Week - 4 Part 2
Problem Solving tips
Solution of Quadratic Equations
Mathematics for Data Science - 1

1 Drawing a parabola:

Drawing a parabola by finding the X-intercept of the quadratic functions (i.e., by finding the roots of the quadratic equation corresponding to the quadratic function) is quite useful and interesting. For example, to draw parabola represented by the quadratic function $f(x) = x^2 - 6x - 16$ we can proceed as:

(a) The coefficient of x^2 is positive which means the parabola opens upward (i.e. opens towards the positive direction of Y-axis) .

(b) As the parabola opens upward, we get the minimum value of function and the point where the minimum value is attained is nothing but the vertex of the parabola. The coordinate of the vertes is $(-\frac{b}{2a}, -\frac{b^2}{4a} + c) \equiv (-\frac{-6}{2\times 1}, -\frac{(-6)^2}{4\times 1} + (-16)) \equiv (3, -25)$.

(c) Now the next question is: will the parabola intersect the $X-$ axis, touch the $X-$ axis, or none of the above cases will occur? For that purpose, we can use the discriminant $\sqrt{b^2 - 4ac} = \sqrt{6^2 + 4 \times 16} > 0$ (no need of calculation). The discriminant is greater than zero which means the parabola intersects the X-axis.

(d) If a parabola intersects the $X-$ axis, it intersects the X-axis at least at two points, which means the function has two distinct zeroes.

(e) For the accurate value of zeroes (or X- intercepts) we need to solve $f(x) = 0 \implies x^2 - 6x - 16 = 0$.

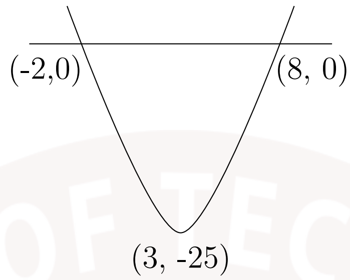
(f) Here $c = -16$ has two factors -8 and 2 which gives $-8 + 2 = -6$. Therefore, the above equation can be solved using factorization method i.e.,

$$x^2 - 6x - 16 = 0 \implies (x - 8)(x + 2) = 0$$

Hence the roots of the quadratic equation are 8 and -2.

(g) Therefore the parabola will intersect the $X-$ axis at $(8, 0)$ and $(-2, 0)$.

(h) Combining all the above information we can draw a rough diagram now as shown below.



We can conclude the following results after watching the above graph.

(a) For the domain $(-\infty, -2) \cup (8, \infty)$ the function has positive values and in the domain $(2, 8)$ the function has negative values.

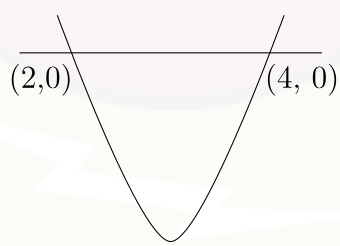
(b) The function $f(x)$ has the range $[-25, \infty)$ or $\mathbb{R} \setminus (-\infty, -25)$.

(c) The same concept can be applied to find the domain of a function f in which the functional value $f(x)$ is greater than some constant c. To solve this kind of questions we can write $f(x) = c$ and solve for x. For example to find the domain when the function $f(x)$ will have value strictly greater than -24, we will solve the equation $f(x) = -24$.

$$f(x) = -24$$

$$x^2 - 6x - 16 = -24$$
$$x^2 - 6x + 8 = 0$$
$$(x-4)(x-2) = 0$$

Using the above procedure we can draw a graph for the function $g(x) = x^2 - 6x + 8 = (x-4)(x-2)$ as shown below.



Now we can conclude that the $g(x)$ has the value strictly greater than zero in $(-\infty, 2) \cup (4, \infty)$. Which means $f(x)$ has the value strictly greater than -24 in $(-\infty, 2) \cup (4, \infty)$.

2 Finding the equation of a parabola:

The equation of parabola can be found using many approaches but the simplest way is assuming it as a function $f(x) = ax^2 + bx + c$. Here we need to find the three parameters a, b, and c and can be found using the given conditions one by one.

(a) Choosing the axes and the origin solves half of the problems for example if parabola passes through origin, then $c = 0$ and we need to find only a and b.

(b) If parabola has the axis of symmetry at $Y-$ axis, then $b = 0$ and we need to find only a and c. (Finding a and c is much easier than finding a and b).

(c) Example: A gateway is constructed in the shape of a parabola. The road through the gateway has a width of 10m and the maximum height of the gateway is 5m as shown in Figure 1. A container truck of width 4m and uniform height tries to pass through the gateway. What is the maximum possible height of the truck so that it just touches the wall of the gateway?

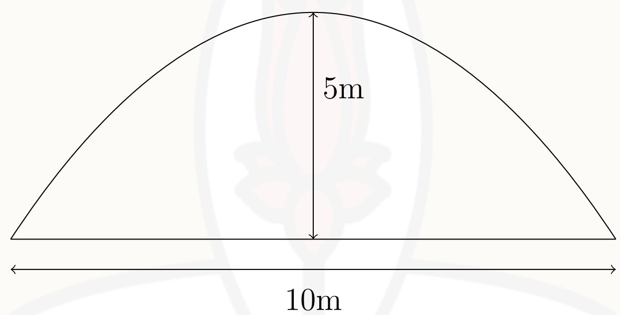
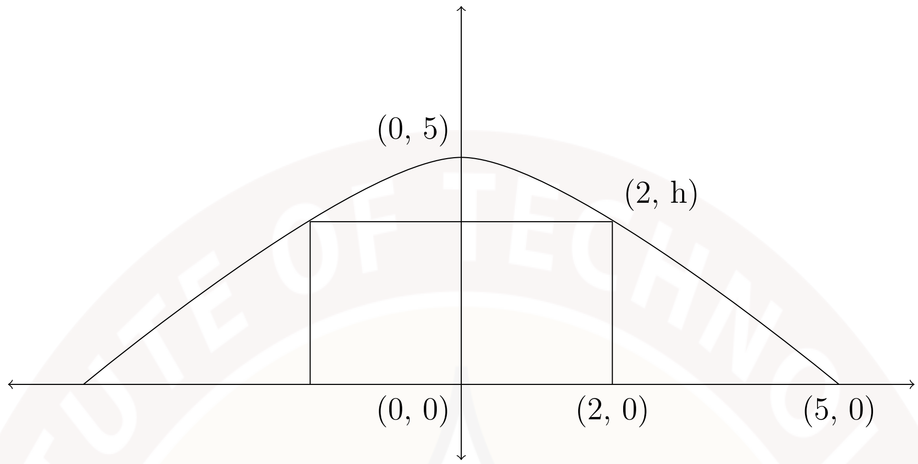


Figure 1

Solution: Let the equation of parabolic gate is represented by $f(x) = ax^2 + bx + c$, and if we assume that the axes is at the middle of gateway, the parabola will be symmetric about Y-axis which means $b = 0$ and $f(x) = ax^2 + bx$. We need to find only a and c now.
As the truck has to pass from the middle of gateway for maximum possible height of truck. The truck will take 2m on both sides of Y-axis. So we just need to find the height of truck we need to find the height of gateway at $x = 2$.



As the parabola passes through the point $(0, 5)$, then solving:

$$f(x) = ax^2 + c = 0$$
$$f(0) = a \times (0)^2 + c = 5$$
$$c = 5$$
$$f(x) = ax^2 + 5$$

The parabola also passes through the point $(5, 0)$. So,

$$f(5) = a \times (5)^2 + 5 = 0 \implies a = -\frac{1}{5}$$

Therefore

$$f(x) = -\frac{1}{5}x^2 + 5$$
$$f(2) = 4.2$$

So the maximum possible height of the truck would be 4.2 m.

3 Things to remember:

(a) The quadratic equation can be solved using Squaring method, Factorization method, and Quadratic formula. But most of the time the factorization is recommended.

(b) Sometimes, using factorization method we can directly find the solution without writing much. For example to find the solution of $x^2 - 5x + 6 = 0$, we can directly say that $x = 3$ and $x = 2$ are the solutions as $-3 \times -2 = 6$ and $-3 + (-2) = -5$ (not even need to write the factor).

(c) Quadratic equation is not recommended to be solved by square method when the roots are not real.

(d) It is mandatory to think which one is the better root among the two roots of a quadratic equation (assuming there are two distinct roots) for a particular context of a given question. For example if the question has the reference of time, distance or age we can not accept negative solutions. If the question has reference of month number or number of people we can not accept the fractional values.

(e) Discriminant has a lot of applications whenever the question is related with the number of roots of a quadratic equation. Even without solving the equation, we can conclude many important properties of the roots using discriminant.