Mathematics for Data Science - 1

Graded assignment solutions

Week - 6

1 Multiple Choice Questions (MCQ):

1. What should be subtracted from the polynomial $P(x) = 6x^4 + 5x^3 + 4x - 4$ to make it divisible by $2x^2 + x - 1$?

   1. $4x$
   2. $4x - 3$
   3. $6x - 3$
   4. $2x - 3$

   Answer: Option 2.

   Solution:

   Using 4 step division algorithm, we find the remainder when $P(x)$ is divided by $2x^2+x-1$. If we subtract the obtained remainder from $P(x)$ then the resultant polynomial will be divisible by $2x^2 + x - 1$.
   Now,

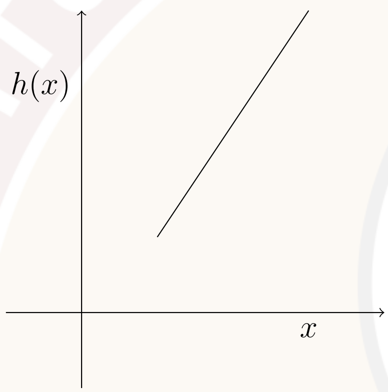
$$
\begin{array}{r|l}
 & 3x^2 + x + 1 \\
\hline
2x^2 + x - 1 & 6x^4 + 5x^3 \qquad + 4x - 4 \\
 & -6x^4 - 3x^3 + 3x^2 \\
\hline
 & 2x^3 + 3x^2 + 4x \\
 & -2x^3 - x^2 + x \\
\hline
 & 2x^2 + 5x - 4 \\
 & -2x^2 - x + 1 \\
\hline
 & 4x - 3
\end{array}
$$

   Therefore, when $P(x)$ is divided by $2x^2 + x - 1$, we get $4x - 3$ as the remainder.
   Hence, $\mathbf{4x - 3}$ should be subtracted from the polynomial $P(x) = 6x^4 + 5x^3 + 4x - 4$ to make it divisible by $2x^2 + x - 1$.

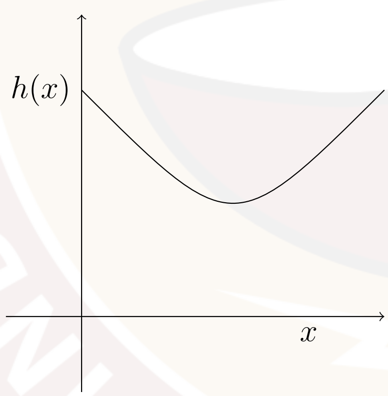
2. Table A-5.1 provides the information regarding some polynomials. Which is the most suitable (not exact) representation of $h(x)$ where $h(x)$ is known to be a polynomial in x, and if $h(x) = \frac{P(x)Q(x)-R(x)S(x)+S(x)P(x)}{P(x)+P(x)Q^2(x)}$?

Polynomial	Degree	Condition
$P(x)$	m	$m > 0$
$Q(x)$	n	$m > 2n > 0$
$R(x)$	k	$k = m - n$
$S(x)$	t	$t = 2n$

Table A-5.1



1.



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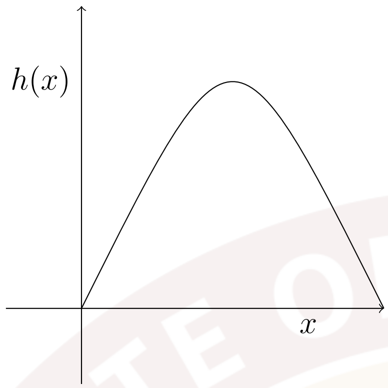
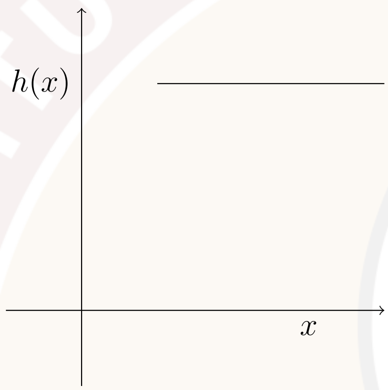
3. 

4. 

Answer: option 4

Solution:

Given, the degree of $P(x)$ is 'm' where $m > 0$, the degree of $Q(x)$ is 'n' where $m > 2n > 0$, the degree of $R(x)$ is 'k' where $k = m - n$, and the degree of $S(x)$ is 't' where $t = 2n$.

Also, $h(x) = \frac{P(x)Q(x) - R(x)S(x) + S(x)P(x)}{P(x) + P(x)Q^2(x)}$ and $h(x)$ is known to be a polynomial. The degree of $h(x)$ will be the difference between the degree of the numerator and the degree of the denominator. The degree of the numerator will be the degree of the term which has the highest degree in the numerator. Similarly, the degree of the denominator will be the degree of the term which has the highest degree in the denominator.

Now, the degree of the polynomial $P(x)Q(x)$ will be '$m + n$', the degree of $R(x)S(x)$ will be '$k + t = m - n + 2n = m + n$', and the degree of $S(x)P(x)$ will be '$t + m = 2n + m$'.

Therefore, the degree of the numerator (polynomial $P(x)Q(x) - R(x)S(x) + S(x)P(x)$) will be '$m + 2n$'.
Similarly, the degree of the denominator (polynomial $P(x) + P(x)Q^2(x)$) will be $m + 2n$.

As $h(x)$ is given to be a polynomial and also the degrees of the polynomials in the numerator and the denominator are same, we can conclude that the degree of $h(x)$ is zero i.e. $h(x)$ should be a constant.
So, option 4 is the most suitable representation of $h(x)$.

Use the following information to solve questions 3-5.

A manufacturing company produces three types of products A, B, and C from one raw material in a single continuous process. This process generates total solid wastes (W) (in kg) as $W(r) = -0.0001r^3 + 0.1r^2 + r$, where r is the amount of raw material used in kg. If instead, the company uses three different batch-processes (one batch process for one product) to produce the above products, then the amount of waste generated because of products A, B, and C are given as $W_A = -0.00001r^4 + 0.015r^3$, $W_B = -0.005r^3 + 0.05r^2$ and $W_C = 0.05r^2$ respectively. (See the Figure A-5.1 for the reference.)

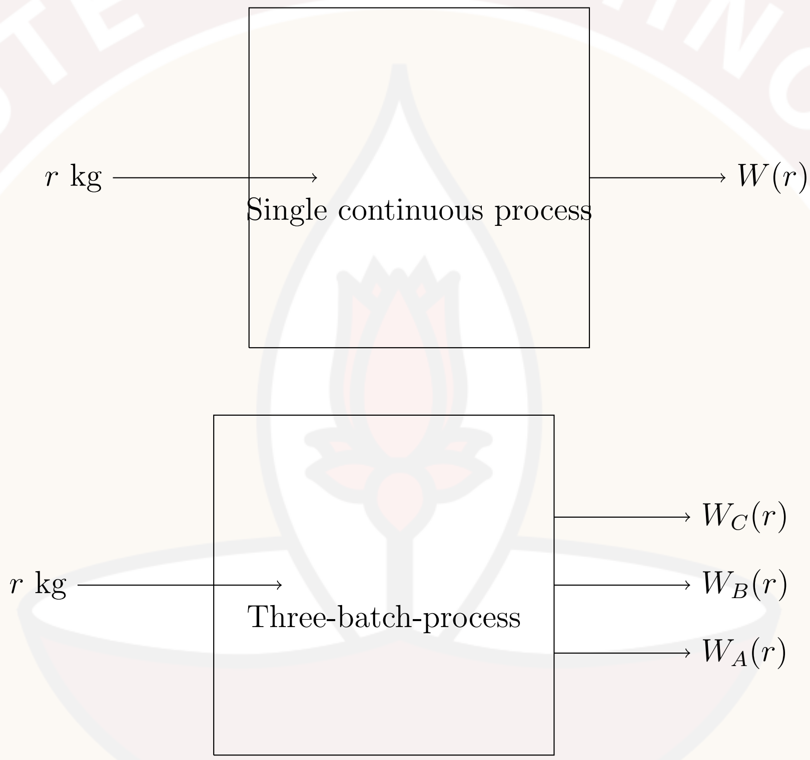


Figure A-5.1

3. What is the total amount of waste generated because of the three different batch-processes?

    1. $-0.00001r^4 + 0.01r^3 + 1.5r^2$
    2. $-0.00001r^4 + 0.015r^3 + 1.5r^2$
    3. $-0.00001r^4 + 0.01r^3 + 0.1r^2$
    4. $-0.00001r^4 + 0.01r^3 + 0.5r^2$

    Answer: Option 3

Solution:

The total amount of waste generated because of the three different batch-processes is

$$\begin{aligned} W_A + W_B + W_C &= -0.00001r^4 + 0.015r^3 - 0.005r^3 + 0.05r^2 + 0.05r^2 \\ &= \mathbf{-0.00001r^4 + 0.01r^3 + 0.1r^2} \end{aligned}$$

4. What is the ratio of the total waste generated by the three-batch-processes with respect to the single continuous process?

   1. $-0.01r$
   2. $-0.1r$
   3. $0.1r$
   4. $0.01r$

   Answer = Option 3.

   Solution:

   The total waste generated by the three-batch-processes is
   $W_A + W_B + W_C = -0.00001r^4 + 0.01r^3 + 0.1r^2$.
   The waste generated in the single continuous process is $W(r) = -0.0001r^3 + 0.1r^2 + r$.
   The ratio of the total waste generated by the three-batch-processes with respect to the single continuous process is

$$\begin{aligned}\frac{W_A + W_B + W_C}{W(r)} &= \frac{-0.00001r^4 + 0.01r^3 + 0.1r^2}{-0.0001r^3 + 0.1r^2 + r} \\ &= \frac{(0.1r)(-0.0001r^3 + 0.1r^2 + r)}{-0.0001r^3 + 0.1r^2 + r} \\ &= \boldsymbol{0.1r}\end{aligned}$$

5. Let the company wastes Rs. 5,000 in waste treatment when it uses the single continuous process by consuming 100 kg of raw material. If instead of continuous process the company uses the three-batch-processes, then how much extra amount (in Rs.) will the company have to pay for waste treatment with respect to the continuous process?

   1. 50,000
   2. 500
   3. 45,000
   4. 5,000
   5. 4,000

   Answer: Option 3

   Solution:

   As the ratio for waste generation (continuous to batch) is 10 we can calculate cost for waste management from batch process will be ten times of the continuous process.
   Therefore the cost for waste management from the batch process will be $5,000 \times 10 = 50,000$.
   So the the extra amount required is $50,000 - 5,000 = 45,000$.

2 Multiple Select Questions (MSQ):

6. Let $P(x)$ and $Q(x)$ be two non zero polynomials of degrees m and n respectively. If $f(x) = P(x) + Q(x)$, $g(x) = P(x)Q(x)$, and $h(x) = P(x)\{P(x)Q(x) + \frac{P(x)}{Q(x)}\}$, where $h(x)$ is known to be a polynomial in x, then choose the set of correct options.

   1. The degree of $f(x)$ is $m + n$.
   2. The degree of $g(x)$ is $m + n$.
   3. The degree of $f(x)$ is $max\{m, n\}$ if $m \neq n$, where $max\{m, n\}$ represents the maximum value from m and n.
   4. The degree of $h(x)$ is m^3.
   5. The degree of $h(x)$ is n^3.
   6. The degree of $h(x)$ is $2m + n$.

   Answer: Options 2, 3, and 6.

   Solution:

   Given, $P(x)$ and $Q(x)$ are two non zero polynomials of degree m and n respectively.
   Also, $f(x) = P(x) + Q(x)$.
   If $m > n$, then the degree of the polynomial $f(x)$ will be m, else if $m < n$, then the degree of the polynomial $f(x)$ will be n, else if $m = n$, then the degree of the polynomial will be less than or equal to m(or n).
   Therefore, we can conclude that the degree of the polynomial $f(x)$ is $max\{m, n\}$ if $m \neq n$, where $max\{m, n\}$ represents the maximum value from m and n.
   Hence, option 1 is incorrect, and option 3 is correct.

   Now, $g(x) = P(x)Q(x)$, the degree of the polynomial $g(x)$ will be the sum of the degrees of the polynomials $P(x)$ and $Q(x)$.
   Therefore, the degree of $g(x)$ is $m + n$. Hence, option 2 is correct.

   Finally, $h(x) = P(x)\{P(x)Q(x) + \frac{P(x)}{Q(x)}\} = (P(x))^2Q(x) + \frac{(P(x))^2}{Q(x)}$.
   The degree of the polynomial $(P(x))^2Q(x)$ will be $2m + n$ and as given that $h(x)$ is a polynomial implies $Q(x)$ divides $(P(x))^2$, so the degree of the polynomial $\frac{(P(x))^2}{Q(x)}$ will be $2m - n$.
   Since $2m + n > 2m - n$, the degree of the polynomial $h(x)$ is $2m + n$. Hence, options 4 and 5 are incorrect, and option 6 is correct.

7. Given a polynomial $P(x) = (2x+5)(1-3x)(x^2+3x+1)$, then choose the set of correct options.

    1. Coefficient of x^5 is 0.
    2. Coefficient of x^3 is -18.
    3. Degree of P is 4.
    4. Coefficient of x^3 is -13.
    5. Degree of P is 7.
    6. All of the above.

    Answer: Options 1 and 3.

Solution:

$$
\begin{aligned}
\text{Given, } P(x) &= (2x+5)(1-3x)(x^2+3x+1) \\
&= (2x+5-6x^2-15x)(x^2+3x+1) \\
&= (5-6x^2-13x)(x^2+3x+1) \\
&= 5x^2-6x^4-13x^3+15x-18x^3-39x^2+5-6x^2-13x \\
&= -6x^4-31x^3-40x^2+2x+5
\end{aligned}
$$

Option 1 is correct, because there is no x^5 term in the polynomial $P(x)$. So, the coefficient of x^5 is 0.
The degree of the polynomial $P(x)$ is 4. Hence, option 3 is correct and option 5 is incorrect.
The coefficient of x^3 is -31. Hence, options 2 and 4 are incorrect.

8. A sheet $ABCD$ of dimensions 10 ft x 3 ft is shown in Figure A-5.2. A box is made by removing two squares of equal dimensions $AEFG$ and $DHIJ$ and two rectangles of equal dimensions $BKLM$ and $CNOP$ respectively.

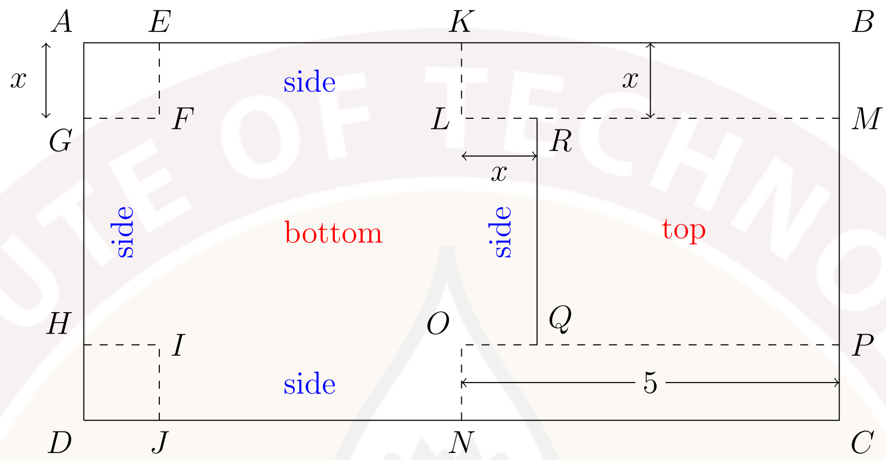


Figure A-5.2

Choose the set of correct options.

1. The volume of the box is $2x^2 - 23x + 30$.
2. The volume of the box is $2x^3 - 13x^2 + 15x$.
3. If $x = 0.5$, then the volume of the box is 5.625 cubic ft.
4. Value of x should always be greater than 0 but less than 1.5.

Answer: Options 2 and 4.

Solution:

From Figure A-5.2, the length of the box will be $EK = AB - KB - AE = 10 - 5 - x = 5 - x$, the breadth of the box will be $GH = AD - AG - HD = 3 - x - x = 3 - 2x$, and the height of the box will be $AE = x$.
Therefore, the volume of the box V given by length $\times$ breadth $\times$ height will be

$$V = (5 - x)(3 - 2x)(x)$$
$$V = (15 - 3x - 10x + 2x^2)(x)$$
$$V = 2x^3 - 13x^2 + 15x$$

Hence, options 1 is incorrect, and option 2 is correct.

If $x = 0.5$, then the volume of the box

$$
\begin{aligned}
V &= 2x^3 - 13x^2 + 15x \\
V &= 2(0.5)^3 - 13(0.5)^2 + 15(0.5) \\
V &= 2(0.625) - 13(0.25) + 7.5 \\
V &= 1.25 - 3.25 + 7.5 = 5.5
\end{aligned}
$$

Hence, option 3 is incorrect.

Now, to create a box, length of every side of box should always have a positive value. Therefore, $x > 0$ and $5 - x > 0 \implies x < 5$ and $3 - 2x > 0 \implies x < 1.5$. Combining all the conditions we get $x \in (0, 1.5)$.

3 Numerical Answer Type (NAT):

9. A curious student created a performance profile of his favourite cricketer as $R = -x^5 + 6x^4 - 30x^3 + 80x^2 + 70x + c$, where R is the total cumulative runs scored by the cricketer in x matches. He picked three starting values shown in Table A-5.2 and tried to find the value of c. If he uses Sum Squared Error method, then what will be the value of c?

No. of matches	Total score
1	120
2	285
3	361

Table A-5.2

Answer: -2

Solution:

Let us calculate the predicted cumulative runs scored by the player in the first three matches.
Substituting $x = 1, 2, 3$ in the given function, we get

$$\begin{aligned}
R(1) &= -(1)^5 + 6(1)^4 - 30(1)^3 + 80(1)^2 + 70(1) + c \\
&= -1 + 6 - 30 + 80 + 70 + c \\
&= 125 + c \\
R(2) &= -(2)^5 + 6(2)^4 - 30(2)^3 + 80(2)^2 + 70(2) + c \\
&= -32 + 96 - 240 + 320 + 140 + c \\
&= 284 + c \\
R(3) &= -(3)^5 + 6(3)^4 - 30(3)^3 + 80(3)^2 + 70(3) + c \\
&= -243 + 486 - 810 + 720 + 210 + c \\
&= 363 + c
\end{aligned}$$

Now, let us find the sum squared error of cumulative score for these three matches.

$$\begin{aligned}
\text{SSE} &= \sum_{n=1}^{3}(R(n)-y_n)^2, \text{where } y_n \text{ is the total cumulative score in } n \text{ matches.}\\
&= (R(1)-y_1)^2 + (R(2)-y_2)^2 + (R(3)-y_3)^2\\
&= (125+c-120)^2 + (284+c-285)^2 + (363+c-361)^2\\
&= (5+c)^2 + (c-1)^2 + (2+c)^2\\
&= 25+10c+c^2+c^2-2c+1+4+4c+c^2\\
&= 3c^2+12c+30
\end{aligned}$$

We have to find the value of c such that SSE becomes minimum, this is equal to the minimum value of the quadratic equation $3c^2+12c+30$.
We know that the minimum value of any quadratic function of form $f(x) = Ax^2+Bx+D$, occurs at $x = \frac{-B}{2A}$. Here, $A = 3, B = 12$
So, the minimum value of the quadratic equation $3c^2+12c+30$, occurs at
$c = \frac{-B}{2A} = \frac{-12}{2(3)} = -2$
Therefore, the minimum SSE is obtained when the value of c is $\mathbf{-2}$.

10. What is the minimum value of $x-$coordinate for the points of intersection of functions $f(x) = -x^5 + 5x^4 - 7x - 2$ and $g(x) = -x^5 + 5x^4 - x^2 - 2$?

Answer: 0

Solution:

At the points of intersection, observe that $f(x) = g(x)$.
Here, $f(x) = -x^5 + 5x^4 - 7x - 2$ and $g(x) = -x^5 + 5x^4 - x^2 - 2$.
Equating the functions we get,

$$\begin{aligned} -x^5 + 5x^4 - 7x - 2 &= -x^5 + 5x^4 - x^2 - 2 \\ -7x &= -x^2 \\ x^2 - 7x &= 0 \\ x(x-7) &= 0 \\ \implies x &= 0 \text{ (or) } x = 7 \end{aligned}$$

Therefore, the minimum value of x- coordinate for the points of intersection of functions $f(x)$ and $g(x)$ is **0**.