Week - 6
Practice Assignment
Graphs of polynomials
Mathematics for Data Science - 1

NOTE:
There are some questions which have functions with discrete valued domains (such as month or year). For simplicity, we treat them as continuous functions.

Syllabus Covered:

- Graphs of Polynomials: Identification and Characterization
- Zeroes of Polynomial Functions
- Graphs of Polynomials: Multiplicities
- Graphs of Polynomials: Behavior at X-intercepts
- Graphs of Polynomials: End behavior
- Graphs of Polynomials: Turning points
- Graphs of Polynomials: Graphing & Polynomial creation

1 Multiple Select Questions (MSQ):

1. Figure: M1W7PA-7.1 shows the graph of polynomial $p(x)$. Choose the set of correct options.

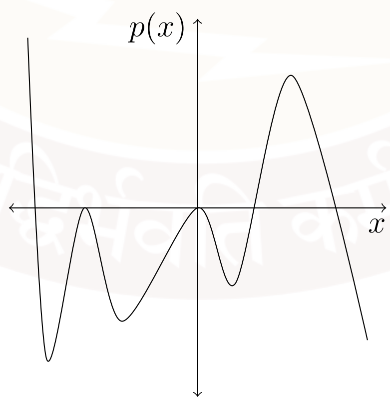


Figure: M1W7PA-7.1

- ◯ The degree of $p(x)$ is minimum 5.
- ◯ **The degree of $p(x)$ is minimum 7.**
- ◯ **x^4 could be a factor of $p(x)$.**
- ◯ $p(x)$ is an odd function.
- ◯ Multiplicity of a positive root of $p(x)$ can be even.
- ◯ **Multiplicities of zero and at least one negative root could be the same.**

Solution:

Option (b): Correct
Let a_1, a_2, a_3, a_4, and a_5 be the points at which the value of $p(x) = 0$ are as shown in Figure: M1W7PAS-7.1. At points a_1, a_4, a_5, the curve crosses in a linear fashion hence the degree should be 1, which accounts for total 3 degrees. At points a_2 and a_3, the curve bounces back, therefore it can have at least 2 degrees each, which accounts for 4 degrees together.
Therefore all together the degree of $p(x)$ is minimum 7.

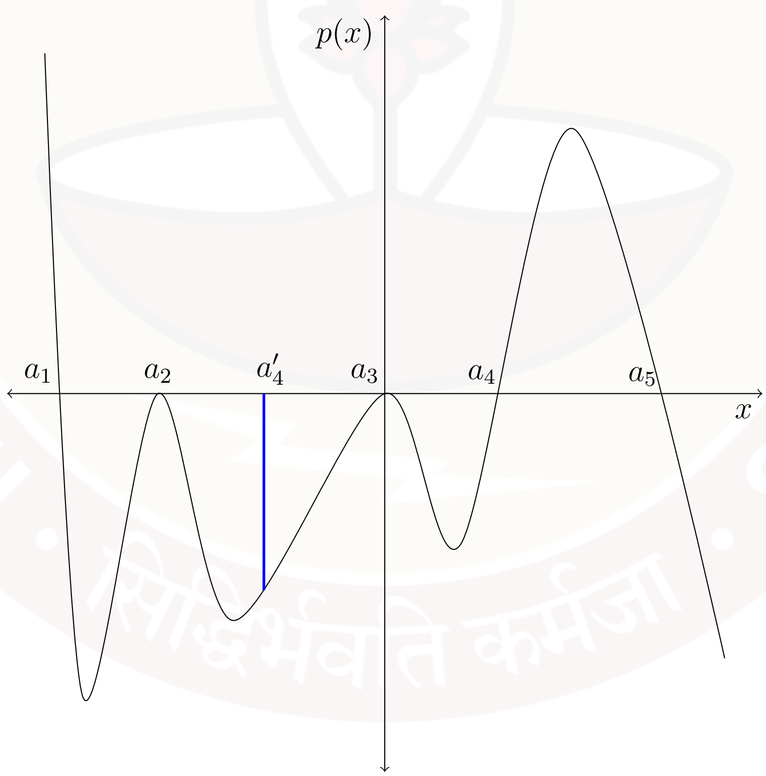


Figure: M1W7PAS-7.1

Option (c): Correct
Since at $x = 0$, the curve bounces back therefore, at point $(0, 0)$ the factor will be of the form x^n where n is an even number. Hence when $n = 4$, x^4 could be a factor of $p(x)$.

Option (d): Incorrect
A function is odd when $f(-x) = -f(x)$ for all $x \in \mathbb{R}$ which means the graph is symmetric with respect to the origin. But it is not the case as in Figure: M1W7PAS-7.1. Therefore, $p(x)$ is not an odd function.

Option (e): Incorrect
Multiplicities of a positive root of $p(x)$ cannot be even because at points a_4 and a_5 the curve crosses in a linear fashion hence the multiplicity should be 1.

Option (f): Correct
At a_2 the root is negative and at a_3 it has zero root and the curve bounces back at both point. Therefore at those points the factor will be of the form x^n where n is an even natural number and they can be same.

2. Choose the correct options.

   ◯ Every function must be either an odd function or an even function.

   ◯ **A function is an even function if $f(x) = f(|x|)$.**

   ◯ **$f(x) = 0$, for all $x \in \mathbb{R}$, is an even function.**

   ◯ Every even degree polynomial is an even function.

Solution:

Option (a): Incorrect
Some functions could be neither odd nor even. For example, $f(x) = x^3 + x^2$ then $f(-x) = -x^3 + x^2$.
It is not an even function because $f(-x) \neq f(x)$ and not an odd function because $f(-x) \neq -f(x)$.

Option (b): Correct
Given, $f(x) = f(|x|) \implies f(x) = f(x)$ or $f(x) = f(-x)$.
Function is an even function when $f(x) = f(-x)$.

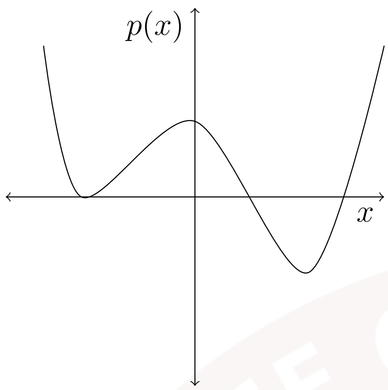
Option (c): Correct
As $f(x) = 0$, $f(-x) = 0$ and $f(x) = f(-x)$ therefore it is an even function.

Option (d): Incorrect
Because it may have other terms which will influence the nature of the function.
Example:$f(x) = x^4 + x^3$
It is an example of even degree polynomial but it is not even function because $f(-x) \neq f(x)$ where $f(-x) = x^4 - x^3$

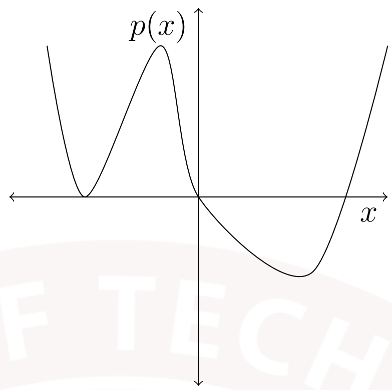
3. The polynomial $p(x) = a_n x^n + a_{n-1}x^{n-1} + ... + a_0$ has the following properties:

- $p(x)$ is an even degree polynomial.
- $p(x)$ has at least one positive real root and at least one negative real root.
- $(x-2)^n,\ max(n) = 2$ is a factor of $p(x)$.
- $p(0) \neq 0$

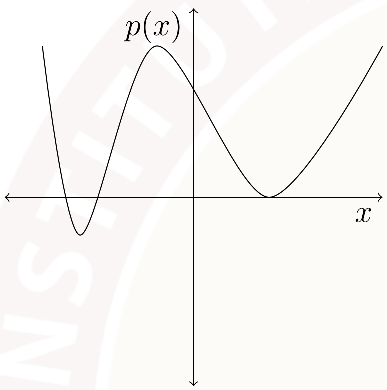
From the options given, choose the the possible representations of $p(x)$.
[Ans: Options C, E]



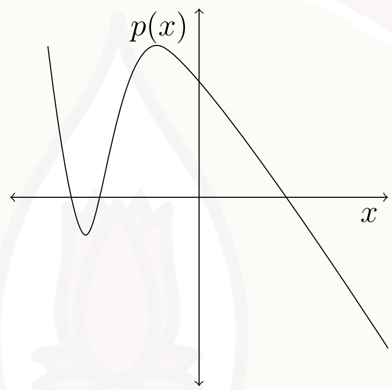
A.



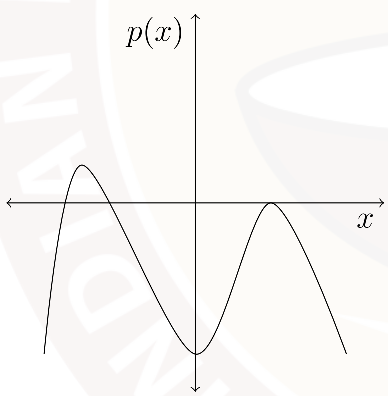
B.



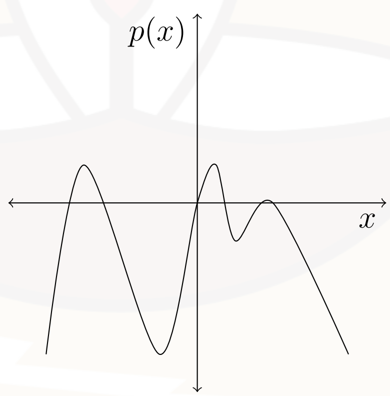
C.



D.



E.



F.

Figure: M1W7PA-7.2

Solution:

According to the condition $(x-2)^n$, $max(n) = 2$, options C, E and F qualify to be right options. But option F does not fulfil the condition $p(0) \neq 0$ condition. Hence options C and E are correct.

2 Multiple Choice Questions (MCQ):

4. Suppose a cubic polynomial f intersects the X-axis at $x = 1$ and $x = -2$. Moreover, $f(x) < 0$ when $x \in (0, 1)$, and $f(x) > 0$ when $x \in (-2, 0)$. Find out the equation of the polynomial.

   - ○ Inadequate information.
   - ○ $a(x^3 - x^2 - 2x),\ a > 0$
   - ○ $\boldsymbol{a(x^3 + x^2 - 2x),\ a > 0}$
   - ○ $a(x^3 + 3x^2 - x - 3),\ a < 0$

   Solution:

   A cubic polynomial can have at most three roots. It is given that f intersects the X-axis at $x = 1$ and $x = -2$ which accounts for the two factors $(x - 1)$ and $(x - (-2))$. Therefore, the equation is of the form $f(x) = a(x - b)(x - 1)(x + 2)$ where a and b are constants.
   Based on the end behavior of $f(x)$ two possible rough diagrams are shown in Figure: M1W7PAS-7.2.

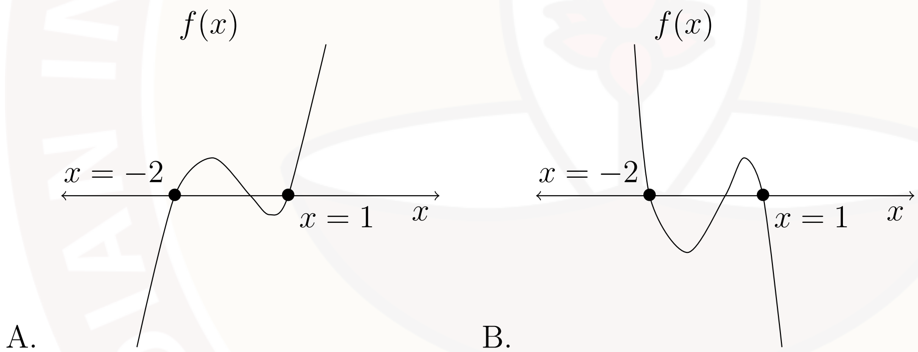


Figure: M1W7PAS-7.2

   Also $f(x) < 0$ when $x \in (0, 1)$, $f(x) > 0$ when $x \in (-2, 0)$ so only A in Figure: M1W7PAS-7.2 can represent the function. When $x \to \infty$, $f(x) = \infty$, and $x \to -\infty$, $f(x) = -\infty$, shows that $a > 0$.

   Clearly, the function is changing the sign at $x = 0$ which means $(x - 0)$ is a factor of $f(x)$ which means $b = 0$. Therefore,
   $f(x) = a(x)(x - 1)(x + 2) \implies \boldsymbol{f(x) = a(x^3 + x^2 - 2x).}$

5. The volume of a box V, varies with some variable x as $V(x) = x^3 + 12x^2 + 39x + 28$ cubic metres. If $(x + a)$ metre is the measurement of one side of the box, then choose the correct option for a.

   - ◯ $a = -1, 5, 3$
   - ◯ $a = 1, 5, 3$
   - ◯ $a = -7, -4, 1$
   - ◯ $a = 7, 2, 2$
   - ◯ $\boldsymbol{a = 7, 1, 4}$
   - ◯ $a = 28, 1, 1$

Solution:

We know that the volume of a box is determined by multiplying the lengths of sides of the box. If $(x + a)$ is the measurement of one side, then it will be a factor of the volume polynomial.
By hit and trial method, one of the roots of $V(x) = x^3 + 12x^2 + 39x + 28$ is -1. Hence $(x + 1)$ is one of the factors of the cubic polynomial $V(x)$. On dividing $V(x)$ by $(x + 1)$, we get $x^2 + 11x + 28$ which on factorization gives the other factors $(x + 7)$ and $(x + 4)$. Thus the possible values of a are 1, 4, and 7.

Use the following information for questions 6 and 7.
Ankita has to travel to various locations for advertising her company's products. The company reimburses her expenses such as accommodation, food etc. The company also blacklists an employee whenever the employee's expenditure in a given month exceeds ₹ 9000. The accounts department fits the data of Ankita's monthly expenditure to a polynomial $E(x)$ (in ₹) where x is the number of months since her joining the company. The polynomial fit is known to be applicable for a period of two years.

6. If $E(x) - 9000 = a(x-4.5)(x-12)(x-20),\ a > 0$, then how many times has Ankita been black listed in two years?

   - ◯ 7
   - ◯ 4
   - ◯ **11**
   - ◯ 3
   - ◯ 15
   - ◯ 9

   Solution:

   Company blacklists an employee whenever employee's expenditure in given month is more than 9000 which means if $E(x) > 9000 \implies E(x) - 9000 > 0$.
   On solving,

   $$\begin{aligned} E(x) - 9000 &> 0 \\ a(x-4.5)(x-12)(x-20) &> 0 \end{aligned}$$

   Clearly the zeros of the polynomial are 4.5, 12, and 20. The above polynomial is a cubic polynomial (odd degree polynomial) and $a > 0$, then using end behavior a rough plot is shown in Figure: M1W7PAS-7.3.

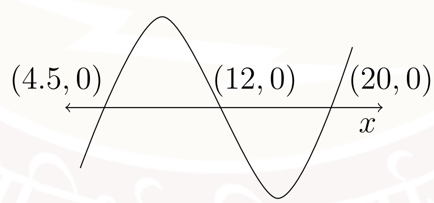
   

   Figure: M1W7PAS-7.3

   From the Figure: M1W7PAS-7.3 clearly $E(x) - 9000 > 0$ when $x \in (4.5, 12)$ and $x \in (20, 24]$ i.e, when the number of months are $x = 5,\ 6,\ 7,\ 8,\ 9,\ 10,\ 11,\ 21,\ 22,\ 23,$ and 24. Therefore, Ankita will be black listed 11 times in two years.

7. Choose the information that is not required to solve question number 6.

   - ◯ End behavior of $E(x)$.
   - ◯ Zeroes of the function.
   - ◯ Degree of the polynomial $E(x)$.
   - ◯ **Exact value of $\boldsymbol{a}$.**

   Solution:

   The end behavior of a polynomial function is the behavior of the graph of $E(x)$ as x approaches $+\infty$ or $-\infty$. So, the knowledge of sign of the leading coefficient is used to predict the end behavior of the function. In question number 6, the fact that $a > 0$ was crucial to solve the problem.

   Degree of the polynomial $E(x)$, is used to determine the maximum number of solutions it can have and also the number of times it will cross the $X-$axis when graphed. In question number 6 the degree is 3.

   Zeroes of the function are critical to determine where the function touches or crosses $X-$axis. In question number 6, three zeroes were given.

   Exact value of a in the question number 6 will simply increase the $y-$coordinate of the vertex value, which is not of concern in the above problem when $a > 0$. Thus this information is not required to solve question number 6.

8. An equipment shows the reading $y(x)$ upon applying load x (in tonnes). Starting from $x = 0$ tonne, the load is steadily increased and thus the reading $y(x)$ is also observed to increase. The first stage of failure is observed at a certain load x_1 after which increasing the load results in a decrease in the reading. The load is continually increased after the first stage of failure and the second stage of failure occurs at load x_2 where the reading reaches 80 and the equipment stops working.

   Use the information provided below and find the maximum load x_2 (in tonnes) that can be applied to this equipment so that it does not stop working.

   Useful information:

   (a) $y(x) - 80 = ax(bx^2 + cx + d)(x + 1)(x - 4)$

   (b) $c^2 - 4bd < 0$

   (c) $ab < 0$

   Choose the correct option.

   - ◯ 0
   - ◯ -1
   - ◯ **4**.
   - ◯ None of the above.

   Solution:

   As the reading y is a dependent function of x, it can not be less than zero as load x cannot be negative. Initially the equipment reading $y(x)$ increases as the load x increases but it starts to decrease as load increases after the 1^{st} failure i.e, when load $x = x_1$ and stops working when $y(x) = 80$. (see Figure: M1W7PAS-7.4).

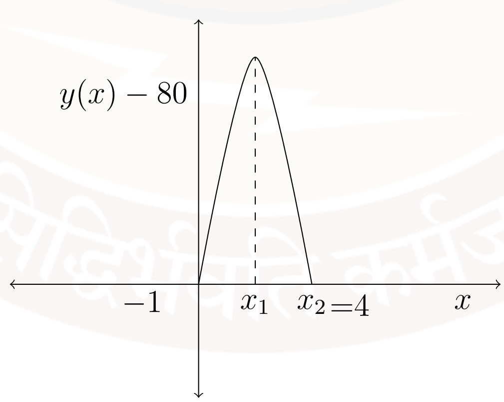


Figure: M1W7PAS-7.4

So the equipment works when $y(x) \geq 80 \implies y(x) - 80 \geq 0$.
On solving,

$$\begin{aligned} y(x) - 80 &\geq 0 \\ ax(bx^2 + cx + d)(x+1)(x-4) &\geq 0 \end{aligned}$$

Since the polynomial is of degree 5, it could have at most 5 roots.
Finding the roots,

$$\begin{aligned} x &= 0 \\ x &= -1 \\ x &= 4 \end{aligned}$$

For other two roots,

$$bx^2 + cx + d = 0$$

Given that $c^2 - 4bd < 0$ which indicates $bx^2 + cx + d$ has no real root. The curve represented by $y(x) - 80$ is shown in the Figure: M1W7PAS-7.4. Clearly, $y(x) - 80 \geq 0$ when $x \leq 4$. Thus maximum load will be $x_2 = 4$ (in tonnes) that can be applied to this equipment so that it does not stop working.

3 Numerical Type Questions (NAT):

9. Let A be the interval $[\alpha, \beta]$, where α and β are the smallest and the largest roots respectively of $f_1(x) = x^4 - 3x^3 - 9x^2 - 3x - 10$. If B is the largest proper subset of A such that elements of B are integers, then what is the cardinality of B? [Ans: 8]

Solution:

By hit and trial method, one of the roots is -2, thus the factor $(x + 2)$ when divides $f_1(x)$ gives $x^3 - 5x^2 + x - 5$. Now we can write it as $x^2(x - 5) + 1(x - 5)$, thus we get $(x - 5)(x^2 + 1)$.
$(x^2 + 1)$ has no real root, therefore the interval of A is all real values in $[-2,\ 5]$. Since B is the largest proper subset of A and it contains only integers, thus the elements of B will be -2, -1, 0, 1, 2, 3, 4, 5 and its cardinality is 8.

10. A train follows a path along the curve $y = x^3 + 12x^2 + 3x$ and Riya is travelling on a path $y = 0$. How many places can Riya catch the train? [Ans: 3]

Solution:

Riya takes the path $y = 0$ which means she is travelling along the $X-$axis. So, Riya can catch the train at $x-$intercepts of the curve $y = x^3 + 12x^2 + 3x$. On factorizing we get $x(x^2 + 12x + 3)$. So x is one factor and other factors can be obtained from $x^2 + 12x + 3$. The discriminant of $x^2 + 12x + 3$ is given by $b^2 - 4ac = 12 \times 12 - 4 \times 1 \times 3 = 132 > 0$, thus it will have 2 real and distinct roots different from 0. So altogether Riya can catch the train at 3 places.