Mathematics for Data Science - 1

Practice Assignment Solutions

Week-4

1. Multiple Choice Questions (MCQ):

1. What will be the equation of the tangent to the curve $f(x) = 2x^2 + 9x + 20$ at point (-3,11)?

   - ◯ $y = 3x$
   - ◯ $\boldsymbol{y = -3x + 2}$
   - ◯ $y = -3x + 20$
   - ◯ $y = -\frac{x}{3} + 2$
   - ◯ $y = \frac{x}{3} + 20$
   - ◯ $y = -\frac{x}{3}$

Solution:

A rough diagram is given in the Figure PS-4.1 .

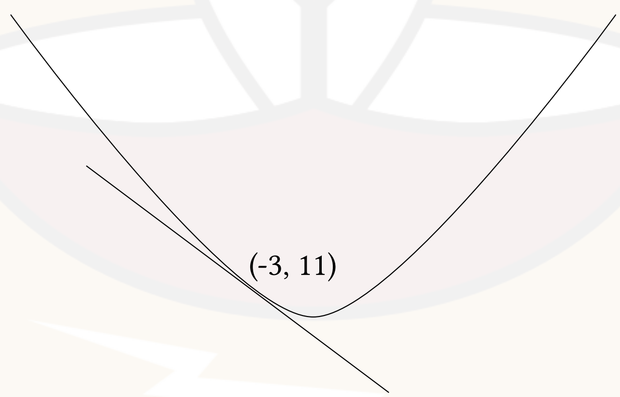


Figure PS-4.1

Let the equation of the tangent be $y = mx + c$, where m is the slope of the tangent line.
Note that m is also the slope of f at (-3, 11).
The slope of any quadratic function $g(x) = ax^2 + bx + c$, where $a \neq 0$ at x will be $2ax + b$.

Therefore, at $x = -3$,

$$m = 2ax + b \implies m = 2 \times 2 \times (-3) + 9 \implies m = -3$$

Since the tangent passes through the point (-3, 11), it should satisfy the equation of the tangent.

$$y = mx + c \implies 11 = -3 \times (-3) + c \implies c = 2.$$

So, the equation of the tangent will be $\boldsymbol{y = -3x + 2}$.

2. Find the length of the line segment on the straight line $y = 2$ bounded by the curve $y = 4x^2$.

- ○ $\frac{1}{\sqrt{2}}$
- ○ $\boldsymbol{\sqrt{2}}$
- ○ $1 + \sqrt{2}$
- ○ $1 + \frac{1}{\sqrt{2}}$

Solution:

Given $y = 4x^2$. Observe that, on comparing the above with the the general form of a quadratic function $f(x) = ax^2 + bx + c$, we have $b = 0$ which means Y-axis is the axis of symmetry. Also $c = 0 \implies$ the curve represented by this function will pass through the origin.
$-b/2a = 0$ and at $x = 0 \implies y = 0$ which means the vertex is at the origin and $a > 0 \implies$ the parabola is upward opened.
$y = 2$ is a constant function and it will pass through the point (0, 2). A rough diagram is given in the Figure PS-4.2

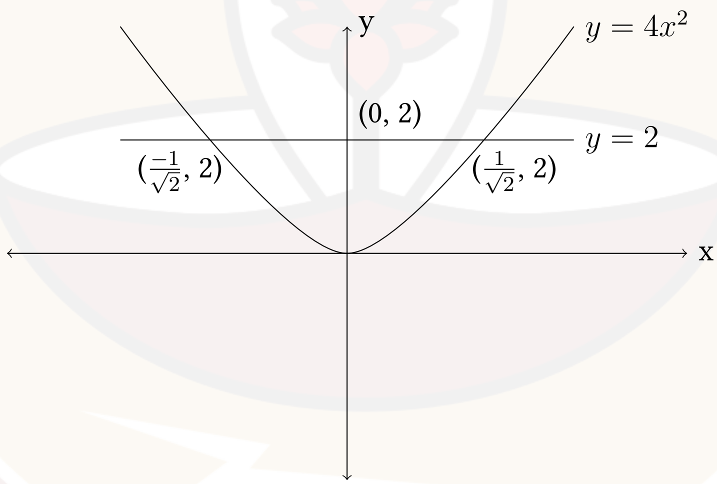
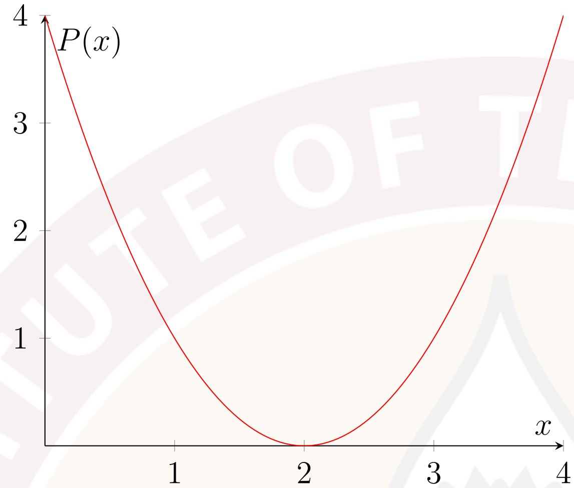
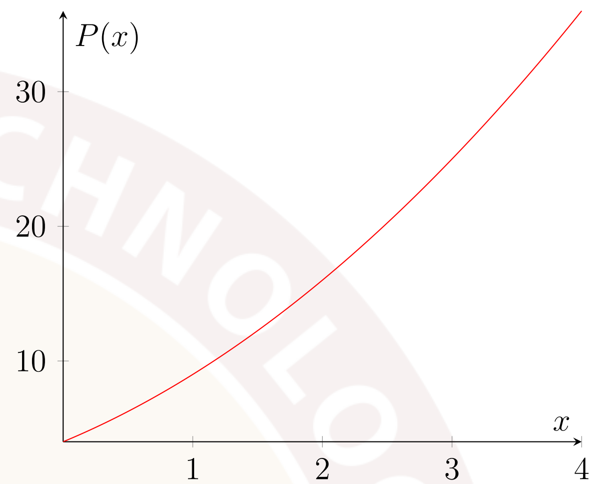


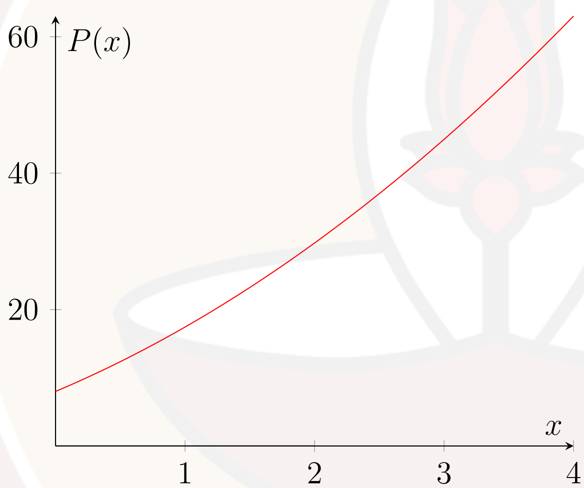
Figure PS-4.2

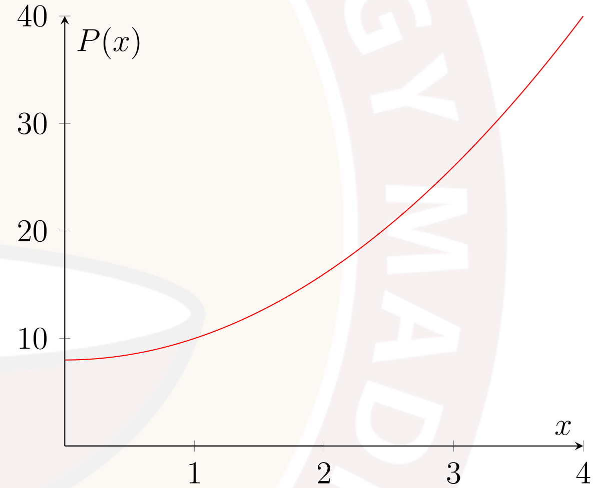
At the intersection points, $4x^2 = 2 \implies x = \pm\frac{1}{\sqrt{2}}$ which means the intersection points will be $(\frac{-1}{\sqrt{2}}, 2)$ and $(\frac{1}{\sqrt{2}}, 2)$.
Observe that these intersecting points will be the end points of the required line segment on the straight line $y = 2$.
Therefore, the length of the line segment on the straight line $y = 2$ bounded by the curve $y = 4x^2$ will be $\sqrt{(2-2)^2 + (\frac{1}{\sqrt{2}} - (\frac{-1}{\sqrt{2}})^2} = \sqrt{0 + (\frac{2}{\sqrt{2}})^2} = \sqrt{0 + (\sqrt{2})^2} = \boldsymbol{\sqrt{2}}$.

3. Mr. Mehta has two sons. Both sons send money to their father each month separately as $M_1(x) = (x-2)^2$ and $M_2(x) = (x+2)^2$ respectively. If x denotes the month, then choose the curve which best represents the total amount (P(x)) received by Mr. Mehta every month.

- ◯ 

- ◯ 

- ◯ 

- ◯ 

Solution:

Given,

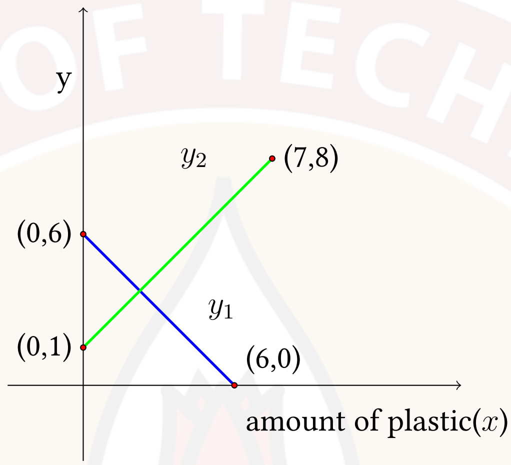
$$M_1(x) = (x-2)^2$$
$$M_2(x) = (x+2)^2.$$

So, the total amount received by Mr. Mehta is:

$$P(x) = M_1(x) + M_2(x) = (x-2)^2 + (x+2)^2 = x^2 - 4x + 4 + x^2 + 4x + 4$$
$$\Rightarrow P(x) = 2x^2 + 8.$$

In $P(x),\ b = 0$ which means Y-axis will be the axis of symmetry of the curve $p(x)$.
Now, the curve shown in the first option is not symmetric about the line $x = 0$. So, option 1 is incorrect.
The curve in the second option, passes through the origin but that is not the case for $P(x)$ as $x = 0 \implies P(x) = 8$. So, option 2 is incorrect.
The curve in the third option, does not pass through (4, 40). So, option 3 is also incorrect.
Now, the curve in the last option will pass through the points (0, 8), (1, 10), and (4,40). So, the curve in the fourth option will be the best curve that represents the total amount received by Mr.Mehta every month.

4. A civil engineer found that the durability d of the road she is laying depends on two functions y_1 and y_2 as follows: $d = ay_1y_2$ where $a > 0$. Functions y_1 and y_2 depend on the amount of plastic (x) mixed in bitumen, and their variations are shown in the graph given below. Find the values of functions y_1 and y_2 such that the durability of the road is maximum.



Solution:

Given, the durability of the road $d = ay_1y_2$.
From the given graph, the equations of the lines:

$$y_1 = 6 - x$$
$$y_2 = x + 1$$
$$\implies d = ay_1y_2 = a(6-x)(x+1) = -ax^2 + 5ax + 6a$$

Here $a > 0 \implies -a < 0$ which means the curve represented by d is open downward and the durability d of the road is the maximum at $x = \frac{-b}{2a} = \frac{-5a}{2(-a)} = \frac{5}{2}$.
Therefore, the value of $y_1 = 6 - x = 6 - \frac{5}{2} = \frac{7}{2}$ and the value of $y_2 = x + 1 = \frac{5}{2} + 1 = \frac{7}{2}$.

5. Let A be the set of all points on the curve defined by the function $f_1(x) = x^2 - x - 42$ and let B be the set of all points on the curve f_2 defined by the reflection of the curve f_1 with respect to X axis. If C is the set of all points on the axes then choose the correct option regarding the cardinality of set D where $D = (A \cap B) \cup (A \cap C) \cup (B \cap C)$.

   - ◯ infinite.
   - ◯ 8
   - ◯ **4**
   - ◯ 6
   - ◯ 2
   - ◯ zero.

   Solution:

   For the function $f_1(x) = x^2 - x - 42$, $a > 0 \implies$ opening upward, $-\frac{b}{2a} = \frac{1}{2} \implies x = \frac{1}{2}$ is the axis of symmetry.
   $x = 0 \implies f_1(0) = -42$ so, it will pass through the point (0, -42).
   The reflection of $f_1(x)$ with respect to $X-$ axis i.e. $f_2(x)$ will pass through the point (0, 42).
   For intersection points of both curves:
   Both the curves will be intersecting on same place on $X-$ axis as they are mirror image of each other around $X-$ axis. A rough diagram is given in the Figure PS-4.3

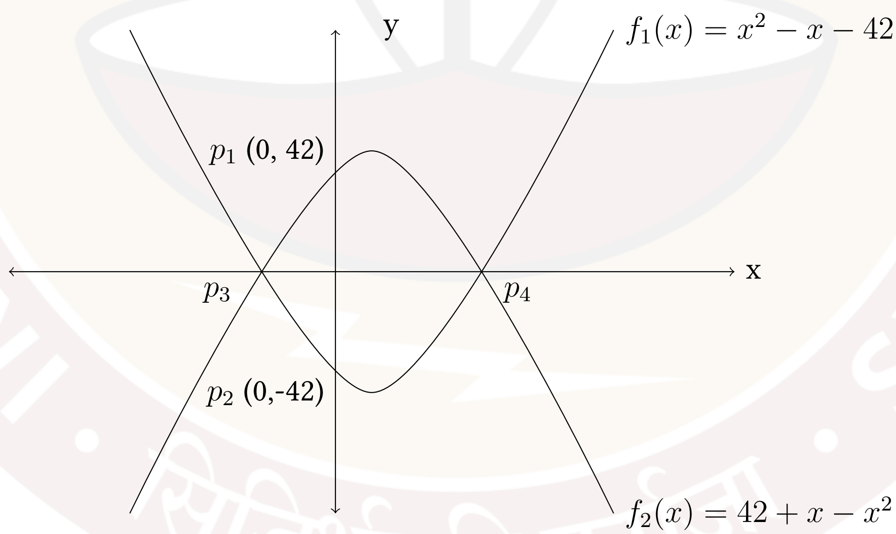


Figure PS-4.3

   Since A is the set of all points on the curve f_1, B will be the set of all points on the curve f_2 and C will be the set of all points on the X-axis or Y-axis.

From Figure PS-4.3,
$A\cap\ B$ is the set of all points which are on f_1 and f_2. Therefore, $A\cap\ B$ = $\{p_3, p_4\}$.
$A\cap C$ is the set of all points which are on the curve f_1 and on the X-axis or Y-axis. Therefore, $A\cap\ C$ = $\{p_3, p_4, p_2\}$.
$B\cap\ C$ is the set of all points which are on the curve f_2 and on the $X-$axis or $Y-$axis. Therefore, $B\cap\ C$ = $\{p_3, p_1, p_4\}$.
Now, $D = (A\cap\ B)\cup\ (A\cap\ C)\cup\ (B\cap\ C) = \{p_1, p_1, p_1, p_4\}$ and therefore, the cardinality of D is 4.

6. Let $f_1(x) = x^2 - 25$. Let A be the set of all points inside the region by the curves representing $f_1(x)$ and its reflection $f_2(x)$ with respect to $X-$ axis (excluding the points on curve). Choose the correct option.

- ◯ The cardinality of A is 2.
- ◯ The cardinality of A is 4.
- ◯ **$\boldsymbol{Y-}$ coordinates of the points in set A belong to the interval $\boldsymbol{(-25, 25)}$.**
- ◯ $Y-$ coordinates of the points in set A belong to the interval $[-25, 25]$.
- ◯ $X-$ coordinates of the points in set A belong to the interval $[-5, 5]$.
- ◯ $X-$ coordinates of the points in set A will be all real numbers because f_1 is a quadratic function.

Solution:

A rough diagram is shown in the Figure PS-4.4

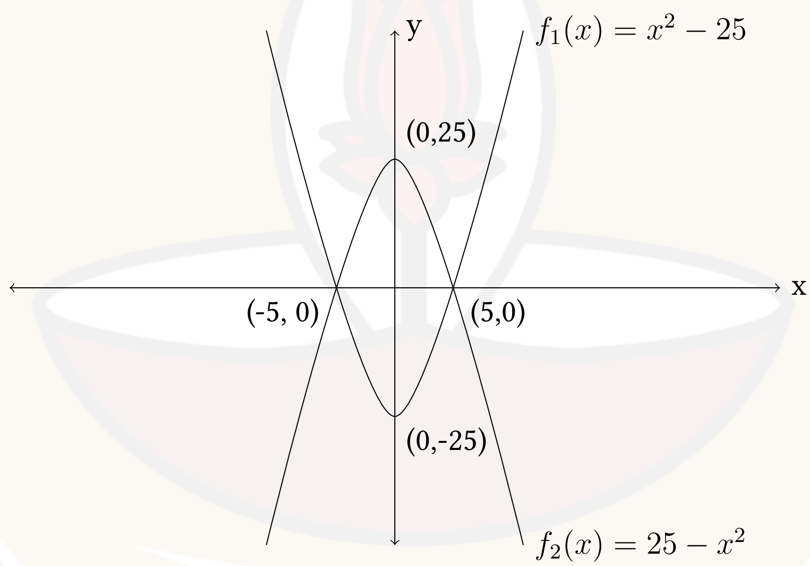


Figure PS-4.4

From the Figure PS-4.4, observe that the set A is infinite, because the region between the two curves f_1 and f_2 has infinitely many points. Therefore, the cardinality of A is not finite. So, options 1 and 2 are wrong.
Also, the region is in between the lines y = +25 and y = -25. Therefore, Y-coordinates of all the points in set A lie between -25 and +25 (-25 and +25 are excluded because they are points on the curves). So, option 3 is correct and option 4 is incorrect because -25 and +25 are included.

Also, the points in A are in between the lines x = -5 and x = +5 (-5 and +5 are excluded because they are points on the curves). Therefore, the X-coordinates of the points in set A belong to the interval (-5, 5). So, options 5 and 6 are incorrect.

2. Multiple Select Questions (MSQ):

1. **Choose the correct set of options regarding the function $\mathbf{f(x) = x^2 + 6x + 8}$**

   - ◯ $y = -3$ is the axis of symmetry.
   - ◯ **-2 and -4 are the zeroes of the above function**.
   - ◯ The maximum value of the above function is -1.
   - ◯ **Slope of the function at $(-3, -1)$ is zero**.
   - ◯ **$2x + 6$ is the slope of this curve at any given x**.
   - ◯ The function is symmetric around $x = 3$.

Solution:

Given, $f(x) = x^2 + 6x + 8$.
The axis of symmetry of $f(x)$ is $x = \frac{-b}{2a} = \frac{-6}{2} = -3$.
Therefore, $x = -3$ is the axis of symmetry of curve $f(x)$. So, options 1 and 6 are incorrect.

For zeros:

$$f(-2) = (-2)^2 + 6(-2) + 8 = 4 - 12 + 8 = 0$$
$$f(-4) = (-4)^2 + 6(-4) + 8 = 16 - 24 + 8 = 0$$

Hence, -2 and -4 are the zeros of the given function. So, option 2 is correct.

As $f(x)$ is an upward parabola, the maximum value of the function is + ∞ at $x = +\infty$. So, option 3 is incorrect.

Now, at $x = -3$, $f(x) = f(-3) = (-3)^2 + 6(-3) + 8 = 9 - 18 + 8 = -1$.
Therefore, the point (-3, -1) is the vertex of the given function. Also, the slope of the function at vertex is always 0. So, option 4 is correct.

We know that the slope of any given quadratic function $g(x) = ax^2 + bx + c; a, b, c \in \mathbb{R}$ at point $(x, g(x))$ is $2ax + b$. Here, $a = 1, b = 6$ and $c = 8$
Therefore, the slope of $f(x)$ is $2x + 6$ at any given x. So, option 5 is correct.

2. A quadratic function f is such that its value decreases over the interval $(-\infty, -2)$ and increases over the interval $(-2, \infty)$, and $f(0) = f(-4) = 23$. Then, f can be

   - ◯ $-3x^2 - 12x + 23$
   - ◯ $\mathbf{3x^2 + 12x + 23}$
   - ◯ $5(x-2)^2 + 3$
   - ◯ $\mathbf{5(x+2)^2 + 3}$
   - ◯ $\mathbf{ax^2 + 4ax + 23,\ a > 0}$
   - ◯ $ax^2 + 4ax + 23,\ a < 0$

Solution:

Given, the values of f decreases over $(-\infty, -2)$ and increases over interval $(-2, \infty)$. Also, $f(0) = f(-4) = 23$.
The curve f is roughly shown in the Figure PS-4.5.

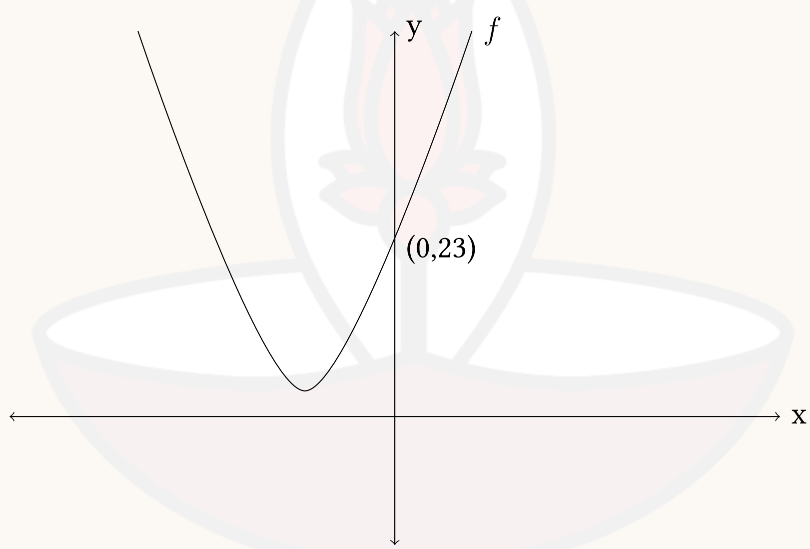


Figure PS-4.5

Suppose $f(x) = ax^2 + bx + c$, for any $a, b, c \in \mathbb{R}$.
We have $f(0) = 23 = a(0)^2 + b(0) + c = c \Rightarrow c = 23$.
Now, $f(-4) = 23 = a(-4)^2 + b(-4) + 23 \Rightarrow 16a - 4b = 0 \Rightarrow b = 4a$.
As the curve f which is shown in the Figure PS-4.5 is an upward parabola, the value of a should be positive.

Therefore, the quadratic function that satisfies the given conditions will be of the form $f(x) = ax^2 + 4ax + 23$, for all $a > 0$. So, option 5 is correct and option 6 is incorrect.

If $a = 3$, then f can be $3x^2 + 12x + 23$. So, option 2 is correct.

If $a = 5$, then f can be $5x^2 + 20x + 23 = 5(x + 2)^2 + 3$. So, option 4 is correct.

In option 1, the leading coefficient of the given function is $-3 = a < 0$. So, it is incorrect.

3. Suppose one root of a quadratic equation of the form $ax^2 + bx + c = 0$, with $a, b, c \in \mathbb{R}$, is $2 + \sqrt{3}$. Then choose the correct set of options.

   - ○ **There can be infinitely many such quadratic equations.**
   - ○ There is no such quadratic equation.
   - ○ There is a unique quadratic equation satisfying the properties.
   - ○ $\boldsymbol{x^2 - 4x + 1 = 0}$ **is one such quadratic equation**.
   - ○ $x^2 - 2x - 3 = 0$ is one such quadratic equation.

Solution:

Given, $2 + \sqrt{3}$ is a root of $ax^2 + bx + c = 0$. One root of the quadratic equation is known. The other root can be any real number k.
For each value of k we will have a different quadratic equation. Therefore, there can be infinitely many quadratic equations that have $2 + \sqrt{3}$ as a root. So, option 1 is correct and options 2,3 are incorrect.

Now, option 4 is correct because the function value (at $x = 2 + \sqrt{3}$) is

$$(2 + \sqrt{3})^2 - 4(2 + \sqrt{3}) + 1 = 4 + 4\sqrt{3} + 3 - 8 - 4\sqrt{3} + 1 = 0$$
$$\Rightarrow 2 + \sqrt{3} \text{ is a root of } x^2 - 4x + 1 = 0.$$

Option 5 is incorrect because the function value (at $x = 2 + \sqrt{3}$) is

$$(2 + \sqrt{3})^2 - 2(2 + \sqrt{3}) - 3 = 4 + 4\sqrt{3} + 3 - 4 - 2\sqrt{3} - 3 = 2\sqrt{3} \neq 0$$
$$\Rightarrow 2 + \sqrt{3} \text{ is not a root of } x^2 - 2x - 3 = 0.$$

4. A company's profits are known to be dependent on the months of a year. The profit pattern (in lakhs of Rupees) from January to December is $P(x) = -2x^2 + 25x$. Here, x represents the month number, starting from 1 (for January) and ending at 12 (for December). On this basis, choose the correct option.

   - ◯ **The maximum profit in a month is Rs.78 lakhs.**
   - ◯ The maximum profit in a month is Rs.78.125 lakhs.
   - ◯ The maximum profit in a month is Rs.77 lakhs.
   - ◯ **The maximum profit is recorded in June.**
   - ◯ The profit in December is 144 lakhs.
   - ◯ None of the above.

Solution:

The profit of the company is given as $P(x) = -2x^2 + 25x$. Observe $P(x)$ is downward open. So, the maximum profit will be recorded at vertex.
The X-Coordinate of the vertex is $x = \frac{-b}{2a} = \frac{-25}{2(-2)} = 6.25$
So, the vertex lies between the lines $x = 6$ and $x = 7$
Therefore, the maximum profit will be recorded in the month of June($x = 6$) or July($x = 7$).
The profit(in lakhs of Rupees) in June is

$$P(6) = -2(6)^2 + 25(6) = -72 + 150 = 78$$

and profit(in lakhs of Rupees) in July is

$$P(7) = -2(7)^2 + 25(7) = -98 + 175 = 77$$

Therefore, the maximum profit of Rs.78 lakhs is recorded in the month of June. So, options 1 and 4 are correct.

The profit (in lakhs of Rupees) in December is

$$P(12) = -2(12)^2 + 25(12) = -288 + 300 = 12$$

So, option 5 is incorrect.

5. Raghav sells 2000 packets of bread for Rs. 20000 each day, and makes a profit of Rs. 4,000 per day. He finds that if the cost price increases by Rs. x per packet, he can increase the selling price by Rs. $2x$ per packet. However, when this price increase happens, he loses $200x$ of his customers. Choose the correct options.

   - ◯ **For the maximum profit per day, cost price is Rs. 12 per packet**.
   - ◯ For the maximum profit per day, cost price is Rs. 4 per packet.
   - ◯ For the maximum profit per day, the sale price increases by Rs. 4 per packet.
   - ◯ For the maximum profit per day, Raghav will lose 400 customers.
   - ◯ **The maximum difference in profit per day could be Rs. 3200**.
   - ◯ The maximum difference in profit per day could be Rs. 7200.

Solution:

The selling price of bread $\frac{20000}{2000}$ = 10 Rupees per packet.
We know that, selling price - cost price = profit $\Rightarrow$ 20000 - cost price = 4000 $\Rightarrow$ cost price per day= 16000.
Therefore, the cost price is = $\frac{16000}{2000}$ = 8 Rupees per packet.

Now, if the cost price of each packet increases to $8+x$ and the selling price of each packet is increased to $10+2x$, then the customers left will be $2000-200x$.

So, the total profit (say P) in terms of x:

profit = (selling price of each packet- cost price of each packet) x (number of customers)

$$\implies P(x) = \{(10+2x)-(8+x)\} \times (2000-200x)$$
$$\implies P(x) = (2+x)(2000-200x)$$
$$\implies P(x) = 4000+1600x-200x^2.$$

The maximum profit occurs at $x = -\frac{b}{2a}$ = $-\frac{1600}{2(-200)}$ = 4.

Hence, for the maximum profit per day:

cost price per packet = $8+x$ = 8 + 4 = **12**.
sale price per packet = $10+2x$ = 10 + 8 = **18**
The customers he loses = $200x$ = 200(4) = **800**.
Maximum profit = $4000+1600x-200x^2$ = $4000+1600(4)-200(4)^2$ = **7200**

Therefore, maximum difference in profit = 7200 - 4000 = **3200 Rupees**.
So, the options 1 and 5 are correct.

3. Numerical answer type(NAT):

1. A farmer has a wire of length 576 metres. He uses it to fence his rectangular field to protect it from animals. If he fences his field with four rounds of wire, and the field has the maximum area possible to accommodate such a fencing, what is the area (in square metres) of the field?

   Solution:

   Suppose, the length of the rectangular field is 'l' metres and breadth of the rectangular field is 'm' metres. So, the perimeter of the rectangular field will be $2(l+m)$.
   Now, as he fences his field with four rounds of wire, we have four times the perimeter of the field which, in turn, is equal to the length of the wire. i.e,

$$4(2(l+m)) = 576$$
$$\Rightarrow l+m = \frac{576}{8} = 72$$
$$\Rightarrow m = 72 - l$$

Area of field (A) = lm

$$\Rightarrow A = l(72-l) = 72l - l^2$$

The maximum area of the field (A_{max})= $-\frac{b^2}{4a} + c$

$$A_{max} = -\frac{72^2}{4\times(-1)} + 0$$

$$\boldsymbol{\Rightarrow A_{max} = 1296 \text{ square metres}}$$

2. Consider the quadratic function $f(x) = x^2 - 2x - 8$. Two points P and Q are chosen on this curve such that they are 2 units away from the axis of symmetry. R is the point of intersection of axis of symmetry and the $X - axis$. And S is the vertex of the curve. Based on this information, answer the following:
   (a) What is the height of $\triangle PQR$ taking PQ as the base?
   (b) What is the height of $\triangle PQS$ taking PQ as the base?

Solution:

The axis of symmetry of $f(x)$ is $x = \frac{-b}{2a} = \frac{-(-2)}{2(1)} = 1$ and two units away points will be $x = 1 + 2 = 3$ and $x = 1 - 2 = -1$.
At $x = 3 \Rightarrow f(x) = -5$ and at $x = -1 \Rightarrow f(x) = -5$. Also, the vertex of the curve is (1, -9).
A rough diagram can be drawn with this information as shown in Figure PS-4.6.

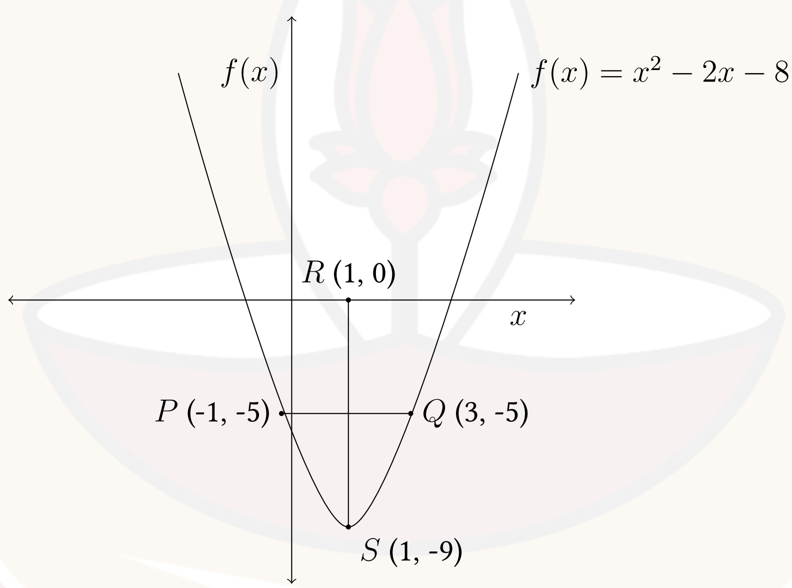


Figure PS-4.6

(a) From the above figure, the height of $\triangle PQR$ taking PQ as the base will be the distance between lines $y = 0$ and $y = -5$ and that is equal to 0 - (-5) = 5 units.

(b) From the above figure, the height of $\triangle PQS$ taking PQ as the base will be the distance between lines $y = -5$ and $y = -9$ and that is equal to (-5) - (-9) = 4 units.