Mathematics for Data Science - 1
Graded Assignment Solutions
Week 3 - March Qualifier, ’21

1 MULTIPLE CHOICE QUESTIONS:

1. Which of the following equations represents the general form of a straight line? (Ans: a)

   General form of a straight line ;
   $Ax + By + c$

   - ✓ ◯ $5x + 3y + 2 = 0$
   - ◯ $5x^2 + 3y + 3 = 0$ - not a straight line
   - ◯ $y = 3x + 2$ - slope form : $y = mx + c$
   - ◯ $x/2 + y/3 = 1$ - intercept form : $\frac{x}{a} + \frac{y}{b} = 1$

2. Distance between the lines $3y - 2x - 4 = 0$ and $4x - 6y + 7 = 0$ is (Ans: b)

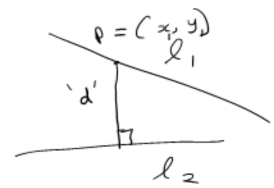
   - ◯ $\frac{15}{2\sqrt{13}}$
   - ✓ ◯ $\frac{1}{2\sqrt{13}}$
   - ◯ $\frac{15}{\sqrt{13}}$
   - ◯ $\frac{1}{\sqrt{13}}$

Soln. : To find the distance between two straight lines, consider a point on one of the lines and then find perpendicular distance of that point from the second line.

Consider $3y - 2x - 4 = 0$. To find a point on this line, put $y = 0$ $\Rightarrow$ $0 - 2x - 4 = 0$ $\Rightarrow$ $x = -2$

i.e. $(-2, 0)$ is a point on this line

Now, perpendicular distance 'd' of the point $(-2, 0)$ from the line $4x - 6y + 7 = 0$ is to be found.



distance 'd' of the point $p = (x_1, y_1)$ from the line $Ax + By + c = 0$ is : $d = \frac{|Ax_1 + By_1 + c|}{\sqrt{A^2 + B^2}}$

for the line $4x - 6y + 7 = 0$, $A = 4$, $B = -6$, $c = 7$

$\therefore d = \frac{|4 \times (-2) + -6 \times 0 + 7|}{\sqrt{4^2 + (-6)^2}} = \frac{|-8+7|}{\sqrt{16+36}} = \frac{1}{2\sqrt{13}}$

distance between two lines can also be found using the formula $d = \frac{|c_2 - c_1|}{\sqrt{A^2 + B^2}}$ if the equations are in the form $Ax + By + c_1 = 0$
$Ax + By + c_2 = 0$

3. In a fluid flow domain, stream lines and equipotential lines are perpendicular to each other. If the equation of a stream line is given by: $7x + 8y - 2 = 0$ and the point of intersection with the corresponding equipotential line is marked as $(3, 6)$ then the equation of the corresponding equipotential line is: (Ans: d)

   - ○ $7x - 8y - 5 = 0$
   - ○ $8x - 7y + 36 = 0$
   - ○ $8x + 7y + 18 = 0$
   - ✓ ○ $24x - 21y + 54 = 0$

   ADDITIONAL INFO: Stream lines represent the directions of flow and the equipotential lines join the points with equal velocity potential.

THIS QUESTION WILL BE REMOVED FROM WEEK 3 ASSIGNMENT.

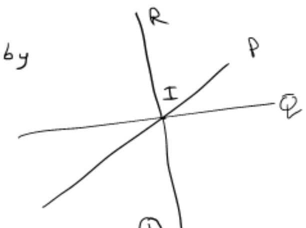
4. Consider the two lines $P := 3x - 4y + 5 = 0$ and $Q := 4x + 5y - 45 = 0$. There is another line $R := Q + \lambda P = 0$ passing through the intersection of these two lines. Value of the constant λ is ten times the length of the perpendicular distance of the line P from the origin $(0, 0)$. The exact equation of the line R is given by: (Ans: b)

- ○ $7x - 8y + 5 = 0$
- ✓○ $34x - 35y + 5 = 0$
- ○ $19x - 20y - 20 = 0$
- ○ $5x + 6y - 55 = 0$

ADDITIONAL INFO: In general, the equation of a line passing through the intersection of two lines $P := a_p x + b_p y + c_p = 0$ and $Q := a_q x + b_q y + c_q = 0$ is $Q + \lambda P = 0$, where the constant λ is real and can be determined based on additional conditions.

ADDITIONAL INFO: Equation of a line R passing through the intersection of two line P and Q is given by

$Q + \lambda P = 0$ where λ is real



Here, $Q + \lambda P = (a_q + \lambda a_p)x + (b_q + \lambda b_p) y + c_q + \lambda c_p = 0$

$= a_R x + b_R y + c_R$

$a_R = a_q + \lambda a_p$, $b_R = b_q + \lambda b_p$, $c_R = c_q + \lambda c_p$ —(1)

Let I be the intersection point with the coordinates (x_i, y_i)

Let us assume lines P, Q intersect exactly at one point I i.e. they are not parallel.

Solving for x, y from $P := 0$, $Q := 0$ we would obtain coordinates of I

$a_p x + b_p y + c_p = 0$

$a_p x = -c_p - b_p y$

$x = \dfrac{-c_p - b_p y}{a_p}$

$a_q x + b_q y + c_q = 0$

substituting $x = \dfrac{-c_p - b_p y}{a_p}$

$\dfrac{a_q}{a_p}(-c_p - b_p y) + b_q y + c_q = 0$

$\Rightarrow -\dfrac{a_q c_p}{a_p} + y\left[b_q - \dfrac{b_p a_q}{a_p}\right] + c_q = 0$

$\therefore y = \dfrac{c_p \dfrac{a_q}{a_p} - c_q}{b_q - b_p \dfrac{a_q}{a_p}} = \dfrac{c_p a_q - c_q a_p}{b_q a_p - b_p a_q}$

and $x = \dfrac{-c_p - b_p y}{a_p}$

$= \dfrac{-c_p - b_p\left(\dfrac{c_p a_q - c_q a_p}{b_q a_p - b_p a_q}\right)}{a_p} = \dfrac{-c_p b_q a_p + \cancel{c_p b_p a_q} - \cancel{b_p c_p a_q} + b_p c_q a_p}{a_p(b_q a_p - b_p a_q)}$

$= \dfrac{\cancel{a_p}}{\cancel{a_p}}\left(\dfrac{-c_p b_q + b_p c_q}{b_q a_p - b_p a_q}\right)$

$\therefore I = \left(\dfrac{-c_p b_q + b_p c_q}{b_q a_p - b_p a_q}, \dfrac{c_p a_q - c_q a_p}{b_q a_p - b_p a_q}\right)$

where $b_q a_p - b_p a_q \neq 0$

Now, line R must also pass through I

$a_R x + b_R y + c_R = 0$

substituting coordinates of I in the above equation,

$a_R\left[-c_P b_Q + b_P c_Q\right] + b_R\left[c_P a_Q - c_Q a_P\right] + c_R\left(b_Q a_P - b_P a_Q\right) = 0$

substituting the values of a_R, b_R and c_R from ① we can show that a linear relation holds between the coefficients a's, b's, c's of the 3 lines for any real λ.

Q4 Soln. : $P: = 3x - 4y + 5 = 0$, $Q: 4x + 5y - 45 = 0$

$R: = Q + \lambda P = 0 = (4 + 3\lambda)x + (5 - 4\lambda)y - 45 + 5\lambda$

To find λ,

Let distance of the line P from the origin $(0, 0)$ be 'd'

distance of a line $Ax + By + C = 0$ from a point (x_0, y_0) is given by

$$\frac{|Ax_0 + By_0 + C|}{\sqrt{A^2 + B^2}}$$

so,
$$d = \frac{|3\times 0 - 4\times 0 + 5|}{\sqrt{3^2 + 4^2}} = \frac{5}{\sqrt{25}} = \frac{5}{5} = 1$$

given that $\lambda = 10d = 10$

substituting $\lambda = 10$ in $R: = 0$ we get,

$R: = (4 + 30)x + (5 - 40)y - 45 + 50 = 0$

$\therefore R: = 34x - 35y + 5 = 0$

Answer the questions Q5, Q6 based on the following passage: In general, if p is the perpendicular distance of a line ℓ from the origin, then the equation of ℓ can be written in the form: $x\cos\alpha + y\sin\alpha = p$ where α is the angle (measured in the anticlockwise direction) made by the perpendicular, from the origin to ℓ, with the x-axis. Hints: $\sin 30 = 1/2, \sin 120 = \frac{\sqrt{3}}{2}, \cos 30 = \frac{\sqrt{3}}{2}, \cos 120 = -1/2$ (all angles are measured in degrees).

5. The equation of a line is given by $\frac{\sqrt{3}}{2}x + \frac{1}{2}y = 5$. Value of the angle (measured in the anticlockwise direction) made by the line with the x- axis (in degrees) is? (Ans: a)

   ✓ ○ 120

   ○ 60

   ○ 150

   ○ 30

$\frac{\sqrt{3}}{2}x + \frac{1}{2}y = 5$

$\sqrt{3}x + y = 10$

slope $m = -\sqrt{3} = \tan\theta \Rightarrow \theta = 120$ degrees

6. The perpendicular from the origin intersects the line $\frac{\sqrt{3}}{2}x + \frac{1}{2}y = 5$ at the coordinates: (Ans: b)

   ○ $(\frac{5}{2}, \frac{5\sqrt{3}}{2})$

   ✓ ○ $(\frac{5\sqrt{3}}{2}, \frac{5}{2})$

   ○ $(\frac{-5\sqrt{3}}{2}, \frac{5}{2})$

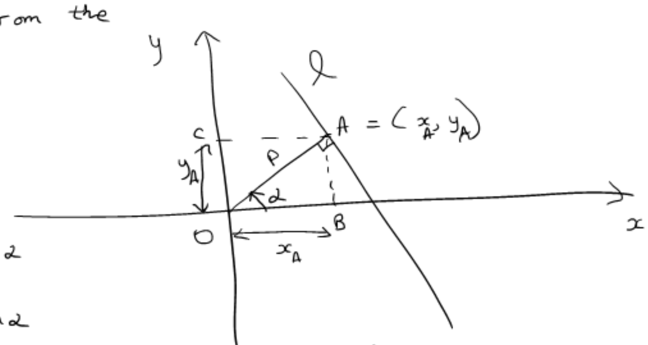
   ○ $(\frac{5\sqrt{3}}{2}, \frac{-5}{2})$

ADDITIONAL INFO:

Consider a line $\ell := ax + by + c = 0$

p = distance of the perpendicular from the origin O to the line ℓ

Let A be a point on ℓ with coordinates (x_A, y_A).



$OB = x_A$, $OC = y_A$

$\cos\alpha = \frac{OB}{OA} = \frac{x_A}{p} \Rightarrow x_A = p\cos\alpha$

$\sin\alpha = \frac{AB}{OA} = \frac{y}{p} \Rightarrow y_A = p\sin\alpha$

We know that perpendicular distance of O from a line ℓ is,

$\frac{|a\times 0 + b\times 0 + c|}{\sqrt{a^2+b^2}} = \frac{|c|}{\sqrt{a^2+b^2}} = OA = p$

Now, A is a point on ℓ.

$x_A\cos\alpha + y_A\sin\alpha = p\cos^2\alpha + p\sin^2\alpha = p(\cos^2\alpha + \sin^2\alpha) = p$

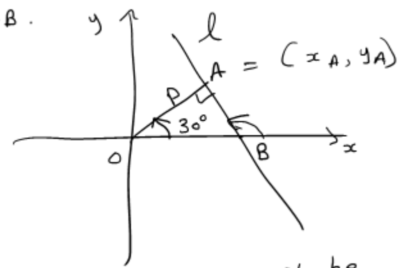
$(\because \cos^2\alpha + \sin^2\alpha = 1$ is a trigonometric identity$)$

Q 5 soln. :
based on passage

$l: \frac{\sqrt{3}}{2}x + \frac{1}{2}y = 5$ is of the form $x\cos\alpha + y\sin\alpha = p$

$\therefore p = 5, \quad \cos\alpha = \frac{\sqrt{3}}{2} \Rightarrow \alpha = 30$ degrees

Consider the right-angled triangle ΔOAB.



$180 = \angle ABO + 30 + 90$

($\because$ sum of interior angles of a triangle must be 180 degrees)

$\Rightarrow \angle ABO = 180 - 30 - 90$

$= 60$

$\therefore$ angle made by the line l with x-axis must be $180 - 60 = 120$ degrees

($\because$ angle in a line must be 180)

Q 6 soln. : perpendicular from origin O to the line l intersects it at the point A.

we know that, $x_A = p\cos\alpha = 5 \times \cos 30 = \frac{5\sqrt{3}}{2}$

$y_A = p\sin\alpha = 5 \times \sin 30 = 5 \times \frac{1}{2}$

$\therefore A = (x_A, y_A) = \left(\frac{5\sqrt{3}}{2}, \frac{5}{2}\right)$

2 MULTIPLE SELECT QUESTIONS:

7. Let X be the set of all straight lines in the coordinate plane. Let us define a relation R on X as follows, $R := \{(l_1, l_2) \in X \times X \mid \text{The lines } l_1 \text{and } l_2 \text{ intersect at least at one point}\}$. Which of the following statements is/are true? (Ans: a, c)

   ✓ ○ R is symmetric.

   ○ R is transitive.

   ✓ ○ R is reflexive.

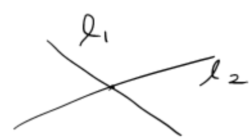
   ○ R is an equivalence relation.

Q 7 Soln. : Given that $R := \{(l_1, l_2) \in X \times X \mid l_1 \text{ and } l_2 \text{ intersect at least once}\}$

(a) if two lines l_1, l_2 intersect at least at one point, then l_2, l_1 also intersect i.e. order of the pair (l_1, l_2) is not significant. Hence, $(l_1, l_2) \in R$ and $(l_2, l_1) \in R$. $\therefore$ R is symmetric.

or

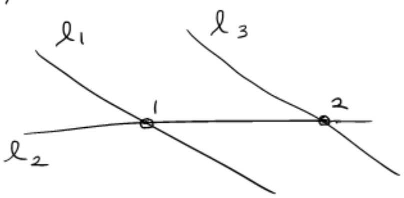
(b) Consider 3 lines l_1, l_2, l_3. Assume that l_1, l_2 intersect at a point.



Assume that l_2, l_3 also intersect each other.

Is it always true that l_1, l_3 must intersect at least once ?

one example is $\Rightarrow$



l_1, l_2 intersect at 1,
l_2, l_3 intersect at 2,
but l_1, l_3 never intersect.

So, at least for one combination of l_1, l_2, l_3, l_1 and l_3 do not intersect.

$\therefore$ R is not transitive

(d) A relation is an equivalence relation if and only if it satisfies symmetric, reflexive and transitive properties. Here, R is not transitive. Hence, R is not an equivalence relation.

(c) Consider one element l in the set X of straight lines. l intersect with itself at all the points. Thus, R is reflexive. { reflexive property : $(l, l) \in R$

8. Consider three points on the xy- coordinate plane A = $(3, -7)$, B = $(6, -14)$ and C = $(-9, 21)$. Which of the following statements is/are true? (Ans: a, c, d)

✓ ○ If we consider these three points to be the vertices of a triangle, then the area is 0.

○ If we consider these three points to be the vertices of a triangle, then the area is 168 square units.

✓ ○ The points A, B and C are collinear.

✓ ○ In general, if the area of a triangle considering any three points be zero then the three points are collinear.

Soln. : $A = (3, -7) = (x_A, y_A)$, $B = (6, -14) = (x_B, y_B)$, $C = (-9, 21) = (x_C, y_C)$

Area of the triangle ΔABC

$$= \frac{1}{2} \left| x_A (y_B - y_C) + x_B (y_C - y_A) + x_C (y_A - y_B) \right|$$

$$= \frac{1}{2} \left| 3(-14-21) + 6(21-(-7)) + (-9)(-7-(-14)) \right|$$

$$= \frac{1}{2} \left| -105 + 168 - 63 \right| = 0 \text{ units}$$

Hence, option (a) is correct and option (b) is wrong.

option (c) : To check if A, B, C are collinear, compare the slopes of the line segments AB, BC and AC. If they are all of equal value, then the 3 points must be on the same straight line.

$$\text{slope of } AB = \frac{-14-(-7)}{6-3} = \frac{-7}{3}$$

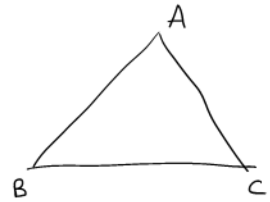
$$\text{slope of } BC = \frac{21-(-14)}{-9-6} = \frac{35}{-15} = \frac{-7}{3}$$

$$\text{slope of } AC = \frac{21-(-7)}{-9-3} = \frac{28}{-12} = -\frac{7}{3}$$

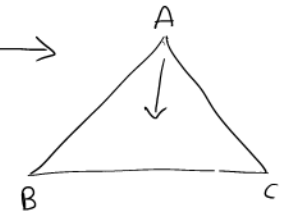
All the slopes are equal to $-7/3$. Hence, A, B, C are collinear. option (c) is also correct.

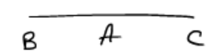
Option (d) : If the area of a triangle is zero, then the 3 points are collinear.

Consider 3 points A, B, C $\longrightarrow$



If the area of the $\Delta ABC = 0$, then the third vertex must lie on the opposite side.





Thus, the points A, B, C must be collinear. option (d) is also correct.

Additional Info : The statement in option (d) can be proved with mathematical rigour using the concept of linear Algebra.

$$\text{area of } \Delta ABC = \frac{1}{2}\left| x_A(y_B - y_C) + x_B(y_C - y_A) + x_C(y_A - y_B)\right|$$

The expression inside the modulus | | can be written in a matrix determinant form shown below

$$\begin{array}{c|ccc|} & \text{Col.1} & \text{Col.2} & \text{Col.3} \\ \text{Row 1} & x_A & x_B & x_C \\ \text{Row 2} & y_A & y_B & y_C \\ \text{Row 3} & 1 & 1 & 1 \end{array} = D$$

. without changing the value of D we can perform some Row and Column operations such that,

$$D = \begin{vmatrix} x_A & 1 & 1 \\ y_A & \dfrac{y_B - y_A}{x_B - x_A} & \dfrac{y_C - y_B}{x_C - x_B} \\ 1 & 0 & 0 \end{vmatrix}$$

$$= x_A(0 - 0) + 1\left(\frac{y_C - y_B}{x_C - x_B} - 0\right) + 1\left(0 - \frac{y_B - y_A}{x_B - x_A}\right)$$

If area of $\Delta ABC = 0$, then $D = 0 \Rightarrow \dfrac{y_C - y_B}{x_C - x_B} = \dfrac{y_B - y_A}{x_B - x_A}$.

$\Rightarrow$ slope of line segment BC = slope of line segment BA

Thus, the points A, B, C must be collinear.

NOTE : This additional info is not part of this qualifier course. It will be discussed in Maths Term 2.

9. You have been closely monitoring your bike's mileage recently. Here is a table showing two columns representing the amount paid for fuel and the corresponding mileage in Km. You have noted down the distance travelled each time when the fuel meter falls back to a fixed reference mark. By computing the SSE for each, identify which one of the given options is the best fit (with the least error)? Consider y to be the amount paid and x to be the corresponding distance in Km. (Ans: c)

(y_i) (x_i) $i = 1$ to 7

Amount paid (₹)	Distance (Km)
70	20
50	15
40	14
20	8
80	28
100	40
90	35

Table 1

- ○ $y = 2x + 35$
- ○ $y = x + 32$
- ✓ ○ $y = 1.5x + 36$ is the best fit.
- ○ $y = 3.5x + 5$

ADDITIONAL INFO: Do you see how laborious the calculations can get for such an elementary data set? Can you think of better ways to minimize the error function?

Solution : For a straight line $y = mx + c$, SSE is given by

$$SSE = \sum_{i=1}^{n} (y_i - \hat{y})^2 = \sum_{i=1}^{n} (y_i - mx_i - c)^2$$

where, n = total number of recordings of data. Here, $n = 7$

For each of the options compute SSE and then find the best straight line fit by identifying the option with the least SSE.

option 1 : $SSE = (70 - 2\times 20 - 35)^2 + (50 - 2\times 15 - 35)^2 + (40 - 2\times 14 - 35)^2 + (20 - 2\times 8 - 35)^2 + (80 - 2\times 28 - 35)^2 + (100 - 2\times 40 - 35)^2 + (90 - 2\times 35 - 35)^2$

Similarly find SSE for all the 4 options and observe that option (c) : $y = 1.5x + 36$ has the least SSE.

Additional Info : $SSE = \sum_{i=1}^{n} (y_i - y)^2$

$$= \sum_{i=1}^{n} (y_i - mx_i - c)^2 \quad - (1)$$

It is laborious to compute SSE for every option and compare the values. There must be a better way to find the values of m and c such that SSE is minimum.

If SSE is reduced in this form,

$$SSE = k_1^2 + (k_m m - d_m)^2 + (k_c c - d_c)^2 \quad - (2)$$

where, k_1, k_m, d_m, k_c, d_c are real constants, then SSE is minimum when $m = \frac{d_m}{k_m}$ and $c = \frac{d_c}{k_c}$.

In many practical cases, it may not be possible to reduce SSE given by (1) to the form given by (2) as there would be additional terms containing mc.

Thus, to minimize SSE in (1), multivariate differential calculus can be used. This topic is out of scope of the Maths Term 2. Qualifier syllabus. It will be discussed in

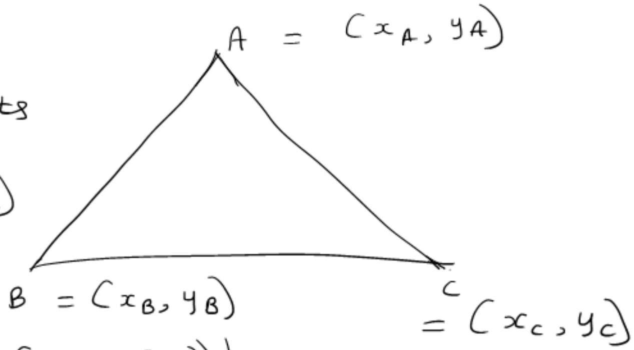
3 NUMERICAL ANSWER TYPE:

10. The area of a triangle is 5 square units. Two of its vertices are $(2, 1)$ and $(3, -2)$. The third vertex with coordinates (x, y) lies on the line $y = x + 3$. Given the condition that $7 < x + y < 11$, what is the value of $x + y$? $= 10$ units (Ans: 10)

Soln. :

Let $A = (2,1)$, $B = (3,-2)$

and $C = (x,y)$



Given, area $\Delta ABC = 5$ sq. units

$$= \frac{1}{2} \left| x_A (y_B - y_C) + x_B (y_C - y_A) + x_C (y_A - y_B) \right|$$

$$= \frac{1}{2} \left| 2(-2-y) + 3(y-1) + x(1-(-2)) \right|$$

$$= \frac{1}{2} \left| -4 - 2y + 3y - 3 + 3x \right| = \frac{1}{2} \left| y + 3x + 3 \right| = 5$$

$$\Rightarrow \left| y + 3x - 7 \right| = 10$$

$y + 3x - 7 = \pm 10$ since the expression within $| \ |$ can take both positive and negative value.

Case 1

$$y + 3x - 7 = 10$$

$$y + 3x = 17 \quad - (1.1)$$

Point $C = (x, y)$ also lies on the line $y = x + 3$

solving $y + 3x = 17$ &

$$y = x + 3 \quad - (2)$$

multiplying (2) by 3,

$$3y = 3x + 9 \quad - (3)$$

(1.1) + (3),

$$y + \cancel{3x} + 3y = 17 + \cancel{3x} + 9$$

$$4y = 26 \Rightarrow y = 26/4$$

$$x = y - 3 = \frac{26}{4} - 3 = \frac{14}{4}$$

$$\therefore \quad x + y = \frac{14 + 26}{4} = 10$$

Case 2

$$y + 3x - 7 = -10$$

$$y + 3x = -3 \quad - (1.2)$$

(1.2) + (3),

$$y + \cancel{3x} + 3y = -3 + \cancel{3x} + 9$$

$$4y = 6$$

$$y = \frac{6}{4}$$

$$x = y - 3 = \frac{6}{4} - 3 = \frac{-6}{4}$$

$$x + y = \frac{-6 + 6}{4} = 0$$

Since $7 < x + y < 11$,

$\therefore$ $x + y = 10$ is the answer.