Residual Spatial Autocorrelation Test with Moran's *I*

Outline:

- What is spatial autocorrelation
- Definition of the link matrix
- Why autocorrelation in regression residuals?
- Moran’s scatterplot
- General structure of test statistic and its distribution

What is Spatial Autocorrelation

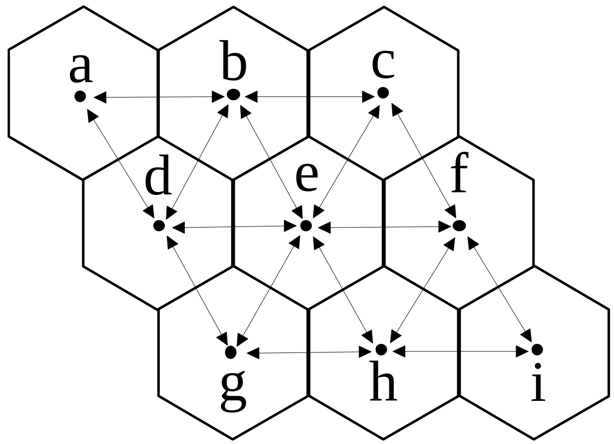
- The concept of an ***internal spatial relationship*** (order) is trickier than in the time-series situation, where the order comes naturally (the future depends on the past)
- In spatial analysis, the relationships among the observations are [a] ***multidirectional***, [b] ***multilateral*** and [c] ***not equally spaced***.
  There are also more observations at the ***edge of the study area*** and factors like ***spatial extend of the regions*** and ***underlying densities*** become important.
- Geographical theories provide many ***concepts of spatial relationships*** within a single variable. For instance:
  - simple distances between points or representative points of areas

- neighborhood relationships between areas (rook's or queen's specification in square tessellations, higher order neighborhood relationship of regions several neighbors apart)
- traffic flows or migration patterns
- spatial hierarchies (hub and spokes)
- diffusion processes etc.

The Spatial Link Matrix

- The spatial connectivity matrix operationalizes the underlying structure of the potential spatial relationships among the observations
- For potential distance relationships we have the distance matrix (known from road atlases, perhaps using spherical distances)
- For potential neighborhood relationships we must use a binary spatial connectivity matrix

  Example: Encoding a spatial tessellation as a binary connectivity matrix



Spatial tessellation of 9 hexagonal cells with underlying connectivity structure

$$\Leftrightarrow \quad \begin{array}{c} \\ a \\ b \\ c \\ d \\ e \\ f \\ g \\ h \\ i \end{array} \begin{array}{c} \begin{array}{ccccccccc} a & b & c & d & e & f & g & h & i \end{array} \\ \begin{pmatrix} \mathbf{0} & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & \mathbf{0} & 1 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & \mathbf{0} & 0 & 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & \mathbf{0} & 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 1 & \mathbf{0} & 1 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 & \mathbf{0} & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 & 0 & \mathbf{0} & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 1 & \mathbf{0} & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & \mathbf{0} \end{pmatrix} \end{array}$$

Binary 9×9 *spatial connectivity matrix*
B

- An element $b_{ij}=1$ denotes that the tiles *i* and *j* are ***adjacent*** and an element $b_{ij}=0$ signifies that the tiles *i* and *j* are not common neighbors
- The connectivity matrix $\mathbf{B}$ has $n\times n$ elements and it is ***symmetric***
- An area is not connected to itself. Thus all ***diagonal*** elements are zero
- For study areas with a ***large number*** of regions the generation of the connectivity matrix $\mathbf{B}$ (or distance matrix) must be left to a GIS program.
- Problems occur if we have island and holes in our study area. Usually machine generated connectivity matrices must be polished manually.

- Most of the elements are zero. There are efficient storage modes for sparse matrices. For empirical map patterns an area in the interior has on average 6 neighbors.
- The binary spatial link matrices $\mathbf{B}$ are usually coded: For instance, if each row is divided by its row sum n_i we get the row-sum standardized link matrix:

$$\mathbf{W} = \begin{pmatrix} \mathbf{0} & 1/2 & 0 & 1/2 & 0 & 0 & 0 & 0 & 0 \\ 1/4 & \mathbf{0} & 1/4 & 1/4 & 1/4 & 0 & 0 & 0 & 0 \\ 0 & 1/3 & \mathbf{0} & 0 & 1/3 & 1/3 & 0 & 0 & 0 \\ 1/4 & 1/4 & 0 & \mathbf{0} & 1/4 & 0 & 1/4 & 0 & 0 \\ 0 & 1/6 & 1/6 & 1/6 & \mathbf{0} & 1/6 & 1/6 & 1/6 & 0 \\ 0 & 0 & 1/4 & 0 & 1/4 & \mathbf{0} & 0 & 1/4 & 1/4 \\ 0 & 0 & 0 & 1/3 & 1/3 & 0 & \mathbf{0} & 1/3 & 0 \\ 0 & 0 & 0 & 0 & 1/4 & 1/4 & 1/4 & \mathbf{0} & 1/4 \\ 0 & 0 & 0 & 0 & 0 & 1/2 & 0 & 1/2 & \mathbf{0} \end{pmatrix}$$

Why do we observe autocorrelation in regression residuals?

- Regression residuals $\mathbf{e}$ capture the unexplained part of the regression model
- (1) Misspecification Rational: if we are missing relevant variables in the set of independent variables $\mathbf{X}$, which exhibit a spatial pattern, then their spatial pattern will spill over into the regression residuals $\mathbf{e}$.
- (2) Spatial Process Rational: The spatial object exhibit some ***spatial exchange relationships***, e.g.,
    - interaction flows,

- competition effects or
- agglomerative advantages

then the regression residuals $\mathbf{e}$ will become spatially autocorrelated.
These exchange relationships among the observations cannot be captured by the independent variables, but manifest in a ***covariance/correlation*** matrix $\mathbf{\Omega}(\rho)$ among the observations.
The parameter ρ is the ***autocorrelation level*** and measuring the strength of the spatial process. For

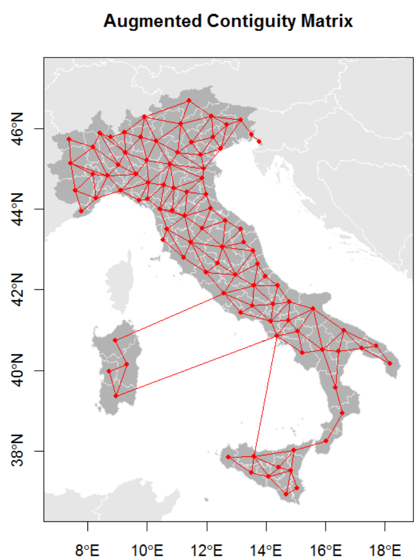
$$\rho = \begin{cases} > 0 & \text{postive autocorrelation} \\ \;\; 0 & \text{spatial independence} \\ < 0 & \text{negative autocorrelation} \end{cases}$$

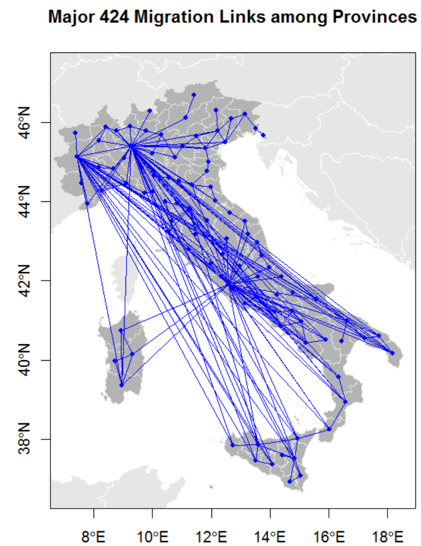
All these processes $\mathbf{\Omega}(\rho)$ dependent on the ***coded*** spatial link matrix $\mathbf{W}$ among the spatial objects.

- <u>(3) Spatial Scale/Aggregation Rational:</u> If areal objects are split into parts and these split parts are merged with adjacent areal objects then these aggregated objects share parts of the information, which they inherited from the split objects.
  This induces spatial autocorrelation among adjacent spatial objects.
  Implication: One needs to get the spatial scale of analysis right.

Spatial Link Matrix for Italian Provinces

- Several spatial link matrices $\mathbf{B}$ are conceivable:



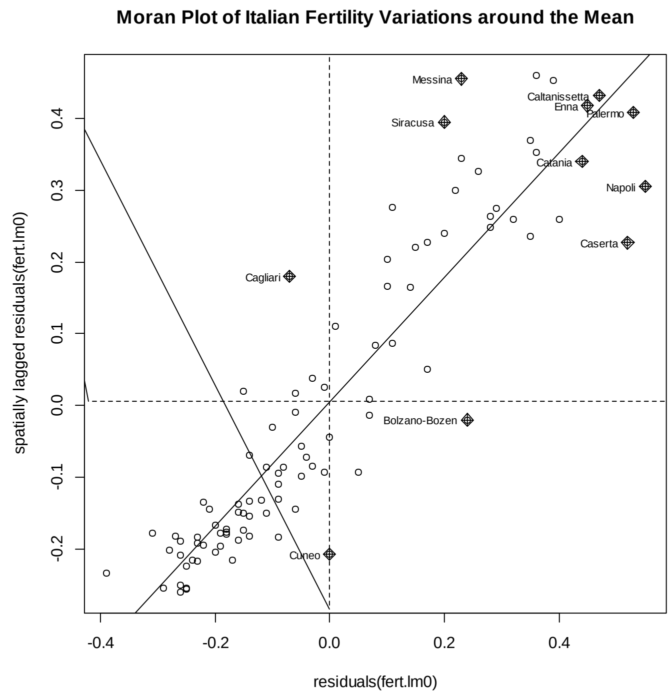


- Notice the artificial links between Sicily as well as Sardinia and the mainland.

Moran Scatterplot

- In the Moran Scatterplot a residual e_i at location i is plotted against the average residual $\bar{e}_{j|i}$ at its neighboring locations $j \in Neighbors\ of\ i$:

$$\bar{e}_{j|i} = \frac{\sum_{j \in Neighbors\ of\ i} b_{ij} \cdot e_j}{n_i} = \sum_{j \in Neighbors\ of\ i} w_{ij} \cdot e_j$$

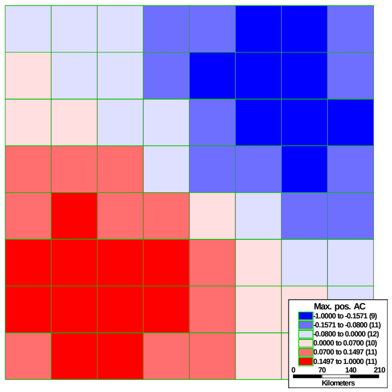


The Moran's *I* Spatial Autocorrelation Statistic

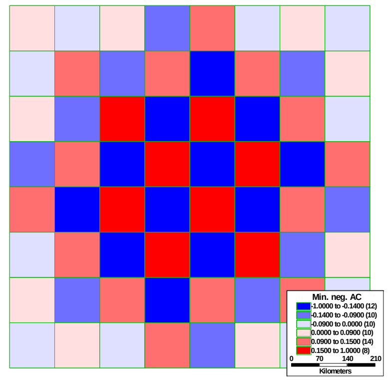
- In essence, the Moran's I statistic can be conceived as a correlation coefficient between a residual e_i at the reference location i and the residuals e_j at its neighboring locations:
  - If on average the residuals at the locations are similar to the residuals are their neighboring locations, then we observed local positive autocorrelation. E.g., dominate patches of positive or negative residuals.

- On the other hand, if the residuals at the locations are dissimilar to the residuals at the neighboring locations, then we observe negative local autocorrelation and a spatial outlier. E.g., pattern of residuals that alternate in their signs.

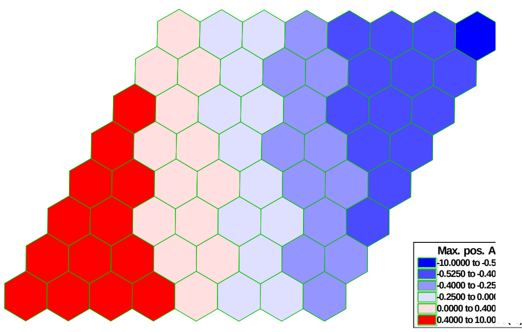
- Examples of extreme map patterns for different tessellations:



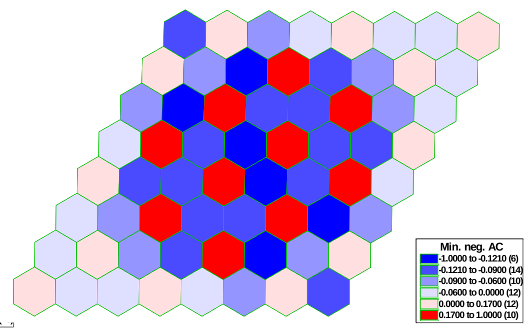
$I_{\max} = 0.9570$



$I_{\min} = \text{-}1.0304$



$I_{\max} = 0.9775$



$I_{\min} = \text{-}0.5271$

- Structure of the Moran's I statistic for regression residuals is:

$$I = \frac{n \cdot \sum_{i=1}^{n} \sum_{j=1}^{n} e_i \cdot w_{ij} \cdot e_i}{\underbrace{\left(\sum_{i=1}^{n} \sum_{j=1}^{n} w_{ij}\right)}_{\text{sum of all elements in } \mathbf{W}} \cdot \underbrace{\sum_{i=1}^{n} e_i^2}_{\text{sum of squared residuals}}}$$

Distribution of Moran's *I* and Test for Spatial Autocorrelation

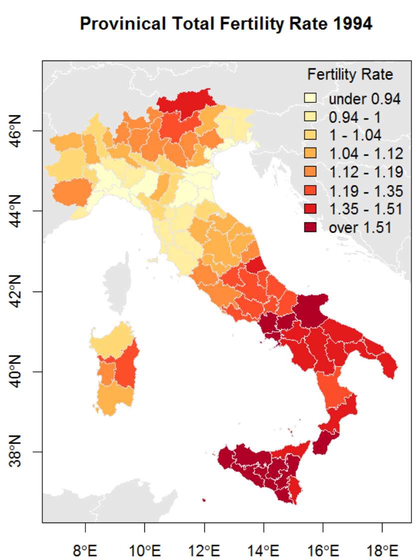
- In general, for well-behaved spatial link matrices $\mathbf{B}$ with more than 50 spatial objects and normal distributed residuals the Moran's *I* will be approximately normal distributed.
- However, its expected value $E(I|H_0)$ and variance $Var(I|H_0)$ under the null hypothesis of spatial independence depend on the used link matrix $\mathbf{B}$, how it is coded, the estimated regression parameters $\widehat{\boldsymbol{\beta}}$, and the used set of independent variables $\mathbf{X}$.
  Note, most software packages ignore these variations in the expected value $E(I|H_0)$ and variance $Var(I|H_0)$.
- Using its expectation and variance, the observed value of Moran's I_{obs} can be transformed into a standard normal distributed $z(I_{obs})$ variable:

$$z(I_{obs}|H_0) = \frac{I_{obs} - E(I|H_0)}{\sqrt{Var(I|H_0)}} \sim N(0,1)$$

- Extreme values of $z(I_{obs}|H_0)$ in the tails of the standard normal distribution indicate significant spatial autocorrelation.

Italian Fertility Example continued

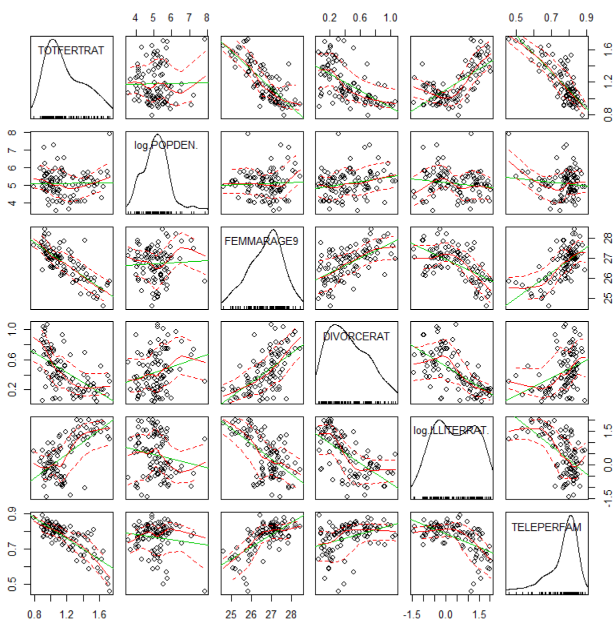
- First, spatial autocorrelation is analyzed for just the variation around the mean. We expect a high degree of spatial autocorrelation because this model is misspecified.
- Once relevant independent variables are added to the model the spatial autocorrelation should drop because some spatial a pattern in the independent variables coincides with the spatial pattern in the dependent variable.
- The variation around the mean $e_i = y_i - \bar{y}$ can be modeled with are regression equation that just consists of an intercept term: **`residuals(lm(y~1))`**.
- Map pattern of the observed dependent variable ***Total Period Fertility*** and the autocorrelation of the regression residuals around the mean is $e_i = y_i - \bar{y}$:

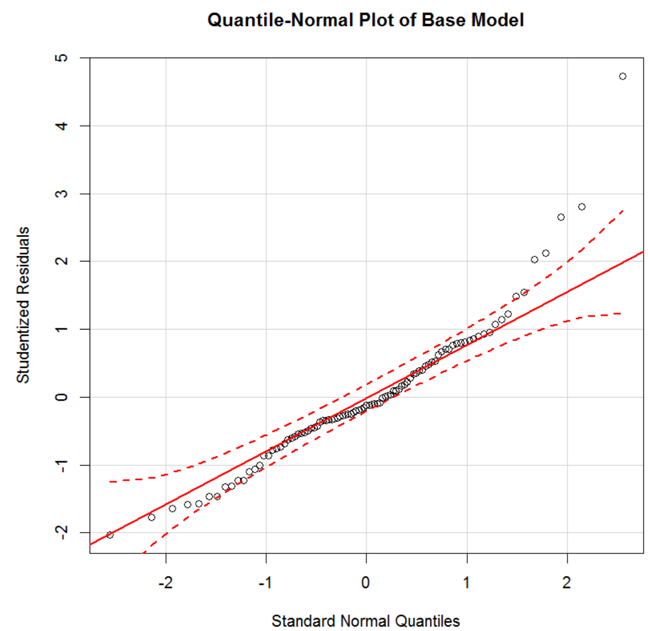



```
          Global Moran's I for regression residuals

model: lm(formula = TOTFERTRAT ~ 1, data = prov.df)
       weights: nb2listw(prov.nb, style = "S")

Moran I statistic standard deviate = 12.7804, p-value < 2.2e-16
alternative hypothesis: greater
sample estimates:
Observed Moran's I        Expectation          Variance
       0.853201213       -0.010638298       0.004568551
```

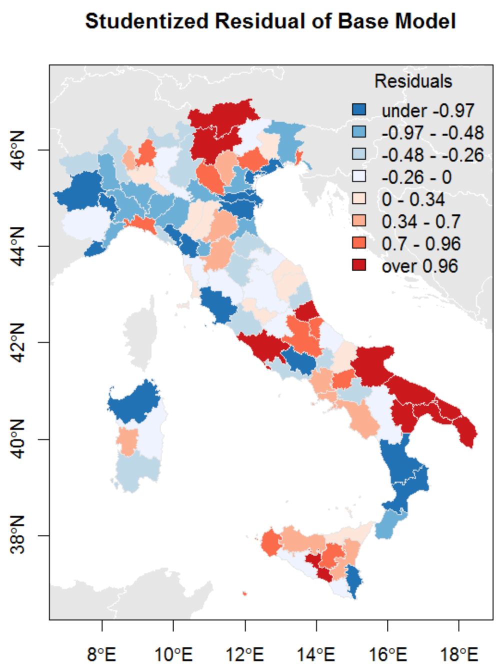
- The applied ***regression model*** to explain the Total Fertility Rate is

```
lm(formula = TOTFERTRAT ~ FEMMARAGE9 + DIVORCERAT + log(ILLITERRAT) +
    TELEPERFAM, data = prov.df)

Coefficients:
                Estimate Std. Error t value Pr(>|t|)
(Intercept)      4.78139    0.48606   9.837 6.23e-16 ***
FEMMARAGE9      -0.09647    0.02050  -4.706 9.11e-06 ***
DIVORCERAT      -0.11839    0.05772  -2.051   0.0431 *
log(ILLITERRAT)  0.03072    0.01707   1.799   0.0753 .
TELEPERFAM      -1.28499    0.18078  -7.108 2.69e-10 ***

Residual standard error: 0.1047 on 90 degrees of freedom
Multiple R-squared: 0.8051,    Adjusted R-squared: 0.7965
F-statistic: 92.96 on 4 and 90 DF,  p-value: < 2.2e-16
```





- The map pattern of the residuals $e_i = y_i - \hat{y}_i$ and their autocorrelation level are




```
        Global Moran's I for regression residuals

model: lm(formula = TOTFERTRAT ~ FEMMARAGE9 +
       DIVORCERAT + log(ILLITERRAT) + TELEPERFAM,
       data = prov.df)
       weights: nb2listw(prov.nb, style = "S")

Moran I standard deviate = 4.7288, p-value =1.129e-06
alternative hypothesis: greater
sample estimates:
Observed Moran's I       Expectation          Variance
       0.276027992      -0.032044882       0.004244308
```

- Notes:
  - ***Accounting*** for the ***exogenous variables*** has substantially reduced the autocorrelation level (⇒misspecification perspective).
  - Residuals are best mapped by a ***bipolar*** map theme.

- The two most ***extreme possible spatial map patterns*** for the Italian provinces and their autocorrelation levels