Week - 4
Assignment
Quadratic Equations
Mathematics for Data Science - 1
Full marks: 25

NOTE:
There are some questions which have functions with discrete valued domains (such as month or year). For simplicity, we treat them as continuous functions.

1 Multiple Choice Questions (MCQ):

1. Find out the points where the curve $y = 4x^2 + x$ and the straight line $y = 2x - 3$ intersect with each other. **1 mark**

   - ◯ $(\frac{3}{2}, 0)$ and $(\frac{3}{2}, \frac{21}{2})$.
   - ◯ Only at the origin.
   - ◯ **The curve and the straight line do not intersect.**
   - ◯ $(1, -1)$ and $(1, 5)$.

   Solution: The following Figure M1W5AS-1 shows that the curves $y = 4x^2 + x$ and $y = 2x - 3$ are not intersecting with each other at any point.

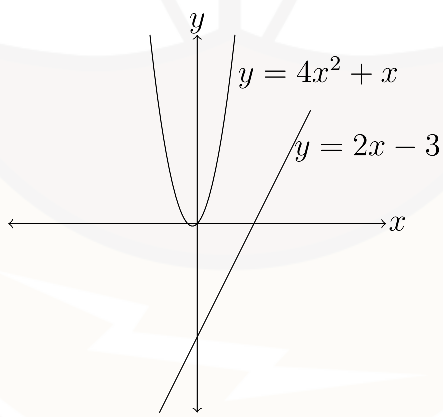


Figure: M1W5AS-1

Suppose that the curves $y = 4x^2 + x$ and $y = 2x - 3$ are intersecting at the point (a, b). Let us try to find the point. As both curves are passing through the point (a, b), it should satisfy both the equations. So we have $b = 4a^2 + a$ and $b = 2a - 3$. Which implies,

$$4a^2 + a = 2a - 3$$

i.e.

$$4a^2 - a + 3 = 0$$

The discriminant of the obtained quadratic is

$$(-1)^2 - 4(4)(3) = 1 - 48 = -47 < 0$$

As the discriminant is negative, the quadratic equation has no real root. So there cannot be any point (a, b) on the real plane at which these curves intersect each other.

2. Three friends A, B and C started their journey at 08:00 am from the points $(2, 31)$, $(3, 51)$, and $(6, 0)$ respectively. A followed the path along the curve $3x^2 + 2x + 15$ and B followed the path along the curve $2x^2 + 10x + 3$. They all meet at 11:00 am at a point P whose x coordinate is greater than 4. If C followed a straight-line path, and one unit is equal to 1 km then what was the speed of C? **3 marks**

   - ◯ 31.26 km/hr
   - ◯ 32 km/hr
   - ◯ **45 km/hr**
   - ◯ 45.5 km/hr
   - ◯ $\frac{4}{3}$ km/hr
   - ◯ $\frac{3}{4}$ km/hr

Solution: A and B followed the the path along the curve $3x^2+2x+15$ and $2x^2+10x+3$ respectively and met at the point P. Hence P is one of the points of intersections of these two curves. Let the coordinate of the point P be (a, b). Hence we have,

$$\begin{aligned} 3a^2 + 2a + 15 &= 2a^2 + 10a + 3 \\ a^2 - 8a + 12 &= 0 \\ (a-6)(a-2) &= 0 \end{aligned}$$

Hence there are two possible values for a: 2 and 6. It is given in the problem that the x coordinate of the point P is greater than 4. Hence a must be 6. Now substituting the value of a in any one of the equations we get the value of b as 135.

Hence coordinate of the point P is $(6, 135)$.

C starts its journey from the point $(6, 0)$ and reaches the point P, whose coordinate is $(6, 135)$, along a straight-line path. So the distance covered by C is 135 units. Now in the problem it is given that one unit is equal to 1 km. So in 3 hours (i.e., 8:00 am to 11:00 am) C has covered 135 km.

So the speed of C is $\frac{135}{3} = 45$ km/hr.

3. Consider the curve of the quadratic function $y = (x - \frac{1}{p})(x + \frac{1}{q})$, with $p, q \neq 0$. Suppose the distance between its x intercepts is 2, and the y intercept is at a distance 1 from the origin. Then which of the following equations is correct for this given curve? **1 mark**

 - ◯ $y = (x - \frac{1}{3})(x - \frac{5}{3})$
 - ◯ $y = (x - 1)^2$
 - ◯ $\boldsymbol{y = (x - 1)(x + 1)}$
 - ◯ None of the above.

Solution: Given the quadratic function $y = (x - \frac{1}{p})(x + \frac{1}{q})$.

$$x \text{ intercepts} : (\tfrac{1}{p}, 0) \text{ and } (-\tfrac{1}{q}, 0).$$
$$y \text{ intercept} : (0, -\tfrac{1}{pq})$$

From the given information in the problem we have,

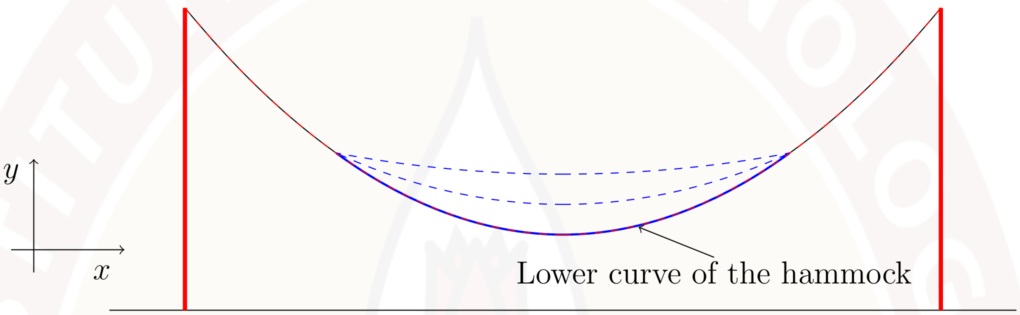
$$\frac{1}{p} + \frac{1}{q} = 2 \qquad (1)$$
$$\frac{1}{pq} = 1 \qquad (2)$$

From equation (2) we have, $q = \frac{1}{p}$. By substituting this in equation (1) we get,

$$\begin{aligned} \frac{1}{p} + p &= 2 \\ \frac{1 + p^2}{p} &= 2 \\ 1 + p^2 &= 2p \\ p^2 - 2p + 1 &= 0 \\ (p - 1)^2 &= 0 \\ p &= 1 \end{aligned}$$

Hence, $q = \frac{1}{p} = 1$. So the equation of the given curve is $y = (x - 1)(x + 1)$.

4. A hammock is a cloth swing as shown in the picture below. The height (y) from ground of any point of the lower curve of the hammock varies with respect to the horizontal distance (x) from some fixed origin. The origin is in the plane of y and x. Consider the following equations and choose the correct option. (It is expected that the hammock does not touch the ground.)
A: $y = x^2 + 6x + 8$
B: $y = x^2 + 4x + 8$
C: $y = x^2 + 4x + 2$ **6 marks**



- ◯ Only A can represent the hammock.
- ◯ **Only B can represent the hammock.**
- ◯ Only C can represent the hammock.
- ◯ Both A and B can represent the hammock.
- ◯ Both A and C can represent the hammock.
- ◯ None of these equations is appropriate to represent the hammock.

Solution: As it is expected that the hammock does not touch the ground, the y-coordinate of the vertex of the parabola should be strictly greater than 0.

$$\text{Vertex of A} : (-3, -1)$$
$$\text{Vertex of B} : (-2, 4)$$
$$\text{Vertex of C} : (-2, -2)$$

So only for B, the y-coordinate of the vertex is strictly greater than 0. Hence, only B can represent the hammock.

5. A bird is hunting for fish in a pond. She swoops down from a height and picks up a fish from the water surface and flies back up, all along a path $y = 4x^2 - (2k+2)x + 1$, where $k \geq 0$, y is its height from the water surface, and x is the horizontal distance from a fixed origin which is in the plane of the path. Then, the value of k can be: **3 marks**

   - ◯ 3
   - ◯ 2
   - ◯ **1**
   - ◯ 5

   Solution: As the bird swoops down from a height and picks up a fish from the water surface and flies back up, all along a path $y = 4x^2 - (2k+2)x + 1$, there should be only one real root of the equation $4x^2 - (2k+2)x + 1 = 0$. Hence the discriminant should be 0. So we have,

$$\begin{aligned}
(-(2k+2))^2 - 4(4)(1) &= 0 \\
4k^2 + 4 + 8k - 16 &= 0 \\
4k^2 + 8k - 12 &= 0 \\
k^2 + 2k - 3 &= 0 \\
(k+3)(k-1) &= 0
\end{aligned}$$

   The value of k can be 1 or -3. As k is given to be non-negative, we have $\boldsymbol{k = 1}$.

6. If the height of a right-angled triangle is 2 cm more than its base, and the hypotenuse is 4 cm more than its base, then what is the height (cm) of the triangle? **1 mark**

   - ◯ 10
   - ◯ **8**
   - ◯ 6
   - ◯ 4
   - ◯ 3
   - ◯ None of the above.

   Solution: Let the length of the base of the given triangle be x cm. The height will be $x + 2$ cm and the length of the hypotenuse will be $x + 4$ cm. As it is a right angled triangle, using the Pythagorean theorem we have,

$$
\begin{aligned}
x^2 + (x+2)^2 &= (x+4)^2 \\
x^2 + x^2 + 4 + 4x &= x^2 + 16 + 8x \\
x^2 - 4x - 12 &= 0 \\
(x-6)(x+2) &= 0
\end{aligned}
$$

   As the length of base cannot be negative, x must be positive. Hence $x = 6$.
   So the height of the triangle is 8 cm.

2 Multiple Select Questions (MSQ):

7. A company opens two new branches A and B in 2010. A and B make yearly profits in lakhs as $P_1(y) = 10y - y^2$ and $P_2(y) = y^2 - 6y$ respectively, where y is the number of years after opening the branch. Let loss be represented as $-ve$ of profit. Then choose the correct set of options from the following. **3 marks**

   ○ **The range of profit made by branch B for the first 9 years is $[-9, 27]$.**

   ○ **The range of profit made by branch A for the first 10 years is $[0, 25]$.**

   ○ Till 2016 branch A was making a loss.

   ○ **In 2018, both companies made the same profit.**

   ○ **Going by the trajectory of branch A, 2020 is the suitable time to shut down the branch for avoiding a loss.**

   ○ B never goes into a loss.

Solution: Firstly observe that we are given two quadratic functions of y. So y will be plotted along the horizontal axis and $p(y)$ will be plotted along the vertical axis. Also observe in Figure: M1W5AS-7 that the parabola represented by the function $p_2(y)$ opens towards the positive direction of y-axis, i.e. open upwards, as the coefficient of y^2 is positive and the parabola represented by the function $p_1(y)$ opens towards the negative direction of y-axis, i.e. open downwards, as the coefficient of y^2 is negative in this case.

- $p_2(y)$ will be minimum at the vertex of the parabola represented by the corresponding curve. The coordinate of the vertex is $(3, -9)$. Moreover, $p_2(y)$ will be increasing for $y > 3$. So in the first 9 years the maximum value of $p_2(y)$ will be at $y = 9$, and $p_2(9) = 27$. Hence, the range of profit made by branch B for the first 9 years is $[-9, 27]$. So the first option is correct.
- $p_1(y)$ will be minimum at the vertex of the parabola represented by the corresponding curve. The coordinate of the vertex is $(5, 25)$. Moreover, $p_1(y)$ increases till $y = 5$ starting from $y = 0$, and then again decreases again for $y > 5$. $p_1(10) = p_1(0) = 0$. Hence, the range of profit made by the branch A for the first 10 years is $[0, 25]$. So the second option is also correct.
- A is never making a loss. So the third option is not correct.
- If both the branches have to make same profit at some point then $p_1(y)$ must be equal with $p_2(y)$ for some y.

$$\begin{aligned} 10y - y^2 &= y^2 - 6y \\ 2y^2 - 16y &= 0 \\ y(y-8) &= 0 \end{aligned}$$

  Hence at $y = 0$ and $y = 8$ the profit will be same. At $y = 0$ both the branches are opening so there is no profit for both of them, so $p_1(0) = 0 = p_2(y)$. At $y = 8$, i.e in 2018, both the companies made the same profit. Hence the fourth option is correct.

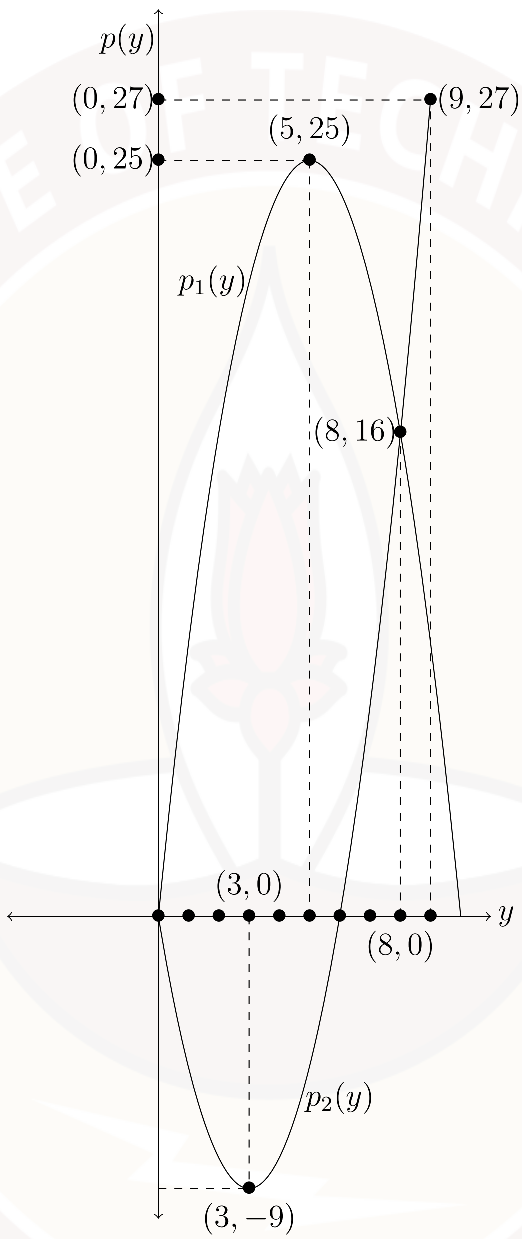


Figure: M1W5AS-7

- We have already calculated that $p_1(10) = 0$ and for $y > 10$, $p_1(y) < 0$. So after 2020 the branch A will be making a loss. Hence going by the trajectory of branch A, 2020 is the suitable time to shut down the branch for avoiding a loss. Hence the fifth option is also correct.

- For $0 \leq y \leq 6$, $p_2(y) \leq 0$. Hence till 2016 the branch B is making a loss. So the last option is not the correct one.

8. Choose the correct set of options regarding the quadratic equation $(5m+9)^2x^2-(3n+4)x+1=0$ where $m,n\in\mathbb{Z}$. **3 marks**

- ◯ Both roots are equal if and only if $10m-3n+14=0$.
- ◯ **Both roots are equal if $\boldsymbol{10m-3n+14=0}$.**
- ◯ **Both roots are equal if $\boldsymbol{m=n=-2}$.**
- ◯ Discriminant is 91 if $m=n=-1$.
- ◯ Both roots will always be distinct and real if $m=n$.
- ◯ Discriminant is 91 if $m=n=4$.

Solution: In order to talk about the roots of the quadratic equation $(5m+9)^2x^2-(3n+4)x+1=0$ where $m,n\in\mathbb{Z}$, we have to calculate the discriminant of $(5m+9)^2x^2-(3n+4)x+1$.

Discriminant of $(5m+9)^2x^2-(3n+4)x+1$ is

$$\begin{aligned}&=(-3n-4)^2-4(5m+9)^2(1)\\&=(3n+4)^2-(2(5m+9))^2\\&=(3n+4+10m+18)(3n+4-10m-18)\\&=(3n+10m+22)(3n-10m-14)\end{aligned}$$

Both the roots of the equation will be equal if the discriminant is 0. Then either $(3n+10m+22)=0$ or $(3n-10m-14)=0$.
So both the roots are equal if $3n-10m-14=0$, i.e. $10m-3n+14=0$. But from this we can not say that if both the roots are equal then $10m-3n+14=0$, because both the roots can be equal even if $3n+10m+22$ is 0 and $10m-3n+14\neq 0$, as even in this case the discriminant will be 0.
So the first option is not correct, whereas the second one is correct.

If $m=n=-2$, then $(10m-3n+14)=(-20+6+14)=0$, So the discriminant will be 0. Hence both roots are equal if $m=n=-2$, i.e **the third option is also correct.**

If $m=n=-1$, then the discriminant is $(-3-10+22)(-3+10-14)=-63$. So **the fourth option is not correct.**
We have already seen that if $m=n=-2$ then the roots are equal. So **the fifth option is also not correct.**
If $m=n=4$, then the discriminant is $(12+40+22)(12-40-14)=-3108\neq 91$. Hence **the sixth option is also not correct.**

3 Numerical Answer Type (NAT):

9. In order to cover a fixed distance of 48 km, two vehicles start from the same place. The faster one takes 2 hrs less and has a speed 4 km/hr more than the slower one. Using the given information, answer the following questions. **1 mark+ 1 mark**

   (a) What is the speed (in km/hr) of the slower vehicle? [Ans: 8]

   (b) What is the time (in hrs) taken by the faster one? [Ans: 4]

   Solution: Let the speed of the slower vehicle be x km/hr. The time taken by the slower one to cover 48 km is $\frac{48}{x}$ hr. The speed of the faster one is $x+4$ km/hr. So the time taken by the faster one to cover 48 km is $\frac{48}{x+4}$. It is given than the faster one takes 2 hrs less than the slower one to cover the distance. So we have,

$$\begin{aligned}\frac{48}{x} - \frac{48}{x+4} &= 2\\ \frac{x+4-x}{x(x+4)} &= \frac{2}{48}\\ \frac{x(x+4)}{4} &= 24\\ x^2+4x-96 &= 0\\ (x+12)(x-8) &= 0\end{aligned}$$

   As the speed has to be positive, x must be 8.
   Hence the speed of the slower vehicle is 8 km/hr.
   The speed of the faster one is 12 km/hr. So the time taken by the faster one is $\frac{48}{12} = 4$ hrs.

10. Let the ratio of the length to the breadth of a flag be 3:2. Let the cost of the cloth required to make the flag be Rs. 4 per square metre and the cost of stitching along its perimeter be Rs. 2 per metre. If the cost of making (the cost of cloth and the cost of stitching) 6 such flags is Rs. 24, then answer the following questions.
1 mark+ 1 mark

(a) How much length (in metre) has to be stitched along the perimeter to make 6 flags in total ? [Ans: 10]

(b) What is the total area (in square metres) of 6 flags? [Ans: 1]

Solution: Let the length and the breadth of the flag be $3x$ and $2x$ metre. So the cloth required to make a flag is $6x^2$ square metre. The length of the perimeter is $10x$ metre. Hence the cost of the cloth required is $4 \times 6x^2 = 24x^2$ Rs. and the cost of stitching is $2 \times 10x = 20x$ Rs.
The cost of making of 6 flag is 24 Rs. So the cost of making 1 flag is 4 Rs.
Hence we have,

$$\begin{aligned} 24x^2 + 20x &= 4 \\ 6x^2 + 5x - 1 &= 0 \\ (x+1)(6x-1) &= 0 \end{aligned}$$

x cannot be negative as the length and breadth cannot be negative. So $x = \frac{1}{6}$. Therefore the length of a flag is $\frac{1}{2}$ metre and the breadth of a flag is $\frac{1}{3}$. Hence, the perimeter is 5/3 and the area of a flag is $\frac{1}{6}$ square metre.
Hence 10 metre has to be stitched along the perimeter to make 6 flags in total and the total area of 6 flags is 1 square metres.