CS F351: Theory of Computation 12^{th} **Nov, 2022**

29: Pumping Lemma for Context Free Languages

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Pumping Lemma for CFLs

For any CFL L, $\exists\ k \in \mathbb{N}$, such that for any word $w \in L$ with $|w| > k$, there exists a partition w = uvxyz such that following holds:

- $|vy| > 0$; i.e. the part to be pumped should be non empty.
- $uv^ixy^iz \in L$, $\forall i \in \mathbb{N}$
- $|vxy| \leq k$.

Note: Pumping Lemma is a necessary condition for a language to be Context-Free. It is not sufficient.

Example: $L = \{a^nb^nc^n \mid n \geq 0\}$ is not a CFL, as it fails the pumping lemma.
Proof:

- Assume, for a contradiction, that L is a CFL. Hence, Pumping Lemma for CFL should hold for L, since it is a necessary condition for a language to be CFL.
- let G be a grammar for this language. WLOG, consider G to be in CNF. Assume for the size of G is bounded above by p.
- Since we want to pick a long enough word, it should have length more than 2p, as G is in CNF. Let $k = 2p + 1$.
- Let $w = a^kb^kc^k$ be the word in L such that $|w| > 2p$ (i.e. $|w| \geq k$).
- To arrive at a contradiction, we need to show that for any spilt of this word of the form $uvxyz, \exists i$ such that $uv^ixy^z \notin L$.
- Following cases of word split can arise:

  Case I: Any part to be pumped (i.e v or y) is word made of single letter. In such case, that part can be pumped one more time so that we get a new word that does not lie in L. This is true for all combinations of v, y. An example in which v is a word of all a's is given below.

$$w = a^{j1}.a^{j2}.a^{j3}.a^{j4}.a^{(k-j1-j2-j3-j4)}b^kc^k$$

$$w = u.v.x.y.z$$

The new word formed after pumping is:

$$w' = u.v^2.x.y^2.z = a^{(k+j2+j4)}b^k c^k$$

Similarly, it can be proved that any word w whose partition v or y is word made with 1 letter will produce a word $w' \notin L$.

Case II:Any part to be pumped (i.e. v or y) is word made of 2 letters. In such case, that part can be pumped one more time so that we get a new word that does not lie in L. This is true for all combinations of v, y. An example in which v is a word of a's and b's is given below.

$$w = a^{j1}.a^{k-j1}b^{j2}.b^{j3}.b^{j4}.b^{k-j2-j3-j4}.c^k$$

$$w = u.v.x.y.z$$

The new word formed after pumping is:

$$w' = u.v^2.x.y^2.z = a^k b^{j2} a(j - j1) b^k c^k \notin L$$

Similarly, it can be proved that any word w whose partition v or y is word made with 2 letters will produce a word $w' \notin L$.

Case III: Any part to be pumped (i.e. v or y) is word made of 3 letters. In such case, that part can be pumped one more time so that we get a new word that does not lie in L. This is true for all combinations of v, y. An example in which y is a word of a's, b's and c's is given below.

$$w = a^{j1}.a^{k-j1}b^{j2}.b^{j3}.b^{j4}.b^{k-j2-j3-j4}.c^k$$

$$w = u.v.x.y.z$$

The new word formed after pumping is:

$$w' = u.v^2.x.y^2.z = a^{(k+j2)}b^k c^{j4} a^{(k-j1-j2-j3)} b^k c^k \notin L$$

Similarly, it can be proved that any word w whose partition v or y is word made with 3 letters will produce a word $w' \notin L$.

- Hence for any spilt of this word of the form $uvxyz$, $\exists i$ such that $uv^ixy^iz \notin L$.
- But, this is a contradiction, as we assumed that L is a CFL. Therefore, it can be concluded that our assumption was wrong.
- Hence, L is not a CFL. Hence proved.