer alternative definitions for the mid-point of an order sequence of data values.

- The median is also called the 50% quantile, i.e., second quartile.
- The median location is defined as that data point with the rank $[i] = (n+1)/2$.
- The median minimizes the ***absolute distances*** to all data values:

$$\sum_{i=1}^{n} |x_i - X_{median}| < \sum_{i=1}^{n} |x_i - \theta| \text{ for any } \theta \neq X_{median}$$

$$\text{with } |x_i - X_{median}| \equiv \sqrt{(x_i - X_{median})^2}$$

This can be used to find, for instance, the optimal location (on a line) to which the overall distance is the smallest (see Euclidian median definition in BBR, p. 136)

The Arithmetic Mean

- The arithmetic mean is commonly known as just the mean.
- This statistic denoted by $\bar{X}$ and calculated by $\bar{X} = \frac{1}{n} \cdot \sum_{i=1}^{n} x_i$, where n is the number of sample observations.
- The mean is valid only for metric data.
- Assume that we place the data at their proper location on a scale. Data points have identical weights. Then the mean is the pivot point where the scale is in balance:

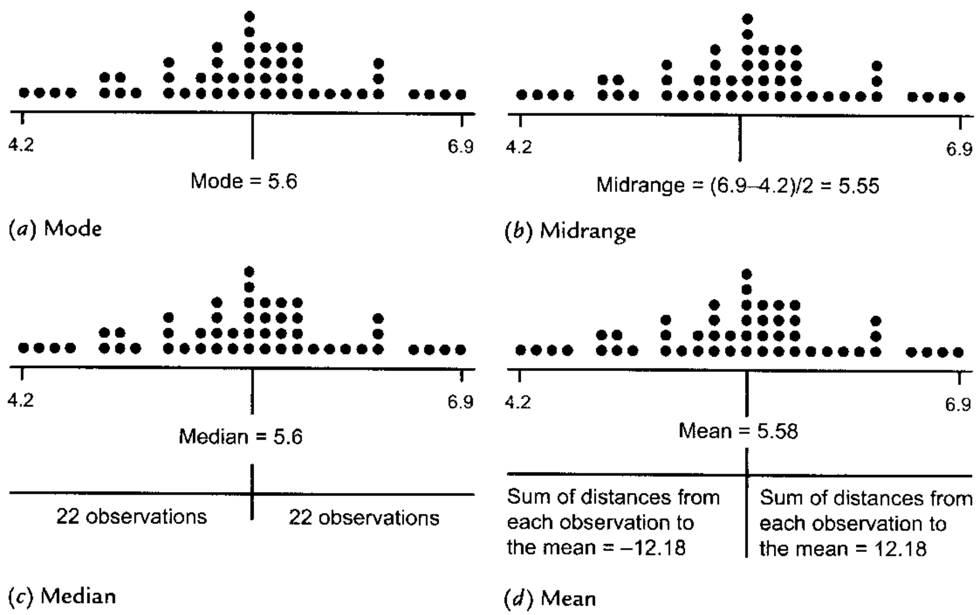


FIGURE 3-2. Measures of central tendency using the DO data.

- Some properties of the mean:

- The ***sum of the differences*** around the mean is zero: $\sum_{i=1}^{n}(x_i - \bar{X}) = 0$.
  To see this property note:

$$\begin{aligned}\sum_{i=1}^{n}(x_i - \bar{X}) &= \sum_{i=1}^{n} x_i - n \cdot \bar{X} \\ &= \sum_{i=1}^{n} x_i - n \cdot \frac{\sum_{i=1}^{n} x_i}{n} \\ &= 0\end{aligned}$$

  Therefore, it is the center of gravity of the data, i.e., the pivot at which the scale is balanced.

- The ***squared deviations*** of the data around the mean is smaller than for any other central tendency statistic (see the math review and animation **OUTLIERANIMATION.MP4**):

$$\sum_{i=1}^{n}(x_i - \bar{X})^2 < \sum_{i=1}^{n}(x_i - \theta)^2 \text{ for any } \theta \neq \bar{X}$$

Comparison of properties of the measures of central tendency

- If a distribution is both symmetric and uni-modal, then the mode, median and mean are all identical.
  This property can be used to test for skewness:
  - For ***positively skewed*** uni-modal distributions we have: $x_{mode} < x_{median} < \bar{x}$

    - For ***negatively skewed*** uni-modal distributions we have: $\bar{x} < x_{median} < x_{mode}$

- Advantages of the ***mode***:
    - easy to calculate
    - can be used for categorical data
- Disadvantage of the ***mode***:
    - The mode may not be representative for the central tendency in the data, if it is located in one tail.
    - For multimodal data its definition is somewhat arbitrary.
- Advantages of the ***median***:
    - Unaffected by extreme observation or skewed data distributions. It is more informative in these cases than the mean. This is why you frequently find the median in reports.
    - Fixed distance units of the measurement scale are not required.
- Disadvantages of the ***median***:
    - Not as stable from sample to sample because it depends on just one data value (for odd sample sizes) or two data values (for even sample sizes).
    - Has disadvantages in mathematical derivations because it is based a discontinuous function. The absolute function is not smooth.
    - Computationally it requires that the data are sorted first, which takes extra time.
- Advantages of the ***mean***:
    - easy to calculate with several useful properties in mathematical statistics.

  - relates well to many parameters of underlying theoretical population distributions.
  - It provides a stable estimate from sample to sample
- Disadvantages of the ***mean***:
  - sensitive to skewed distributions and outliers

Weighted Mean

- The weighted mean is used for data which are aggregated into G homogenous groups. Each representative group value x_g has a weight $w_g \geq 0$ equal to the size of the group:

$$\bar{x}_{weig} = \frac{1}{\sum_{g=1}^{G} w_g} \cdot \sum_{g=1}^{G} w_g \cdot x_g$$

- The arithmetic mean is a special case of the weighted mean by setting $w_g = 1$ and $G = n$.

Trimmed Mean and Winsorized Mean

- The trimmed mean ***discards*** an equal proportion of data (say 5%, 10%, or 20%) from ***both tails*** of the distribution. Subsequently, the mean is taken from the remaining observations.
- If $x_{[1]} \leq x_{[2]} \leq \cdots \leq x_{[n]}$ represents the ordered sample values then the trimmed mean is given by

$$\bar{x}_\tau = \frac{1}{n - 2 \cdot k} \cdot \sum\nolimits_{i=k+1}^{n-k} x_{[k]}$$

where k is the smallest integer greater than or equal to $n \cdot \tau$ and $0 \leq \tau \leq 0.5$.

- Advantage of the ***Trimmed Mean***:
  - It becomes robust against the effects of outliers and skewed distribution while still focusing on the most representative data point.
  - For $\tau = 0$ we get the arithmetic mean $\bar{x}$ and for $\tau \cong 0.5$ we get the median x_{median}.
- Disadvantages of the ***Trimmed Mean***:
  - We disregard a certain percentage of observation from the tails of our sample and, therefore, suppress information.
  - The data must be sorted in order to calculate the trimmed mean.
  - There are no fixed rules on how many observations we would need to trim away from either end of the distribution. Therefore, it is important to first visually inspect the distribution.
- Winsorized mean: Similar to the trimmed mean, however, each of the smallest dropped observations is replaced by $x_{[k+1]}$ and ***vice versa*** for the largest dropped observations by $x_{[n-k]}$.

$$\bar{x}_{\tau}^{W} = \frac{1}{n} \cdot \left((k+1) \cdot \left(x_{[k+1]} + x_{[n-k]} \right) + \sum_{i=k+2}^{n-k-1} x_{[i]} \right)$$

This has the advantage that the sample size is not reduced. The winsorized mean is calculate with the `trim` option in the R function `mean(x, trim = 0, na.rm = FALSE)`.

The Geometric Mean

- The average rate of change r_i per time period is calculated over several successive time period change rates as the ***geometric mean***:

$$r = \sqrt[n]{r_1 \cdot r_2 \cdot \cdots \cdot r_n}$$
$$= \left(\prod_{i=1}^{n} r_i\right)^{\frac{1}{n}}$$
$$= \exp\left(\frac{\sum_{i=1}^{n} \ln(r_i)}{n}\right)$$

  Note, all rates must be positive, i.e., $r_i > 0$ where $r_i = y_i / y_{i-1}$.

- Example: A savings account accumulates variable interest by year over a three years period.

Account Beginning of Year	Account End of Year	Annual Growth Rate
$ 100	$ 115	$\frac{115}{100} = 1.15\%$
$ 115	$ 110	$\frac{110}{115} \cong 0.96\%$
$ 110	120	$\frac{120}{110} \cong 1.09\%$

The average annual growth rate is $\sqrt[3]{\frac{120}{100}} \cong 1.063\%$ or $\sqrt[3]{1.15\% \cdot 0.96\% \cdot 1.09\%} \cong 1.063\%$

The Harmonic Mean

- The harmonic mean is used to calculate the average for rate variables with a fixed common denominator, that is, $x_i\ miles/one\ hour$ or $x_i\ miles/one\ gallon$, where each measured value refers to the ***same distance*** segment travelled.
- The harmonic mean is

$$\bar{x}_{harm} = \frac{n}{\sum_{i=1}^{n} \frac{1}{x_i}}.$$

- For instance:

  [a] if a vehicle travels the ***first distance*** at $60\ \ miles/h$ and an ***identical second distance*** at $40\ \ miles/h$, then it would have covered both equal distances at $\frac{2}{\frac{1}{60}+\frac{1}{40}} = 48\ \ miles/h$;

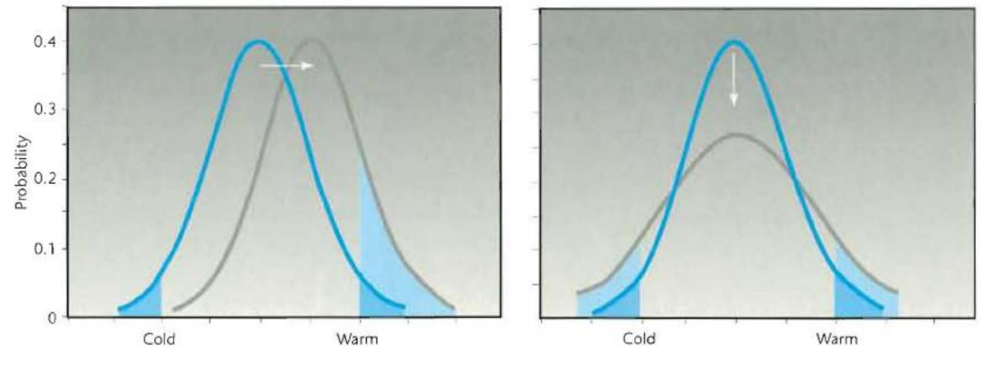
  [b] on the other hand, if a vehicle would have travelled two ***equal time intervals*** at $60\ \ miles/h$ and at $40\ \ miles/h$, respectively, then the average speed of $50\ \ miles/h$ would have covered the same distance.

Measures of Variability

Introduction

- The central tendency gives us information of the approximate location of a distribution. However, it does ***not*** provide any ***information*** about its spread.
- All measures of spread discussed here assume that we have a variable on the ***metric measurement scale***.
- The ***spread*** (also called ***dispersion*** or ***variability***) is the degree to which all individual data points are ***jointly distributed around*** the central tendency.
  It provides an additional ***summary*** measure of the underlying distribution of our data.
- The spread can range from anywhere between ***highly clustered*** around the central tendency to ***extremely dispersed*** around the central tendency
- The ***smaller the spread*** the more ***representative*** the central tendency is for observed data points.
- Small spread translated into high accuracy as long as the central tendency is unbiased (lacks a systematic error). Recall BBR pp. 29-31.
- Example: The climate change debate not is not only concerned with an overall increase in temperature, it is also apprehensive about an increase in the variability of the temperature. This implies that a small shift in temperature in combination with a higher variability will

increase the risk for extreme weather events:



Overall Range

- The range just depends just on the ***two most extreme*** observations:
  $\text{range}(X) = \max(X) - \min(x) = x_{[n]} - x_{[1]}$.
- It is, therefore, highly sensitive to ***sample variations*** and ***outliers***.

Interquartile Range

- To calculate the interquartile range, 25% of the observations at the bottom and the top of a distribution are discarded.
- The range of the remaining ***50% of the inner observations*** defines the interquartile range $=$ $x_{[75\%]} - x_{[25\%]}$.
- Trimming away 50% of the data in the distribution's tail may discard much information and, therefore, may not represent the variability well.
  It has been suggested to work with statistics with less than 50% trim.

Variation around the Mean: Variance and Standard Deviation

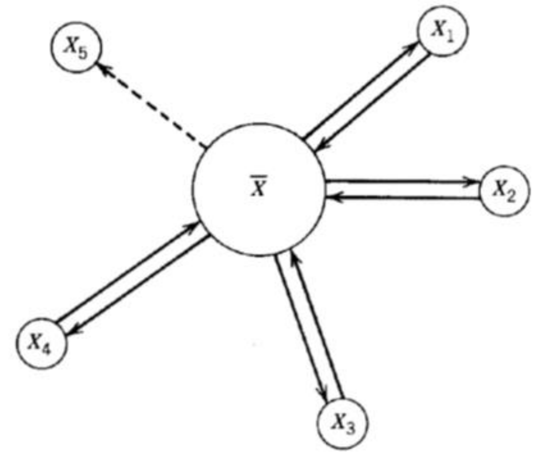
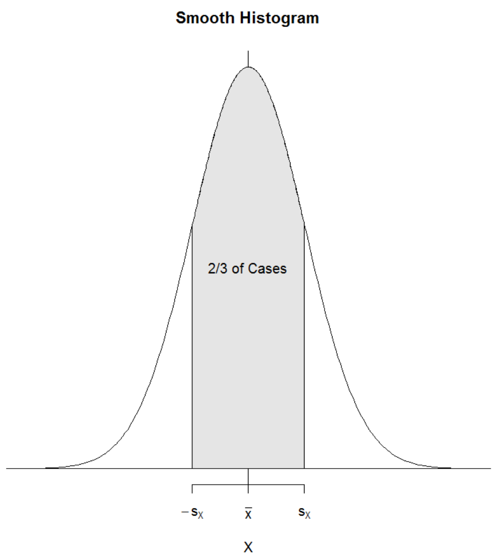


FIGURE 2.29 Diagrammatic representation of the calculation of mean and variance from five observations. Mean $\bar{X}$ is calculated from all observations. Variance is calculated from differences between observations and the mean. When four differences have been found, the fifth difference is known.

- If we would have all N observations from the underlying population of a random variable X with the population mean μ_X then the ***population variance*** σ_X^2 is calculated by $\sigma_X^2 = \frac{1}{N} \cdot \sum_{i=1}^{N}(x_i - \mu)^2$.
- In contrast, for a given sample with an estimated mean $\bar{x}$ and n sample observations the ***estimated variance*** s_X^2 is calculated by $s_X^2 = \frac{1}{n-1} \cdot \sum_{i=1}^{n}(x_i - \bar{x})^2$.
  Note the use of $n-1$ in the denominator and the estimated mean $\bar{x}$ in the squared deviations.
- Once we know the mean $\bar{x}$ and $n-1$ observations, the last observation can be calculated due to the ***zero-sum restriction***: $x_n = \sum_{i=1}^{n-1} x_i - n \cdot \bar{x}$. This is known as a loss of ***degrees of freedom***.
- The ***measurement unit*** of the variance is in terms of ***squared units*** of the original variable X. Squared units are difficult to interpret; therefore, by taking the square root of the variance we get a measurement unit that matches that of our original variable.
- The square root of the variance is called ***standard deviation***: $\sigma_X = \sqrt{\sigma_X^2}$ for the population and $s_X = \sqrt{s_X^2}$ for the sample, respectively.



- For approximately ***symmetric*** distributions with a noticeable tendency to ***cluster around their mean***, roughly $2/3$ of the observations will be not more than one standard deviation to the left and right away from the mean.
- However, this rule of thumb will not hold for highly skewed distributions or multimodal distributions.

Variation around the Mean: Mean Absolute Deviation

- Recall, we have seen for the mean that $\sum_{i=1}^{n}(x_i - \bar{x}) = 0$.
- Consequently, positive and negative variations around the mean ***cancel out***,
- To overcome this problem the ***mean of absolute differences*** could be used: $MAD = \frac{1}{n} \cdot \sum_{i=1}^{n}|x_i - \bar{x}|$.
- MAD is in the same ***measurement unit*** as the original variable X.
- Remember, if the median x_{median} is used instead of the mean $\bar{x}$ then $\sum_{i=1}^{n}|x_i - x_{median}|$ minimizes the sum of the absolute distances between all x_is and the median.

Properties of the Mean and Variance Estimation Rules

- ***Estimation rules*** are briefly called ***estimators*** in the statistical literature.
- The sample mean $\bar{x}$ is an estimator for the population mean μ and the sample variance s_X^2 is an estimator for the population variance σ_X^2.

- A estimator preferably should satisfy specific criteria:
  - Unbiased: On average over many different samples from the same population, estimation rule should give values ***equal*** to their associated ***true population parameter***.
  - Efficient: ***Average deviation*** of a sample estimator around the true population parameter should be as ***small*** as possible (will be addressed in a later chapter).
- The concept of ***biasedness*** is somewhat academic because, in order to assess a potential bias of an estimator, we would need to know the true population parameter.
- However, one can show with mathematical arguments that some estimation rules will lead to biased estimates, whereas another estimation rule will not be biased.
- The average of an estimator over many independent samples of the same length from a population is called in statistical terms the ***expected value*** $E[\,]$

  The expected value for the estimation rule $\frac{\sum_{i=1}^{n}(x_i-\bar{x})^2}{n}$ for the spread is

$$E\left[\frac{\sum_{i=1}^{n}(x_i-\bar{x})^2}{n}\right] = \underbrace{\frac{(n-1)}{n}}_{<1}\cdot\sigma_X^2 \text{ whereas } E\left[\frac{\sum_{i=1}^{n}(x_i-\bar{x})^2}{n-1}\right] = \sigma_X^2.$$

  Therefore, $\boldsymbol{\frac{1}{n-1}}\cdot\sum_{i=1}^{n}(x_i-\bar{x})^2$ is an unbiased estimation rule for σ_X^2 whereas $\boldsymbol{\frac{1}{n}}\cdot\sum_{i=1}^{n}(x_i-\bar{x})^2$ slightly under-estimates the variance .
- Two ***heuristic explanations*** for the use of $n-1$ in the denominator can be employed:

(i) Because the estimator $\bar{x}$ ***minimizes the sum of the squared deviations***, that is, $\min_\theta \sum_{i=1}^{n}(x_i-\theta)^2$, for the sample observations,
we always have $\sum_{i=1}^{n}(x_i-\bar{x})^2 \leq \sum_{i=1}^{n}(x_i-\mu_X)^2$ (compared to the true population expectation μ_X).
To correct for this ***shrinkage*** in the numerator of the sample variance estimator, the denominator must also be slightly reduced from n to $n-1$ in order to avoid an under-estimation of the variance.

(ii) Due to ***zero-sum restriction*** in $\sum_{i=1}^{n}(x_i-\bar{x})=0$ one observation cannot vary freely anymore. We therefore loose one ***degree of freedom*** (see p 13 of the notes).

- For a ***large*** number of sample observations the bias becomes negligibly small because $n \approx n-1$ and therefore $\lim_{n\to\infty}\frac{n-1}{n} \approx 1$.

Problem of the Trimmed Variance Estimator

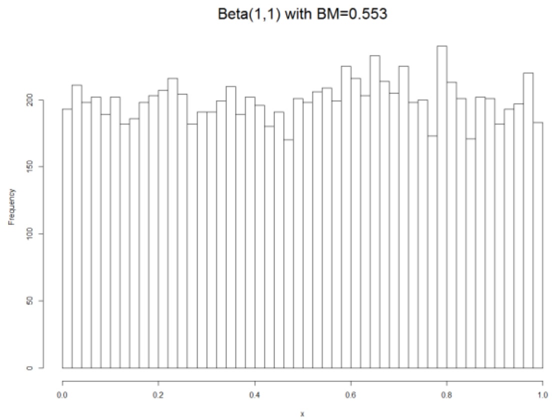
- While the trimmed mean $\bar{x}_\tau$ and the winsorized mean $\bar{x}_\tau^W$ will not differ much, the trimmed variance substantially underestimates the spread, because it ignores any tail observations.
- A winsorized statistic is similar to a trimmed statistic; however, the number of dropped observations at the bottom and the top of the distribution are ***padded*** by the smallest and largest remaining observations in the sample.
- The padding with smallest and largest feasible values in the winsorized variance adjusts for this underestimation by including valid but extreme observations.

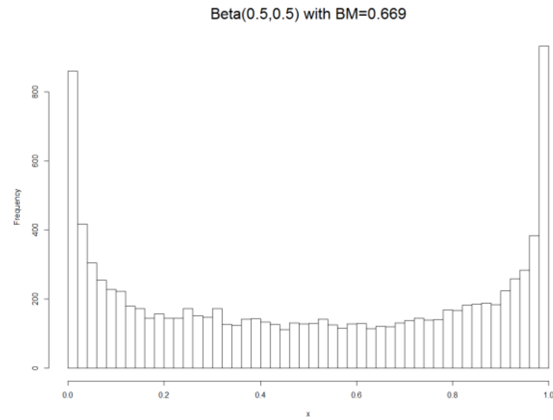
Other Measures of Spread and Shape of Distributions

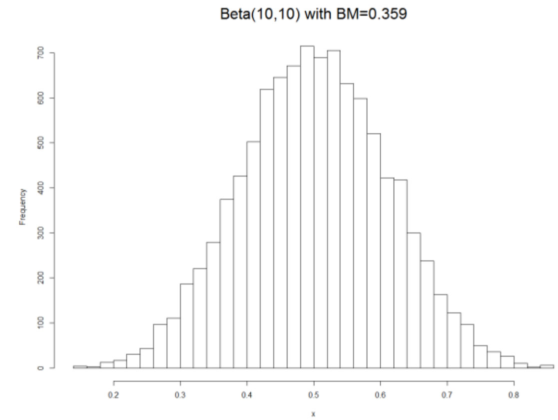
- Third moment measure: A measure for the ***skewness*** is $\text{skew}(X) = \frac{\sum_{i=1}^{n}(x_i-\bar{x})^3}{\left(\sum_{i=1}^{n}(x_i-\bar{x})^2\right)^{\frac{3}{2}}}$, which is zero for symmetric distributions and positive or negative for positively or negatively skewed distributions, respectively.
  Underlying structure of the equation:
  - The third power (and any odd power) in the numerator preserves the sign of the deviation around the mean. Therefore, the sign of overall sum determine the direction of skewness.
  - Outlying observations substantially inflate the numerator due to the third power.
  - The denominator expression $\left(\sum_{i=1}^{n}(x_i-\bar{x})^2\right)^{\frac{3}{2}}$ standardizes the numerator so that the skewness estimator becomes independent of the variance.
    In essence, this is the numerator of the variance estimator taken to the power of $3/2$.
- An alternative measure of skewness is Pearson's $\gamma = \frac{3\cdot(\bar{x}-x_{median})}{s}$
- Fourth moment measure: A measure for the ***kurtosis*** is $kurt(X) = \frac{\sum_{i=1}^{n}(x_i-\bar{x})^4}{\left(\sum_{i=1}^{n}(x_i-\bar{x})^2\right)^{\frac{4}{2}}} - 3$. It follow the same construction principle as the skewness, however, it uses the fourth power.
  - In order to make the kurtosis for the normal distribution (used frequently as reference distribution) equal to zero the value 3 is subtracted.
  - For an even power of 4 all deviation around the mean are positive.

- Compared to a normal distribution, if a distribution has more mass in the tails than in the center, then the kurtosis becomes larger than 0,
  and if it has more mass in the center relative to the tails, then it becomes less than 0.
- The denominator $\left(\sum_{i=1}^{n}(x_i-\bar{x})^2\right)^{\frac{4}{2}}$ again standardizes the numerator, however, now the power $4/2$ is used.
- The kurtosis becomes ***difficult to interpret for skewed distributions***, because one tail is heavy whereas the other is short.
- Note, there are difference software implementations for the skewness and kurtosis estimators. See the online help for the functions **`e1071::skewness( )`** and **`e1071::kurtosis( )`** in the library **`e1071`** and their options.

- <u>The Bimodality Index</u>
  - The bimodality index is defined as $\mathrm{BM}=(\mathrm{skewness}^2+1)/\left[\mathrm{kurtosis}+\frac{3\cdot(\mathrm{n}-1)^2}{(\mathrm{n}-2)\cdot(\mathrm{n}-3)}\right]$.
  - This is implemented in the user-specified function **`BiModalityIndex( )`** in the R-code **`Chapter03SampleCode.r`**.
  - It works best for nearly symmetric but potentially multimodal distributions.
  - For a ***uniform*** distribution its value is ≈ 0.55.
  - For ***bimodal*** or multimodal distributions this index will be larger than 0.55.
  - For a ***unimodal*** distribution this index is less than 0.55.
  - Some examples using a ***beta***-distribution:







- Variation relative to the mean:
    - If two distributions have different means and a positive support (all observations x_i must be for theoretical reasons $x_i \geq 0$) then the standard deviations may not be directly comparable:
    - A larger mean allows for more spread toward the zero-origin point of the distribution.
    - To adjust for the induced potential extra variability, the ***coefficient of variation*** is used:

$$CV = \frac{s}{\bar{x}} \; \forall x_i \geq 0 \,.$$