CS F351: Theory of Computation **16 Sep, 2022**

8: More Closure Properties

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More Closure Properties

0.1 Monoid

A tuple $(S, \bullet, e)$ is known as a *monoid*, if:

- S is a set.
- $\bullet$: is *associative*, i.e., $\forall u, v, w \in S, (w \bullet u) \bullet v = w \bullet (u \bullet v)$
- $e \in S$ is the identity element, i.e., $\forall w \in S, e \bullet w = w \bullet e = w$

Eg: $(\Sigma^\star, \bullet, e)$ is a monoid.
Eg: $(\mathbb{N}, +, 0)$ is a monoid. [Also commutative.]
Eg: $(\{e, a\}, \bullet, e)$ is *NOT* a monoid. [Because, if we consider $'a'$ and $'a'$, we get the concatenation $a \bullet a = aa$, which is not a part of the set $\{e, a\}$, since it has to be $SxS \to S$.]

Specification: (trying to describe what is required in words)

$$\exists x \exists y.(x < y \wedge R_a(x) \wedge R_b(y))$$

$R_a(x)$ **:** Tells us that at the position $'x'$ in the word, we have the letter $'a'$.
$R_b(y)$ **:** Tells us that at the position $'y'$ in the word, we have the letter $'b'$.

0.2 Homomorphism

A function $h : S_1 \to S_2$ is called a (monoid) homomorphism from $(S_1, \bullet_1, e_1)$ to $(S_1, \bullet_1, e_1)$, if:

- $h(e_1) = e_2$.
- $\forall u, v \in S_1, h(u \bullet_1 v) = h(u) \bullet_2 h(v)$.

Eg: $h : \{a, b\}^* \to \mathbb{N}$ is defined as $h(w) = |w|$ is a homomorphism from $(\Sigma^*, \bullet, e)$ to $(\mathbb{N}, +, 0)$.

Theorem 1. *Let h is a homomorphism from $(\Sigma^*, \bullet, e)$ to $(\Gamma^*, \bullet, e)$. Then:*

1. *If $L \subset \Sigma^*$ is regular, then $h(L) \subseteq \Gamma^*$ is regular. [$h(L) := \{h(w) | w \in L\}$]*
2. *If $L \subset \Gamma^*$ is regular, then $h^-1(L) \subseteq \Gamma^*$ is regular. [$h(L) := \{w \in \Sigma | h(w) \in L\}$]*