Week - 6
Assignment
Mathematics for Data Science - 1

NOTE:
There are some questions which have functions with discrete valued domains (such as month or year). For simplicity, we treat them as continuous functions.

1 Multiple Select Questions (MSQ):

Consider the following graphs:

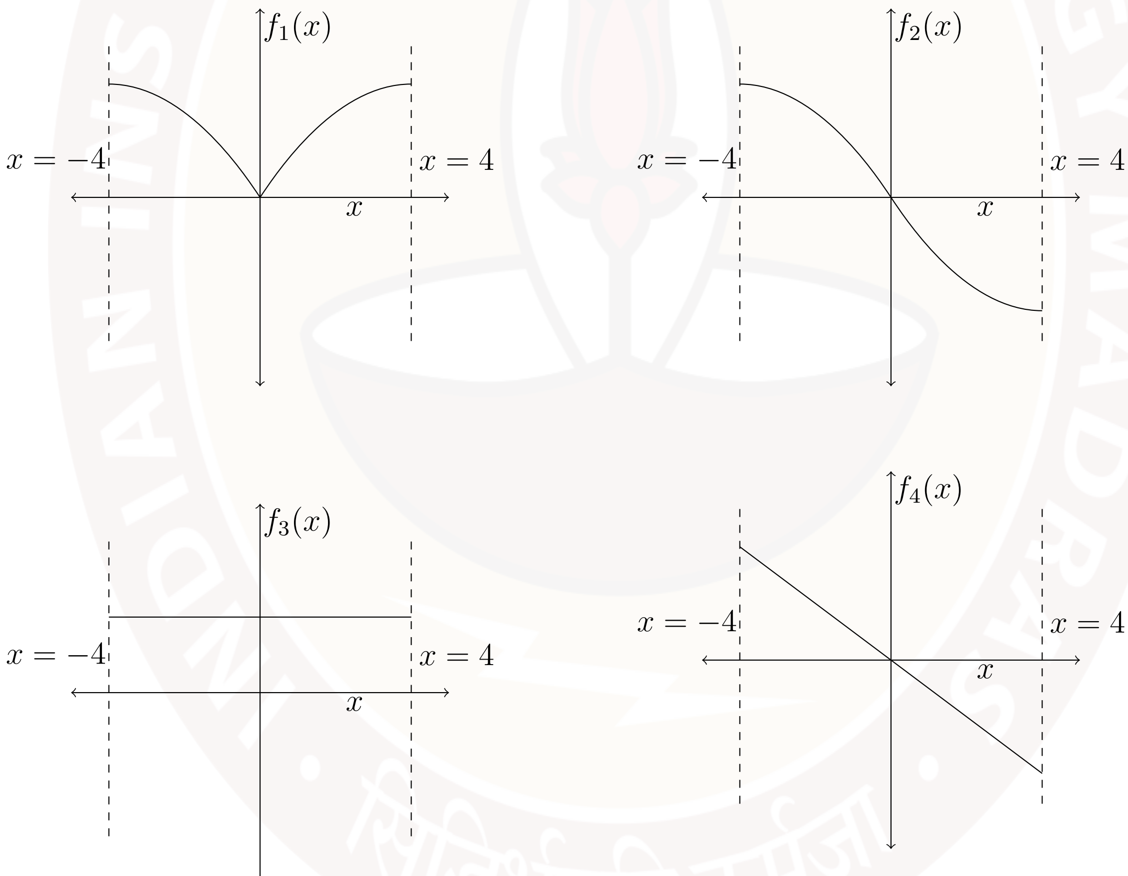
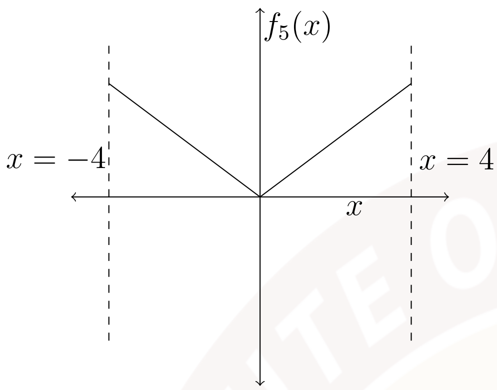
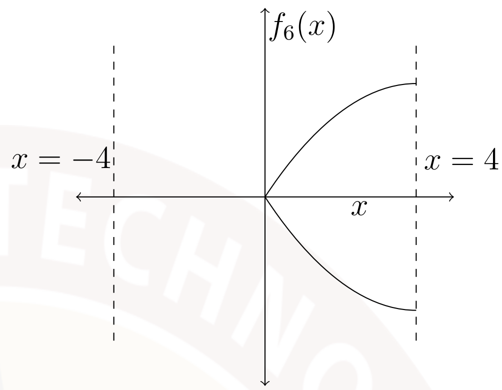


Figure: A-7.4





Domain for each one is $[-4, 4]$.

1. Find out the correct answers.
   - ○ **f_1 is an even function in the given domain.**
   - ○ f_2 is an even function in the given domain.
   - ○ **f_3 is an even function in the given domain.**
   - ○ f_4 is an even function in the given domain.
   - ○ **f_5 is an even function in the given domain.**
   - ○ f_6 is an even function in the given domain.

2. Find out the correct answers.
   - ○ f_1 is an odd function in the given domain.
   - ○ **f_2 is an odd function in the given domain.**
   - ○ f_3 is an odd function in the given domain.
   - ○ **f_4 is an odd function in the given domain.**
   - ○ f_5 is an odd function in the given domain.
   - ○ f_6 is an odd function in the given domain.

   Solution:
   A function $f(x)$ is said to be an even function if $f(-x) = f(x)$ and is said to be an odd function if $f(-x) = -f(x)$. Now, $f(-x) = f(x)$ if and only if the graph of the function $f(x)$ is symmetric about Y- axis and $f(-x) = -f(x)$ if and only if the graph of the function $f(x)$ is symmetric about the origin.
   Since the graphs of $f_1(x), f_3(x)$ and $f_5(x)$ are symmetric about Y- axis, $f_1(x), f_3(x)$ and $f_5(x)$ are even functions in the given domain.
   Similarly, the graphs of $f_2(x)$ and $f_4(x)$ are symmetric about origin, $f_2(x)$ and $f_4(x)$ are odd functions in the given domain.
   In the Figure: A-7.6, since we get two outputs corresponding to one input, it follows that $f_6(x)$ is not a function.

2 Multiple Choice Questions (MCQ):

A box has width which is 3 cm more than double the length, and has height which is 2 cm more than thrice the length of the box. The box is filled with small bricks each of whose lengths is one fourth of the length of the box. The width of each brick is 1 cm more than six times the length of the brick, and the height of each brick is 1 cm more than $\frac{8}{3}$ times the length of the brick.

3. Assuming x (in cm) to be the length of the box, what is the volume (in cubic cm) of the box?

   ◯ $6x^3 + 6$

   ◯ $6x^3 + 5x^2 + 6x$

   ◯ $\boldsymbol{6x^3 + 13x^2 + 6x}$

   ◯ $6x^2 + 13x^3 + 6$

4. The maximum number of bricks can be kept in the box is

   ◯ 4

   ◯ 6

   ◯ 12

   ◯ **24**

Solution:

Let x be the length of the box (in cm). So width of the box will be $2x + 3$ and height will be $3x + 2$.
Also, length of a brick $= \frac{x}{4}$,
width of a brick $= 6 \times \frac{x}{4} + 1 = \frac{3x}{2} + 1 = \frac{3x+2}{2}$,
height of a brick $= \frac{8}{3} \times \frac{x}{4} + 1 = \frac{2x}{3} + 1 = \frac{2x+3}{3}$.
So, volume of a brick = length $\times$ width $\times$ height
i.e volume of a brick$(V_{br}) = \frac{x}{4} \times (\frac{3x}{2} + 1) \times (\frac{2x}{3} + 1) = \frac{x(3x+2)(2x+3)}{24}$.
Also, volume of the box $(V_b) = x \times (2x + 3) \times (3x + 2) = 6x^3 + 13x^2 + 6x$.
Now, the maximum number of bricks $= \frac{V_b}{V_{br}} = \frac{x(2x+3)(3x+2)}{\frac{x(3x+2)(2x+3))}{24}} = 24$.

5. The consumption of new plastics in year x after opening a company is given as a polynomial $N(x)$ (in tonnes). The company also recycles the used plastics and regenerates them for use. The regenerated amount of plastic in year x after opening the company is given as the polynomial $R(x)$ (in tonnes). These polynomials are known to be applicable for the first 15 years of the company's functioning.
Use the following notes to solve the question:

- $N(x) = -0.005x^4 + 0.2x^3 - 3x^2 + 14x + 70$
- $R(x) = 0.005x^4 - 0.1x^3 + x^2 + x$
- $P(x) = 0.01x^4 - 0.3x^3 + 4x^2 - 13x - 70$ has exactly two real roots.
- $Q(x) = 0.01x^3 - 0.2x^2 + 2x + 7$ has exactly one real root and it is negative.
- $Q(x)$ is a factor of $P(x)$.

When will the company regenerate more plastic than it would have consumed? (Years from opening the company.)

- ◯ After 4 years.
- ◯ After 6 years.
- ◯ After 8 years.
- ◯ **After 10 years.**
- ◯ Never
- ◯ None of the above.

Solution:
We have to find the value of x for which the polynomial $R(x) - N(x) > 0$.
Now, from the given data $R(x) - N(x) = P(x)$ and $Q(x)$ divides $P(x)$.
So,

$$\begin{aligned} P(x) &= 0.01x^4 - 0.3x^3 + 4x^2 - 13x^2 - 70 \\ &= (0.01x^4 - 0.2x^3 + 2x^2 + 7x) + (-0.1x^3 + 2x^2 - 20x - 70) \\ &= x(0.01x^3 - 0.2x^2 + 2x + 7) - 10(0.01x^3 - 0.2x^2 + 2x + 7) \\ &= (x - 10)(0.01x^3 - 0.2x^2 + 2x + 7) \end{aligned}$$

Now, as given $P(x) = 0.01x^4 - 0.3x^3 + 4x^2 - 13x - 70$ has exactly two real roots and $Q(x)$ has exactly one real root which is negative.
Hence, $P(x)$ has only one positive real root which is $x = 10$.
That means after 10 years, the company will regenerate more plastic than its consumption of new plastic.

6. Given that $p(x) = (2x^2+mx+8)(4x^2+nx+1)$, M is the set of values of m, N is the set of values of n, and $C = M \cap N$. If $p(x)$ always has four real distinct root, then choose the correct option.

   - ◯ $M = \{z \mid z \in (-\infty, -4) \cup (4, \infty)\}$
   - ◯ $\boldsymbol{C = \{z \mid z \in (-\infty, -8) \cup (8, \infty)\}}$
   - ◯ $M = \{8, -8\}$
   - ◯ $C = \{4, -4\}$
   - ◯ $M = \{z \mid z \in \mathbb{R}\}$

   Solution:
   Given $p(x) = (2x^2 + mx + 8)(4x^2 + nx + 1)$.
   So, $p(x)$ has degree 4.
   Now, $p(x) = (2x^2 + mx + 8)(4x^2 + nx + 1) = 0$
   $\implies (2x^2 + mx + 8) = 0$ or $(4x^2 + nx + 1) = 0$.
   Now, for four distinct real root of the $p(x)$, the discriminant $D_1, D_2 > 0$ for both quadratic equations $(2x^2 + mx + 8) = 0$ and $(4x^2 + nx + 1) = 0$ respectively.
   Now, discriminant of the quadratic equation $(2x^2 + mx + 8) = 0$ is

$$D_1 = m^2 - 4 \times 2 \times 8 > 0$$

$$\implies m^2 > 64$$

$$\implies |m| > 8.$$

   In interval form $m \in (-\infty, -8) \cup (8, \infty)$.
   Since M is the set of values of m so $M = \{z \mid z \in (-\infty, -8) \cup (8, \infty)\}$.
   Similarly, discriminant of the quadratic equation $(4x^2 + nx + 1) = 0$ is

$$D_2 = n^2 - 4 \times 4 \times 1 > 0$$

$$\implies n^2 > 16$$

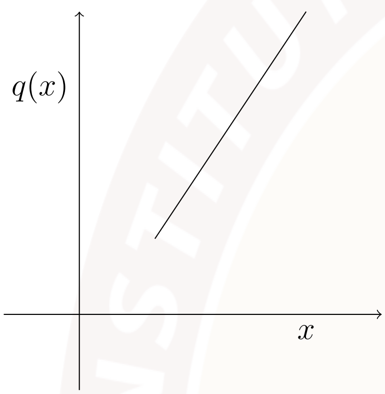
$$\implies |n| > 4.$$

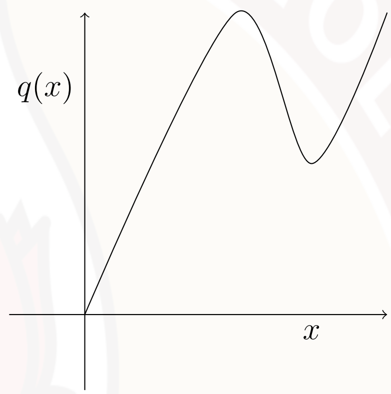
   In interval form $n \in (-\infty, -4) \cup (4, \infty)$.
   Since N is the set of values of n so $N = \{z \mid z \in (-\infty, -4) \cup (4, \infty)\}$.
   Now, $C = M \cap N = ((-\infty, -8) \cup (8, \infty)) \cap ((-\infty, -4) \cup (4, \infty)) = (-\infty, -8) \cup (8, \infty)$.
   In set notation form $C = \{z \mid z \in (-\infty, -8) \cup (8, \infty)\}$.

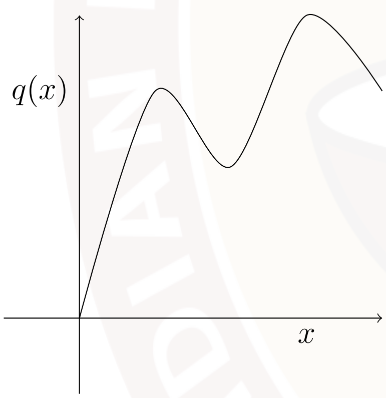
Suppose there are three polynomial functions with the following properties:

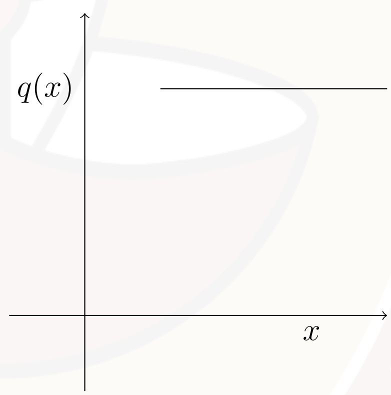
- $deg\ f(x) > deg\ g(x)$
- $deg\ f(x) = deg\ h(x) = 5$
- $h(x)$ divides $f(x)$.

7. Which of the following graphs is most suitable to represent $q(x) = \frac{f(x)}{h(x)}$?
[Answer: 4th Option]









8. What will be the maximum possible degree of the remainder when $f(x)+g(x)$ is divided by $h(x)$?
   - ◯ 5
   - ◯ 4
   - ◯ $\boldsymbol{deg\ g(x)}$
   - ◯ $(deg\ g(x)) - 1$

Solution (a):
Since $h(x)$ divides $f(x)$, the remainder is zero. Further $q(x)$ is the quotient. It follows that

$$f(x) = q(x)h(x) + 0.$$

Now,

$$deg\ q(x)h(x) = deg\ q(x) + deg\ h(x) = deg\ f(x) = deg\ h(x) \implies deg\ q(x) = 0.$$

So, $q(x)$ is a constant function i.e $q(x) = c$ where c is a constant.
Now, in options (a), (b) and (c), the value of $q(x)$ varies with x. So, these options are false.
In option (d), the value of $q(x)$ remains constant as x is varying. So, this option is true.
Solution (b):
Since $h(x)$ divides $f(x)$ and $deg\ g(x) < deg\ f(x) = deg\ h(x)$.
So, while dividing $f(x) + g(x)$ by $h(x)$, $g(x)$ will be the remainder.
Hence, the maximum possible degree of the remainder when $f(x) + g(x)$ is divided by $h(x)$ is $deg\ g(x)$.

3 Numerical Answer Type (NAT):

9. A company tracks their profits with respect to number of years (t) from the year of establishment. At the end of the second year (i.e. $t = 2$) the company registers neither profit nor loss. The same situation arises at the end of fifth and seventh year. If the equation relating the profit (in thousands) and the number of years t, is a cubic polynomial in t, with leading coefficient being 20. What will be the profit (in thousands) at the end of 10 years?

   Answer: 2400

   Solution:

   Let $p(t)$ be the cubic polynomial denoting the profit of the company with respect to the number of years(t) from the year of establishment.
   Since $p(t)$ is cubic polynomial, the maximum number of possible roots is 3.
   At the end of second year (i.e $t = 2$) the company register neither profit nor loss. So $p(2) = 0$.
   Similarly $p(5) = p(7) = 0$.
   Hence, 2, 5 and 7 are zeroes of the given polynomial and

   $$p(t) = a(t-2)(t-5)(t-7) = a(t^3 - 14t^2 + 59t - 70)$$

   where a is stretch factor.
   Since 20 is leading coefficient of the given polynomial $p(t)$, we have $a = 20$.
   Hence, $p(t) = 20(t-2)(t-5)(t-7)$.
   Now, $p(10) = 20 \times 8 \times 5 \times 3 = 2400$.
   Hence, the profit of the company after 10 years is 2400.

A roller coaster ride follows the curve defined by the polynomial function $f(x) = 9x^3 + 9x^2 - x - 1$ (considering some fixed horizontal reference plane) in the domain $[-1, 0.5]$.

10. How many bends are there in the roller coaster?
Answer: 2

11. In the given domain, how many times does the roller coaster reach the same height as that of the starting point (i.e. for $x = -1$)(taking $x = -1$ in to the account)?
Answer: 3

Solution:
Given

$$\begin{aligned} f(x) &= 9x^3 + 9x^2 - x - 1 \\ &= 9x^2(x+1) - (x+1) \\ &= (x+1)(9x^2 - 1) \\ \implies f(x) &= (x+1)(3x+1)(3x-1). \end{aligned}$$

So, $f(x)$ has three zeros -1, $\frac{1}{3}, -\frac{1}{3}$ which are in the given domain.
The rough sketch of the graph of $f(x)$ is shown in the Figure M1W7AS- 7.1

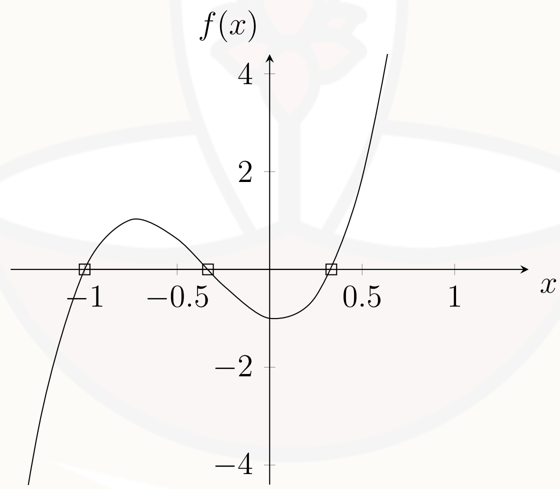


Figure M1W7AS-7.1

From Figure M1W7AS- 7.1 it is clear that that there are two turning points and the number of X- intercepts are 3.
So, for (a): there are two bends in the roller coaster and
for (b): in the given domain, the roller coaster reaches three times the same height as that of the starting point.

A chemical substance A is the reactant in a chemical reaction which gets converted into a product B. The concentrations (in mol/L) of A and B depend on the reaction time t as $C_A(t) = t^3 - t^2 - 21t + 45$ and $C_B(t) = t^3 + t^2 + 22t$ respectively as shown in Figure.

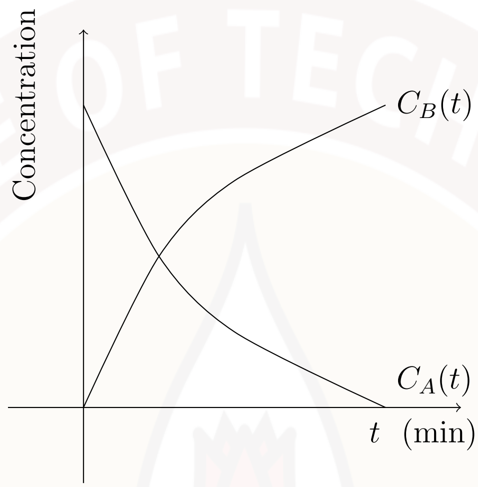


Fig A-7.11

12. At what time t, since the reaction started, the concentrations of A and B would be same?
Answer: 1
Solution
If concentrations of A and B are equal
then

$$\begin{aligned} & C_A(t) = C_B(t) \\ \implies & t^3 - t^2 - 21t + 45 = t^3 + t^2 + 22t \\ \implies & 2t^2 + 43t - 45 = 0 \\ \implies & 2t^2 - 2t + 45t - 45 = 0 \\ \implies & 2t(t-1) + 45(t-1) = 0 \\ \implies & (t-1)(2t+45) = 0 \end{aligned}$$

So, $t = 1$ or $t = -\frac{45}{2}$ but t represents time so t can not be negative. Hence, after 1 min of starting the reaction, the concentrations of A and B become equal.

13. Find the concentration (in mol/L) of either substance when their concentrations become equal.
Answer: 24
Solution
Since at $t = 1$ concentrations of A and B are equal, value of C_A at $t = 1$ is $1-1-21+45 = 24$.

14. Find the concentration (in mol/L) of the product when the concentration of the reactant becomes zero.
Answer: 102
Solution
The concentration of the reactant becomes zero when

$$\begin{aligned}
& C_A(t) = 0 \\
\implies & t^3 - t^2 - 21t + 45 = 0 \\
\implies & t^3 - 3t^2 + 2t^2 - 6t - 15t + 45 = 0 \\
\implies & t^2(t-3) + 2t(t-3) - 15(t-3) = 0 \\
\implies & (t-3)(t^2 + 2t - 15) = 0 \\
\implies & (t-3)(t^2 + 5t - 3t - 15) = 0 \\
\implies & (t-3)^2(t+5) = 0 \\
\implies & t = 3 \ or \ t = -5
\end{aligned}$$

But, t represents time and time can not be negative. So, at $t = 3$ concentration of the reactant becomes zero.
Now, concentration of the product at $t = 3$ is $C_B(3) = 3^3 + 3^2 + 22 \times 3 = 102$.