Series and series representation

Practical Time Series Analysis

Thistleton and Sadigov

Objectives

- Recall infinite series and their convergence
- Examine geometric series
- Represent rational functions as a geometric series

Sequence and series

- Sequence $\{a_n\}$ is list of numbers in definite order

$$a_1, a_2, a_3, \dots a_n, \dots$$

- If the limit of the sequence exists, i..e,

$$\lim_{n\to\infty} a_n = a$$

then we say the sequence is convergent.

Examples

- $a_n = \frac{n}{n+1}$

$$\boxed{\frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \dots, \frac{n}{n+1}, \dots \to 1}$$

- $a_n = 3^n$

$$3, 9, 27, \dots, 3^n, \dots$$

- $a_n = \sqrt{n}$

$$1, \sqrt{2}, \sqrt{3}, \dots, \sqrt{n}, \dots$$

- $a_n = \frac{1}{n^2}$

$$\boxed{1, \frac{1}{4}, \frac{1}{9}, \dots, \frac{1}{n^2}, \dots \to 0}$$

Partial sums

- Partial sums of a sequence $\{a_n\}$ are defined as

$$s_n = a_1 + a_2 + \cdots + a_n$$

- $s_1 = a_1$
- $s_2 = a_1 + a_2$
- $s_3 = a_1 + a_2 + a_3$

$$\vdots$$

Series

- If the partial sums $\{s_n\}$ is convergent to a number s, then we say

  the infinite series $\sum_{k=1}^{\infty} a_k$ is convergent, and is equal to s.

$$\sum_{k=1}^{\infty} a_k = \lim_{n\to\infty} s_n = \lim_{n\to\infty}(a_1 + a_2 + \cdots + a_n) = s$$

- Otherwise, we say $\sum_{k=1}^{\infty} a_k$ is divergent.

Some convergent series

- $\sum_{k=1}^{\infty} \frac{1}{2^k} = 1$
- $\sum_{k=1}^{\infty} \frac{1}{k^2} = \frac{\pi^2}{6}$
- $\sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k} = \ln(2)$

Some divergent series

- $\sum_{k=1}^{\infty} 3^k$

- $\sum_{k=1}^{\infty}(2k+1)$

- $\sum_{k=1}^{\infty} \frac{1}{k}$

Absolute convergence

- Series is absolutely convergent if

$$\sum_{k=1}^{\infty} |a_k|$$

is convergent.

- Absolute convergence implies convergence.

Convergence tests

- Integral test
- Comparison test
- Limit comparison test
- Alternating series test
- Ratio test
- Root test

Geometric series

- Geometric sequence

$$\{ar^{n-1}\}_{n=1}^{\infty} = \{a, ar, ar^2, ar^3, \dots\}$$

- Geometric series

$$\boxed{\sum_{k=1}^{\infty} ar^{k-1} = \frac{a}{1-r} \text{ if } |r| < 1.}$$

- $\sum_{k=1}^{\infty} \frac{1}{2^k} = \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \cdots = \frac{\frac{1}{2}}{1-\frac{1}{2}} = 1$

  since $a = \frac{1}{2}, r = \frac{1}{2}$.

Series representation

- Series representation for $\frac{1}{1-x}$ $where\, a = 1, r = x$.

$$\boxed{\frac{1}{1-x} = 1 + x + x^2 + x^3 + \cdots}$$

$\boxed{\text{if } |x| < 1.}$

Series representation cont.

- Series representation for $\frac{1}{(1-x)\left(1-\frac{x}{2}\right)}$

$$\boxed{\frac{1}{(1-x)\left(1-\frac{x}{2}\right)} = \frac{2}{1-x} + \frac{-1}{1-\frac{x}{2}} = \sum_{k=0}^{\infty}\left(2-\frac{1}{2^k}\right)x^k}$$

If $|x| < 1$ and $\left|\frac{x}{2}\right| < 1$, i.e., if $|x| < 1$.

Complex functions

Assume z is a complex number

$$\frac{a}{1-z} = a + az + az^2 + \cdots = \sum_{k=1}^{\infty} az^{k-1}$$

if $|z| < 1$.

What We’ve Learned

- The definition of infinite series and their convergence
- Geometric series is convergent if the multiplier has norm less than 1
- How to represent some rational functions as a geometric series

Backward shift operator

Practical Time Series Analysis

Thistleton and Sadigov

Objectives

- Define and utilize backward shift operator

Definition

- $X_1, X_2, X_3, \ldots$
- Backward shift operator is defined as

$$BX_t = X_{t-1}$$

- $B^2 X_t = BBX_t = BX_{t-1} = X_{t-2}$
- $B^k X_t = X_{t-k}$

Example – Random Walk

$$X_t = X_{t-1} + Z_t$$

$$X_t = BX_t + Z_t$$

$$\boxed{(1 - B)X_t = Z_t}$$

$$\phi(B)X_t = Z_t$$ polynomial operator

where

$$\boxed{\phi(B) = 1 - B}$$

Example – MA(2) process

$$X_t = Z_t + 0.2Z_{t-1} + 0.04Z_{t-2}$$

$$X_t = Z_t + 0.2BZ_t + 0.04B^2Z_t$$

$$X_t = (1 + 0.2B + 0.04B^2)\, Z_t$$

$$X_t = \beta(B)Z_t$$

where

$$\beta(B) = 1 + 0.2B + 0.04B^2$$

Example – AR(2) process

$$X_t = 0.2X_{t-1} + 0.3X_{t-2} + Z_t$$

$$X_t = 0.2BX_t + 0.3B^2X_t + Z_t$$

$$(1 - 0.2B - 0.3B^2)\, X_t = Z\_t$$

$$\phi(B)X_t = Z_t$$

where

$$\phi(B) = 1 - 0.2B - 0.3B^2$$

MA(q) process (with a drift)

$$X_t = \mu + \beta_0 Z_t + \beta_1 Z_{t-1} + \cdots + \beta_q Z_{t-q},$$

Then,

$$X_t = \mu + \beta_0 Z_t + \beta_1 B^1 Z_t + \cdots + \beta_q B^q Z_t,$$

$$X_t - \mu = \beta(B) Z_t,$$

where

$$\boxed{\beta(B) = \beta_0 + \beta_1 B + \cdots + \beta_q B^q.}$$

AR(p) process

$$X_t = \phi_1 X_{t-1} + \phi_2 X_{t-2} + \cdots + \phi_p X_{t-p} + Z_t$$

Then,

$$X_t - \phi_1 X_{t-1} - \phi_2 X_{t-2} - \cdots - \phi_p X_{t-p} = Z_t$$

$$X_t - \phi_1 B X_t - \phi_2 B^2 X_t - \cdots - \phi_p B^p X_t = Z_t$$

$$\phi\,(B) X_t = Z_t,$$

Where

$$\boxed{\phi\,(B) = 1 - \phi_1 B - \phi_2 B^2 - \cdots - \phi_p B^p.}$$

What We've Learned

- The definition of the Backward shift operator

- How to utilize backward shift operator to write MA(q) and AR(p) processes

Introduction to Invertibility

Practical Time Series Analysis

Thistleton and Sadigov

Objectives

- Learn invertibility of a stochastic process

Two MA(1) models

- Model 1

$$X_t = Z_t + 2Z_{t-1}$$

- Model 2

$$X_t = Z_t + \frac{1}{2}Z_{t-1}$$

Theoretical Auto Covariance Function of Model 1

$$\gamma(k) = Cov\,[X_{t+k}, X_t] = Cov\,[Z_{t+k} + 2Z_{t+k-1}, Z_t + 2Z_{t-1}]$$

If $k > 1$, then $t + k - 1 > t$, so all Z's are uncorrelated, thus $\gamma(k) = 0$.

If $k = 0$, then

$$\gamma(0) = Cov\,[Z_t + 2Z_{t-1}, Z_t + 2Z_{t-1}] =$$

$$Cov[Z_t, Z_t] + 4Cov[Z_{t-1}, Z_{t-1}] = \sigma_Z^2 + 4\sigma_Z^2 = 5\sigma_Z^2.$$

If $k = 1$, then

$$\gamma(1) = Cov\,[Z_{t+1} + 2Z_t, Z_t + 2Z_{t-1}] = Cov\,[2Z_t, Z_t]$$
$$= 2\sigma_Z^2$$

If $k < 0$, then

$$\gamma(k) = \gamma(-k)$$

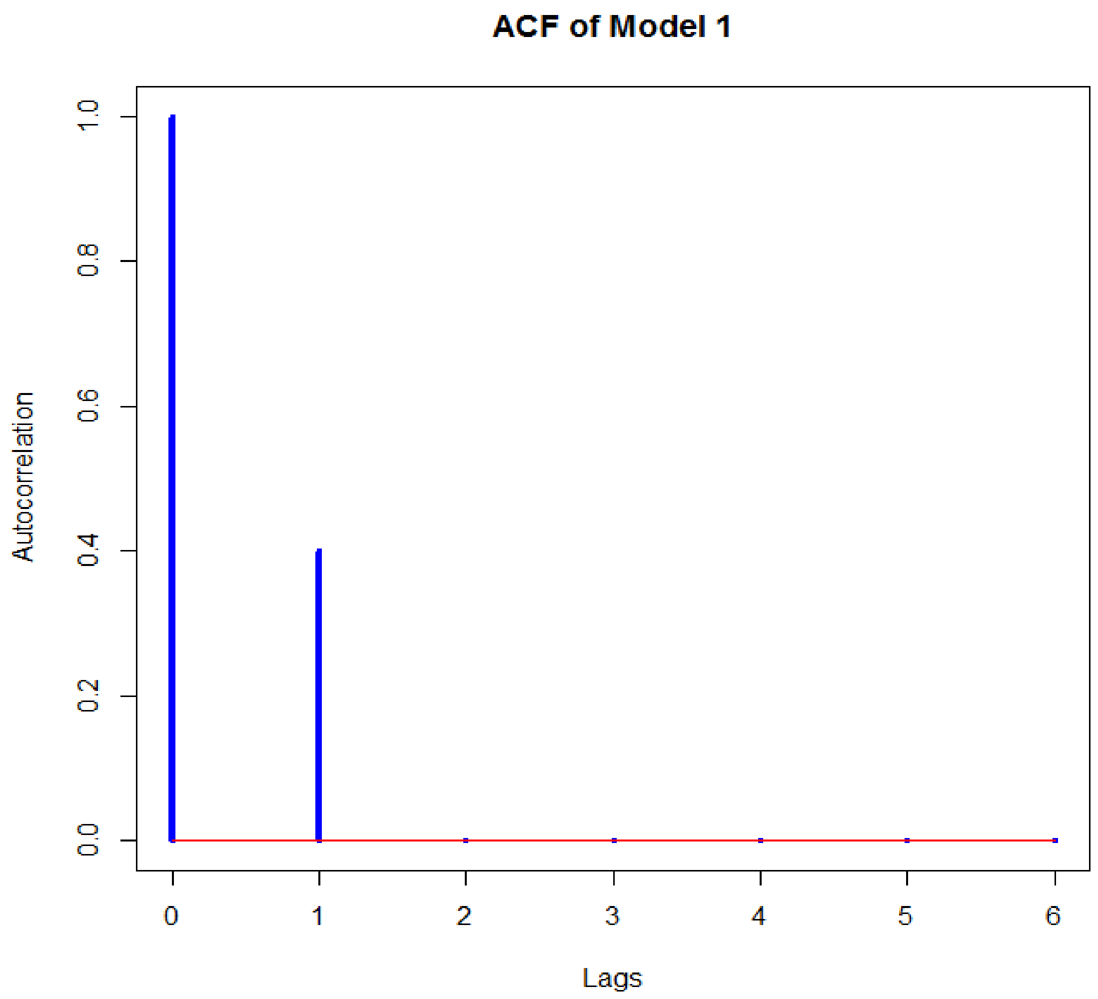
Auto Covariance Function and ACF of Model 1

$$\boxed{\gamma(k) = \begin{cases} 0, & k > 1 \\ 2\sigma_Z^2, & k = 1 \\ 5\sigma_Z^2, & k = 0 \\ \gamma(-k), & k < 0 \end{cases}}$$

Then, since $\rho(k) = \frac{\gamma(k)}{\gamma(0)}$,

$$\boxed{\rho(k) = \begin{cases} 0, & k > 1 \\ \frac{2}{5}, & k = 1 \\ 1, & k = 0 \\ \rho(-k), & k < 0 \end{cases}}$$

ACF



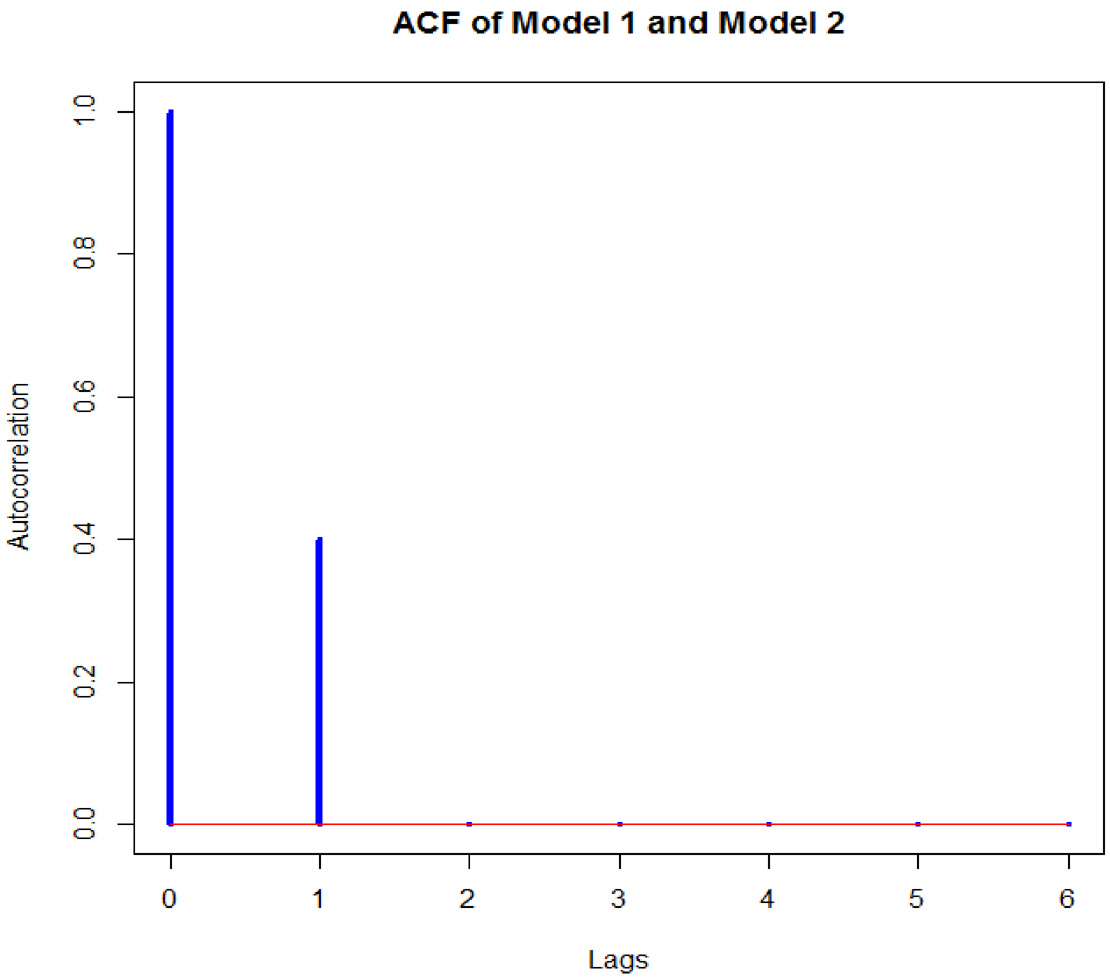
ACF of Model 2

$$\rho(1) = \frac{\gamma(1)}{\gamma(0)} = \frac{Cov\left[Z_{t+1} + \frac{1}{2}Z_t, Z_t + \frac{1}{2}Z_{t-1}\right]}{Cov[Z_t + \frac{1}{2}Z_{t-1},\ \ Z_t + \frac{1}{2}Z_{t-1}]} = \frac{\frac{1}{2}}{1 + \frac{1}{4}} = \frac{2}{5}.$$

Thus we obtain the same ACF:

$$\rho(k) = \begin{cases} 0, & k > 1 \\ \frac{2}{5}, & k = 1 \\ 1, & k = 0 \\ \rho(-k), & k < 0 \end{cases}$$

ACFs are same!



Inverting through backward substitution

MA(1) process

$$X_t = Z_t + \beta Z_{t-1},$$

$$Z_t = X_t - \beta Z_{t-1} = X_t - \beta(X_{t-1} - \beta Z_{t-2}) = X_t - \beta X_{t-1} + \beta^2 Z_{t-2}$$

In this manner,

$$Z_t = X_t - \beta X_{t-1} + \beta^2 X_{t-2} - \beta^3 X_{t-3} + \cdots$$

i.e.,

$$X_t = Z_t + \beta X_{t-1} - \beta^2 X_{t-2} + \beta^3 X_{t-3} - \cdots$$

We ‘inverted’ MA(1) process to AR(∞).

Inverting using Backward shift operator

$$X_t = \beta(B)Z_t$$

where

$$\beta(B) = 1 + \beta B$$

Then, we find Z_t by inverting the polynomial operator $\beta(B)$:

$$\beta(B)^{-1}X_t = Z_t$$

Inverse of $\beta(B)$

$$\boxed{\beta(B)^{-1} = \frac{1}{1+\beta B} = 1 - \beta B + \beta^2 B^2 - \beta^3 B^3 + \cdots}$$

Here we expand the inverse of the polynomial operator as a 'rational function where βB is a complex number'.

Thus we obtain,

$$\beta(B)^{-1} X_t = 1 - \beta X_{t-1} + \beta^2 X_{t-2} - \beta^3 X_{t-3} + \cdots$$

$$Z_t = \sum_{n=0}^{\infty} (-\beta)^n X_{t-n}$$

In order to make sure that the sum on the right is convergent (in the mean-square sense), we need $|\beta| < 1$.

There is an optional reading titled "Mean-square convergence" where we explain this result.

Invertibility - Definition

Definition:

$\{X_t\}$ is a stochastic process.

$\{Z_t\}$ is innovations, i.e., random disturbances or white noise.

$\{X_t\}$ is called **invertible**, if $Z_t = \sum_{k=0}^{\infty} \pi_k X_{t-k}$ where $\sum_{k=0}^{\infty} |\pi_k|$ is convergent.

Model 1 vs Model 2

- Model 1 is **not invertible** since

$$\sum_{k=0}^{\infty} |\pi_k| = \sum_{k=0}^{\infty} 2^k, \qquad \mathit{Divergent}$$

- Model 2 is **invertible** since

$$\sum_{k=0}^{\infty} |\pi_k| = \sum_{k=0}^{\infty} \frac{1}{2^k}, \qquad \mathit{Geometric\ Series}, \qquad \mathit{Convergent}$$

Model choice

- For ‘invertibility’ to hold, we choose Model 2, since $\left|\frac{1}{2}\right| < 1$.

- This way, ACF uniquely determines the MA process.

What We’ve Learned

- Definition of invertibility of a stochastic process

- Invertibility condition guarantees unique MA process corresponding to observed ACF

Invertibility and stationarity conditions

Practical Time Series Analysis

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Objectives

- Articulate invertibility condition for MA(q) processes
- Discover stationarity condition for AR(p) processes
- Relate MA and AR processes through duality

MA(q) process

$$X_t = \beta_0 Z_t + \beta_1 Z_{t-1} + \cdots \beta_q Z_{t-q}$$

Using Backward shift operator,

$$X_t = \left(\beta_0 + \beta_1 B + \cdots + \beta_q B^q\right) Z_t = \beta(B) Z_t$$

We obtain innovations Z_t in terms of present and past values of X_t,

$$Z_t = \beta(B)^{-1} X_t = (\alpha_0 + \alpha_1 B + \alpha_2 B^2 + \cdots) X_t$$

For this to hold, “complex roots of the polynomial $\beta(B)$ must lie outside of the unit circle where B is regarded as complex variable”.

Invertibility condition for MA(q)

MA(q) process is invertible if the roots of the polynomial

$$\beta(B) = \beta_0 + \beta_1 B + \cdots + \beta_q B^q$$

all lie outside the unit circle, where we regard B as a complex variable (not an operator).

(Proof is done using mean-square convergence, see optional reading)

EX: MA(1) process

- $X_t = Z_t + \beta Z_{t-1}$

- $\beta(B) = 1 + \beta B$

- In this case only one (real) root $B = -\frac{1}{\beta}$
- $\left|-\frac{1}{\beta}\right| > 1 \; \Rightarrow |\beta| < 1.$

- Then, $Z_t = \sum_{k=0}^{\infty}(-\beta)^k B^k \, X_t = \sum_{k=0}^{\infty}(-\beta)^k X_{t-k}$

Example – MA(2) process

$$X_t = Z_t + \frac{5}{6} Z_{t-1} + \frac{1}{6} Z_{t-2}$$

Then,

$$X_t = \beta(B) Z_t$$

Where

$$\beta(B) = 1 + \frac{5}{6} B + \frac{1}{6} B^2$$

Example cont.

$$1 + \frac{5}{6}z + \frac{1}{6}z^2 = 0$$

$$z_1 = 2, z_2 = 3$$

Example cont.

$$\beta(B)^{-1} = \frac{1}{1+\frac{5}{6}B+\frac{1}{6}B^2} = \frac{3}{1+\frac{1}{2}B} - \frac{2}{1+\frac{1}{3}B}$$

$$\beta(B)^{-1} = \sum_{k=0}^{\infty}\left[3\left(-\frac{1}{2}\right)^k - 2\left(-\frac{1}{3}\right)^k\right]B^k$$

$$Z_t = \sum_{k=0}^{\infty} \left[3\left(-\frac{1}{2}\right)^k - 2\left(-\frac{1}{3}\right)^k \right] B^k X_t$$

$$Z_t = \sum_{k=1}^{\infty} \pi_k B^k X_t = \sum_{k=1}^{\infty} \pi_k X_{t-k}$$

Where

$$\pi_k = 3\left(-\frac{1}{2}\right)^k - 2\left(-\frac{1}{3}\right)^k$$

MA(2) process $\Longrightarrow$ AR(∞) process

Stationarity condition for AR(p)

AR(p) process

$$X_t = \phi_1 X_{t-1} + \phi_2 X_{t-2} + \cdots + \phi_p X_{t-p} + Z_t$$

is (weakly) stationary if the roots of the polynomial

$$\phi\,(B) = 1 - \phi_1 B - \phi_2 B^2 - \cdots - \phi_p B^p.$$

all lie outside the unit circle, where we regard B as a complex variable (not an operator).

AR(1) process

$$X_t = \phi_1 X_{t-1} + Z_t \implies (1 - \phi_1 B) X_t = Z_t$$

$$\phi(B) = 1 - \phi_1 B$$

$$\phi(z) = 1 - \phi_1 z = 0 \implies z = \frac{1}{\phi_1}$$

$$|z| = \left|\frac{1}{\phi_1}\right| > 1 \;\Rightarrow |\phi_1| < 1$$

Thus, when $|\phi_1| < 1$, the AR(1) process is stationary.

$$X_t = \frac{1}{1 - \phi_1 B} Z_t = (1 + \phi_1 B + \phi_1 B^2 - \cdots) Z_t$$

$$= \sum_{k=0}^{\infty} \phi_1^k Z_{t-k}$$

Another look at ϕ_1

Take Variance from both side,

$$Var[X_t] = Var\left[\sum_{k=0}^{\infty} \phi_1^k Z_{t-k}\right] = \sum_{k=0}^{\infty} \phi_1^{2k} \sigma_Z^2 = \sigma_Z^2 \sum_{k=0}^{\infty} \phi_1^{2k}$$

which is a convergent geometric series if $\left|\phi_1^2\right| < 1$, i.e.,

$$\boxed{|\phi_1| < 1.}$$

AR(p) process $\Longrightarrow$ MA(∞) process

Duality between AR and MA processes

Under invertibility condition of MA(q),

$$\text{MA}(q) \Longrightarrow \text{AR}(\infty)$$

Under stationarity condition of AR(p)

$$\text{AR}(p) \Longrightarrow \text{MA}(\infty)$$

What We’ve Learned

- Invertibility condition for MA(q) processes
- Stationarity condition for AR(p) processes
- Duality MA and AR processes

Mean Square Convergence

Practical Time Series Analysis

Thistleton and Sadigov

Objectives

- Learn mean-square convergence
- Formulate necessary and sufficient condition for invertibility of MA(1) process

Mean-square convergence

Let

$$X_1, X_2, X_3, \ldots$$

be a sequence of random variables (i.e. a stochastic process).

We say X_n converge to a random variable X in the mean-square sense

if

$$E[(X_n - X)^2] \to 0 \; as \; n \to \infty$$

MA(1) model

We inverted MA(1) model

$$X_t = Z_t + \beta Z_{t-1}$$

as

$$Z_t = \sum_{k=0}^{\infty} (-\beta)^k X_{t-k}$$

Infinite sum above is convergent in mean-square sense under some condition on β.

Auto covariance function

$$\gamma(k) = \begin{cases} 0, & k > 1 \\ \beta\sigma_Z^2, & k = 1 \\ (1+\beta^2)\sigma_Z^2, & k = 0 \\ \gamma(-k), & k < 0 \end{cases}$$

Series convergence

Lets find $\beta's$ that partial sum

$$\sum_{k=0}^{n}(-\beta)^k X_{t-k}$$

converges to Z_t in mean-square sense.

$$E\left[\left(\sum_{k=0}^{n}(-\beta)^{k}X_{t-k}-Z_{t}\right)^{2}\right]=E\left[\left(\sum_{k=0}^{n}(-\beta)^{k}X_{t-k}\right)^{2}\right]-2E\left[\sum_{k=0}^{n}(-\beta)^{k}X_{t-k}\,Z_{t}\right]+E[Z_{t}^{2}]$$

$$=E\left[\sum_{k=0}^{n}\beta^{2k}X_{t-k}^{2}\right]+2E\left[\sum_{k=0}^{n-1}(-\beta)^{2k+1}X_{t-k}X_{t-k+1}\right]-2E[X_{t}Z_{t}]+\sigma_{Z}^{2}$$

$$=\sum_{k=0}^{n}\beta^{2k}E\left[X_{t-k}^{2}\right]-2\sum_{k=0}^{n-1}\beta^{2k+1}\,E[X_{t-k}X_{t-k+1}]-2E[Z_{t}^{2}$$

To get

$$E\left[\left(\sum_{k=0}^{n}(-\beta)^k X_{t-k} -\right.\right.$$

i.e.,

$$\left|-\frac{1}{\beta}\right| > 1$$

i.e., zero of the polynomial

$$\beta(B) = 1 + \beta B$$

Lies outside of the unit circle.

What We've Learned

- Definition of the mean square convergence

- Necessary and sufficient condition for invertibility of MA(1) process

Difference equations

Practical Time Series Analysis

Thistleton and Sadigov

Objectives

- Recall and solve difference equations

Difference equation

- General term of a sequence is given, ex: $a_n = 2n + 1$. So,

$$3, 5, 7, \dots$$

- General term not given, but a relation is given, ex:

recursive relationship

$$a_n = 5a_{n-1} - 6a_{n-2}$$

- This is a difference equation (recursive relation)

How to solve difference equations?

- We look for a solution in the format

$$a_n = \lambda^n$$

- For the previous problem,

$$\lambda^n = 5\lambda^{n-1} - 6\lambda^{n-2}$$

We simplify

$$\lambda^2 - 5\lambda + 6 = 0$$

- Auxiliary equation or characteristic equation.

- $\lambda = 2, \lambda = 3$
- $a_n = c_1 2^n + c_2 3^n$
- With some initial conditions, say $a_0 = 3, a_1 = 8$.

We get

$$\begin{cases} c_1 + c_2 = 3 \\ 2c_1 + 3c_2 = 8 \end{cases}$$

Thus,

$$c_1 = 1, c_2 = 2.$$

Solution

$$a_n = 2^n + 2 \cdot 3^n$$

Is the solution of 2nd order difference equation

$$a_n = 5a_{n-1} - 6a_{n-2}$$

k-th order difference equation

$$a_n = \beta_1 a_{n-1} + \beta_2 a_{n-2} + \cdots + \beta_k a_{n-k}$$

Its characteristic equation

$$\lambda^k - \beta_1 \lambda^{k-1} - \cdots - \beta_{k-1}\lambda - \beta_k = 0$$

Then we look for the solutions of the characteristic equation. Say, all k solutions are distinct real numbers, $\lambda_1, \lambda_2, \ldots, \lambda_k$, then

$$a_n = c_1 \lambda_1^n + c_2 \lambda_2^n + \cdots + c_n \lambda_k^n$$

Coefficients $c_j's$ are determined using initial values.

Example - Fibonacci sequence

Fibonacci sequence is defined as follows:

$$1, 1, 2, 3, 5, 8, 13, 21, \ldots$$

i.e., every term starting from the 3rd term is addition of the previous two terms.

Question: What is the general term, a_n, of the Fibonacci sequence?

Formulation

We are looking for a sequence $\{\mathrm{a_n}\}_{\mathrm{n=0}}^{\infty}$, such that

$$a_n = a_{n-1} + a_{n-2}$$

where $a_0 = 1, a_1 = 1$.

Characteristic equation becomes

$$\lambda^2 - \lambda - 1 = 0$$

Then $\lambda_1 = \frac{1-\sqrt{5}}{2}$ and $\lambda_2 = \frac{1+\sqrt{5}}{2}$.

Thus

$$a_n = c_1\left(\frac{1-\sqrt{5}}{2}\right)^n + c_2\left(\frac{1+\sqrt{5}}{2}\right)^n$$

Use initial data

$$\begin{cases} c_1 + c_2 = 1 \\ c_1\left(\frac{1-\sqrt{5}}{2}\right) + c_2\left(\frac{1+\sqrt{5}}{2}\right) = 1 \end{cases}$$

General term of Fibonacci sequence

We obtain

$$c_1 = \frac{5-\sqrt{5}}{10} = -\frac{1}{\sqrt{5}}\left(\frac{1-\sqrt{5}}{2}\right)$$

$$c_2 = \frac{5+\sqrt{5}}{10} = \frac{1}{\sqrt{5}}\left(\frac{1+\sqrt{5}}{2}\right)$$

$$a_n = -\frac{1}{\sqrt{5}}\left(\frac{1-\sqrt{5}}{2}\right)^{n+1} + \frac{1}{\sqrt{5}}\left(\frac{1+\sqrt{5}}{2}\right)^{n+1}$$

Relation to differential equations

k −th order linear ordinary equation

$$y^{(k)} = \beta_1 y^{(k-1)} + \cdots \beta_{k-1} y + \beta_k$$

Solution format $y = e^{\lambda t}$ gives characteristic equation

$$\lambda^k - \beta_1 \lambda^{k-1} - \cdots - \beta_{k-1}\lambda - \beta_k = 0$$

Then we solve the characteristic equation.

What We’ve Learned

- Definition of difference equations and how to solve them

Yule-Walker Equations

Practical Time Series Analysis

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Objectives

- Introduce Yule – Walker equations
- Obtain ACF of AR processes using Yule – Walker equations

Procedure

- We assume stationarity in advance (a priori assumption)
- Take product of the AR model with X_{n-k}
- Take expectation of both sides
- Use the definition of covariance, and divide by $\gamma(0) = \sigma_X^2$
- Get difference equation for $\rho(k)$, ACF of the process
- This set of equations is called Yule-Walker equations
- Solve the difference equation

Example

We have an AR(2) process

$$X_t = \frac{1}{3}X_{t-1} + \frac{1}{2}X_{t-2} + Z_t \ldots (*)$$

Polynomial

$$\phi(B) = 1 - \frac{1}{3}B - \frac{1}{2}B^2$$

has real roots $\frac{-2 \pm \sqrt{76}}{6}$ both of which has magnitude greater than 1, so roots are outside of the unit circle in $\mathbb{R}^2$. Thus, this AR(2) process is a stationary process.

Example cont.

Note that if $E(X_t) = \mu$, then

$$E(X_t) = \frac{1}{3}E(X_{t-1}) + \frac{1}{2}E(X_{t-2}) + E(Z_t)$$
$$\mu = \frac{1}{3}\mu + \frac{1}{2}\mu$$
$$\mu = 0$$

Multiply both side of $(*)$ with X_{t-k}, and take expectation

$$E(X_{t-k}X_t) = \frac{1}{3}E(X_{t-k}X_{t-1}) + \frac{1}{2}E(X_{t-k}X_{t-2}) + E(X_{t-k}Z_t)$$

Example cont.

Since $\mu = 0$, and assume $E(X_{t-k}Z_t) = 0$,

$$\gamma(-k) = \frac{1}{3}\gamma(-k+1) + \frac{1}{2}\gamma(-k+2)$$

Since $\gamma(k) = \gamma(-k)$ for any k,

$$\gamma(k) = \frac{1}{3}\gamma(k-1) + \frac{1}{2}\gamma(k-2)$$

Divide by $\gamma(0) = \sigma_X^2$

$$\boxed{\rho(k) = \frac{1}{3}\rho(k-1) + \frac{1}{2}\rho(k-2)}$$

This set of equations is called Yule-Walker equations.

Solve the difference equation

We look for a solution in the format of $\rho(k) = \lambda^k$.

$$\lambda^2 - \frac{1}{3}\lambda - \frac{1}{2} = 0$$

Roots are $\lambda_1 = \frac{2+\sqrt{76}}{12}$ and $\lambda_2 = \frac{2-\sqrt{76}}{12}$, thus

$$\rho(k) = c_1\left(\frac{2+\sqrt{76}}{12}\right)^k + c_2\left(\frac{2-\sqrt{76}}{12}\right)^k$$

Finding c_1, c_2

Use constraints to obtain coefficients

$$\boxed{\rho(0) = 1 \Rightarrow c_1 + c_2 = 1}$$

And for $k = p - 1 = 2 - 1 = 1,$

$$\rho(k) = \rho(-k)$$

Thus,

$$\rho(1) = \frac{1}{3}\rho(0) + \frac{1}{2}\rho(-1) \Rightarrow \rho(1) = \frac{2}{3} \Rightarrow c_1\left(\frac{2+\sqrt{76}}{12}\right) + c_2\left(\frac{2-\sqrt{76}}{12}\right) = \frac{2}{3}$$

Solve the system for c_1, c_2

$$\begin{cases} c_1 + c_2 = 1 \\ c_1\left(\dfrac{2+\sqrt{76}}{12}\right) + c_2\left(\dfrac{2-\sqrt{76}}{12}\right) = \dfrac{2}{3} \end{cases}$$

Then,

$$c_1 = \frac{4+\sqrt{6}}{8} \text{ and } c_2 = \frac{4-\sqrt{6}}{8}$$

ACF of the AR(2) model

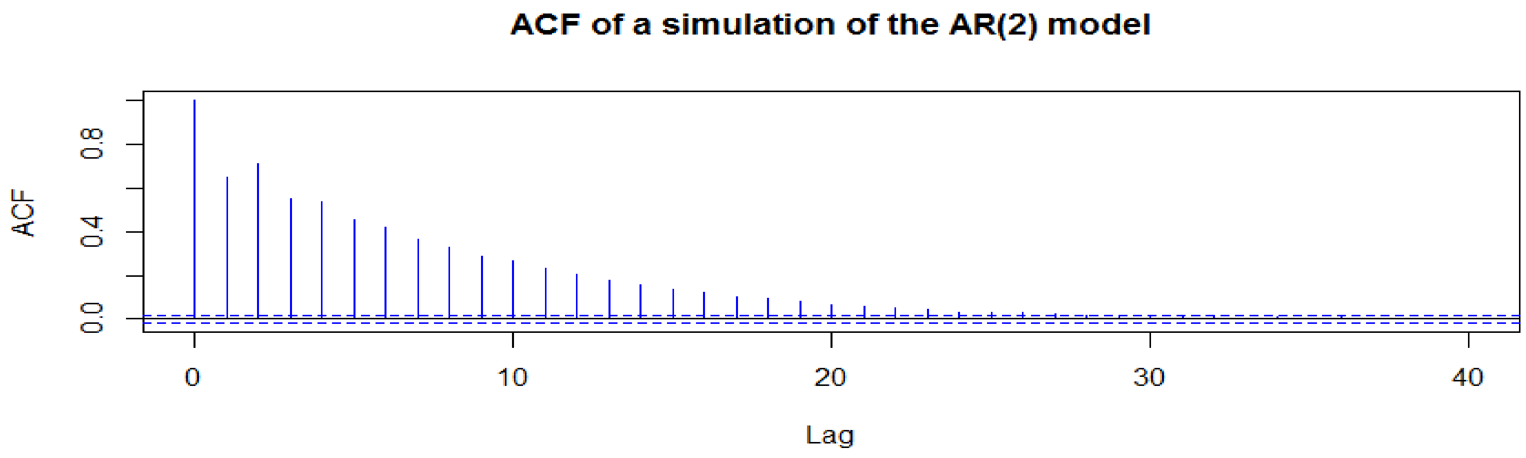
$For\, any\, k \geq 0,$

$$\rho(k) = \frac{4+\sqrt{6}}{8}\left(\frac{2+\sqrt{76}}{12}\right)^{k} + \frac{4-\sqrt{6}}{8}\left(\frac{2-\sqrt{76}}{12}\right)^{k}$$

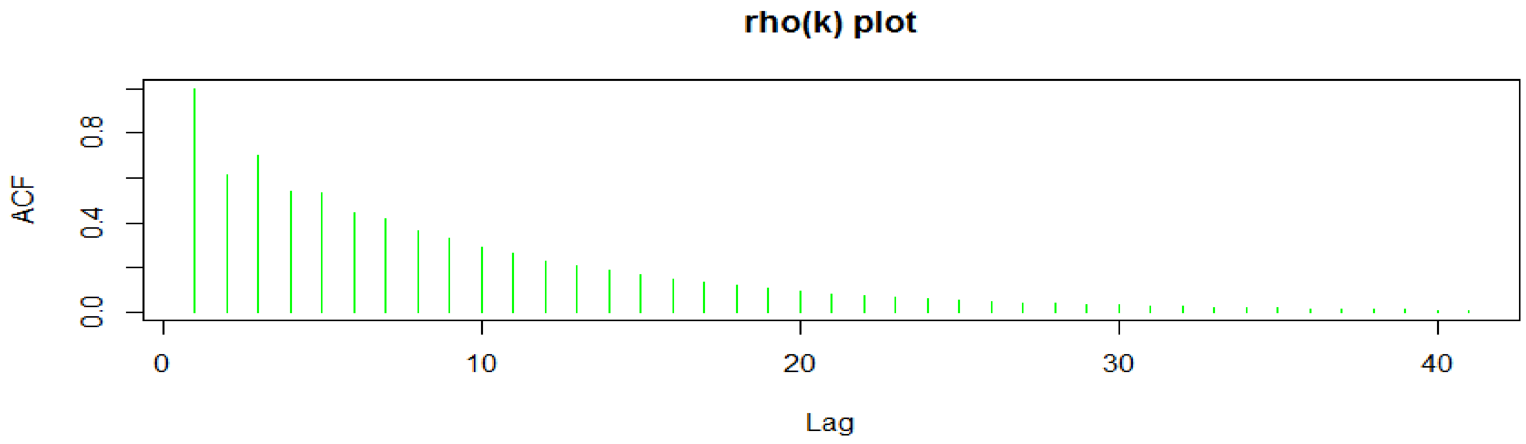
And

$$\rho(k) = \rho(-k)$$

Simulation



ACF of a simulation of the AR(2) model
ACF
0.0
0.4
0.8
0
10
20
30
40
Lag



rho(k) plot
ACF
0.0
0.4
0.8
0
10
20
30
40
Lag

What We’ve Learned

- Yule- Walker equations is set of difference equations governing ACF of the underlying AR process

- How to find the ACF of an AR process using Yule-Walker equations