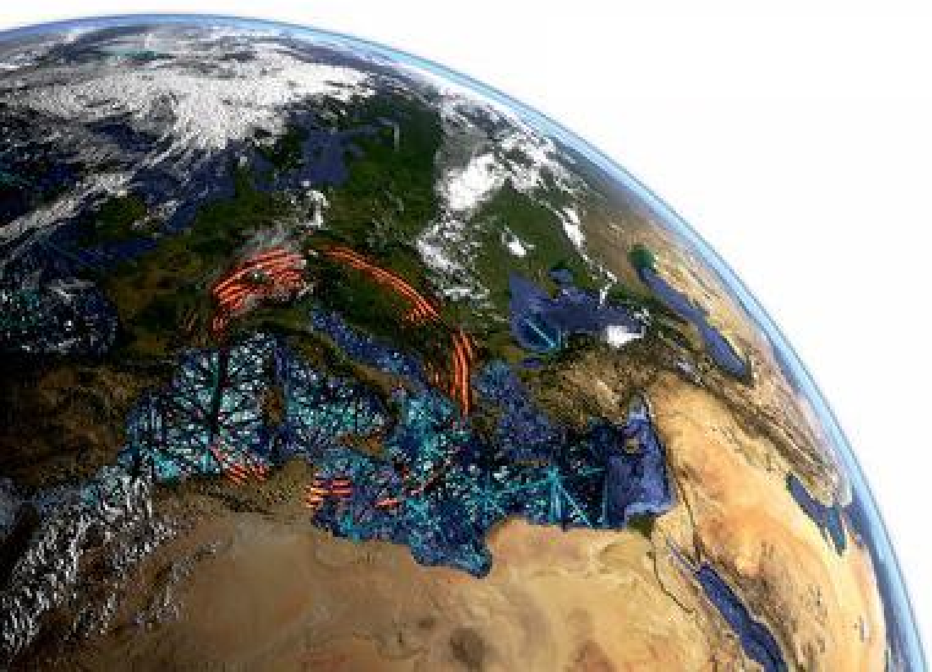
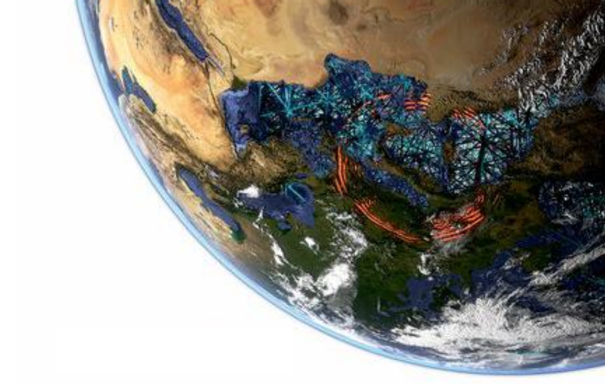
OPTIMALLY ACCURATE SECOND-ORDER TIME-DOMAIN FINITE DIFFERENCE SCHEME FOR THE ELASTIC EQUATION OF MOTION : 1D CASE

Robert J.Geller & Nozomu Takeuchi
GJI (1998)



Taufiqurrahman
Advanced Computational Seismology
9 November 2016

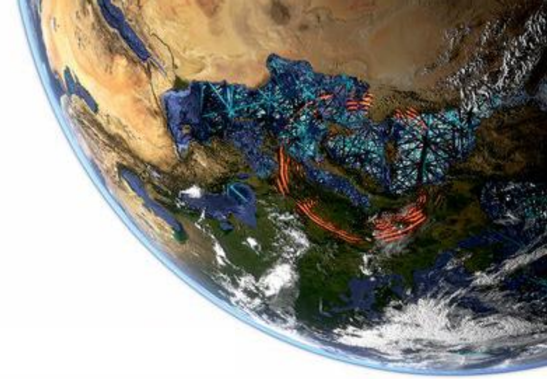
Picture: Computational Seismology - Heiner Igel (2016)



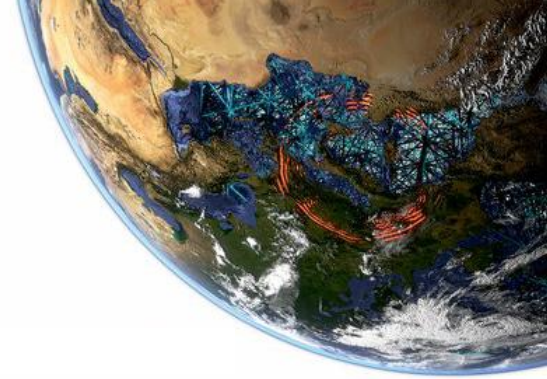
Agenda

- Background
- FD Operators
- Homogeneous & Heterogeneous Case
- 'Predictor-Corrector' Scheme
- Numerical Examples
- Discussion
- Python Code (Homogeneous 1D Case)

Introduction



- in the need of efficient & accurate methods for computing synthetic seismograms, i.e. *waveform inversion technique*
- proposed FD scheme delivers almost 2 orders of magnitude greater accuracy than conventional FD scheme, while only requiring only 2 times as much CPU time

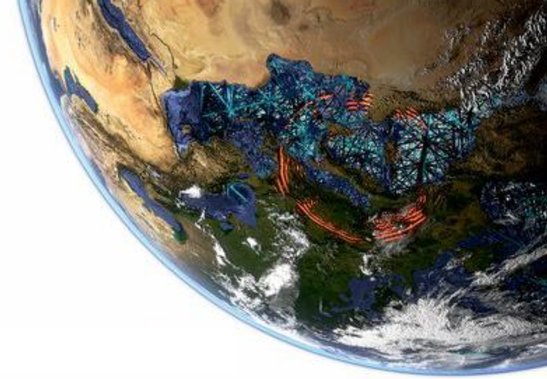


Idea

- Standard techniques of numerical analysis provide estimates of ***the error of discretized numerical operators*** rather than the error of the numerical solutions (i.e. synthetic seismograms) computed using these operators
- i.e. the error of 3-point central FD operator of 2nd-derivative is 2nd-term of Taylor series:

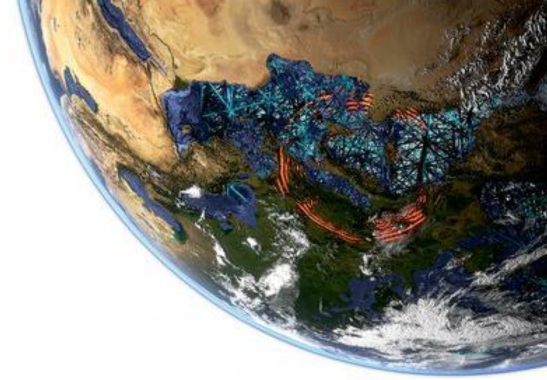
$$\frac{1}{\Delta z^2}\left[u(z-\Delta z)-2u(z)+u(z+\Delta z)\right]=\frac{d^2u}{dz^2}+\boxed{\frac{\Delta z^2}{12}\,\frac{d^4u}{dz^4}}+\ldots.$$

- The error of synthetic seismograms computed by conventional schemes be proportional to *O(**Δx²**).*



Idea

- in 1995, Geller & Takeuchi (cited as **GT95**) developed a method (in frequency-domain) to make formal estimates of ***the error of synthetic seismograms***
- Their results allow numerical operators to be *'tuned'* to produce optimally accurate numerical schemes, i.e. correcting numerical solutions using estimated basic error ***('predictor-corrector' scheme)***
- in 1998, they modified this optimal operator into time-domain



FD Operators

- The time-domain equation of motion for 1D case:

$$\rho \frac{\partial^2 u}{\partial t^2} - \frac{\partial}{\partial z}\left(\mu \frac{\partial u}{\partial z}\right) = f$$

density displacement elastic modulus external force

- Then the FD equation of motion for 1D case:

$$(A_{mMnN} - K_{mMnN})c_{nN} = f_{mM}$$

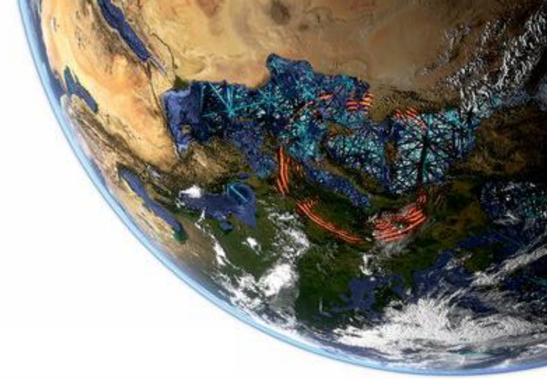
FD operator for $\rho(\partial^2 u/\partial t^2)$

FD operator for $\partial/\partial z[\mu(\partial u/\partial z)]$

discretized displacement $u(n\Delta z, N\Delta t)$

discretized external force

Homogeneous 1D Operators



- The conventional FD operator ***A0*** and ***K0*** are as follows:

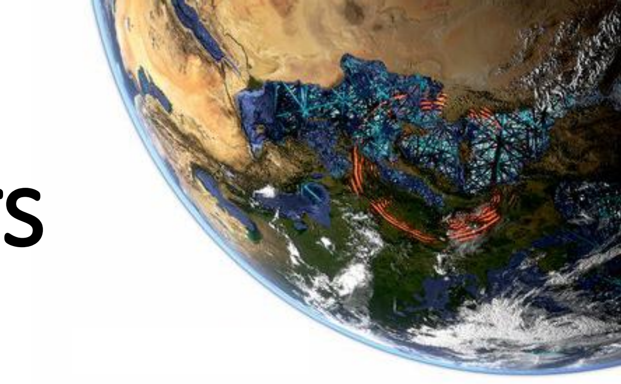
$$\mathbf{A}^0 = \left(\frac{\rho}{\Delta t^2}\right) \times \begin{array}{c|ccc|} t+\Delta t & & 1 & \\ t & & -2 & \\ t-\Delta t & & 1 & \\ \hline & z-\Delta z & z & z+\Delta z \end{array}$$

```
A0   = rho / (dt**2) * np.matrix\
('  0.   1.   0.;\
    0.  -2.   0.;\
    0.   1.   0.')
```

$$\mathbf{K}^0 = \left(\frac{\mu}{\Delta z^2}\right) \times \begin{array}{c|ccc|} t+\Delta t & & & \\ t & 1 & -2 & 1 \\ t-\Delta t & & & \\ \hline & z-\Delta z & z & z+\Delta z \end{array}$$

```
K0   = mu / (dx**2) * np.matrix\
('  0.   0.   0.;\
    1.  -2.   1.;\
    0.   0.   0.')
```

Note : A0 and K0 have the numerical dispersion (operator error) of normal 3-point 2nd-derivative operators



Homogeneous 1D Operators

- The *error* for the conventional operator is:

$$(\delta A^0_{mMnN} - \delta K^0_{mMnN}) c_{nN} = \frac{\Delta t^2}{12} \rho \left(\frac{\partial^4 u}{\partial t^4} \right) - \frac{\Delta z^2}{12} \mu \left(\frac{\partial^4 u}{\partial z^4} \right) \bigg|_{t=t_M, z=z_m}$$

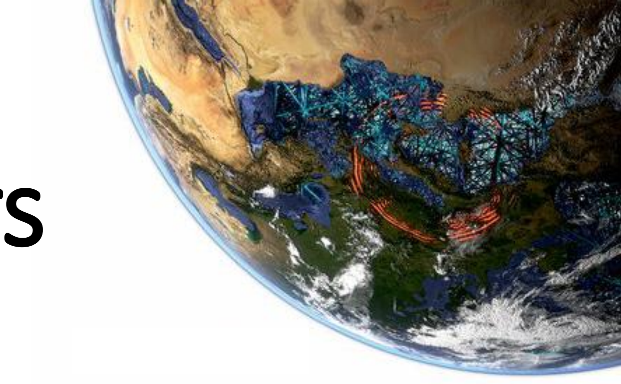
- where:

$$\delta A^0_{mMnN} = A^0_{mMnN} - A^{\text{exact}}_{mMnN} \qquad \delta K^0_{mMnN} = K^0_{mMnN} - K^{\text{exact}}_{mMnN}$$

dA = A0 - A   dK = K0 - K

- But, how can we estimate *exact* ***A*** and ***K*** operator?

Homogeneous 1D Operators



- **GT95**: taking Fourier transform, operator error in frequency domain given by:

B and L term is Fourier transform of A and K

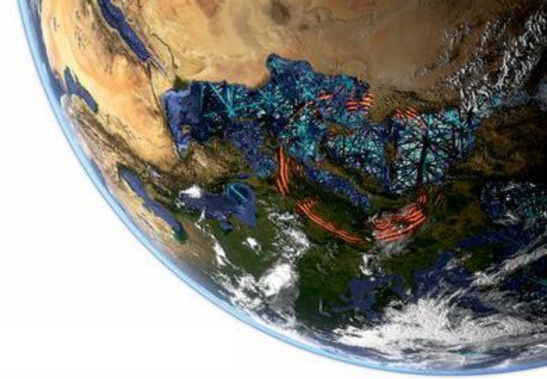
$$(\delta B^0_{mn} - \delta L^0_{mn})\, d_n = \boxed{\frac{\omega^2 \Delta t^2}{12}}\, \rho\omega^2 u - \boxed{\frac{\Delta z^2}{12}\frac{d^2}{dz^2}} \left(\mu \frac{d^2 u}{dz^2}\right)\Bigg|_{z=z_m}$$

- RHS is 'basic error' of the operator, this not equal zero. This should zero when *eigenfunction* and *eigenfrequency* is equal . Modified operator then:

$$(\delta B_{mn} - \delta L_{mn})\, d_n = \boxed{\left(\frac{\omega^2 \Delta t^2}{12} - \frac{\Delta z^2}{12}\frac{d^2}{dz^2}\right)} \left[\rho\omega^2 u + \mu \frac{d^2 u}{dz^2}\right]\Bigg|_{z=z_m}$$

!= 0

Homogeneous 1D Operators



- Taking inverse Fourier transform, representation of basic error in time domain

$$(\delta A_{mMnN} - \delta K_{mMnN})c_{nN} = \left\{ \rho \left[\frac{\Delta t^2}{12}\left(\frac{\partial^4 u}{\partial t^4}\right) + \frac{\Delta z^2}{12}\left(\frac{\partial^4 u}{\partial z^2 \partial t^2}\right)\right] - \mu \left[\frac{\Delta z^2}{12}\left(\frac{\partial^4 u}{\partial z^4}\right) + \frac{\Delta t^2}{12}\left(\frac{\partial^4 u}{\partial t^2 \partial z^2}\right)\right]\right\}\Bigg|_{t=t_M, z=z_m}$$

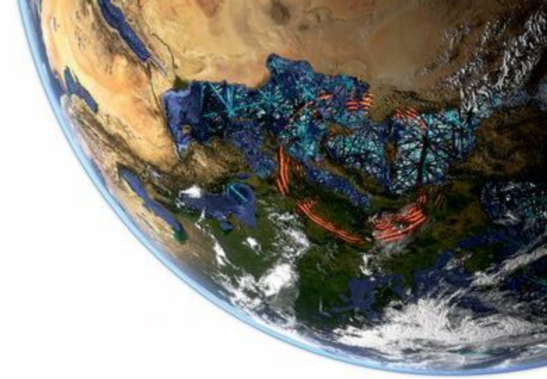
$$= \left\{ \boxed{\frac{\Delta t^2}{12}\frac{\partial^2}{\partial t^2}} \left[\rho \frac{\partial^2 u}{\partial t^2} - \mu \frac{\partial^2 u}{\partial z^2}\right] + \boxed{\frac{\Delta z^2}{12}\frac{\partial^2}{\partial z^2}} \left[\rho \frac{\partial^2 u}{\partial t^2} - \mu \frac{\partial^2 u}{\partial z^2}\right]\right\}\Bigg|_{t=t_M, z=z_m},$$

Note : Basic error of modified operators is derivatives of the equation of motion in brackets

- error in space & time considered as **single quantity**!

d0=(dA-dK)

Homogeneous 1D Operators



- *'Omitting'* the details of the derivation, the modified FD operator ***A*** and ***K*** are as follows:

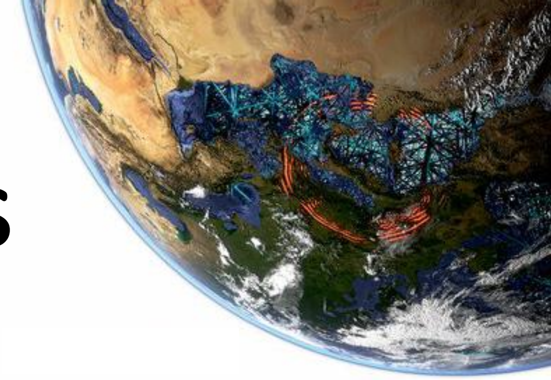
$$
\mathbf{A} = \left(\frac{\rho}{\Delta t^2}\right) \times
\begin{array}{c|ccc}
t+\Delta t & 1/12 & 10/12 & 1/12 \\
t & -2/12 & -20/12 & -2/12 \\
t-\Delta t & 1/12 & 10/12 & 1/12 \\
\hline
 & z-\Delta z & z & z+\Delta z
\end{array}
\qquad
\mathbf{K} = \left(\frac{\mu}{\Delta z^2}\right) \times
\begin{array}{c|ccc}
t+\Delta t & 1/12 & -2/12 & 1/12 \\
t & 10/12 & -20/12 & 10/12 \\
t-\Delta t & 1/12 & -2/12 & 1/12 \\
\hline
 & z-\Delta z & z & z+\Delta z
\end{array}
$$

*Note: sum of horizontal for **A** & sum of vertical for **K**, will obtain the conventional FD operator*

- smear out the discretized 2nd-time-derivative operator in space & time, so *numerical dispersion* (error of the phase velocity) of the discretized equation of motion is *zero* to 2nd-order in $\boldsymbol{\Delta z^2}$ & $\boldsymbol{\Delta t^2}$

dA = A0 - A
dK = K0 - K
d0 = (dA-dK)

Heterogeneous 1D Operators

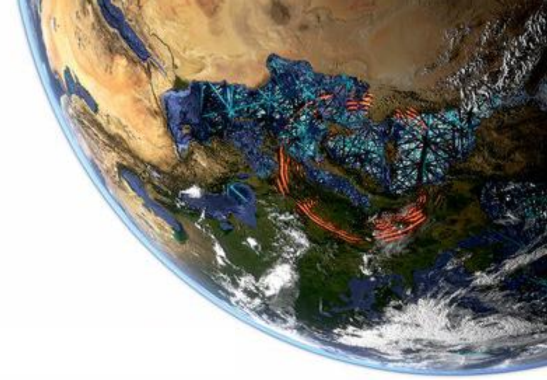


- With same fashion as homogeneous case, FD operator ***A0*** and ***K0*** are as follows:

$$
\mathbf{A}^0 = \left(\frac{1}{\Delta t^2}\right) \times
\begin{array}{c|ccc|}
t+\Delta t & & \rho_m & \\
t & & -2\rho_m & \\
t-\Delta t & & \rho_m & \\
\hline
 & z-\Delta z & z & z+\Delta z
\end{array}
\qquad
\mathbf{K}^0 = \left(\frac{1}{2\Delta z^2}\right) \times
\begin{array}{c|ccc|}
t+\Delta t & & & \\
t & (\mu_{m-1}+\mu_m) & -(\mu_{m-1}+2\mu_m+\mu_{m+1}) & (\mu_m+\mu_{m+1}) \\
t-\Delta t & & & \\
\hline
 & z-\Delta z & z & z+\Delta z
\end{array}
$$

- The modified FD operator ***A*** and ***K*** are as follows:

$$
\mathbf{A} = \left(\frac{1}{12\Delta t^2}\right) \times
\begin{array}{c|ccc|}
t+\Delta t & \rho_m & 10\rho_m & \rho_m \\
t & -2\rho_m & -20\rho_m & -2\rho_m \\
t-\Delta t & \rho_m & 10\rho_m & \rho_m \\
\hline
 & z-\Delta z & z & z+\Delta z
\end{array}
\qquad
\mathbf{K} = \left(\frac{1}{24\Delta z^2}\right) \times
\begin{array}{c|ccc|}
t+\Delta t & (\mu_{m-1}+\mu_m) & -(\mu_{m-1}+2\mu_m+\mu_{m+1}) & (\mu_m+\mu_{m+1}) \\
t & 10(\mu_{m-1}+\mu_m) & -10(\mu_{m-1}+2\mu_m+\mu_{m+1}) & 10(\mu_m+\mu_{m+1}) \\
t-\Delta t & (\mu_{m-1}+\mu_m) & -(\mu_{m-1}+2\mu_m+\mu_{m+1}) & (\mu_m+\mu_{m+1}) \\
\hline
 & z-\Delta z & z & z+\Delta z
\end{array}
$$



'Predictor-Corrector' Scheme

- Using ***1st-order Born approximation*** i.e. *assume scattering event is neglected*.
- We want correcting numerical solutions using estimated basic error ***(d0)***, as steps follow:

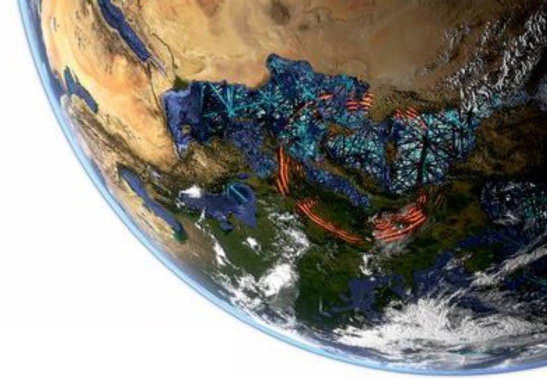
1. Predict wavefield at ***t+Δt*** using ***A0***, ***K0***, ***A*** and ***K***
2. Estimate basic error ***d0***
3. Compute ***δc*** (correction to the ***u*** at ***t+Δt***)
4. Correct wavefield after each time step

   $$c_{n(N+1)} = c^{0}_{n(N+1)} + \delta c_{n(N+1)}$$

5. Remap time level

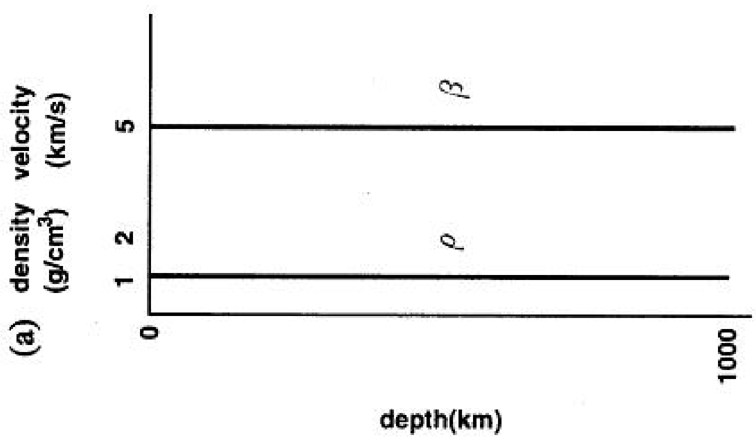
   $$c^{0}_{n(N+1)} = c_{n(N+1)}$$

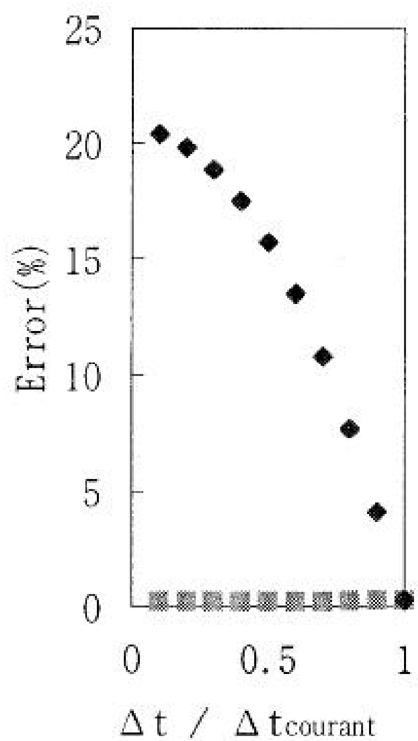
```
dA = A0 - A
dK = K0 - K
d0 = (dA-dK)

odp[i] =     opold[i - 1: i + 1] * d0[:, 0]\
       +        op[i - 1: i + 1] * d0[:, 1]\
       +     opnew[i - 1: i + 1] * d0[:, 2]

opnew = opnew + odp

opold, op = op, opnew
```



Numerical Examples 1

- Consider a homogeneous medium 1D case. Spatial grid dz = 1 km. Source used is Ricker wavelet (central f = 10 s). Source is at z = 500 km, receiver at z 300 km.

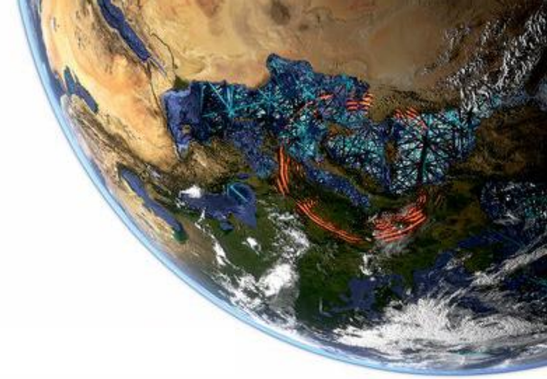




Error of waveforms:
- conventional (diamonds)
- optimal (square)

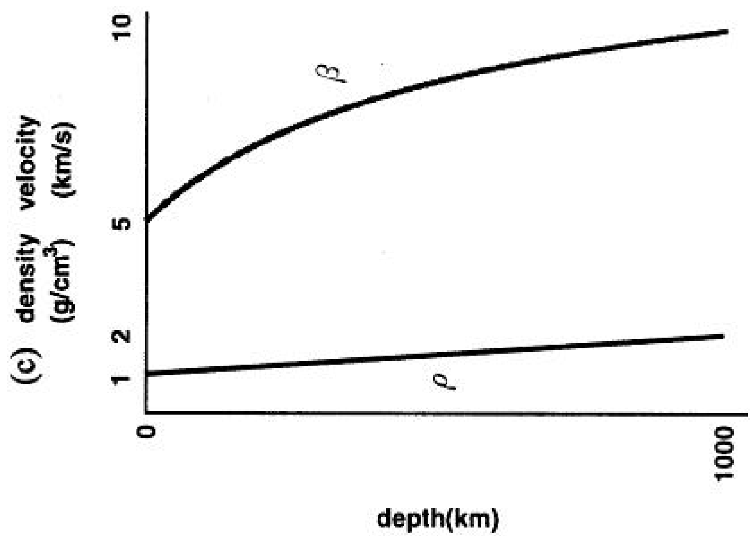
Time step normalized by Courant stability limit

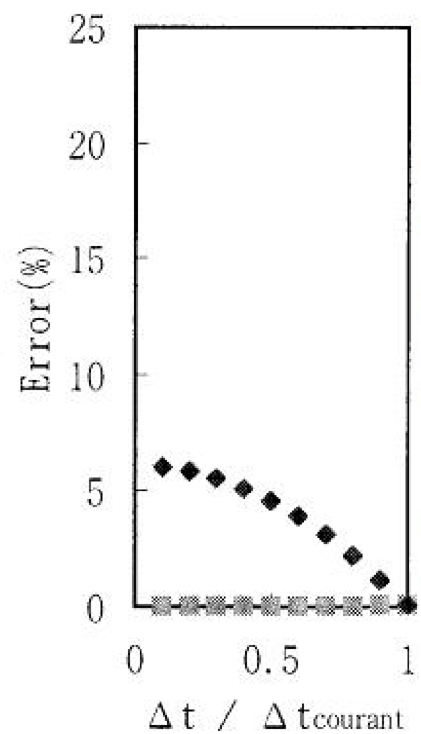
Error of modified operator is uniform regardless the choice of dt and dx



Numerical Examples 2

- Consider a heterogeneous medium 1D case. Spatial grid upper dz = 0.5 km, lower dz = 1 km. Source used is Ricker wavelet (central f = 10 s). Source is at z = 500 km, receiver at z 300 km.





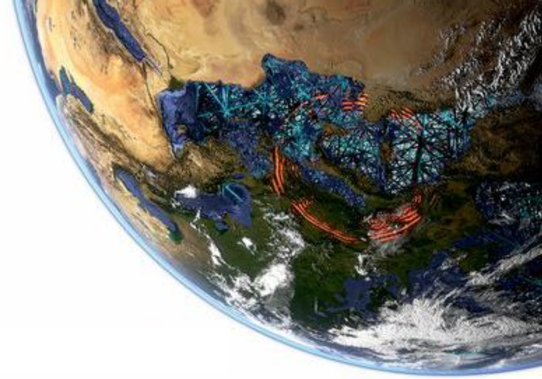
Error of waveforms:
- conventional (diamonds)
- optimal (square)

Time step normalized by Courant stability limit

in Inhomogeneous grid, number of nodes per wavelength is constant throughout the medium

Error of modified operator is uniform regardless the choice of dt and dx

Accuracy vs. CPU Time

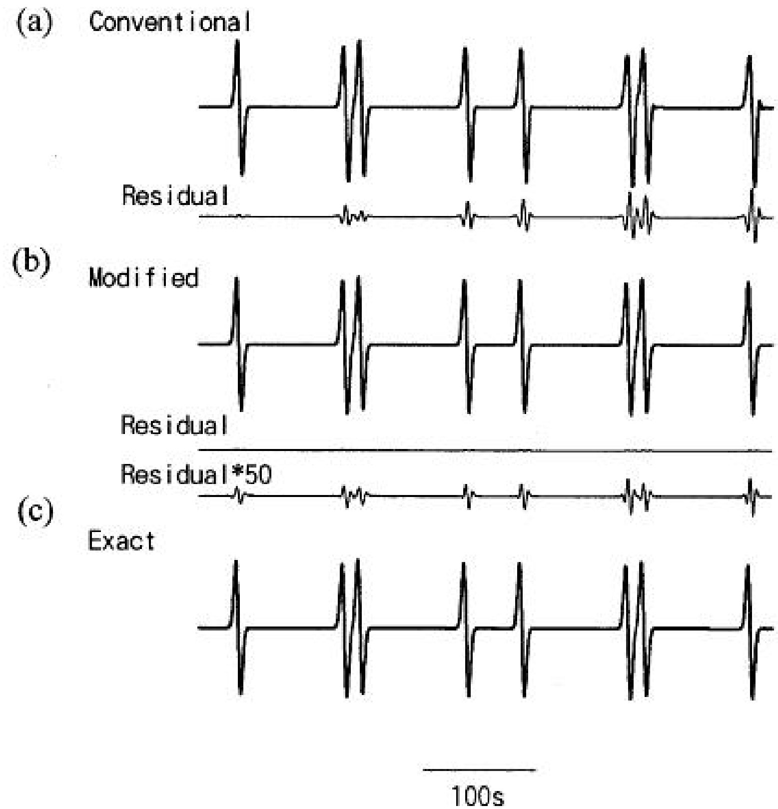


- Comparison of the error & CPU time. Medium length is 1000 km and time length is 500s.

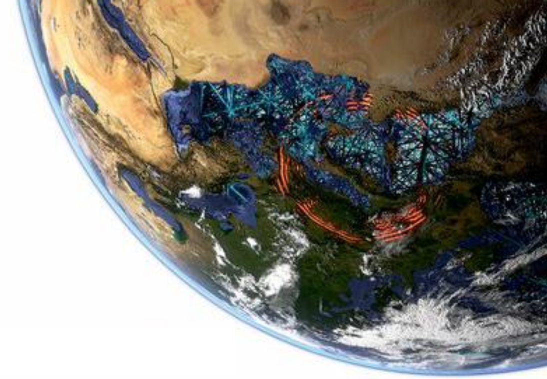
Table 1. CPU time required on Super-SPARC (60 MHz, 1 CPU) and error for conventional and modified operators.

	CPU time			Error		
	Convent.	Mod	Ratio	Convent.	Mod	Ratio
500 grids 5000 steps	1 s	2 s	**0.5**	22%	0.32%	**69**
1000 grids 10000 steps	5 s	10 s	**0.5**	5.6%	0.054%	**104**
10000 grids 100000 steps	455 s	921 s	**0.49**			

Note : proposed FD scheme delivers almost 2 orders of magnitude greater accuracy than conventional FD scheme, while only requiring only 2 times as much CPU time

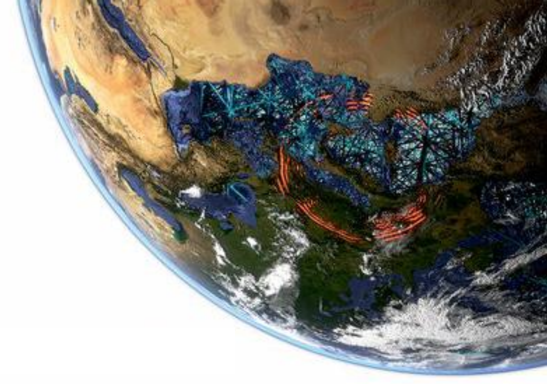


Discussion

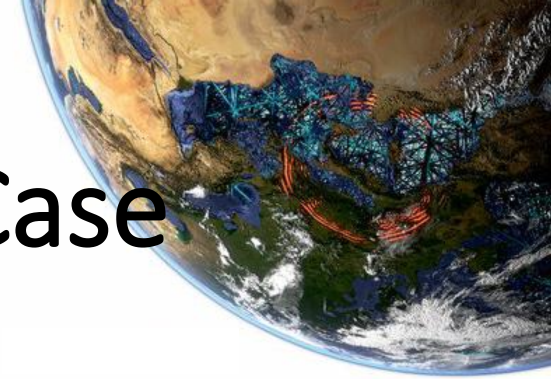


- The net error of the synthetics due to combined effects of temporal and spatial discretization must be considered as a **single quantity**
- Error can be minimized by *'tuning'* the operators so that the errors due to spatial and temporal discretization come as close as possible to **cancelling each other**

References



- Geller, Robert. J., Takeuchi, Nozomu. ***Optimally Accurate Second-Order Time-Domain Finite Difference Scheme for The Elastic Equation of Motion: One-Dimensional Case***. GJI. 1998.
- Igel, Heiner. ***Computational Seismology***. Oxford Publication Press. 2016.



Py Code (1D Homogeneous Case

- Download python code (ipynb file) from my github:

https://github.com/git-taufiqurrahman/Advanced-Computational-Seismology-2016

- Clone or download ***hom1d_with_optimal_operator+analytical.ipynb*** to your Jupyter.nb working directory
- Open the file and run all

Thank You!