Week-8
Mathematics for Data Science - 1
Exponential and Logarithm
Assignment

1 Multiple Choice Questions(MCQ)

1. If $18^x - 12^x - (2 \times 8^x) = 0$, then the value of x is.

   1. $\frac{\ln 2}{\ln 3 - \ln 2}$
   2. $\frac{\ln 18}{\ln 12 - \ln 8}$
   3. $\ln 2$
   4. $\ln 18$

   Answer: Option 1

Solution:

$$18^x - 12^x - (2 \times 8^x) = 0$$

Domain = $\mathbb{R}$ as all the terms are exponential functions.

we can write: $2^x . 9^x - 2^x . 6^x - 2^x(2 \times 4^x) = 0$

2^x is a positive number. Dividing by $2^x \Rightarrow$

$$9^x - 6^x - (2 \times 4^x) = 0 \quad \text{—— (1)}$$

$\boxed{\text{Let } a = 3^x \text{ and } b = 2^x}$

then $9^x = (3^2)^x = 3^{2x} = (3^x)^2 = a^2$

$6^x = 2^x 3^x = ba$

$4^x = (2^x)^2 = a^2$

Therefore equation (1) would be

$$a^2 - ab - 2b^2 = 0$$

$$\Rightarrow \quad a^2 - 2ab + ab - 2b^2 = 0$$

$$a(a-2b) + b(a-2b) = 0$$

$$(a-2b)(a+b) = 0$$

If $a - 2b = 0 \Rightarrow a = 2b \Rightarrow 3^x = 2 \times 2^x$

taking log $\Rightarrow$ $x \log 3 = \log 2 + x \log 2$

$$\boxed{x = \frac{\log 2}{\log 3 - \log 2}} \text{ — Answer.}$$

If $a + b = 0$

$3^x + 2^x = 0 \Rightarrow$ Not possible

2. Suppose three distinct persons A, B and C are standing on the X- axis of the XY- plane (as shown in the figure M1W9G-1) and the distance between B and A is same as the distance between C and B. The coordinates of A, B and C are $(\log_5 3, 0)$, $(\log_5(3^x - \frac{9}{2}), 0)$ and $(\log_5(3^x - \frac{9}{4}), 0)$ respectively. What is the distance between C and B?
(MCQ), (Ans:(a))

1. $\log_5(2)$ units.
2. $\log_5(\frac{5}{4})$ units.
3. $\log_5(\frac{3}{2})$ units
4. $\log_5(\frac{7}{3})$ units.

$$AB = \sqrt{\left\{\log_5(3^x - 9/2) - \log_5 3\right\}^2 + 0} = \log_5 \frac{(3^x - 9/2)}{3} \Rightarrow 3^x > 9/2$$

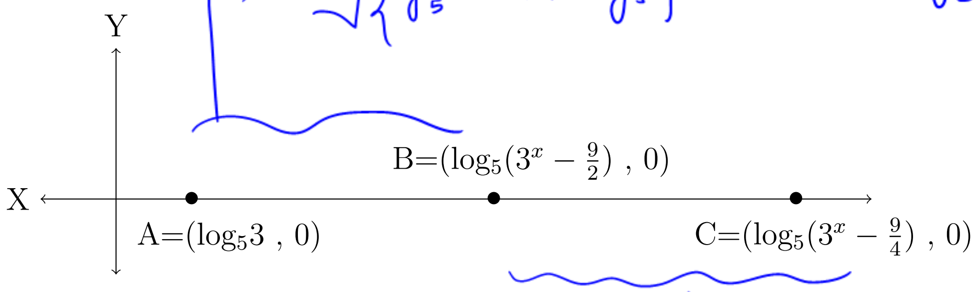


Figure M1W9G-1

$$BC = \sqrt{\left\{\log_5(3^x - 9/4) - \log_5(3^x - 9/2)\right\}^2 + 0}$$

$$= \log_5 \left\{\frac{3^x - 9/4}{3^x - 9/2}\right\} \Rightarrow 3^x - 9/4 > 0$$

Answer: Option 3

Given: $AB = BC$

$$\log_5\left\{\frac{3^x - 9/2}{3}\right\} = \log_5\left\{\frac{3^x - 9/4}{3^x - 9/2}\right\}$$

$$\frac{3^x - 9/2}{3} = \frac{3^x - 9/4}{3^x - 9/2} \Rightarrow (3^x - 9/2)^2 = 3(3^x - 9/4)$$

$$\Rightarrow (3^x)^2 + (9/2)^2 - 2(3^x)(9/2) = 3(3^x) - 3(9/4)$$

take $3^x = a \Rightarrow$ $a^2 + (\frac{9}{2})^2 - 9a = 3a - 3(9/4) \Rightarrow a^2 - 12a + \frac{81}{4} + 27/4 = 0$

$$\Rightarrow a^2 - 12a + \frac{108}{4} = 0 \Rightarrow a^2 - 12a + 27 = 0 \Rightarrow (a-3)(a-9) = 0$$

If $a = 3 \Rightarrow 3^x = 3 \Rightarrow x = 1$ but $3^1 \not> 9/2$. Therefore, $\boxed{x = 2}$

If $a = 9 \Rightarrow 3^x = 9 \Rightarrow x = 2$ Now $3^2 = 9 > 9/2$

$$BC = \log_5 \left\{ \frac{3^x - 9/4}{3^x - 9/2} \right\} = \log_5 \left\{ \frac{3^2 - 9/4}{3^2 - 9/2} \right\} = \log_5 \left\{ \frac{3/4}{1/2} \right\}$$

$$\boxed{BC = \log_5 \left\{ 3/2 \right\}} \Longrightarrow \text{Answer.}$$

3. In a city, a rumour is spreading about the safety of corona vaccination. Suppose N number of people live in the city and $f(t)$ is the number of people who **have not** yet heard about the rumour after t days. Suppose $f(t)$ is given by $f(t) = Ne^{-kt}$, where k is a constant. If the population of the city is 1000, and suppose 40 have heard the rumor after the first day. After how many days (approximately) half of the population would have heard the rumor?

    1. 20
    2. 17
    3. 13
    4. 12

    Answer: Option 2

After first day $\Rightarrow$ $t=1$

40 have heard therefore, $1000 - 40 = 960$ people have not heard i.e.,

$$960 = Ne^{-kt}$$

$$t=1 \Rightarrow \quad 960 = 1000e^{-k} \Rightarrow \boxed{e^{-k} = \frac{960}{1000}}$$

Half of population will heard then $f(t) = \frac{1000}{2} = 500$

therefore,

$$500 = 1000e^{-kt} \Rightarrow \frac{500}{1000} = \left(e^{-k}\right)^t$$

$$\frac{500}{1000} = \left(\frac{960}{1000}\right)^t$$

taking log:

$$\ln\left\{\frac{500}{1000}\right\} = t \ln\left\{\frac{960}{1000}\right\}$$

$$\ln(1/2) = t\ln(.96) \Rightarrow t = \frac{\ln(0.5)}{\ln(.96)}$$

$$t \simeq 16.97$$

$\boxed{\text{t can be condisered as 17 days.}}$

4. Consider the function $f(x) = \log_2(12 + 4x - x^2)$. The range of f is

1. $(-\infty, 4]$
2. $(-\infty, \infty)$
3. $(0, \infty)$
4. $(0, \log 12]$

Answer: Option 1

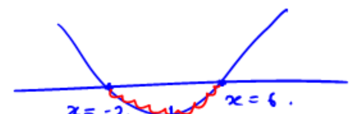
Domain:

$12 + 4x - x^2 > 0$.

$x^2 - 4x - 12 < 0$

$(x-6)(x+2) < 0$

$\boxed{x \in (-2, 6)}$



at $x = 2$

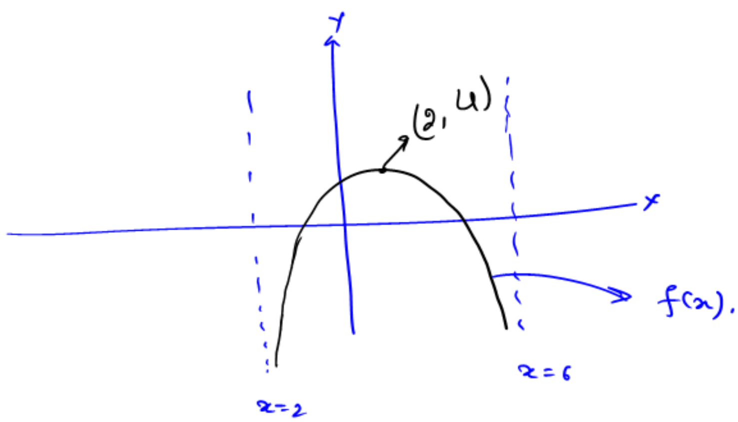
$12 + 4x - x^2 = 12 + 8 - 4 = 16$

$f(x) = \log_2(16) = 4$

→ $x = -2$ and $x = 6$ will work as assymptotes:

→ As at $x = 0$, $\log x$ tends towards $-\infty$, similarly at $x = -2$ and $x = 6$, $f(x)$ will tends towords $-\infty$.

→ $f(x)$ will be symmetric Arround $x = 2$.



$\boxed{\text{Range: } (-\infty, 4]}$ — Ans.

Use the following information for the questions 5 and 6.

Consider the function $f(x) = \frac{2e^x}{3e^x+1}$ from $\mathbb{R}$ to $\mathbb{R}$.

5. Which of the following is true about f?
   1. f is not a one to one function.
   2. f is a one to one function.
   3. Range of f is $\mathbb{R}$.
   4. f is a bijective function.

   Answer: Option 2

6. The inverse of f would be
   1. $\ln(\frac{2x}{2-3x})$
   2. $\ln(\frac{2x}{2x-x})$
   3. $\ln(\frac{x}{2-3x})$
   4. $\ln(\frac{x}{2x-x})$

   Answer: Option 3

$$f(x) = \frac{2e^x}{3e^x+1}.$$

To find one to one nature:

take $x_1 > x_2$.

$$f(x_1) = \frac{2e^{x_1}}{3e^{x_1}+1}$$

$$f(x_2) = \frac{2e^{x_2}}{3e^{x_2}+1}$$

Let $f(x_1) > f(x_2)$

then $\frac{\cancel{2}e^{x_1}}{3e^{x_1}+1} > \frac{\cancel{2}e^{x_2}}{3e^{x_2}+1}$

$$\cancel{3e^{(x_1+x_2)}} + e^{x_1} > \cancel{3e^{(x_1+x_2)}} + e^{x_2}$$

$$e^{x_1} > e^{x_2}$$

we know that: e^x is an exponetial and increasing function.

therefore if $e^{x_1} > e^{x_2} \Rightarrow x_1 > x_2$

which is true with our assumption.

Therefore, $f(x)$ is an increasing function and thats why one to one function.

Now for Range: $f(x) = \frac{2e^x}{\underbrace{3e^x+1}_{\text{always positive}}}$ $2e^x$ always positive $\Big\}\rightarrow$ $f(x)$ is always positive

which means, codain $\neq$ Range.

$f(x)$ is not Onto function.

As $f(x)$ is one to one function, inverse of $f(x)$ is possible:

$$f(x) = \frac{2e^x}{3e^x+1}$$

Replace x by $f^{-1}(x)$ and $f(x)$ by x:

$$x = \frac{2e^{f^{-1}(x)}}{3e^{f^{-1}(x)}+1}$$

$$3x\, e^{f^{-1}(x)} + x = 2e^{f^{-1}(x)}$$

$$3x\, e^{f^{-1}(x)} - 2e^{f^{-1}(x)} = -x$$

$$e^{f^{-1}(x)}\left\{3x-2\right\} = -x$$

$$e^{f^{-1}(x)} = \frac{x}{2-3x}$$

$$\boxed{f^{-1}(x) = \ln\left\{\frac{x}{2-3x}\right\}} \text{ — Answer.}$$

2 Multiple Select Questions (MSQ)

Use the following information for the questions 7 and 8.

The amount of gold (in kilograms) sold by a jeweler on the mth day of 2019 is given by the function $f(m) = \log_{10}(m+1) - \frac{1}{2}\log_{m+1}(0.01)$ (where $m = 1$ corresponds to the 1st January, 2019, and $m = 365$ corresponds to the 31st December, 2019). Find the correct set of options.

7. If $m > n > 9$, then choose the correct option(s).

    1. $f(m) > f(n)$
    2. $f(m) < f(n)$
    3. $f(m) = f(n)$
    4. $f(m) \leq f(n)$

    Answer: Option 1

8. Choose the correct option(s).

    1. The jeweler sold at least 540 kg gold in 2019.
    2. The jeweler sold at least 730 kg gold in 2019.
    3. The jeweler sold at least 2 kg gold daily throughout the year 2019.
    4. The jeweler sold at least 10 kg gold daily throughout the year 2019.

    Answer: Options 2 and 3

Solution

Given $f(m) = \log_{10}(m+1) - \frac{1}{2}\log_{m+1}(0.01)$

$= \log_{10}(m+1) - \frac{1}{2}\log_{m+1} 10^{-2}$

$= \log_{10}(m+1) - (-2) \times \frac{1}{2}\log_{m+1} 10$

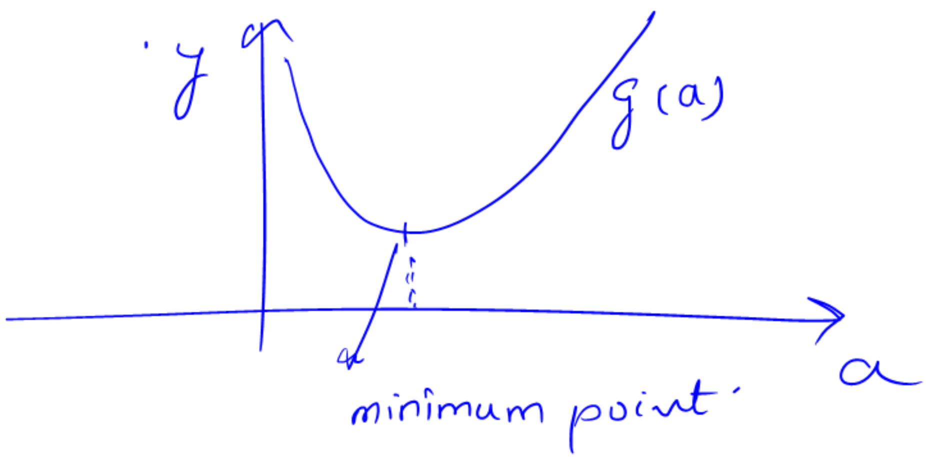
$= \log_{10}(m+1) + \frac{1}{\log_{10}(m+1)}$

Let $\log_{10}(m+1) = a$ then $f(m) = a + \frac{1}{a} = g(a)$ (let)

If $a \to \infty$ the $g(a) \to \infty$

if $a \to 0$ then $g(a) \to \infty$ as $\frac{1}{a} \to \infty$.

therefore the curve will look like

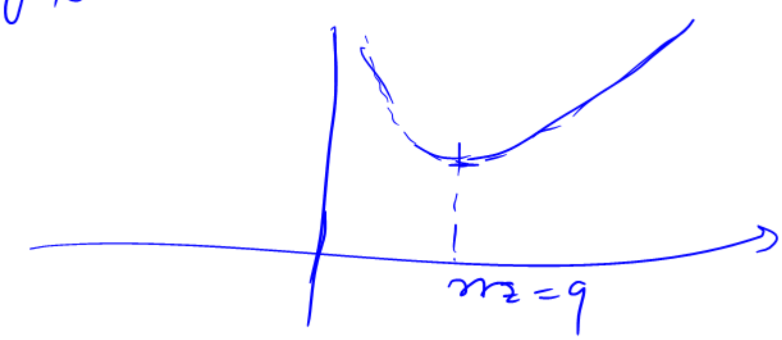


The minimum point will occure at $a = 1$.

We can use Desmos to see the behaviour.

$a = 1 \Rightarrow \log_{10} m+1 = 1 \Rightarrow m = 9$

Therefore:



Therefore After $m=9$ $f(m)$ is an increasing function that's why $f(m) > f(n)$ for $m > n > 9$. → Answer Question 7.

The minimum value of $f(m)$ would be at $m=9 \Rightarrow$

$$f(m)\Big|_{min} = \log_{10}(9+1) + \frac{1}{\log_{10}(9+1)}$$

$$= 2$$

$\Rightarrow$ Therefore jeweller sells atleast 2 kg gold per day. → Answer.

$\Rightarrow$ And a year contains 365 days, therefore atleast $365 \times 2 = 730$ kg gold will be sold in a year. → Answer.

9. The stock market chart of a tourism company (A) is shown roughly in the Figure M1W9G-2. This company was listed in February $(x = 2)$ and experiences a logarithmic fall after the COVID−19 outbreak which is given by $y = -a\ log(x - h) + a$. x represents the number of months since the beginning of the year and y represents the stock price in ₹(1000). During the 10^{th} month the pharmacy company announced that the vaccine is made for the COVID−19. Thereafter, the stock price of the company (A) is raised exponentially $y = 10^{\frac{x}{b}} - b$. Choose the correct set of options. (Note: a is any positive real number, b is a positive integer and h is a constant.)

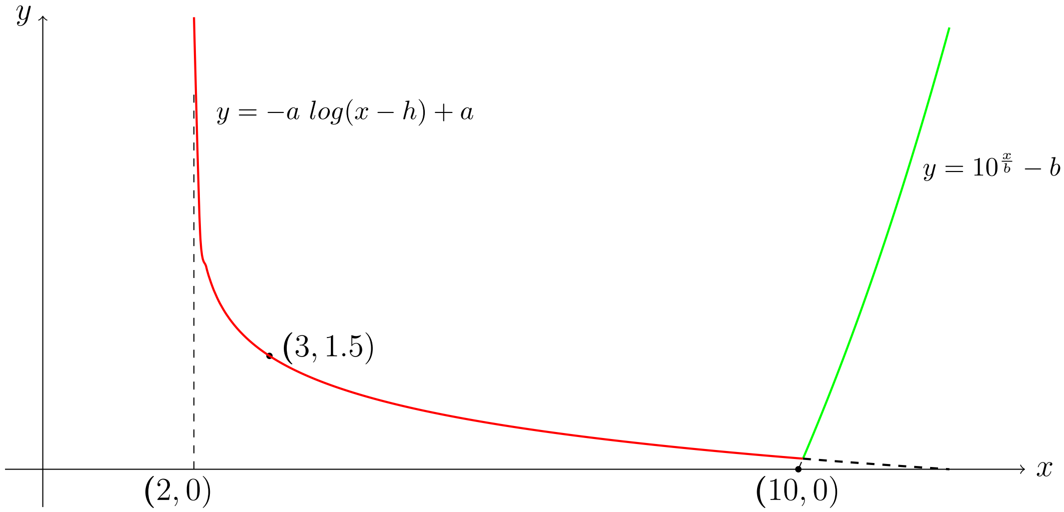


Figure M1W9G-2

1. For logarithmic fall the value of $a = 1.5$ and $h = 2$.
2. For exponential rise passing through $(10, 0)$ the value of $b = 10$.
3. The stock price in 12^{th} month is ₹4000.
4. If the vaccine was not made and the stock price just followed the same logarithmic function through out, then the investor would have lost his/her entire investment on the 12^{th} month.

Answer: Options 1, 2, and 4

The assymptote will occure when $x - h = 0$

$\Rightarrow x = h = 2.$

At $x = 3 \Rightarrow y = -a \log(3-2) + a = 1.5$

$= -a \log 1 + a = 1.5$

$= 0 + a = 1.5 \Rightarrow \boxed{a = 1.5}$

Given $y = 10^{x/b} - b = 0$ at $x = 10$

$\Rightarrow 10^{10/b} - b = 0 \Rightarrow 10^{10/b} = b$

Taking log at base 10 $\Rightarrow \frac{10}{b} \log_{10} 10 = \log_{10} b$

$\Rightarrow \frac{10}{b} = \log_{10} b \Rightarrow 10 = b \log_{10} b$

Taking Antilog :

$\Rightarrow 10^{10} = b^b \Rightarrow \boxed{b = 10}$

for 12^{th} month :

$y = -a \log_{10}(12-2) + a = -1.5 \log_{10} 10 + 1.5$

$y = 0$

option (4) is correct.

10. If $m^{\log_3 2} + 2^{\log_3 m} = 16$. Then, what is the value of m ?
(NAT)
Answer: 27

$$m^{\log_3 2} + 2^{\log_3 m} = 16$$

$$2^{\log_3 m} + 2^{\log_3 m} = 16$$

$$2\left\{2^{\log_3 m}\right\} = 16$$

$$2^{\log_3 m} = 8 = 2^3$$

$$\log_3 m = 3$$

$$m = 3^3 = 27 \text{ Ans.}$$