Mathematics for Data Science - 1
Graded Assignment
Week 10

1 MULTIPLE CHOICE QUESTIONS:

1. An undirected graph G has 20 vertices and the degree of each vertex is at least 3 and at most 5. Which of the following statements is true regarding the graph G?
   (a) The minimum number of edges that the graph G can have is 60.
   (b) The maximum number of edges that the graph G can have is 100.
   (c) The maximum number of edges that the graph G can have is 60.
   (d) The minimum number of edges that the graph G can have is 30.

Sol:- we know that,
Sum of degree of all the vertices is twice the number of edges in the graph.

* Consider the minimum possible case: Suppose every vertex in the graph G has degree 3.

Now,
Sum of degree of all vertices = $20 \times 3 = 60$
$\Rightarrow$ Minimum number of edges that the graph G can have is $\frac{60}{2} = 30$.

* Consider the maximum possible case: Suppose every vertex in the graph G has degree 5.

Now,
Sum of degree of all vertices = $20 \times 5 = 100$
$\Rightarrow$ Maximum number of edges that the graph G can have is $\frac{100}{2} = 50$.

$\therefore$ option (d) is Correct.

2. If G is a connected undirected graph such that every vertex has degree at most k, and the shortest path between any two vertices has length at most 2, then the number of vertices in G can be at most

(a) $k^2 - 1$

(b) $k^2 + 1$

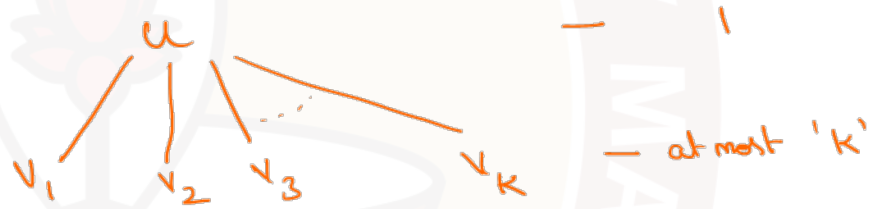
(c) k^2

(d) $k^2 - k$

Sol:- Let 'u' be a Vertex in the graph G.

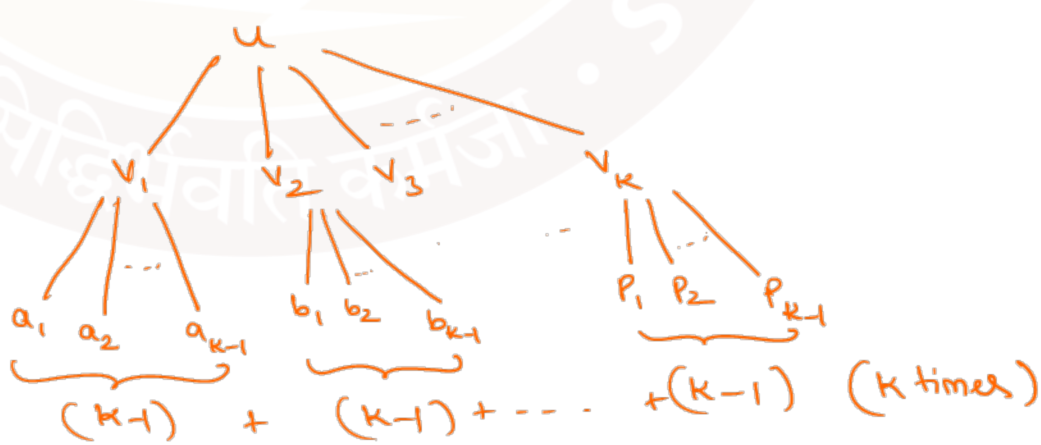
As the degree of every Vertex is at most 'k', the number of Vertices that are adjacent to 'u' can be at most 'k'.

Also, if we draw a BFS tree starting with Vertex 'u', then the depth of the tree cannot be more than 2 because the length of the shortest Path between any two vertices is at most 2.



Now each of v_i can have at most $k-1$ adjacent Vertices because 'u' is already adjacent to each of v_i.

$\therefore$ BFS tree with at most vertices can be



$\therefore$ level 2 of BFS tree can have atmost $k(k-1)$ Vertices.

$\Rightarrow$ the graph G can have atmost $\underset{(\text{level } 0)}{1} + \underset{(\text{level } 1)}{k} + \underset{(\text{level } 2)}{k(k-1)}$

$$= 1 + \cancel{k} + k^2 - \cancel{k}$$

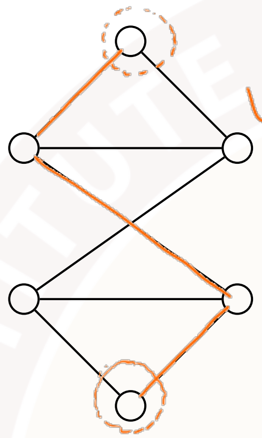
$$= k^2 + 1$$

Hence, the graph can have at most k^2+1 Vertices.

option (b) is correct.

3. Suppose A is the adjacency matrix of a connected undirected graph G.

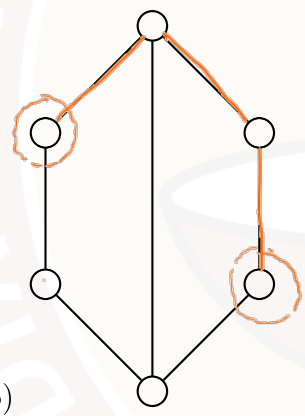
If $A^2 = \begin{bmatrix} 1 & 1 & 1 & 0 & 1 & 1 \\ 1 & 1 & 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 1 & 1 & 0 \\ 0 & 1 & 1 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 & 1 & 1 \\ 1 & 1 & 0 & 1 & 1 & 1 \end{bmatrix}$ and the shortest path between any two vertices has length at most 2, then which of the following may represent the graph G?



(a)

length of shortest path between the highlighted vertices is more than 2.

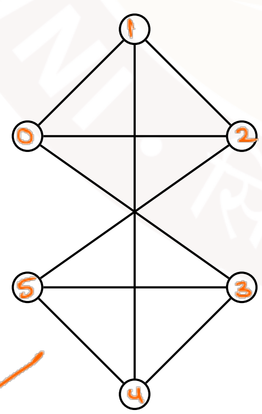
So, option (a) is incorrect.



(b)

length of shortest path between the highlighted vertices is more than 2.

So, option (b) is incorrect.



(c)

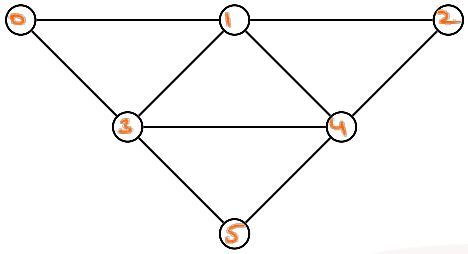
* shortest path between any two vertices is at most 2.

$$* \quad A = \begin{array}{c|c} & \begin{matrix} 0 & 1 & 2 & 3 & 4 & 5 \end{matrix} \\ \begin{matrix} 0 \\ 1 \\ 2 \\ 3 \\ 4 \\ 5 \end{matrix} & \begin{bmatrix} 0 & 1 & 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 & 1 & 0 \end{bmatrix} \end{array}$$

$$A^2 = \begin{array}{c|c} & \begin{matrix} 0 & 1 & 2 & 3 & 4 & 5 \end{matrix} \\ \begin{matrix} 0 \\ 1 \\ 2 \\ 3 \\ 4 \\ 5 \end{matrix} & \begin{bmatrix} 1 & 1 & 1 & 0 & 1 & 1 \\ 1 & 1 & 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 1 & 1 & 0 \\ 0 & 1 & 1 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 & 1 & 1 \\ 1 & 1 & 0 & 1 & 1 & 1 \end{bmatrix} \end{array}$$

$\therefore$ option (c) is correct.

(d)



$$A = \begin{array}{c} \\ 0 \\ 1 \\ 2 \\ 3 \\ 4 \\ 5 \end{array} \begin{array}{c} \begin{array}{cccccc} 0 & 1 & 2 & 3 & 4 & 5 \end{array} \\ \begin{bmatrix} 0 & 1 & 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 1 & 1 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 & 1 & 1 \\ 0 & 1 & 1 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 1 & 0 \end{bmatrix} \end{array}$$

$$A^2 = \begin{array}{c} \\ 0 \\ 1 \\ 2 \\ 3 \\ 4 \\ 5 \end{array} \begin{array}{c} \begin{array}{cccccc} 0 & 1 & 2 & 3 & 4 & 5 \end{array} \\ \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 \end{bmatrix} \end{array}$$

A^2 that we have obtained is different from the given information

so, option (d) is incorrect.

2 MUTIPLE SELECT QUESTIONS:

4. Suppose in a farewell party of IIT Madras Mathematics department, 60 students were present. As in normal parties, handshaking took place and of course no one shook their own hand. The number of students who have made odd number of handshakes is x. Which of the followings can be a possible value of x?

   (a) 6
   (b) 13
   (c) 21
   (d) 28

Sol:- We represent this in a graph model, every Vertex represents a person and edge between a pair of Vertices if the persons representing those Vertices made a handshake.

Now, observe that there are 'x' Vertices in the graph with odd degree and '$60-x$' Vertices with even degree.

* As we know that number of odd degree Vertices in a graph is always even.

$\Rightarrow x$ Can only be even.

$\therefore$ The possible Values of 'x' Can be 6 & 28.

Hence, option (a) & option (d) are Correct.

5. Suppose G is a graph with 6 vertices $0, 1, 2, 3, 4, 5$ and the adjacency matrix of the graph G is $A = \begin{bmatrix} 0 & 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$. Which of the following statements is True?

(a) The graph G is a directed acyclic graph.

(b) From vertex 4, every other vertex is reachable.

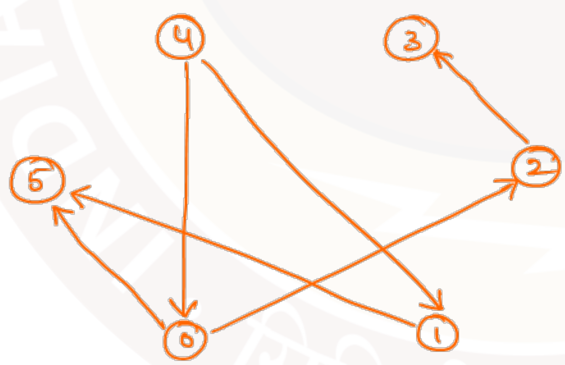
(c) The longest path in the graph G has length 4, in terms of number of edges.

(d) The longest path in the graph is $4 \longrightarrow 0 \longrightarrow 2 \longrightarrow 3$

Sol:- Given,

$$A = \begin{array}{c} \\ 0 \\ 1 \\ 2 \\ 3 \\ 4 \\ 5 \end{array} \begin{array}{c} \begin{array}{cccccc} 0 & 1 & 2 & 3 & 4 & 5 \end{array} \\ \begin{bmatrix} 0 & 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \end{array}$$

we first draw the graph G.

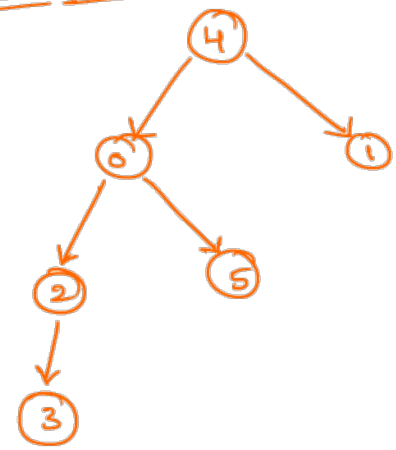


* As the obtained graph does not have any cycles and also it is a directed graph $\Rightarrow$ G is a directed acyclic graph.

So, option (a) is Correct.

* If we draw a BFS tree starting with Vertex '4', then we can get the Vertices that are reachable from Vertex '4'.

BFS tree :-



Every Vertex is present in the BFS tree

$\Rightarrow$ Every Vertex is reachable from Vertex '4'.

So, option (b) is Correct.

* we have

$$A = \begin{bmatrix} 0&0&1&0&0&1 \\ 0&0&0&0&0&1 \\ 0&0&0&1&0&0 \\ 0&0&0&0&0&0 \\ 1&1&0&0&0&0 \\ 0&0&0&0&0&0 \end{bmatrix}$$

from this we find A^2, A^3, A^4.

so,

$$A^2 = \begin{bmatrix} 0&0&0&1&0&0 \\ 0&0&0&0&0&0 \\ 0&0&0&0&0&0 \\ 0&0&0&0&0&0 \\ 0&0&1&0&0&1 \\ 0&0&0&0&0&0 \end{bmatrix}, \quad A^3 = \begin{bmatrix} 0&0&0&0&0&0 \\ 0&0&0&0&0&0 \\ 0&0&0&0&0&0 \\ 0&0&0&0&0&0 \\ 0&0&0&1&0&0 \\ 0&0&0&0&0&0 \end{bmatrix}$$

$$A^4 = \begin{bmatrix} 0&0&0&0&0&0 \\ 0&0&0&0&0&0 \\ 0&0&0&0&0&0 \\ 0&0&0&0&0&0 \\ 0&0&0&0&0&0 \\ 0&0&0&0&0&0 \end{bmatrix}$$

$\therefore$ From A^4, we can conclude that there is no path in the graph G that has length 4. So, option (c) is incorrect.

* From A^3, $4 \longrightarrow 3$ is the longest path in the graph G and the path is $4 \longrightarrow 0 \longrightarrow 2 \longrightarrow 3$.

So, option (d) is Correct.

6. Which of the following statements is(are) true?

   (a) Every directed acyclic graph has a vertex with outdegree 0.

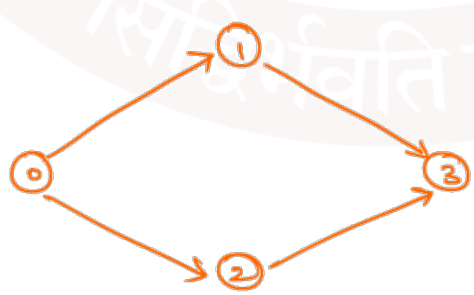
   (b) In a directed acyclic graph, the longest path between a pair of vertices is always unique.

   (c) If an undirected graph G is connected, then the graph representing the transitive closure of the graph G is a complete graph.

   (d) Suppose A is the adjacency matrix of a graph G with n vertices, any non-zero entry (i, j) in the matrix A^k, where $k < n$, indicates that there is a path of length k from vertex i to vertex j.

(a) Suppose every Vertex in a DAG has outdegree > 0.
If the graph has 'n' Vertices, then there will be atleast 'n' outward edges (at least one for each Vertex).
Now,
choose a Vertex 'u', as the outdegree of vertex 'u' is greater than 0 ⇒ there is an outward edge (e) from 'u'.
follow the edge 'e' and proceed to next vertex 'v', even the outdegree (v) > 0 ⇒ there is an outward edge (e') from v.
Now follow e' and proceed to next.
* After n steps, we will definetly end up with a cycle which Contradicts the fact that the graph is a directed acyclic graph.
Hence, Every Vertex in a DAG Cannot have outdegree > 0
⇒ There will be a Vertex which has outdegree 0.
so, option (a) is Correct.

(b)



this is a directed acyclic graph.
longest path from 0 to 3 Can be
0 ⟶ 1 ⟶ 3
0 ⟶ 2 ⟶ 3.
∴ longest path may not be unique.
so, option (b) is incorrect.

(c) the graph representing the transitive closure of any given graph G is constructed by adding edges to the graph G such that if there is a path from vertex 'a' to vertex 'b' in G then we add edge (a,b) to the graph G.

so, if a graph 'H' is connected, then we know that there will be a path from any vertex 'i' to any other vertex 'j' in the graph.

$\Rightarrow$ Hence every edge (i,j) is added to the graph 'H' making 'H' a complete graph.

$\therefore$ the graph representing the transitive closure of a connected graph is complete.

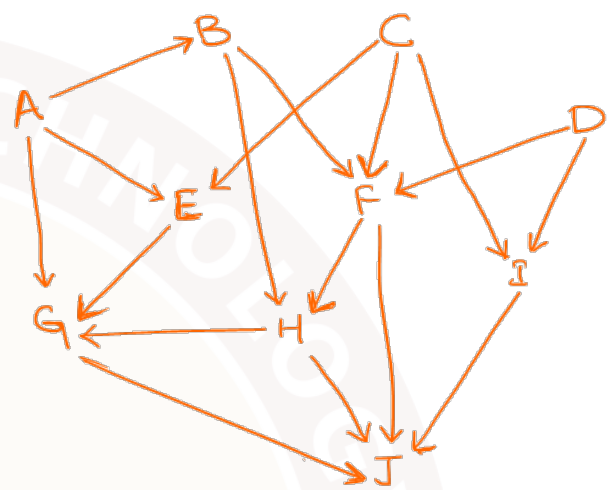
so, option (c) is correct.

(d) we defined A^K in this way (given in the lectures).

option (d) is correct.

USE THE FOLLOWING INFORMATION FOR QUESTIONS [7-8]:
Shreya needs to perform 10 tasks namely $\{A, B, C, D, ...J\}$. Some tasks needs to be performed after performing a particular task. In the below table, column 1 shows the tasks and column 2 shows the sets of tasks that can be performed only after performing the particular task.

A	$\{B, E, G\}$
B	$\{F, H\}$
C	$\{E, F, I\}$
D	$\{F, I\}$
E	$\{G\}$
F	$\{H, J\}$
G	$\{J\}$
H	$\{G, J\}$
I	$\{J\}$
J	$\{\}$



7. Which of the following sequences may represent the possible order in which Shreya can perform the tasks?

(a) $A, C, B, D, E, I, F, H, G, J$

(b) $A, D, C, B, E, I, F, H, G, J$

(c) $C, A, D, E, B, I, F, G, H, J$

(d) $D, C, A, B, E, I, F, H, G, J$

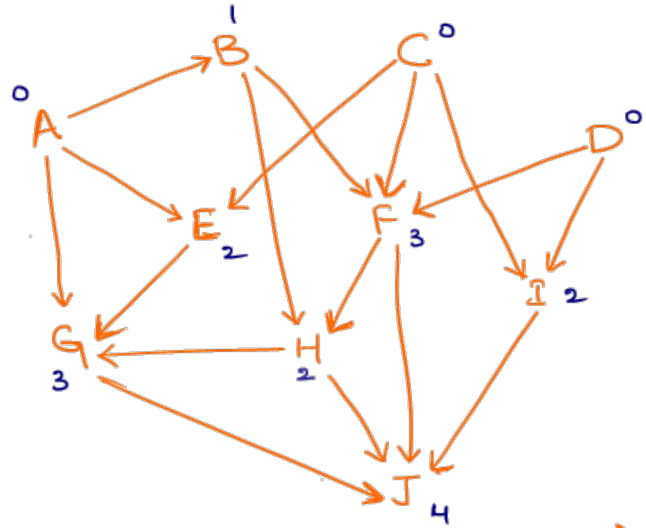
Sol:- Draw a directed graph 'K' that represents the given data such that each vertex represents a task and a directed edge from vertex 'i' to vertex 'j' if task 'j' can be performed only after task 'i'.

* observe that the obtained graph is a DAG (Directed acyclic graph).

Now, if we find topological sequence of this DAG, then in the same order she can perform the tasks.

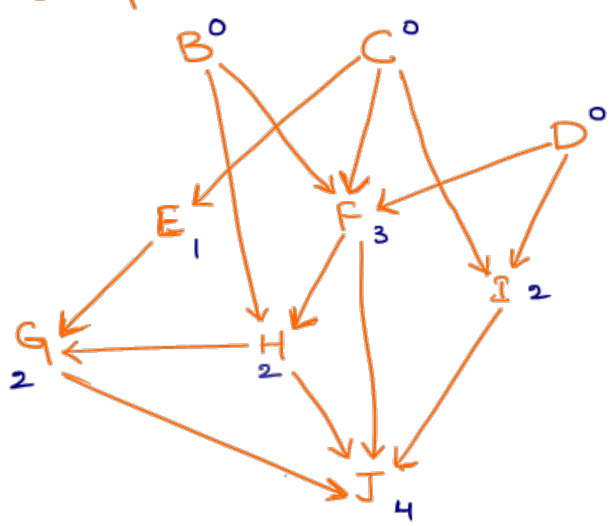
∴ our aim is to find topological sequence of the obtained DAG.

* Find indegree of each Vertex in the graph.



Now, we have three Vertices (A, C, D) that has indegree 0.
∴ we Can choose any one of them i.e, she can perform any of the three tasks

Suppose we choose Vertex 'A', remove it from the DAG and update the indegree of each of the remaining vertex.



* indegree of vertex B has become 0 and indegree(E) = 1, indegree (G) = 2.

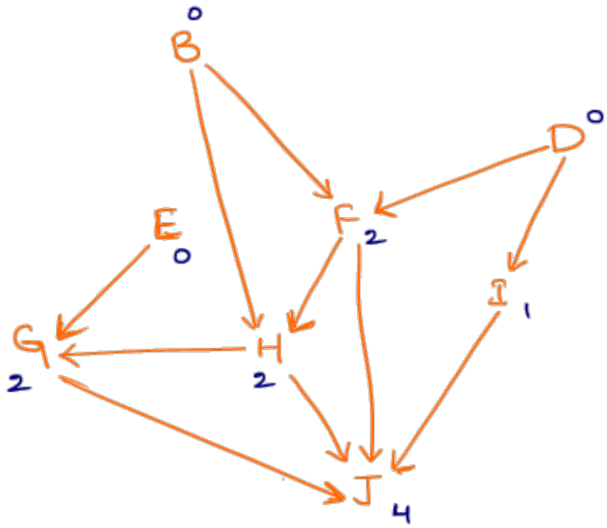
* Topological Sequence :

A,

* Repeat the process till the last Vertex is removed and added to the topological sequence.

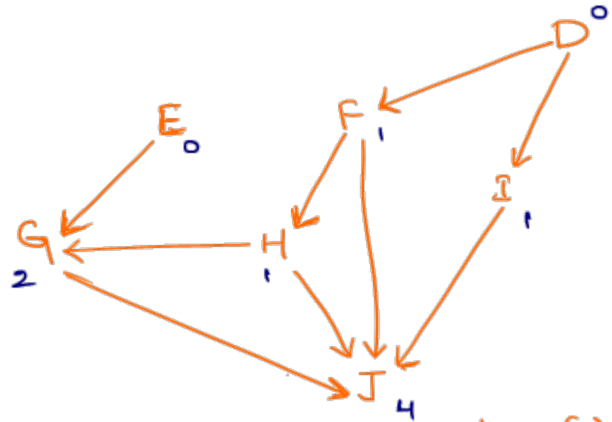
Now, we have again three Vertices (B, C, D) having indegree 0.

So we can choose any one of them. Suppose we choose vertex 'C'.
∴ Remove Vertex 'C' from the graph and update the indegree of each of the Vertex, also add 'C' to the Topological Sequence.



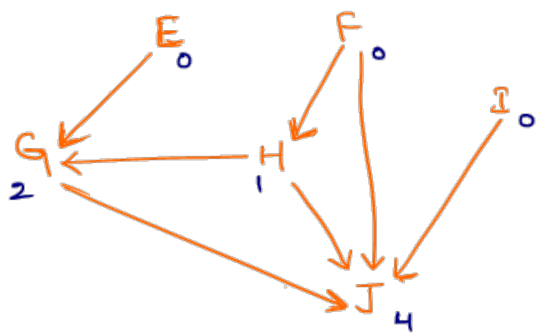
* indegree (E) = 0, indegree (F) = 2, indegree (I) = 1
* Topological Sequence :-

A, C,



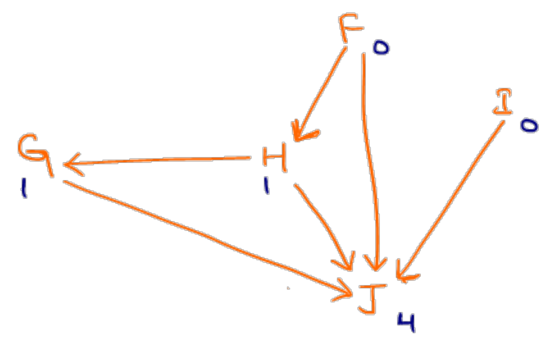
* indegree (F) = 1, indegree (H) = 1
* Topological Sequence :-

A, C, B,



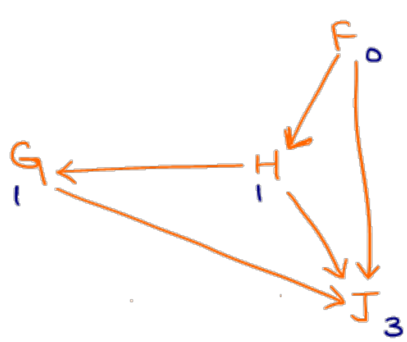
* indegree (F) = 0, indegree (I) = 0
* Topological Sequence :-

A, C, B, D,



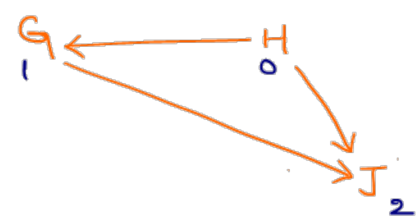
* indegree (G) = 1
* Topological Sequence :-

A, C, B, D, E,



* indegree (J) = 3
* Topological Sequence :-

A, C, B, D, E, I,



* indegree (H) = 0, indegree of (J) = 2
* Topological Sequence :-

A, C, B, D, E, I, F

$G_0 \longrightarrow J_1$

* indegree (G) = 0 , indegree (J) = 1
* Topological Sequence :-
  A, C, B, D, E, I, F, H,

J_0

* indegree (J) = 0
* Topological Sequence :-
  A, C, B, D, E, I, F, H, G,

* Finally we remove 'J' and add it to the topological sequence.

∴ Topological Sequence :- A, C, B, D, E, I, F, H, G, J.

NOTE :- the obtained topological sequence is not unique because instead of starting with A, we can also start with vertex 'C' or vertex 'D'.

∴ shreya can perform tasks in the order A, C, B, D, E, I, F, H, G, J

Hence, option (a) is correct.

* Similarly, A, D, C, B, E, I, F, H, G, J and D, C, A, B, E, I, F, H, G, J are also a possible topological sequence which means shreya can perform tasks in that order too.

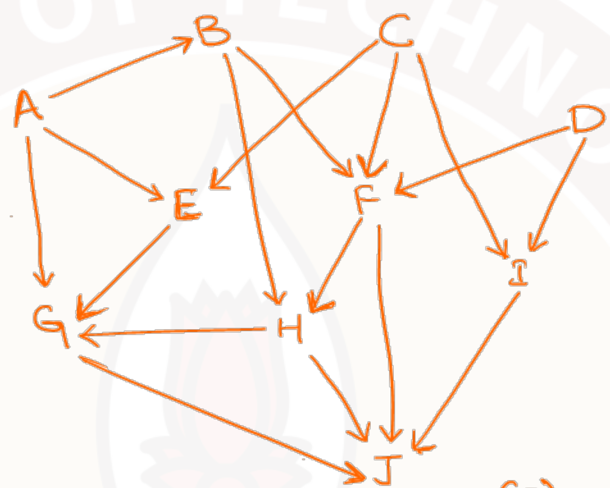
  Hence, options (b), (d) are correct.

* option (c) is incorrect because task 'G' cannot be performed before task 'H'. ∴ that is not a possible order.

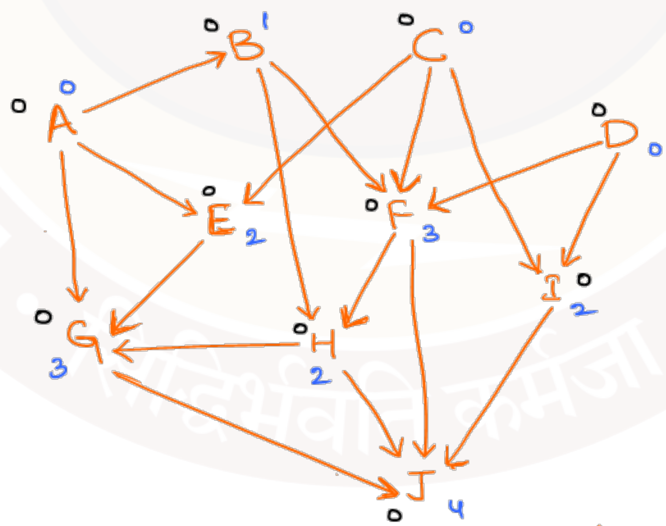
3 NUMERICAL ANSWER TYPE:

8. If each task takes 5 minutes to complete and she performs all the independent tasks simultaneously, then the time(in minutes) taken by Shreya to complete all the tasks is

Sol:- we compute the longest path to each vertex in the DAG (that we got in the previous problem).



* we first initialize longest-path-to (i) = 0 for each vertex 'i' in the DAG.
Also find the indegree (i) for each vertex 'i' in the DAG.



longest-path-to (i).
indegree (i)

Now, we find a vertex 'u' in the DAG which has indegree 0.

* we remove vertex 'u' from the graph and update indegree(i) and longest-path-to(i) for every vertex 'i' that is adjacent to vertex 'u'.

* we update longest-path-to(i) as

$$\text{longest-path-to}(i) = \text{Max}\left\{\text{longest-path-to}(i),\ 1+\text{longest-path-to}(u)\right\}$$

* Repeat the process by finding a new vertex 'v' that has indegree 0, removing it from the graph and update indegree(i) and longest-path-to(i) till all the vertices are removed from the graph.

Finally, after removing all the vertices and updating longest-path-to(i) for all vertices, we get

$$\text{longest-path-to}(A) = \text{longest-path-to}(C) = \text{longest-path-to}(D) = 0.$$

$$\text{longest-path-to}(B) = \text{longest-path-to}(E) = \text{longest-path-to}(I) = 1$$

$$\text{longest-path-to}(F) = 2$$

$$\text{longest-path-to}(H) = 3$$

$$\text{longest-path-to}(G) = 4$$

$$\text{longest-path-to}(J) = 5$$

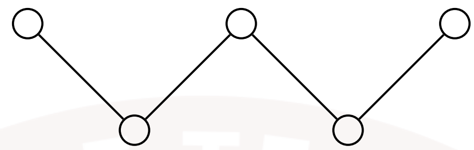
∴ she can first perform tasks 'A','C','D' at same time and next tasks 'B','E','I' at same time followed by task 'F' then by task 'H' then by 'G' and finally task 'J'

⇒ she takes 5 minutes (for tasks A, C, D) + 5 minutes (for tasks B, E, I) + 5 minutes (task F) + 5 minutes (task H) + 5 minutes (task G) + 5 minutes (task J)

$= 5+5+5+5+5+5 = 30$ minutes.

Answer:- 30

9. Suppose R is a relation defined on a set S and it is represented by a graph G that is shown below.



Find the number of edges that need to be added to the graph G such that the new graph obtained after adding the edges represents a transitive relation.

Sol: We have

$\therefore R = \{(1,2), (2,3), (3,4), (4,5)\}$.

The transitive closure of R is obtained by repeatedly adding (a,c) to R for each $(a,b) \in R$ & $(b,c) \in R$.

Now, $(1,2)$ & $(2,3)$ are in R $\Rightarrow (1,3)$ is added to R ——— ①

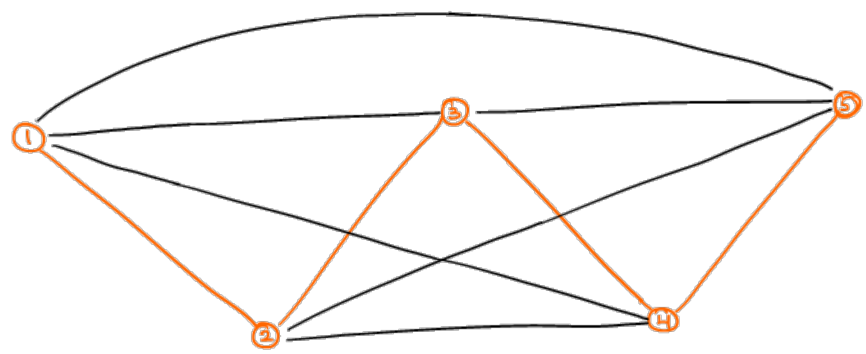
Similarly,
$(2,3)$ & $(3,4)$ in R $\Rightarrow (2,4)$ is added to R ——— ②
$(3,4)$ & $(4,5)$ in R $\Rightarrow (3,5)$ is added to R ——— ③

Now again from ①, ② & ③,
$(1,3)$ & $(3,4)$ are in R $\Rightarrow (1,4)$ is added to R
$(1,3)$ & $(3,5)$ are in R $\Rightarrow (1,5)$ is added to R
$(2,4)$ & $(4,5)$ are in R $\Rightarrow (2,5)$ is added to R.

$\therefore$ the set which represents the transitive closure of R is

$\{(1,2), (2,3), (1,3), (3,4), (2,4), (1,4), (4,5), (1,5), (2,5), (3,5)\}$

So the graph representing this transitive closure of R is



$\therefore$ 6 edges need to be added to the given graph G.

Answer :- 6 .

NOTE :- If a graph is Connected, then the transitive closure of that graph is a Complete graph.