Exercise 5.1 from textbook

The Rhind papyrus is an ancient Egyptian mathematical manuscript that is more than 3500 years old. Problem 79 of the Rhind papyrus poses a problem that can be paraphrased as follows: there are seven houses; in each house lives seven cats; each cat kills seven mice; each mouse has eaten seven spelt seeds; each spelt seed would have produced seven hekat of spelt. What is the sum of all of the named items? Solve this 3500 year old problem.

Solution
There are 7 houses. In each house lives 7 cats. This implies that there are $7 \times 7 = 49$ cats in total. Next, each of the 49 cats kills 7 mice. This implies that there are $49 \times 7 = 343$ mice in total. Again since each of the 343 mice has eaten 7 spelt seeds this again implies that there are $343 \times 7 = 2401$ seeds in total. Finally, each spelt seed would have produced 7 hekat, so there are $2401 \times 7 = 16807$ kehat in total.

Now, the sum of all the named items is $7 + 49 + 2401 + 16807 = 19607$.

Exercise 5.2 from textbook

(a) How many n-tuples $(x_1, x_2, \cdots, x_n)$ are there if the coordinates are required to be integers satisfying $0 \leq x_i < q$?
Ans: There are q possible choices for each x_i, since $0 \leq x_i < q$. Since there are n independent coordinates, the total number of n-tuples is q^n.

(b) Same question as (a), except now there are separate bounds $0 \leq x_i < q_i$ for each coordinate.
Ans: There are q_i choices for each x_i. It follows that the total number of n-tuples is $q_1 q_2 \cdots q_n$.

(c) How many $n \times n$ matrices are there if the entries $x_{i,j}$ of the matrix are integers satisfying $0 \leq x_{i,j} < q$?
Ans: By part (a), total number of $n \times n$ matrices with entries $x_{i,j}$ are q^{n^2}.

(d) Same question as (a), except now the order of the coordinates does not matter. So for example, (0, 0, 1, 3) and (1, 0, 3, 0) are considered the same. (This one is rather tricky.)

Ans: Since the order of the coordinates does not matter, this problem follows the combinatorial principle of combinations with repetition. So, we can think of it as placing n objects into q containers. In other words, this is choosing n items from a total of $n+q-1$ items, which is given by $\binom{q+n-1}{n}$.

(e) Twelve students are each taking four classes, for each class they need two loose-leaf notebooks, for each notebook they need 100 sheets of paper, and each sheet of paper has 32 lines on it. Altogether, how many students, classes, notebooks, sheets, and lines are there?

Ans: Given 12 students, each enrolled in 4 classes, the total class count is $12 \times 4 = 48$. Each class requires 2 notebooks, yielding $48 \times 2 = 96$ notebooks. Each notebook contains 100 sheets, resulting in $96 \times 100 = 9600$ sheets. Each sheet holds 32 lines, leading to $9600 \times 32 = 307,200$ lines. The classroom collectively comprises 12 students, 48 classes, 96 notebooks, 9,600 sheets, and 307,200 lines. So, the total count of all items considered in the classroom is 316956.

Exercise 5.9 from textbook

We know that there are n! different permutations of the set $\{1, 2, \cdots, n\}$

(a) How many of these permutations leave no number fixed?

(b) How many of these permutations leave at least one number fixed?

(c) How many of these permutations leave exactly one number fixed?

(d) How many of these permutations leave at least two numbers fixed?

For each part of this problem, give a formula or algorithm that can be used to compute the answer for an arbitrary value of n, and then compute the value for $n = 10$ and $n = 26$.

Answers

(a) This is the number of derangement. That is, the number of permutations that leave no number fixes. The formula for this permutation, denoted as $!n$ is given by

$$!n = n! \sum_{k=0}^{n} \frac{(-1)^k}{k!}.$$

```python
import math
def derangement(n):
    """Compute the number of derangements (!n) using the inclusion-exclusion principle."""
    return round(
        math.factorial(n) * sum((-1) ** k / math.factorial(k) for k in range(n + 1))
    )

```

Listing 1: Exercise 5.9 (a)- Derangement python Implementation

The summation is approximately $\frac{1}{e}$ for large values of n. So,

- For $n = 10$, we have $!10 = 1334961$
- For $n = 26$, we have $!26 = 148362637348470138328842240$

(b) This is the complement of derangements. So, the number of permutations that leave at least one number fixed is given by $n! - !n$.

- For $n = 10$, we have $10! - !10 = 3628800 - 1334961 = 2293839$
- For $n = 26$, we have $26! - !26 = 254928823778135497255157760$

(c) To count permutations that fix exactly one element and move the others, pick one fixed element from the n choices and permute the remaining 9 elemnts. This is given by $n \times !(n-1)$.

- For $n = 10$, we have $10 \times !9 = 10 \times 133496 = 1334960$
- For $n = 26$, we have $26 \times !25 = 148362637348470127591424000$

(d) The number of permutations that leave at least two numbers fixed is given by $n! - n \cdot !(n-1)$.

- For $n = 10$, we have $10! - 10 \cdot !(9) = 2293840$
- For $n = 26$, we have $26! - 26 \cdot !(25) = 254928823778135507992576000$