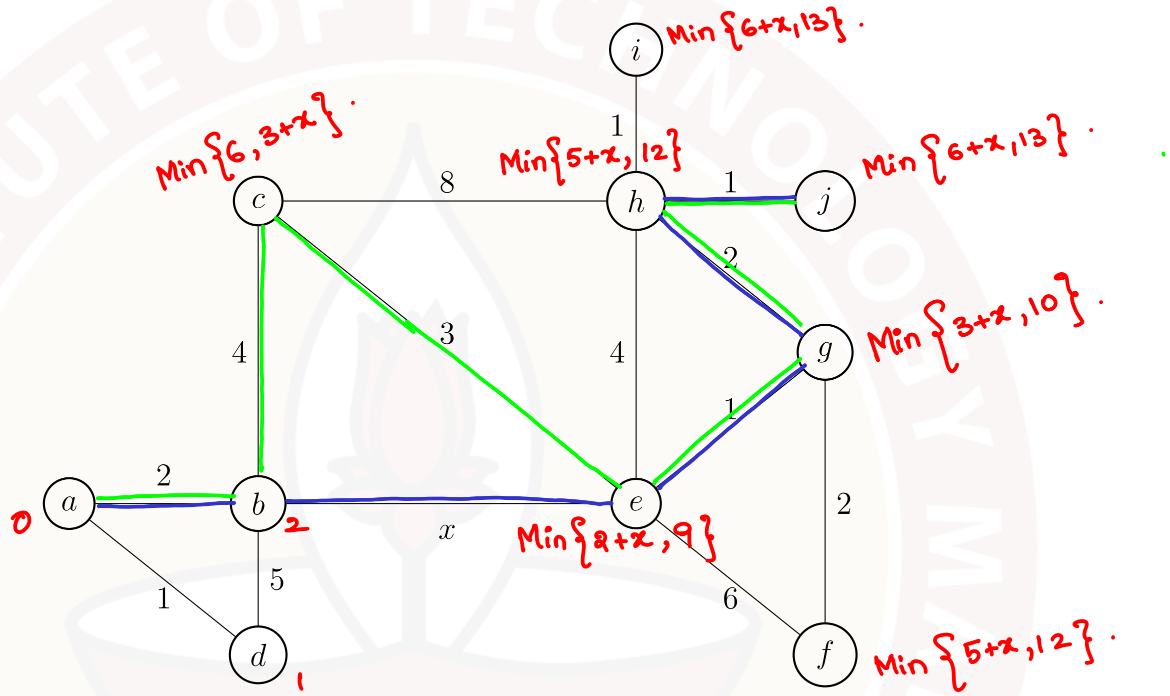
Mathematics for Data Science - 1
Graded Assignment
Week 11

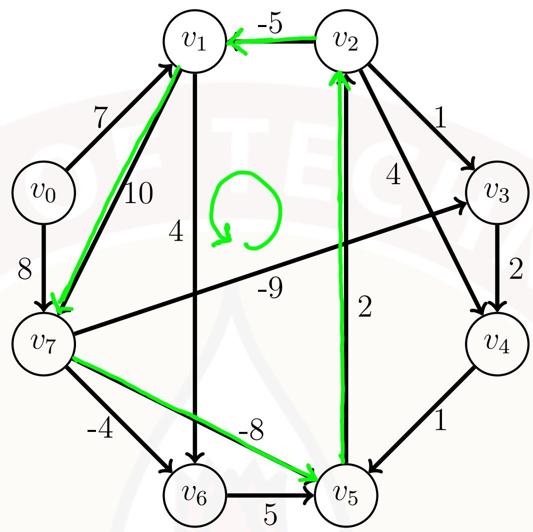
1. An undirected weighted graph G is shown below. Find the set of all positive integer values of x such that if we use Dijkstra's algorithm, there will be a unique shortest path from vertex a to vertex j that contains the edge (b, e).



(a) $\{x|x \in \mathbb{Z}, 0 < x < 8\}$
(b) $\{x|x \in \mathbb{Z}, 0 < x < 7\}$
(c) $\{x|x \in \mathbb{Z}, 0 < x < 6\}$
(d) $\{x|x \in \mathbb{Z}, 0 < x < 9\}$

Sol:- If (b,e) edge is not considered, then the green path shown in the figure will be the shortest path from vertex 'a' to vertex 'j'.
The length of the green path is $2+4+3+1+2+1 = 13$. (1)
Now, it is given that the shortest path from vertex 'a' to vertex 'j' contains edge (b,e)
So from (1), the length of the shortest path should be less than 13.
$\Rightarrow$ The length of [1] blue path $2+x+1+2+1 = x+6 < 13$
$\Rightarrow \boxed{x < 7}$.
Hence, the set of all positive integers values of x is $\{x \mid x \in \mathbb{Z}, 0<x<7\}$

2. An undirected graph G is shown below. Suppose we are trying to perform an algorithm to find the shortest path from vertex v_0 to v_4. Which of the following statements is (are) correct?



(a) Dijkstra's algorithm can be used to find the shortest path from v_0 to v_4.

(b) Bellman-Ford algorithm can be used to find the shortest path from v_0 to v_4 because there are negative weighted edges.

(c) The weight of the shortest path from v_0 to v_4 is 1.

✓(d) Bellman-Ford algorithm cannot be used to find the shortest path from v_0 to v_4 because there is a negative cycle in the given graph.

Sol:- observe that $v_1 \xrightarrow{10} v_7 \xrightarrow{-8} v_5 \xrightarrow{2} v_2 \xrightarrow{-5} v_1$ is a negative cycle

therefore, Bellman-Ford algorithm and Dijkstra's algorithm cannot be used.

3. Which of the following statements is (are) $\boxed{\text{INCORRECT?}}$

(a) In an undirected graph G, if all edges have different positive weights, then the minimum cost spanning tree of graph G is unique.

✓(b) If there is a cycle of weight 0 in a directed graph G, then we cannot find the shortest path using Bellman-Ford algorithm.

(c) Suppose an acyclic undirected graph G with n vertices has $n-1$ edges. Then the graph is connected.

✓(d) In a graph G, every edge with the minimum weight will be in the minimum cost spanning tree.

(a) let $G=(V,E)$ be the original graph.

* Suppose there are two different MCST's T_1 and T_2.
* As both the MCST's are different, the edges in the T_1 and edges in the T_2 are not same. This means that there is at least one edge that belongs to one MCST but not the other.
* Out of all the edges that are present in only one of the MCST, choose the minimum weight edge. Let 'e_1' be the minimum weight edge and also assume that it is in T_1 (we can also assume it in T_2).
* Now, As T_2 is an MCST, adding this edge e_1 to T_2 creates a cycle 'C'
* T_1 is an MCST so it does not contain a cycle, therefore cycle 'C' must have an edge 'e_2' that is not in T_1.
* observe that weight of edge 'e_1' is less than weight of edge 'e_2'.
* Replacing 'e_2' with 'e_1' in T_2 yields a spanning tree with a smaller weight, which contradicts that T_2 is an MCST.

so, option (A) is correct.

(b) Bellman-ford algorithm cannot be performed only if the graph has a negative cycle.

so, Bellman-ford algorithm can be performed on a graph G having a cycle of weight 0.

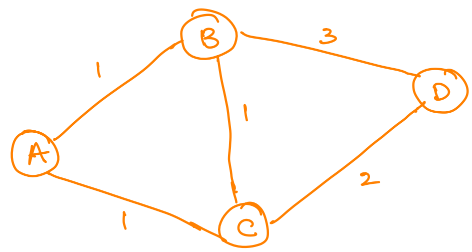
option (b) is incorrect.

(c) An acyclic undirected graph with n vertices and $n-1$ edges is a Tree and a Tree is a minimally connected graph

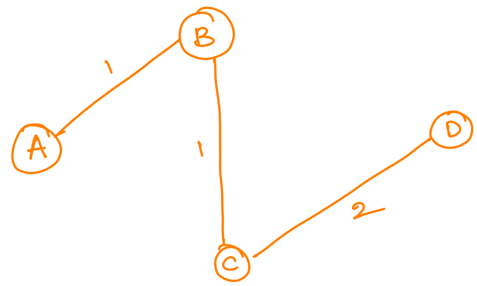
$\Rightarrow$ G is a connected graph.

so, option (c) is correct.

(d) Consider the below graph.



Here, minimum cost spanning tree is



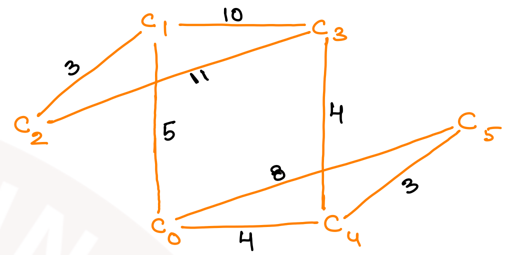
Therefore, edge (A,C) has weight '1' which is the minimum weight in the given graph but it is not in the minimum cost spanning tree.

$\Rightarrow$ Every edge in the graph having the minimum weight may not be in the minimum cost spanning tree.

so, option (d) is incorrect.

Use the following information for questions [4-5]:
A company has branches in each of six cities $C_0, C_1, ..., C_5$. The fare (in thousands of rupees) for a direct flight from C_i to C_j is given by the (i, j)th entry in the following matrix (∞ indicates that there is no direct flight):

$$\begin{array}{c} \\ C_0 \\ C_1 \\ C_2 \\ C_3 \\ C_4 \\ C_5 \end{array} \begin{array}{c} \begin{array}{cccccc} C_0 & C_1 & C_2 & C_3 & C_4 & C_5 \end{array} \\ \begin{bmatrix} 0 & 5 & \infty & \infty & 4 & 8 \\ 5 & 0 & 3 & 10 & \infty & \infty \\ \infty & 3 & 0 & 11 & \infty & \infty \\ \infty & 10 & 11 & 0 & 4 & \infty \\ 4 & \infty & \infty & 4 & 0 & 3 \\ 8 & \infty & \infty & \infty & 3 & 0 \end{bmatrix} \end{array}.$$

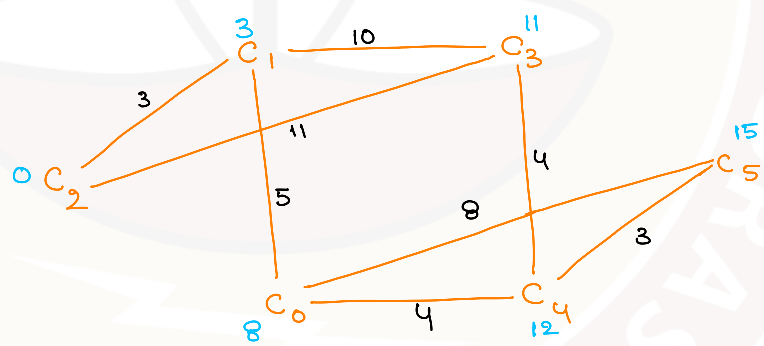


4. An employee of that company wanted to travel from the city C_2 to the city C_5. If he travelled by the cheapest route possible, then the total fare (in rupees) he paid for flight journey was

Sol :- We use Dijkstra's algorithm, to find the cheapest route possible from the city C_2 to the city C_5.

Start by assigning value 0 to C_2 which is the source vertex and compute values of each vertex using the edge weights.

After all the computations, we get the following values:



Finally we get, the value of C_5 is 15 that means he should pay ₹ 15,000 as fare if he has travelled by the cheapest route possible.

Cheapest route from C_2 to C_5 is

$$C_2 \longrightarrow C_1 \longrightarrow C_0 \longrightarrow C_4 \longrightarrow C_5.$$

Answer :- 15

5. If an inspection team member wanted to inspect all the branches of the company starting from C_2 and ending at C_5, visiting each brach exactly once, then which of the following routes should he choose in order to pay minimum fare for flight journey?

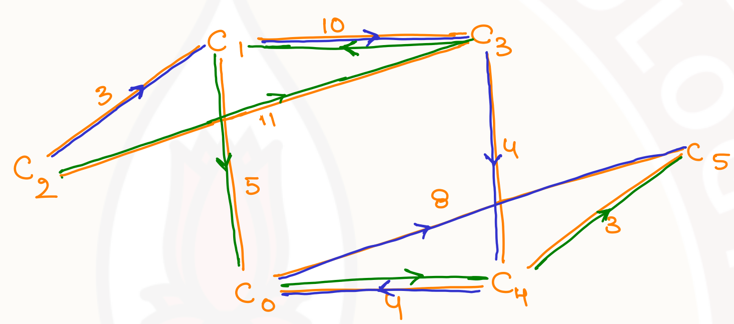
(a) $C_2 \longrightarrow C_3 \longrightarrow C_1 \longrightarrow C_0 \longrightarrow C_4 \longrightarrow C_5$

(b) $C_2 \longrightarrow C_1 \longrightarrow C_3 \longrightarrow C_4 \longrightarrow C_0 \longrightarrow C_5$

(c) $C_2 \longrightarrow C_3 \longrightarrow C_1 \longrightarrow C_4 \longrightarrow C_0 \longrightarrow C_5$

(d) Such a route is not possible.

Sol:- The possible routes are shown in the following graph:



Green route: $C_2 \longrightarrow C_3 \longrightarrow C_1 \longrightarrow C_0 \longrightarrow C_4 \longrightarrow C_5$

Blue route: $C_2 \longrightarrow C_1 \longrightarrow C_3 \longrightarrow C_4 \longrightarrow C_0 \longrightarrow C_5$

* The total fare he should pay if he choose green route is $11+10+5+4+3$
$= 33$.

* The total fare he should pay if he choose blue route is $3+10+4+4+8$
$= 29$.

Therefore, he has to choose blue route inorder to pay minimum fare.

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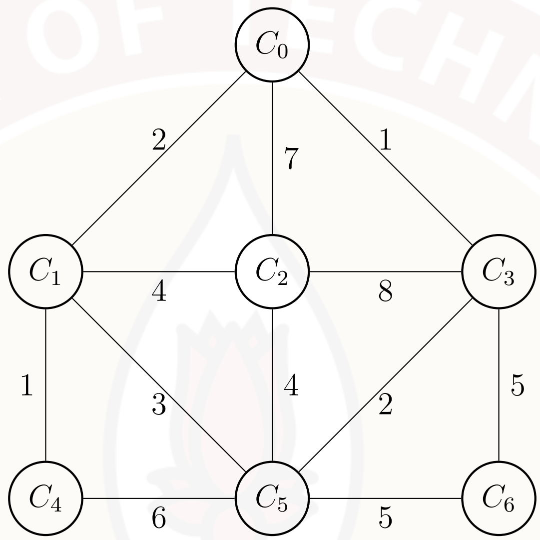
Use the following information for questions [6-7]:

Seven computers $\{C_0, C_2, ...C_6\}$ are linked by a network, and each link has a maintenance cost. The graph below shows how the computers are linked. Each node represents a computer, each edge represents a link between a pair of computers, and weights on the edges represent the maintenance cost (in hundreds of rupees). The goal is to pick a subset of links such that the total maintenance cost is minimum and the computers remain connected through the chosen links.

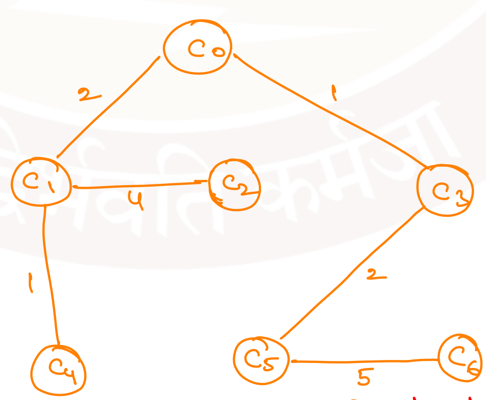


6. What is the total maintenance cost (in hundreds of rupees) of the optimum subset of links?

Sol:- We have to find the cost of minimum cost spanning tree (MCST) of this graph.

Now, perform prim's or Kruskal's algorithm to find MCST.

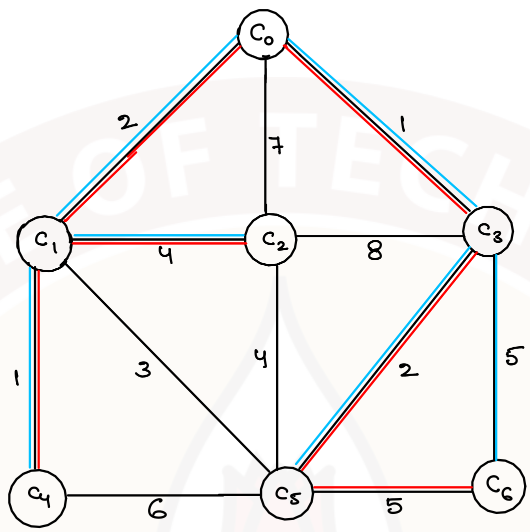
* A possible MCST is.



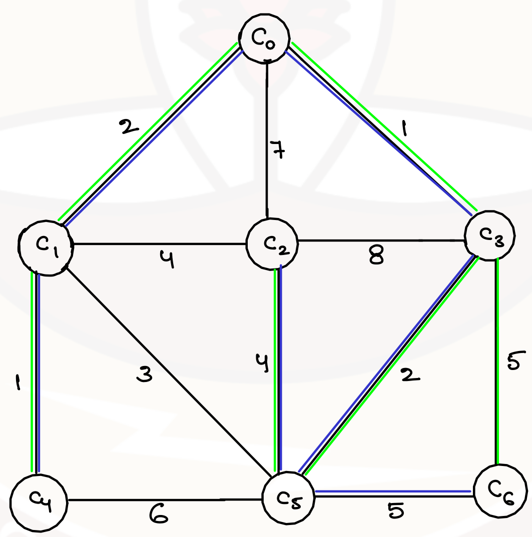
Therefore the total maintenance cost (in hundreds of rupees) of the optimal subsets of links is $1+2+1+2+4+5 = 15$.

Answer :- 15 (NOT 1500)

7. Find the number of different ways of choosing an optimum subset of links for the given graph.



∗ Here the graph with red links and blue links shows two different possible MCST's.



∗ Here the green links and dark blue links shows two other different possible MCST's of the graph.

∴ The number of different ways of choosing the optimal subset of links for the given graph are $2+2=4$.

Answer:- 4.

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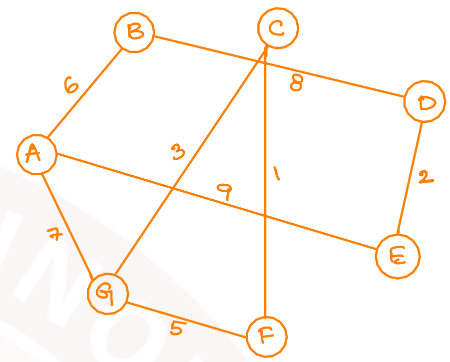
Use the following information for questions [8-9]:
Consider a weighted graph G with 7 vertices $\{A, B, C, D, E, F, G\}$, which is represented by the following adjacency matrix.

$$
\begin{array}{c c}
 & \begin{array}{ccccccc} A & B & C & D & E & F & G \end{array} \\
\begin{array}{c} A \\ B \\ C \\ D \\ E \\ F \\ G \end{array} &
\begin{bmatrix}
0 & 6 & 0 & 0 & 9 & 0 & 7 \\
6 & 0 & 0 & 8 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 3 \\
0 & 8 & 0 & 0 & 2 & 0 & 0 \\
9 & 0 & 0 & 2 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 5 \\
7 & 0 & 3 & 0 & 0 & 5 & 0
\end{bmatrix}
\end{array}.
$$



8. Suppose we perform Prim's algorithm on the graph G starting from vertex A to find an MCST. Then the order in which the vertices are added is

   (a) A, C, F, G, B, D, E

   (b) A, B, D, E, G, C, F

   (c) A, B, G, C, F, D, E

   (d) A, C, F, G, E, D, B

Sol:- Suppose TV is the set of MCST Vertices and TE is the set of MCST edges.

Initialize TV = ϕ and TE = ϕ

Now, given that we start with Vertex 'A'. So, A is added into the set TV.

$\therefore$ TV = $\{A\}$ and TE = ϕ.

choose an edge that is incident to Vertex 'A' and has a minimum weight.

(A, B) edge has weight 6 which is the minimum among the edges (A,B), (A, E), and (A,G).

$\therefore$ edge (A,B) is added to TE and Vertex 'B' is added to TV.

* TV = $\{A, B\}$

TE = $\{(A,B)\}$.

Now, choose an edge that is incident to Vertex 'A' & Vertex 'B' and has a minimum weight.

(A, G) edge has weight 7 which is the minimum among edges (A, E), (A,G), and (B,D).

$\therefore$ edge (A,G) is added to TE and Vertex 'G' is added to TV.

* TV = $\{A, B, G\}$

TE = $\{(A,B), (A,G)\}$.

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* Now we proceed further by choosing the edge which is incident to one of the vertices A, B & G that has minimum weight.

$W((C,G)) = 3$. is the minimum weight edge.

$TV = \{A, B, G, C\}$.

$TE = \{(A,B), (A,G), (C,G)\}$.

* like wise if we proceed further, finally we get

$TV = \{A, B, G, C, F, D, E\}$.

$TE = \{(A,B), (A,G), (C,G), (C,F), (B,D), (D,E)\}$

Therefore the order in which the vertices get added to the set TV is

A, B, G, C, F, D, E .

9. Suppose we perform Kruskal's algorithm on the graph G starting from vertex C to find an MCST. Which of the following edges are not added to the minimum cost spanning tree?

   ✓(a) (A, E)
   (b) (B, D)
   ✓(c) (G, F)
   (d) (A, G)

Sol:- In Kruskal's algorithm, we arrange all the edges in ascending order with respect to weights.

so, (C, F), (D,E), (C,G), (G,F), (A,B), (A,G), (B,D), (A,E) will be the order.

Now, we start constructing the MCST by adding the edges in the ascending order w.r.t. weights.

NOTE: When we add edge make sure that a cycle is not formed.

∴ (C,F), (D,E), (C,G) are added first and (G,F) cannot be added because (C,F), (C,G) & (G,F) forms a cycle.

In the same way, (A,E) is not added because (A,B), (B,D), (D,E) & (A,E) forms a cycle.

Therefore, edges (G,F) and (A,E) are not in the MCST.