Stationarity: Properties and Examples

- For a Weakly Stationary Process, $-1 \leq \rho(t) \leq 1$

This is, of course, perfectly analogous to the property that $-1 \leq \rho \leq 1$ from elementary statistics. If you have had a linear algebra course, this may feel familiar (that is, if you showed that $|x^T y| \leq \|x\|_2 \, \|y\|_2$). We know variances are non-negative, so set up a linear combination

$$V[a\, X_1 + b\, X_2] \geq 0$$

In particular, in the spirit of autocorrelation, set up for lag spacing τ

$$V[a\, X(t) + b\, X(t+\tau)] \geq 0$$

Your probability teacher probably told you (time and time again) that

$$V[X+Y] = V[X] + V[Y] + 2\, cov(X,Y)$$

As well as

$$V[aX] = a^2 V[X]$$

So, immediately,

$$V[a\, X(t) + b\, X(t+\tau)] = a^2 V[X(t)] + b^2 V[X(t+\tau)] + 2ab\, cov\big(X(t), X(t+\tau)\big) \geq 0$$

We are assuming weak stationarity, so replace variance operators with a notation which suggests constants

$$a^2\sigma^2 + b^2\sigma^2 + 2ab\, cov\big(X(t), X(t+\tau)\big) \geq 0$$

Two special cases: (1) Let $a = b = 1$

$$2\,\sigma^2 \geq -2cov\big(X(t), X(t+\tau)\big), \quad \sigma^2 \geq -cov\big(X(t), X(t+\tau)\big)$$

$$1 \geq -\frac{cov\big(X(t), X(t+\tau)\big)}{\sigma^2} = -\frac{\gamma(\tau)}{\gamma(0)} = -\rho(\tau)$$

This gives us

$$\rho(\tau) \geq -1$$

(2) Let $a = 1, b = -1$

It's your turn- take a moment to show

$$\rho(\tau) \leq 1$$

We have already seen a few simple models: noise, random walks, and moving averages. Can we now show that some of our simple models are, in fact, weakly stationary?

Examples

- White Noise

  Is it obvious to you that Gaussian white noise is weakly stationary? Consider a discrete family of independent, identically distributed normal random variables

$$X_t \overset{iid}{\sim} N(\mu, \sigma)$$

The mean function $\mu(t)$ is obviously constant, so look at

$$\gamma(t_1, t_2) = \begin{cases} 0 & t_1 \neq t_2 \\ \sigma^2 & t_1 = t_2 \end{cases}$$

And

$$\boxed{\rho(t_1, t_2) = \begin{cases} 0 & t_1 \neq t_2 \\ 1 & t_1 = t_2 \end{cases}}$$

We are evidently weakly stationary, and could even show strict stationarity if we wanted to.

- Random Walks

  Simple random walks are obviously ***not*** stationary. Think of a walk with N steps built off of IID Z_t where $E[Z_t] = \mu,\ V[Z_t] = \sigma^2$. We would create

$$X_1 = Z_1$$
$$X_2 = X_1 + Z_2$$
$$X_3 = X_2 + Z_3 = X_1 + X_2 + X_3$$
$$\vdots$$
$$X_t = X_{t-1} + Z_t = \sum_{i=1}^{t} Z_i$$

For the mean, using the idea that "the mean of the sum is the sum of the means":

$$E[X_t] = E\left[\sum_{i=1}^{t} Z_i\right] = \sum_{i=1}^{t} E[Z_i] = t \cdot \mu$$

For the variance, using the idea that "the variance of the sum is the sum of the variances *when the random variables are independent*":

$$V[X_t] = V\left[\sum_{i=1}^{t} Z_i\right] = \sum_{i=1}^{t} V[Z_i] = t \cdot \sigma^2$$

(Independent random variables have variances which add. All random variables have means which add).

Even if $\mu = 0$ the variances will still increase along the time series.

- Moving Average Processes, $MA(q)$

A moving average process will create a new set of random variables from an old set, just like the random walk does, but now we build them as, for IID Z_t with $E[Z_t] = 0$ and $V[Z_t] = \sigma_Z^2$

$$MA(q)\ process: \qquad X_t = \beta_0 Z_t + \beta_1 Z_{t-1} + \cdots + \beta_q Z_{t-q}$$

The parameter q tells us how far back to look along the white noise sequence for our average. Since the Z_t are independent, we immediately have (using the usual linear operator results)

$$E[X_t] = \beta_0 E[Z_t] + \beta_1 E[Z_{t-1}] + \cdots + \beta_q E[Z_{t-q}] = 0$$

$$\boxed{V[X_t] = \beta_0^2 V[Z_t] + \beta_1^2 V[Z_{t-1}] + \cdots + \beta_q^2 V[Z_{t-q}] = \sigma_Z^2 \sum_{i=0}^{q} \beta_i^2}$$

The autocovariance isn't all that hard to find either. Consider random variables k steps apart and set up their covariance.

$$cov[X_t, X_{t+k}] = cov\left[\, \beta_0 Z_t + \beta_1 Z_{t-1} + \cdots + \beta_q Z_{t-q}\,,\right.$$
$$\left. \beta_0 Z_{t+k} + \beta_1 Z_{t+k-1} + \cdots + \beta_q Z_{t+k-q}\right]$$

This is a little tricky, but please stay focused. There are two numbers to keep track of, the lag spacing k and the support of the MA process, q.

Now

$$cov[X_t, X_{t+k}] = E[X_t \cdot X_{t+k}] - E[X_t]E[X_{t+k}] = E[X_t \cdot X_{t+k}]$$

Since $E[X_t] = 0$ we really just need

$$
\begin{aligned}
E[X_t \cdot\ X_{t+k}] = E\big[\, &(\beta_0 Z_t + \beta_1 Z_{t-1} + \cdots + \beta_q Z_{t-q}) \\
&\cdot (\,\beta_0 Z_{t+k} + \beta_1 Z_{t+k-1} + \cdots + \beta_q Z_{t+k-q})\big]
\end{aligned}
$$

We can rely on matrix results concerning linear combinations of random variables or just work directly. The patient among us will write out

$$
\begin{aligned}
E[X_t \cdot\ X_{t+k}] = {} & \beta_0\beta_0 E[Z_t\, Z_{t+k}] + \beta_0\beta_1 E[Z_t\, Z_{t+k-1}] + \cdots + \beta_0\beta_q E\big[Z_t\, Z_{t+k-q}\big] \\
& + \beta_1\beta_0 E[Z_{t-1}\, Z_{t+k}] + \beta_1\beta_1 E[Z_{t-1}\, Z_{t+k-1}] + \cdots + \beta_1\beta_q E\big[Z_{t-1}\, Z_{t+k-q}\big] \\
& + \cdots \cdot + \\
& \beta_q\beta_0 E\big[Z_{t-q}\, Z_{t+k}\big] + \beta_q\beta_1 E\big[Z_{t-q}\, Z_{t+k-1}\big] + \cdots \beta_q\beta_q E\big[Z_{t-q}\, Z_{t+k-q}\big]
\end{aligned}
$$

The key to simplifying this is to notice that, since the Z_t are independent, we can say that the expectation of the product is the product of the expectations and so we have

$$
E\big[Z_i \cdot\ Z_j\big] = E[Z_i]E\big[Z_j\big] = \begin{cases} 0 & i \neq j \\ \sigma_Z^2 & i = j \end{cases}
$$

When the lag spacing k is greater than the order of the process q then the subscripts can never be the same (there is no overlap on the underlying $Z_t's$) and we have $cov[X_t, X_{t+k}] = 0$. When the lag spacing is small enough to have contributions, that is if $q - k \geq 0$, you can visualize the sum like this (we just need to keep track of the $\beta's$):

$$
\begin{array}{cccccccccc}
Z_{t-q} & Z_{t-q+1} & \cdots & Z_{t-(q-k)} & \cdots & Z_{t-1} & Z_t & \cdots & & \\
\beta_q & \beta_{q-1} & & \beta_{q-k} & & \beta_1 & \beta_0 & & 0 & 0 \\
& & & & & & & & & \\
& & & Z_{t+k-q} & \cdots & Z_{t+k-k+1} & Z_{t+k-k} & \cdots & Z_{t+k-1} & Z_{t+k} \\
0 & 0 & 0 & \beta_q & & \beta_{k+1} & \beta_k & & \beta_1 & \beta_0
\end{array}
$$

This should make clear that, when $k \leq q$

$$
E[X_t, X_{t+k}\,] = \sigma_Z^2 \cdot \sum_{i=0}^{q-k} \beta_i\, \beta_{i+k} \quad (no\ t\ dependence)
$$

Summing up, then, we have found that

$$\gamma(t_1, t_2) = \gamma(k) = \begin{cases} 0 & k > q \\ \sigma_Z^2 \cdot \displaystyle\sum_{i=0}^{q-k} \beta_i \, \beta_{i+k} & k \leq q \end{cases}$$

We know that the mean function is constant, in fact $\mu(t) = 0$ and the autocovariance function has no t dependence, so we conclude that the *MA(q)* process is (weakly) stationary.

Let's finish this lecture by finding the autocorrelation function. In general

$$\rho(k) = \frac{\gamma(k)}{\gamma(0)}$$

Obviously, then $\rho(0) = 1$. It is easy to see that

$$\gamma(0) = \sigma_Z^2 \cdot \sum_{i=0}^{q} \beta_i \, \beta_i = \sigma_Z^2 \cdot \sum_{i=0}^{q} \beta_i^2$$

Finally

$$\rho(k) = \frac{\sum_{i=0}^{q-k} \beta_i \, \beta_{i+k}}{\sum_{i=0}^{q} \beta_i^2}$$

In the next lecture we will simulate an MA(q) process and validate these results numerically.