Week - 5
Practice Assignment Solution
Quadratic Equations
Mathematics for Data Science - 1

NOTE:

- There are some questions which have functions with discrete-valued domains (such as month or year). For simplicity, we treat them as continuous functions.
- For a given quadratic equation $ax^2 + bx + c = 0, \;\; a \neq 0$:
  - ◯ Sum of roots $= -\frac{b}{a}$.
  - ◯ Product of roots $= \frac{c}{a}$.

1 Multiple Choice Questions (MCQ):

1. What will be the value of parameter k, if the discriminant of equation $4x^2+9x+10k = 0$ is 1?

   ◯ $\frac{82}{80}$
   ◯ $\frac{41}{80}$
   ◯ $\boldsymbol{\frac{1}{2}}$
   ◯ $\frac{41}{160}$
   ◯ 1
   ◯ None of the above.

   Solutions:
   Comparing the given equation $4x^2 + 9x + 10k = 0$ with the standard quadratic equation $ax^2 + bx + c = 0$:

$$a = 4,\; b = 9, \text{ and } c = 10k$$
$$\text{Discriminant } (d) = b^2 - 4ac$$
$$d = 9^2 - 4 \times 4 \times 10k$$
$$1 = 81 - 160k$$
$$\boldsymbol{k = \frac{1}{2}}$$

2. A boat has a speed of 30 km/hr in still water. In flowing water, it covers a distance of 50 km in the direction of flow and comes back in the opposite direction. If it covers this total of 100 km in 10 hours, then what is the speed of flow of the water (in km/hr)?

   - ◯ $5 - 5\sqrt{37}$
   - ◯ $-10\sqrt{6}$
   - ◯ $\mathbf{10\sqrt{6}}$
   - ◯ $20\sqrt{3}$
   - ◯ $-20\sqrt{3}$
   - ◯ 2

Solutions:
Total time taken by the boat = time taken by the boat in the direction of flow + time taken by the boat in the opposite direction of flow.
We know that:

$$time(t) = \frac{distance}{net\ speed}$$

Considering the direction of flow of water to be positive:
The net speed in the direction of flow (v_f) = speed of the boat in still water + speed of flow.
The net speed in the opposite direction of flow (v_b) = speed of the boat in still water - speed of flow.
Let the speed of flow be x then,

$$10 = \frac{50}{v_f} + \frac{50}{v_b}$$

$$10 = \frac{50}{30+x} + \frac{50}{30-x}$$

$$1 = \frac{5}{30+x} + \frac{5}{30-x}$$

$$1 = \frac{5(30-x+30+x)}{(30+x)(30-x)}$$

$$(30+x)(30-x) = 300$$

$$30^2 - x^2 = 300$$

$$x^2 = 600x = \pm 10\sqrt{6}$$

Speed of flow can not be negative therefore, the correct answer is $\mathbf{10\sqrt{6}}$.

3. A stunt man performs a bike stunt between two houses of the same height as shown in Figure 1. His bike (lowest part of the bike) makes an angle of θ at house A with the horizontal at the beginning of the stunt, follows a parabolic path and lands at house B with an angle of $(180 - \theta)$ with the horizontal.

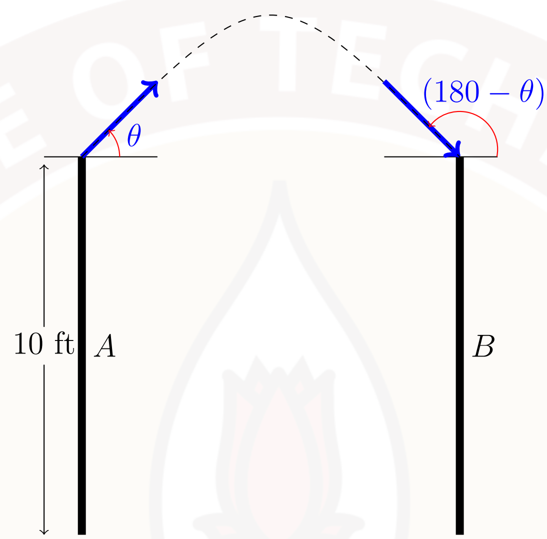

Figure PA-5.1

If the maximum height achieved by the bike is 12.5 ft from the ground and $tan\ \theta = 1$, then find the distance between the two houses.

- ◯ 1 ft
- ◯ 2.5 ft
- ◯ 5 ft
- ◯ **10 ft**
- ◯ 15 ft
- ◯ 20 ft

Solution:

Assuming the top of the house A to be origin, the horizontal direction as $X-$ axis, and the vertical direction as $Y-$ axis.

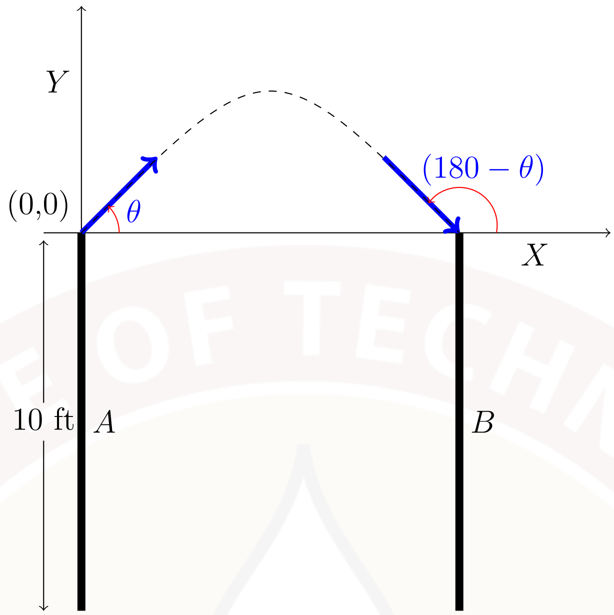

Figure M1W5PAS-3.1

Let the quadratic function representing the above curve be $f(x) = ax^2 + bx + c$.
Since the curve passes through the origin, we have $c = 0$.
The curve is making an angle θ with respect to positive $X-$ axis which means the slope of the tangent at the curve is $\tan\theta$.
We also know that the slope of the curve represented by quadratic function at $x = x$ is $2ax + b$. Therefore,

$$\begin{aligned} 2ax + b &= \tan\theta \\ 2a \times 0 + b &= 1 \\ b &= 1 \end{aligned}$$

The maximum height achieved by the bike is 12.5 ft which means the $y-$ coordinate of the vertex is $12.5 - 10 = 2.5$.
The $x-$ coordinate of the vertex for a curve represented by function $ax^2 + bx + c$ is

$$-\frac{b}{2a} = -\frac{1}{2a}$$

Therefore,

$$
\begin{aligned}
f(x) &= ax^2 + bx + c \\
f(-\frac{1}{2a}) &= 2.5 \\
a \times (-\frac{1}{2a})^2 + 1 \times (-\frac{1}{2a}) + 0 &= 2.5 \\
\frac{1}{4a} - \frac{1}{2a} &= 2.5 \\
-\frac{1}{4a} &= 2.5 \\
a &= -\frac{1}{10}
\end{aligned}
$$

Axis of symmetry,

$$x = -\frac{b}{2a} = -\frac{1}{2 \times (-1/10)} = 5$$

Because of symmetricity, the coordinate of landing point will be (10, 0).
Therefore two houses A and B are 10 ft apart.

2 Multiple Select Question (MSQ):

4. Given that $f_1(x) = -x^2 - 6x$ and $f_2(x) = x^2 + 6x + 10$. Let $f(x)$ be a function such that the domain of $f(x)$ is $[\alpha,\ \beta]$, where $f_1(\alpha) = f_2(\alpha)$ and $f_1(\beta) = f_2(\beta)$, then choose the set of correct options.

   - ◯ Range of $f(x)$ is $[-1, 3]$.
   - ◯ Range of $f(x)$ is $[0, 5]$.
   - ◯ Domain of $f(x)$ is $[-5, 5]$.
   - ◯ **Domain of $\boldsymbol{f(x)}$ is $\boldsymbol{[-5, -1]}$.**
   - ◯ **Inadequate information provided for finding the range of $\boldsymbol{f(x)}$.**
   - ◯ Inadequate information provided for finding the domain of $f(x)$.

Solution:
Since $f_1(\alpha) = f_2(\alpha)$ and $f_1(\beta) = f_2(\beta)$, we have α and β are the abscissa of intersection points of both the curves.
To find the intersection points of the curves represented by $f_1(x)$ and $f_2(x)$:

$$f_1(x) = f_2(x)$$
$$-x^2 - 6x = x^2 + 6x + 10$$
$$2x^2 + 12x + 10 = 0$$

Here,

$$a = 2, b = 12, \text{ and } c = 10$$

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$x = \frac{-12 \pm \sqrt{12^2 - 4 \times 2 \times 10}}{2 \times 2}$$

$$x = \frac{-12 \pm 8}{4} = -1, -5$$

Therefore,

$$\alpha = -5 \text{ and } \beta = -1.$$

Since the Domain of $\boldsymbol{f(x)}$ is $\boldsymbol{[\alpha, \beta]}$ domain of $\boldsymbol{f(x)}$ is $\boldsymbol{[-5, -1]}$.

The figure below gives a rough pictorial representation of $f_1(x)$ and $f_2(x)$ (drawn with smooth lines).
$f(x)$ can have any shape. An example is shown in the figure (drawn with dashed lines) for $f(x)$.

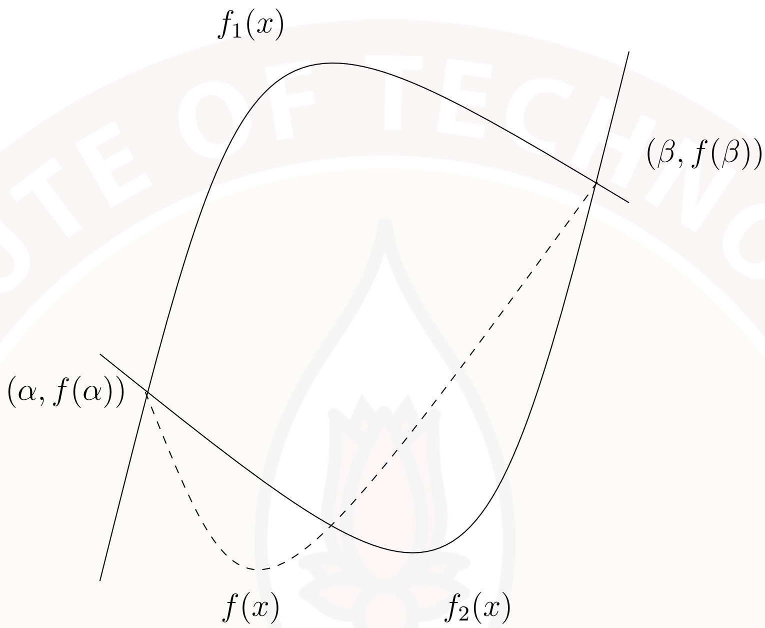

Figure M1W5PAS-4.1

As it is clear from figure that we do not know the minimum and maximum value of $f(x)$, we do not have proper data to comment on the range.

5. If $f(x) = 2x^2 + (5+k)x + 7$, $g(x) = 5x^2 + (3+k)x + 1$, $h_1(x) = f(x) - g(x)$, and $h_2(x) = g(x) - f(x)$, then choose the set of correct options.

- ◯ **Roots for $h_1(x) = 0$ and roots for $h_2(x) = 0$ are real, distinct, and the roots are the same for $h_1(x) = 0$ and $h_2(x) = 0$.**
- ◯ Roots for $h_1(x) = 0$ and roots for $h_2(x) = 0$ are real and distinct but the roots are not the same for $h_1(x) = 0$ and $h_2(x) = 0$.
- ◯ **Sum of roots of quadratic equation $h_1(x) = 0$ will be $\frac{2}{3}$.**
- ◯ **Product of roots of quadratic equation $h_2(x) = 0$ will be -2.**
- ◯ **Axis of symmetry for both the functions $h_1(x)$ and $h_2(x)$ will be the same.**
- ◯ Vertex for both the functions $h_1(x)$ and $h_2(x)$ will be the same.

Solution:
Given that,

$$h_1(x) = f(x) - g(x)$$
$$h_1(x) = -(g(x) - f(x))$$
$$h_1(x) = -h_2(x)$$

Negative sign before any function does not make any changes on zeros of the function. Therefore, roots of $h_1(x) = 0$ and roots of $h_2(x) = 0$ will be same.

Now, for the properties of $h_1(x)$:

$$h_1(x) = f(x) - g(x) = 2x^2 + (5+k)x + 7 - (5x^2 + (3+k)x + 1)$$
$$h_1(x) = -3x^2 + 2x + 6$$
$$d = 2^2 - 4(-3) \times 6 > 0$$

It means the roots of $h_1(x)$ are real and distinct.
The roots of $h_1(x) = 0$ has the same as the roots of $h_2(x) = 0$, which means the roots for $h_2(x) = 0$ will also be real and distinct.

Sum of the roots of $h_1(x) = -3x^2 + 2x + 6$ will be $-\frac{b}{a} = -\frac{2}{(-3)} = \frac{2}{3}$.

Product of the roots of $h_1(x) = -3x^2 + 2x + 6$ will be $\frac{c}{a} = \frac{6}{(-3)} = -2$.

Multiplying a quadratic function by the minus sign does not make any changes in the

axis of symmetry.
The answer to all the above questions can be seen in the given figure.

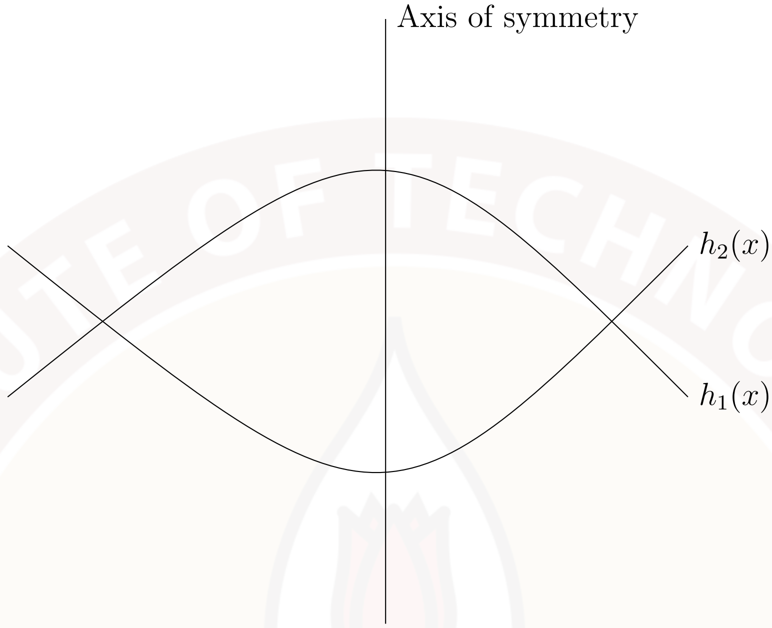

Figure M1W5PAS-5.1

Use following information for questions 6-8.

Vaishali wants to set up a small plate making machine in her village. Table P-5.1 shows the different costs involved in making the plates. Figure 5 shows her survey regarding the demand (number of packets of the plate) versus selling price of plate per packet (in ₹) per day.

Cost type	Cost
Electricity	₹1.5 per packet
Miscellaneous	₹6.5 per packet
Raw material	₹10 per packet

Table P-5.1

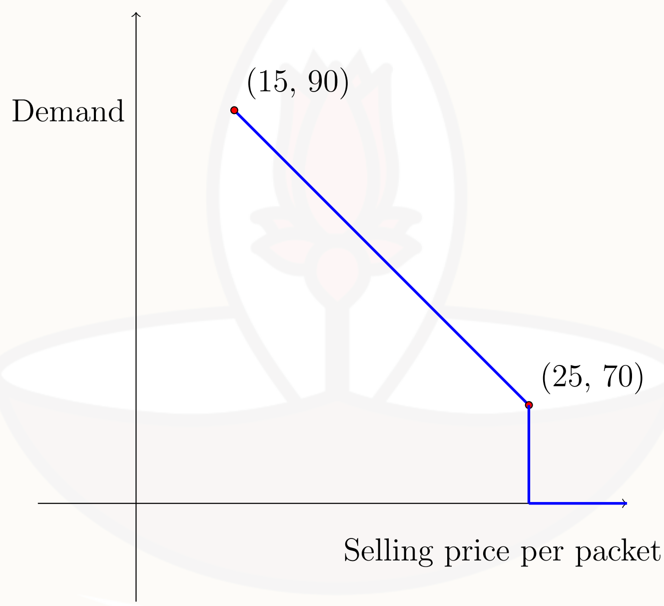

Figure PA-5.2

6. Choose the correct option which shows the profit obtained by Vaishali per day. Here, x is the selling price per packet.
   - ◯ $2(60 - x)$
   - ◯ $x(x - 18)$
   - ◯ $\mathbf{2(x - 18)(60 - x)}$
   - ◯ $2(x + 18)(60 - x)$
   - ◯ Inadequate information.

Solution:
From the figure, it is clear that the demand is dependent on the selling price of plates. Let y be the demand of the numbers of packets, then from two-points form of a line,

$$y - 90 = \frac{70 - 90}{25 - 15}(x - 15)$$
$$y - 90 = -2(x - 15)$$
$$y = -2x + 120$$

From the table, total cost per packet (in ₹)$= 1.5 + 6.5 + 10 = 18$

Per day profit = Demand per day$\times$(Selling price per packet - Cost per packet)

$$\begin{aligned}Profit &= y(x - 18)\\ Profit &= (-2x + 120)(x - 18)\\ Profit &= 2(x - 18)(60 - x)\end{aligned}$$

7. Choose the set of correct options.

   - ◯ **Vaishali should sell a packet with a minimum price of ₹18 so as not to incur any loss.**
   - ◯ Vaishali should sell a packet with a minimum price of ₹12 so as not to incur any loss.
   - ◯ To make maximum profit per day, the selling price per packet should be ₹39.
   - ◯ **To make maximum profit per day, the selling price per packet should be ₹25.**
   - ◯ Vaishali should sell a packet with maximum price of ₹60 so as not to incur any loss.
   - ◯ **Vaishali should sell a packet with a maximum price of ₹25 so as not to incur any loss.**

Solution:
From question 6,

$$Profit = 2(x-18)(60-x)$$

$$Profit = -2x^2 + 156x - 2160 \tag{1}$$

To get minimum selling price with no loss, profit should be zero. Therefore,

$$2(x-18)(60-x) = 0$$
$$x = 18 \text{ or } 60$$

From the graph given in question, it is clear that we can not sell a packet at ₹60, because the demand will be zero.
Therefore, the minimum selling price will be ₹18 per packet.

Since the profit is a quadratic function of the selling price (x) in equation (1) with negative coefficient of x^2.
Therefore, the maximum profit will occur at

$$x = -\frac{b}{2a} = -\frac{156}{2\times(-2)} = 39$$

A rough pictorial representation is shown in Figure below,

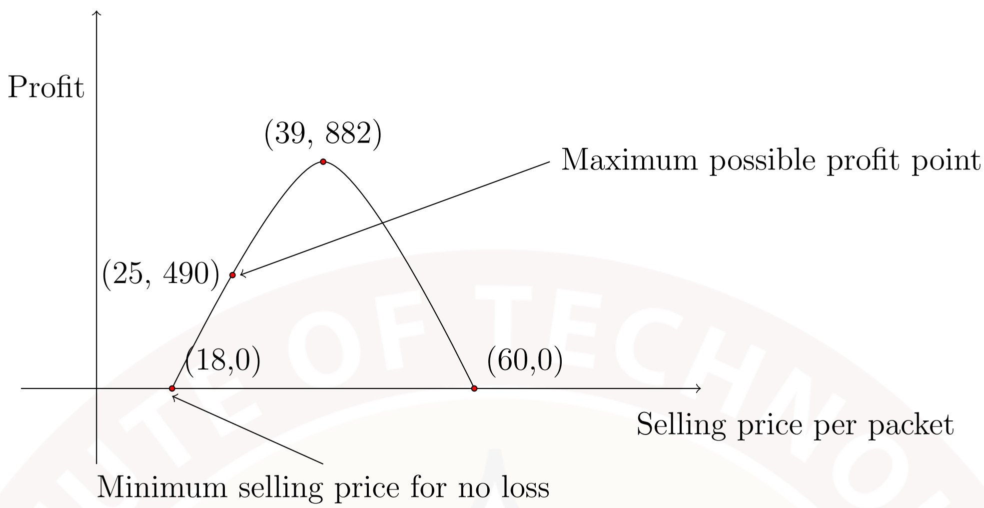

Figure M1W5PAS-7.1

The increase in selling price will result in profit increment till 39. But the maximum acceptable selling price is ₹25, therefore the maximum profit will occur at a selling price of ₹25.

So, from the figure it is clear that the maximum selling price for no loss is ₹60 but we can not increase the price beyond ₹25. Therefore, the maximum profit to incur any loss will be ₹25.

3 Numerical Answer type (NAT):

8. What should be the price of plate per packet (₹) to make a profit of ₹490 per day?
[Hint: $(x-53)$ a factor of $2(-x^2+78x-1325)$.] [Ans: 25]

Solution:
From equation (1) $Profit = -2x^2 + 156x - 2160$

$$-2x^2 + 156x - 2160 = 490$$
$$-2x^2 + 156x - 2650 = 0$$
$$2(-x^2 + 78x - 1325) = 0$$

It is given that $(x-53)$ a factor of $2(-x^2+78x-1325)$. So dividing $2(-x^2+78x-1325)$ by $(x-53)$ we will get $-2x+50$.
Therefore,

$$2(-x^2 + 78x - 1325) = 0$$

$$(x-53)(2x-50) = 0$$

If

$$x - 53 = 0$$

$$x = 53$$

But selling price can not go beyond 25.
Now if,

$$2x - 50 = 0$$
$$x = \frac{50}{2}$$
$$x = 25$$

Therefore, the selling price of plate should be ₹25 .

9. What will be the value of $m + n$ if the sum of the roots and the product of the roots of equation $(5m + 5)x^2 - (4n + 3)x + 10 = 0$ are 3 and 2 respectively?

Solution:

We know that the sum of the roots of an equation $ax^2 + bx + c = 0$ is $\frac{-b}{a}$ and the product of its roots is $\frac{c}{a}$.

Here, $a = 5m + 5, b = -(4n + 3), c = 10$. Substituting these values we get,
The product of the roots of the given equation

$$\frac{c}{a} = \frac{10}{5m + 5} = 2$$

$$5m + 5 = 5$$

$$m + 1 = 1$$

$$\boldsymbol{m = 0}$$

The sum of the roots as

$$\frac{-b}{a} = \frac{-(-(4n + 3))}{5m + 5} = 3$$

$$4n + 3 = 3(5m + 5)$$

For $m = 0$

$$4n + 3 = 3 \times 5$$

$$4n = 12$$

$$\boldsymbol{n = 3}$$

Therefore,

$$m + n = 0 + 3 = 3.$$

10. What will the sum of two positive integers be if the sum of their squares is 369 and the difference between them is 3?. **Solution:**

Let a and b be the two positive integers. Given that

$$a^2 + b^2 = 369 \tag{2}$$
$$a - b = 3 \tag{3}$$

Squaring equation (3) on both sides, we get

$$\begin{aligned}(a-b)^2 &= 3^2\\ a^2 - 2ab + b^2 &= 9\\ 369 - 2ab &= 9\\ 2ab &= 369 - 9\end{aligned}$$

$$2ab = 360$$

Now, to find the sum of the integers

$$\begin{aligned}(a+b)^2 &= a^2 + 2ab + b^2\\ (a+b)^2 &= 369 + 360\\ (a+b)^2 &= 729\\ a+b &= \pm\sqrt{729} = \pm 27\end{aligned}$$

As a and b are positive integers, their sum should also be a positive integer.
Therefore, $\boldsymbol{a+b=27}$.