so’n Feuerball

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Contents

Chapter 1

Setting Up the Model

1.1 Canonical Quantization

1.1.1 Real Scalar Field

Consider a real scalar field ϕ with Lagrangian density ($\eta = \text{diag}(-,+,+,+)$)

$$\mathcal{L} = -\frac{1}{2}(\partial_\mu\phi)(\partial^\mu\phi) - V(\phi) = \frac{1}{2}(\dot{\phi}^2 - \vec{\nabla}^2\phi) - V(\phi) \tag{1.1}$$

with associated Hamiltonian density

$$\mathcal{H} = \pi\dot{\phi} - \mathcal{L} = \frac{1}{2}(\pi^2 + (\vec{\nabla}\phi)^2) + V(\phi) \tag{1.2}$$

where $\pi = \frac{\partial\mathcal{L}}{\partial\dot{\phi}} = \dot{\phi}$. Choose the free scalar field, $V(\phi) = \frac{1}{2}m^2\phi^2$. The equations of motion arising from this is the Klein-Gordon equation

$$(\partial_\mu\partial^\mu - m^2)\phi(t,\vec{x}) = 0\,. \tag{1.3}$$

The equations of motion (1.3) have the general solution

$$\phi(t,x) = \int \frac{\mathrm{d}^3p}{(2\pi)^3}\mathcal{N}_{\vec{p}}\left\{a_{\vec{p}}e^{-i(\omega_{\vec{p}}t-\vec{p}\vec{x})} + b^*_{\vec{p}}e^{i(\omega_{\vec{p}}t-\vec{p}\vec{x})}\right\} \tag{1.4a}$$

$$(\implies)\quad \pi(t,x) = \int \frac{\mathrm{d}^3p}{(2\pi)^3}\mathcal{N}_{\vec{p}}\left\{-i\omega_{\vec{p}}a_{\vec{p}}e^{-i(\omega_{\vec{p}}t-\vec{p}\vec{x})} + i\omega_{\vec{p}}b^*_{\vec{p}}e^{i(\omega_{\vec{p}}t-\vec{p}\vec{x})}\right\}. \tag{1.4b}$$

only subject to the condition $\omega_{\vec{p}} = \sqrt{m^2+p^2}$. Reality of $\phi(t,x)$ implies $a_{\vec{p}} = b_{\vec{p}}$. If one uses

Definition 1.1|1: Poisson Brackets on Field Space

$$\{A,B\} = \int \mathrm{d}^3x\left[\frac{\delta A}{\delta\phi}\frac{\delta B}{\delta\pi} - \frac{\delta A}{\delta\pi}\frac{\delta B}{\delta\phi}\right] \tag{1.5}$$

the field and momentum fields satisfy

$$\{\phi(t,x),\phi(t,y)\} = \{\pi(t,x),\pi(t,y)\} = 0\,,\quad \{\phi(t,x),\pi(t,y)\} = \delta^{(d)}(x-y)\,. \tag{1.6}$$

Quantization is achieved by the replacement

$$i\{\cdot,\cdot\} \to [\cdot,\cdot]\,, \tag{1.7}$$

lifting fields to operators, $\phi \to \hat{\phi}$ and $\pi \to \hat{\pi}$ (though the $\hat{\cdot}$ will be omitted) as well as $\cdot^* \to \cdot^\dagger$. Using $a_{\vec{p}} = b_{\vec{p}}$ together with the normalization $\mathcal{N}^2_{\vec{p}}\omega_{\vec{p}} = \frac{1}{2}$, the fundamental commutator $[\phi(t,x),\pi(t,y)] = i\delta^{(d)}(x-y)$ then implies

Important 1.1|2: Commutators of $a_{\vec{p}}$, $a^\dagger_{\vec{q}}$

$$[a_{\vec{p}}, a_{\vec{q}}] = [a^\dagger_{\vec{p}}, a^\dagger_{\vec{q}}] = 0\,, \quad [a_{\vec{p}}, a^\dagger_{\vec{q}}] = (2\pi)^3\delta^{(3)}(\vec{p}-\vec{q})\,. \tag{1.8}$$

Calculation 1.1|3: Commutators of $a_{\vec{p}}$, $a^\dagger_{\vec{q}}$

Notice the relations

$$\begin{aligned} a_{\vec{p}} &= \frac{1}{2\mathcal{N}_{\vec{p}}}\int \mathrm{d}^3x \left\{\phi(t,x) + \frac{i}{\omega_{\vec{p}}}\pi(t,x)\right\} e^{i(\omega_{\vec{p}}t - \vec{p}\vec{x})} \\ a^\dagger_{\vec{p}} &= \frac{1}{2\mathcal{N}_{\vec{p}}}\int \mathrm{d}^3x \left\{\phi(t,x) - \frac{i}{\omega_{\vec{p}}}\pi(t,x)\right\} e^{-i(\omega_{\vec{p}}t - \vec{p}\vec{x})} \end{aligned} \tag{1.9}$$

The non-vanishing commutator is derived as follows:

$$\begin{aligned} [a_{\vec{p}}, a^\dagger_{\vec{q}}] &= \frac{1}{4\mathcal{N}_{\vec{p}}\mathcal{N}_{\vec{q}}}\int \mathrm{d}^3x\mathrm{d}^3y \left\{-\frac{i}{\omega_{\vec{q}}}[\phi(t,x),\pi(t,y)]e^{i\left((\omega_{\vec{p}}-\omega_{\vec{q}})t-(\vec{p}\vec{x}-\vec{q}\vec{y})\right)}\right. \\ &\qquad \left. + \frac{i}{\omega_{\vec{p}}}[\pi(t,x),\phi(t,y)]e^{-i\left((\omega_{\vec{p}}-\omega_{\vec{q}})t-(\vec{p}\vec{x}-\vec{q}\vec{y})\right)}\right\} \\ &= \frac{1}{4\mathcal{N}_{\vec{p}}\mathcal{N}_{\vec{q}}}\int \mathrm{d}^3x \left\{\frac{1}{\omega_{\vec{q}}}e^{i\left((\omega_{\vec{p}}-\omega_{\vec{q}})t-(\vec{p}-\vec{q})x\right)}\right. \\ &\qquad \left. + \frac{1}{\omega_{\vec{p}}}e^{-i\left((\omega_{\vec{p}}-\omega_{\vec{q}})t-(\vec{p}-\vec{q})x\right)}\right\} \\ &= \frac{(2\pi)^3}{2\mathcal{N}^2_{\vec{p}}\omega_{\vec{p}}}\delta^{(3)}(\vec{p}-\vec{q}) \end{aligned}$$

whereas the vanishing commutators are calculated as

$$\begin{aligned} [a_{\vec{p}}, a_{\vec{q}}] &= \frac{1}{4\mathcal{N}_{\vec{p}}\mathcal{N}_{\vec{q}}}\int \mathrm{d}^3x\mathrm{d}^3y \left\{\frac{i}{\omega_{\vec{q}}}[\phi(t,x),\pi(t,y)]e^{i\left((\omega_{\vec{p}}+\omega_{\vec{q}})t-(\vec{p}\vec{x}+\vec{q}\vec{y})\right)}\right. \\ &\qquad \left. + \frac{i}{\omega_{\vec{p}}}[\pi(t,x),\phi(t,y)]e^{i\left((\omega_{\vec{p}}+\omega_{\vec{q}})t-(\vec{p}\vec{x}+\vec{q}\vec{y})\right)}\right\} \\ &= 0 \end{aligned}$$

and simmilarly for $[a^\dagger_{\vec{p}}, a^\dagger_{\vec{q}}] = 0$.

After this quantization, the fields are written as

$$\phi(t,x) = \int \frac{\mathrm{d}^3p}{(2\pi)^3}\frac{1}{\sqrt{2\omega_{\vec{p}}}}\left\{a_{\vec{p}}e^{-i(\omega_{\vec{p}}t-\vec{p}\vec{x})} + a^\dagger_{\vec{p}}e^{i(\omega_{\vec{p}}t-\vec{p}\vec{x})}\right\} \tag{1.10a}$$

$$\pi(t,x) = \int \frac{\mathrm{d}^3p}{(2\pi)^3}\frac{i\omega_{\vec{p}}}{\sqrt{2\omega_{\vec{p}}}}\left\{-a_{\vec{p}}e^{-i(\omega_{\vec{p}}t-\vec{p}\vec{x})} + a^\dagger_{\vec{p}}e^{i(\omega_{\vec{p}}t-\vec{p}\vec{x})}\right\}\,. \tag{1.10b}$$

To express the Hamiltonian in terms of $a_{\vec{p}}$ and $a^\dagger_{\vec{p}}$ rewrite

$$\phi(t,x) = \int \frac{\mathrm{d}^3p}{(2\pi)^3}\frac{1}{\sqrt{2\omega_{\vec{p}}}}\left\{a_{\vec{p}}e^{-i\omega_{\vec{p}}t} + a^\dagger_{-\vec{p}}e^{i\omega_{\vec{p}}t}\right\}e^{i\vec{p}\vec{x}} \tag{1.11a}$$

$$\pi(t,x) = \int \frac{\mathrm{d}^3p}{(2\pi)^3}\frac{i\omega_{\vec{p}}}{\sqrt{2\omega_{\vec{p}}}}\left\{-a_{\vec{p}}e^{-i\omega_{\vec{p}}t} + a^\dagger_{-\vec{p}}e^{i\omega_{\vec{p}}t}\right\}e^{i\vec{p}\vec{x}}\,. \tag{1.11b}$$

Omit the time dependence for the next calculation, for example by choosing $t = 0$. The Hamiltonian is now easily computed to be

$$
\begin{aligned}
H &= \frac{1}{2}\int \mathrm{d}^3x \frac{\mathrm{d}^3p}{(2\pi)^3}\frac{\mathrm{d}^3q}{(2\pi)^3}\frac{1}{\sqrt{4\omega_{\vec{p}}\omega_{\vec{q}}}} e^{i(\vec{p}+\vec{q})\vec{x}}\Big[-\omega_{\vec{p}}\omega_{\vec{q}}(-a_{\vec{p}}+a^\dagger_{-\vec{p}})(-a_{\vec{q}}+a^\dagger_{-\vec{q}})+ \\
&\qquad + (-\vec{p}\vec{q}+m^2)(a_{\vec{p}}+a^\dagger_{-\vec{p}})(a_{\vec{q}}+a^\dagger_{-\vec{q}})\Big] && (1.12\text{a}) \\
&= \frac{1}{2}\int \frac{\mathrm{d}^3p}{(2\pi)^3}\frac{1}{2\omega_{\vec{p}}}\Big[-\omega_{\vec{p}}^2(\cancel{a_{\vec{p}}a_{-\vec{p}}} - a^\dagger_{-\vec{p}}a_{-\vec{p}} - a_{\vec{p}}a^\dagger_{\vec{p}} + \cancel{a^\dagger_{-\vec{p}}a^\dagger_{\vec{p}}})+ \\
&\qquad + (\vec{p}^2+m^2)(\cancel{a_{\vec{p}}a_{-\vec{p}}} + a^\dagger_{-\vec{p}}a_{-\vec{p}} + a_{\vec{p}}a^\dagger_{\vec{p}} + \cancel{a^\dagger_{-\vec{p}}a^\dagger_{\vec{p}}})\Big] && (1.12\text{b}) \\
&= \int \frac{\mathrm{d}^3p}{(2\pi)^3}\frac{\omega_{\vec{p}}}{2}(a^\dagger_{\vec{p}}a_{\vec{p}} + a_{\vec{p}}a^\dagger_{\vec{p}}) && (1.12\text{c}) \\
&= \int \frac{\mathrm{d}^3p}{(2\pi)^3}\omega_{\vec{p}}(a^\dagger_{\vec{p}}a_{\vec{p}} + \frac{1}{2}[a_{\vec{p}}, a^\dagger_{\vec{p}}]) && (1.12\text{d})
\end{aligned}
$$

Since only the combination $a_{\vec{p}}a^\dagger_{\vec{p}}$ shows up, the explicit time dependence would have dropped out anyways. The commutation relation between H, $a_{\vec{p}}$ and $a^\dagger_{\vec{p}}$ are given by

$$
[H, a_{\vec{p}}] = -\omega_{\vec{p}}a_{\vec{p}}\,, \qquad [H, a^\dagger_{\vec{p}}] = \omega_{\vec{p}}a^\dagger_{\vec{p}} \tag{1.13}
$$

From quantum mechanics it is now clear that H every momentum mode $\vec{p}$ has a discrete spectrum of excitations or energy eigenstates, such that

$$
H|n_{\vec{p}}\rangle = (\omega_{\vec{p}} + E_0)|n_{\vec{p}}\rangle \tag{1.14}
$$

and the operator $a_{\vec{p}}$ ($a^\dagger_{\vec{p}}$) annihilates (creates) excitations,

$$
a_{\vec{p}}|n_{\vec{p}}\rangle = \sqrt{n_{\vec{p}}}|(n-1)_{\vec{p}}\rangle\,, \qquad a^\dagger_{\vec{p}}|n_{\vec{p}}\rangle = \sqrt{(n+1)_{\vec{p}}}|(n+1)_{\vec{p}}\rangle \tag{1.15}
$$

E_0 is the (IR) divergent vacuum energy. The vacuum is defined by $a_{\vec{p}}|0\rangle = 0\ \forall\vec{p}$. The operator

$$
N_{\vec{p}} = a^\dagger_{\vec{p}}a_{\vec{p}} \tag{1.16}
$$

is the number operator for a given momentum mode and its expectation value $n(\vec{p}) = \langle N_{\vec{p}}\rangle$ has the interpretation of the momentum space number density,

$$
N = \int \frac{\mathrm{d}^3p}{(2\pi)^3} n(\vec{p})\,, \qquad n(\vec{p}) = (2\pi)^3\frac{\mathrm{d}N}{\mathrm{d}^3p} \tag{1.17}
$$

1.1.2 Complex Scalar Field

Consider a complex scalar field $\phi = \frac{1}{\sqrt{2}}(\phi_1 + i\phi_2)$ with $\phi_k, k \in \{1,2\}$ two real scalar fields. The Lagrangian of ϕ can be written as the sum of the Lagrangians $\mathcal{L}_k$ of ϕ_k. Similarly for the Hamiltonian

$$
\mathcal{L} = -(\partial_\mu\phi^*)(\partial^\mu\phi) - m^2\phi\phi^* = \sum_k \Big\{-\frac{1}{2}\Big((\partial_\mu\phi_k)(\partial^\mu\phi_k) + m^2\phi_k^2\Big)\Big\} = \sum_k \mathcal{L}_k \tag{1.18}
$$

With the conjugate momenta $\pi_k = \dot{\phi}_k$ the conjugate momentum of ϕ turns out to be

$$
\pi = \frac{\partial\mathcal{L}}{\partial\dot{\phi}} = \dot{\phi}^* = \frac{\pi_1 - i\pi_2}{\sqrt{2}} \tag{1.19}
$$

and the Hamiltonian is therefore

$$
\mathcal{H} = \pi\dot{\phi} + \pi^*\dot{\phi}^* - \mathcal{L} = \dot{\phi}\dot{\phi}^* + (\vec{\nabla}\phi)(\vec{\nabla}\phi^*) + m^2\phi\phi^* = \sum_k \frac{1}{2}(\pi_k^2 + (\vec{\nabla}\phi_k)^2 + m^2\phi_k^2) = \sum_k \mathcal{H}_k \tag{1.20}
$$

Quantization rules are imposed as before on the real scalar field ϕ_k. From this, it is immediately clear that $\phi(t,\vec{x})$ takes the form

$$\phi(t,\vec{x}) = \int \frac{\mathrm{d}^3p}{(2\pi)^3}\frac{1}{\sqrt{2\omega_{\vec{p}}}}\Big\{\frac{a_{\vec{p},(1)}+ia_{\vec{p},(2)}}{\sqrt{2}}e^{-i(\omega_{\vec{p}}t-\vec{p}\vec{x})}+\frac{a^\dagger_{\vec{p},(1)}+ia^\dagger_{\vec{p},(2)}}{\sqrt{2}}e^{i(\omega_{\vec{p}}t-\vec{p}\vec{x})}\Big\} \tag{1.21a}$$

$$\pi(t,\vec{x}) = \int \frac{\mathrm{d}^3p}{(2\pi)^3}\frac{i\omega_{\vec{p}}}{\sqrt{2\omega_{\vec{p}}}}\Big\{-\frac{a_{\vec{p},(1)}-ia_{\vec{p},(2)}}{\sqrt{2}}e^{-i(\omega_{\vec{p}}t-\vec{p}\vec{x})}+\frac{a^\dagger_{\vec{p},(1)}-ia^\dagger_{\vec{p},(2)}}{\sqrt{2}}e^{i(\omega_{\vec{p}}t-\vec{p}\vec{x})}\Big\} \tag{1.21b}$$

It is intuitive to define

$$a_{\vec{p}} = \frac{a_{\vec{p},(1)}+ia_{\vec{p},(2)}}{\sqrt{2}}\,, \qquad b^\dagger_{\vec{p}} = \frac{a^\dagger_{\vec{p},(1)}+ia^\dagger_{\vec{p},(2)}}{\sqrt{2}} \tag{1.22}$$

recovering the looking familiar expression

$$\phi(t,\vec{x}) = \int \frac{\mathrm{d}^3p}{(2\pi)^3}\frac{1}{\sqrt{2\omega_{\vec{p}}}}\Big\{a_{\vec{p}}e^{-i(\omega_{\vec{p}}t-\vec{p}\vec{x})}+b^\dagger_{\vec{p}}e^{i(\omega_{\vec{p}}t-\vec{p}\vec{x})}\Big\} \tag{1.23a}$$

$$\pi(t,\vec{x}) = \int \frac{\mathrm{d}^3p}{(2\pi)^3}\frac{i\omega_{\vec{p}}}{\sqrt{2\omega_{\vec{p}}}}\Big\{-b_{\vec{p}}e^{-i(\omega_{\vec{p}}t-\vec{p}\vec{x})}+a^\dagger_{\vec{p}}e^{i(\omega_{\vec{p}}t-\vec{p}\vec{x})}\Big\} \tag{1.23b}$$

where now, unlike before, explicitly $a_{\vec{p}} \neq b_{\vec{p}}$ is found.

The commutation relations of $a_{\vec{p},(k)}$, $a^\dagger_{\vec{p},(k)}$ are trivially given by

$$[a_{\vec{p},(j)}, a^\dagger_{\vec{q},(k)}] = (2\pi)^3\delta^{(3)}(\vec{p}-\vec{q})\delta_{jk}\,, \qquad [a^{(\dagger)}_{\vec{p},(j)}, a^{(\dagger)}_{\vec{p},(k)}] = 0 \tag{1.24}$$

and lead to

$$[a_{\vec{p}}, b_{\vec{q}}] = \frac{1}{2}[a_{\vec{p},(1)}+ia_{\vec{p},(2)}, a_{\vec{q},(1)}-ia_{\vec{q},(2)}] = 0 \tag{1.25a}$$

$$[a^{(\dagger)}_{\vec{p}}, a^{(\dagger)}_{\vec{q}}] = 0 \tag{1.25b}$$

$$[a_{\vec{p}}, b^\dagger_{\vec{q}}] = \frac{1}{2}[a_{\vec{p},(1)}+ia_{\vec{p},(2)}, a^\dagger_{\vec{q},(1)}+ia^\dagger_{\vec{q},(2)}] = \frac{1}{2}\big([a_{\vec{p},(1)}, a^\dagger_{\vec{q},(1)}]-[a_{\vec{p},(2)}, a^\dagger_{\vec{q},(2)}]\big) = 0 \tag{1.25c}$$

$$[a_{\vec{p}}, a^\dagger_{\vec{q}}] = \frac{1}{2}[a_{\vec{p},(1)}+ia_{\vec{p},(2)}, a^\dagger_{\vec{q},(1)}-ia^\dagger_{\vec{q},(2)}] = \frac{1}{2}\big([a_{\vec{p},(1)}, a^\dagger_{\vec{q},(1)}]+[a_{\vec{p},(2)}, a^\dagger_{\vec{q},(2)}]\big) = (2\pi)^3\delta^{(3)}(\vec{p}-\vec{q}) \tag{1.25d}$$

$$[b_{\vec{p}}, b^\dagger_{\vec{q}}] = \frac{1}{2}[a_{\vec{p},(1)}-ia_{\vec{p},(2)}, a^\dagger_{\vec{q},(1)}+ia^\dagger_{\vec{q},(2)}] = \frac{1}{2}\big([a_{\vec{p},(1)}, a^\dagger_{\vec{q},(1)}]+[a_{\vec{p},(2)}, a^\dagger_{\vec{q},(2)}]\big) = (2\pi)^3\delta^{(3)}(\vec{p}-\vec{q}) \tag{1.25e}$$

From this, again the Hamiltonian is derived to be

$$H = \int \frac{\mathrm{d}^3p}{(2\pi)^3}\omega_{\vec{p}}\Big(a^\dagger_{\vec{p},(1)}a_{\vec{p},(1)}+a^\dagger_{\vec{p},(2)}a_{\vec{p},(2)}+\frac{1}{2}\big([a_{\vec{p},(1)}, a^\dagger_{\vec{p},(1)}]+[a_{\vec{p},(2)}, a^\dagger_{\vec{p},(2)}]\big)\Big) \tag{1.26a}$$

$$= \int \frac{\mathrm{d}^3p}{(2\pi)^3}\omega_{\vec{p}}\Big(a^\dagger_{\vec{p}}a_{\vec{p}}+b^\dagger_{\vec{p}}b_{\vec{p}}+\frac{1}{2}\big([a_{\vec{p}}, a^\dagger_{\vec{p}}]+[b_{\vec{p}}, b^\dagger_{\vec{p}}]\big)\Big) \tag{1.26b}$$

using $a^\dagger_{\vec{p}}a_{\vec{p}} = \frac{1}{2}(a^\dagger_{\vec{p},(1)}-ia^\dagger_{\vec{p},(2)})(a_{\vec{p},(1)}+ia_{\vec{p},(2)})$ and $b^\dagger_{\vec{p}}b_{\vec{p}} = \frac{1}{2}(a^\dagger_{\vec{p},(1)}+ia^\dagger_{\vec{p},(2)})(a_{\vec{p},(1)}-ia_{\vec{p},(2)})$. Whereas $n_{\vec{p}} = \langle a^\dagger_{\vec{p}}a_{\vec{p}}\rangle$ has the interpretation of a particle number density, $\overline{n}_{\vec{p},J} = \langle b^\dagger_{\vec{p}}b_{\vec{p}}\rangle$ is understood as the antiparticle density.

1.2 Particle Spectra from Classical Sources

1.2.1 Real Scalar Field

Follow **CITE: REINHARD: FIELD QUANTIZATION** around eq. 4.140. Define the Pauli-Jordan function $\Delta(x)$ via

$$
\begin{aligned}
i\Delta(x-y) &= \int \frac{\mathrm{d}^3p}{(2\pi)^3}\frac{1}{2\omega_{\vec{p}}}\Big(e^{-i(\omega_{\vec{p}}(x^0-y^0)-\vec{p}(\vec{x}-\vec{y}))} - e^{i(\omega_{\vec{p}}(x^0-y^0)-\vec{p}(\vec{x}-\vec{y}))}\Big) && (1.27)\\
&= \int \frac{\mathrm{d}^3p}{(2\pi)^3}\frac{1}{2\omega_{\vec{p}}}\Big(e^{ip(x-y)} - e^{-ip(x-y)}\Big) && (1.28)\\
&= \int \frac{\mathrm{d}^3p}{(2\pi)^3}\frac{1}{2\omega_{\vec{p}}}\Big(e^{-i\omega_{\vec{p}}(x^0-y^0)} - e^{i\omega_{\vec{p}}(x^0-y^0)}\Big)e^{i\vec{p}(\vec{x}-\vec{y})} && (1.29)\\
&= 2\pi\int \frac{\mathrm{d}^4p}{(2\pi)^4}\epsilon(p^0)\delta(p^2+m^2)e^{-i(p^0(x^0-y^0)-\vec{p}(\vec{x}-\vec{y}))} && (1.30)\\
&= 2\pi\int \frac{\mathrm{d}^4p}{(2\pi)^4}\epsilon(p^0)\delta(p^2+m^2)e^{ip(x-y)} && (1.31)
\end{aligned}
$$

which satisfies (...) $(\partial_\mu\partial^\mu - m^2)\Delta = 0$. $\epsilon(x)$ is the sign function. The retarded and advanced propagators $\Delta_{R,A}(x)$ are given by

$$\Delta_R(x) = \Theta(x^0)\Delta(x)\,, \qquad \Delta_A(x) = \Theta(x^0)\Delta(x)\,. \tag{1.32}$$

which immediately implies $\Delta(x) = \Delta_R(x) - \Delta_A(x)$. These function satisfy

$$(\partial_\mu\partial^\mu - m^2)\Delta_{R,A}(x) = \delta^{(4)}(x) \tag{1.33}$$

Calculation 1.2|2: Greens Functions

Solve $(\partial_\mu\partial^\mu - m^2)D(x) = \delta^{(4)}(x)$. Using

$$D(x) = \int \frac{\mathrm{d}^4p}{(2\pi)^4}\tilde{D}(p)e^{ipx}$$

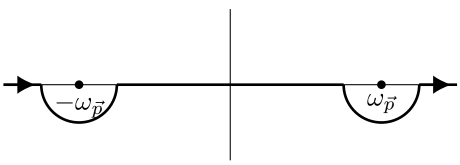
one finds

$$\int \frac{\mathrm{d}^4p}{(2\pi)^4}(-p^2-m^2)\tilde{D}(p) = \delta^{(4)}(x)\tilde{D}(p) = -\frac{1}{p^2+m^2} \tag{1.34}$$

and thus

$$
\begin{aligned}
D(x) &= \int \frac{\mathrm{d}^4p}{(2\pi)^4}\frac{-1}{p^2+m^2}e^{-ipx} && (1.35)\\
&= -\int \frac{\mathrm{d}^3p}{(2\pi)^3}\frac{\mathrm{d}p^0}{2\pi}\frac{1}{(p^0+\omega_{\vec{p}})(-p^0+\omega_{\vec{p}})}e^{i(p^0t-\vec{p}\vec{x})} && (1.36)
\end{aligned}
$$

Definition 1.2|2: Retarded Propagator, Contour



If $t > 0$, close the integration contour in the upper imaginary half plane.

$$\int \frac{\mathrm{d}p^0}{2\pi} \frac{1}{(p^0+\omega_{\vec{p}})(-p^0+\omega_{\vec{p}})} e^{ip^0t} = 2\pi i \Big(\lim_{p^0\to\omega_{\vec{p}}} \frac{1}{2\pi}(p^0-\omega_{\vec{p}}) \frac{e^{ip^0t}}{(p^0+\omega_{\vec{p}})(-p^0+\omega_{\vec{p}})} + \lim_{p^0\to-\omega_{\vec{p}}} \frac{1}{2\pi}(p^0+\omega_{\vec{p}}) \frac{e^{ip^0t}}{(p^0+\omega_{\vec{p}})(-p^0+\omega_{\vec{p}})} \tag{1.37}$$

$$= i\Big(\frac{e^{-i\omega_{\vec{p}}t} - e^{i\omega_{\vec{p}}t}}{2\omega_{\vec{p}}}\Big) \tag{1.38}$$

For $t < 0$, close the integration in the lower half plane, such that there is no residue within the integration contour. This leads to

$$D_R(x) = \frac{1}{i}\Theta(t) \int \frac{\mathrm{d}^3p}{(2\pi)^3} \frac{1}{2\omega_{\vec{p}}} (e^{-i(\omega_{\vec{p}}t - \vec{p}\vec{x})} - e^{i(\omega_{\vec{p}}t-\vec{p}\vec{x})}) = -i\Theta(x^0) \int \frac{\mathrm{d}^3p}{(2\pi)^3}\frac{1}{2\omega_{\vec{p}}}(e^{ipx} - e^{-ipx}) = \Theta(x^0)\Delta(x) \equiv \Delta_R(x) \tag{1.39}$$

Consider now a real scalar field that evolves according to the inhomogeneous Klein-Gordon equation

$$(\partial_\mu \partial^\mu - m^2)\phi = -J \tag{1.40}$$

The solution is constructed by superposition of homogeneous solutions and a particular inhomogeneous solutions. Requesting $\phi \equiv 0$ for vanishing source, one finds

$$\phi_J(x) = -\int \mathrm{d}^4y \Delta_R(x-y) J(y) \tag{1.41a}$$

$$= i\int \mathrm{d}^4y \Theta(x^0-y^0) \int \frac{\mathrm{d}^3p}{(2\pi)^3}\frac{1}{2\omega_{\vec{p}}}(e^{ip(x-y)} - e^{-ip(x-y)}) J(y) \tag{1.41b}$$

$$\overset{x^0 \gtrless y^0}{=} i\int \frac{\mathrm{d}^3p}{(2\pi)^3}\frac{1}{2\omega_{\vec{p}}}(J(p)e^{ipx} - J(-p)e^{-ipx}) \tag{1.41c}$$

$$= i\int \frac{\mathrm{d}^3p}{(2\pi)^3}\frac{1}{2\omega_{\vec{p}}}(J(p)e^{-i(\omega_{\vec{p}}t-\vec{p}\vec{x})} - J(-p)e^{i(\omega_{\vec{p}}t-\vec{p}\vec{x})}) \tag{1.41d}$$

where $J(p) = \int \mathrm{d}^4y J(x) e^{-ipy}$ was used.

Taking the homogeneous solution ϕ_0 as given by (1.10a) into account, the field after the source has vanished is given by

$$\phi(t,x) = \phi_0(t,\vec{x}) + \phi_J(t,\vec{x}) = \int \frac{\mathrm{d}^3p}{(2\pi)^3}\frac{1}{\sqrt{2\omega_{\vec{p}}}} \left\{ \big(a_{\vec{p}} + \frac{iJ(p)}{\sqrt{2\omega_{\vec{p}}}}\big) e^{-i(\omega_{\vec{p}}t-\vec{p}\vec{x})} + \big(a^\dagger_{\vec{p}} - \frac{iJ(-p)}{\sqrt{2\omega_{\vec{p}}}}\big) e^{i(\omega_{\vec{p}}t-\vec{p}\vec{x})} \right\} \tag{1.42}$$

This is described by effectively replacing annihilation and creation operators via

$$a_{\vec{p}} \mapsto a_{\vec{p}} + \frac{iJ(p)}{\sqrt{2\omega_{\vec{p}}}}, \qquad a^\dagger_{\vec{p}} \mapsto a^\dagger_{\vec{p}} - \frac{iJ(-p)}{\sqrt{2\omega_{\vec{p}}}} \tag{1.43}$$

These replacements are of course compatible, considering that for a real source $J(p) = J^*(-p)$. Since $J(p)$ is just a $\mathbb{C}$-number, it does not alter the commutation relations from which the Hamiltonian and number operator are derived. The number density after the source has vanished, starting from the initial vacuum state $|0\rangle$ without any particles, is given by

$$n_{\vec{p},J} = \langle 0| \Big(a_{\vec{p}} + \frac{iJ(p)}{\sqrt{2\omega_{\vec{p}}}}\Big)\Big(a^\dagger_{\vec{p}} - \frac{iJ^*(p)}{\sqrt{2\omega_{\vec{p}}}}\Big)|0\rangle = \frac{1}{2\omega_{\vec{p}}}|J(p)|^2 \tag{1.44}$$

1.2.2 Complex Scalar Field

The derivation for the free scalar field is completely analogous. The identity $J(p) = J^*(-p)$ may not be used anymore. Instead, $J(p)$ contributes to the spectrum of particles, whereas $J(-p)$ contributes to the spectrum of antiparticles, in the following way:

$$a_{\vec{p}} \mapsto a_{\vec{p}} + \frac{iJ(p)}{\sqrt{2\omega_{\vec{p}}}}, \qquad b^\dagger_{\vec{p}} \mapsto b^\dagger_{\vec{p}} - \frac{iJ(-p)}{\sqrt{2\omega_{\vec{p}}}} \tag{1.45}$$

The particle and antiparticle momentum space number densities, or spectra, induces by the source J are now

$$n_{\vec{p},J} = \frac{1}{2\omega_{\vec{p}}}|J(p)|^2\,, \qquad \overline{n}_{\vec{p},J} = \frac{1}{2\omega_{\vec{p}}}|J(-p)|^2 \tag{1.46}$$

1.2.3 Extracting the Source from the Late Time Field

Equation (1.41d) allows use to extract

$$J(p)e^{-i\omega_{\vec{p}}t} = \int \mathrm{d}^3x\Big(- i\omega_{\vec{p}}\phi_J(x) + \big(\partial_t\phi_J(x)\big)\Big)e^{-i\vec{p}\vec{x}} \tag{1.47a}$$

$$J(p) = \int \mathrm{d}^3x\Big(\phi_J(x)\overleftarrow{\partial_t}e^{i(\omega_{\vec{p}}t-\vec{p}\vec{x})} - \phi_J(x)\overrightarrow{\partial_t}e^{i(\omega_{\vec{p}}t-\vec{p}\vec{x})}\Big) \tag{1.47b}$$

$$J(-p)e^{i\omega_{\vec{p}}t} = -\int \mathrm{d}^3x\Big(- i\omega_{\vec{p}}\phi_J(x) - \big(\partial_t\phi_J(x)\big)\Big)e^{i\vec{p}\vec{x}} \tag{1.47c}$$

$$J(-p) = \int \mathrm{d}^3x\Big(\phi_J(x)\overleftarrow{\partial_t}e^{-i(\omega_{\vec{p}}t-\vec{p}\vec{x})} - \phi_J(x)\overrightarrow{\partial_t}e^{-i(\omega_{\vec{p}}t-\vec{p}\vec{x})}\Big) \tag{1.47d}$$

1.3 Matching Hydrodynamics with Field Theory

1.3.1 Expanding around Minimum of Linear σ-model

The Lagrangian density

$$\mathscr{L} = \mathcal{L}_{\text{kin}} - V(\sigma,\vec{\pi}) = -\frac{1}{2}(\partial_\mu\sigma)(\partial^\mu\sigma) - \frac{1}{2}(\partial_\mu\vec{\pi})(\partial^\mu\vec{\pi}) + \frac{1}{2}\mu^2(\sigma^2+\vec{\pi}^2) + \frac{\lambda}{4}(\sigma^2+\vec{\pi}^2)^2 + h\sigma \tag{1.48}$$

can be expanded around the minimum at $\sigma_0 = f_\pi + h\cdot\frac{1}{2\mu^2} + \mathcal{O}(h^2)$ where $f_\pi = \frac{\mu}{\sqrt{\lambda}}$. Performing the substitution $\sigma \mapsto v + \sigma$ and neglecting terms of order $\mathcal{O}(h^2, \sigma^3, \sigma\vec{\pi}^2, (\vec{\pi}^2)^2)$ and higher the potential reads

$$V(\sigma,\vec{\pi}) = -\frac{\mu^4}{4\lambda} + \frac{1}{2}m_\sigma\sigma^2 + \frac{1}{2}m_\pi^2\vec{\pi}^2 \tag{1.49}$$

with pion mass $m_\pi^2 = \frac{h}{f_\pi}$ and sigma mass $m_\sigma^2 = 2\mu^2 + \mathcal{O}(h)$. Defining $\pi^\pm = (1/\sqrt{2})(\pi^1 \mp i\pi^2)$ one gets

$$(\pi^1)^2 + (\pi^2)^2 = |\pi^+|^2 + |\pi^-|^2 = 2\pi^+\pi^- \equiv 2\pi^+\overline{\pi^+} \tag{1.50}$$

The expansion of the Lagrangian around σ_0 breaks the $SO(4)$-symmetry associated to the vector $(\sigma,\vec{\pi})$ and chooses explicitly a minimum within the $SO(4)$-symmetric mexican hat potential. The residual symmetry is $SU(3)$. It features the $SO(2)$ subgroup of symmetry transformations

$$\begin{pmatrix}\pi^1\\ \pi^2\end{pmatrix} \mapsto \begin{pmatrix}\cos\alpha & \sin\alpha\\ -\sin\alpha & \cos\alpha\end{pmatrix}\begin{pmatrix}\pi^1\\ \pi^2\end{pmatrix} \qquad \Longleftrightarrow \qquad \pi^\pm \mapsto e^{\pm i\alpha}\pi^\pm \tag{1.51}$$

Note $\pi^- = \overline{\pi^+}$.

The Lagrangians and energy-momentum tensors $T^{\mu\nu} = 2(\partial\mathscr{L}/\partial g_{\mu\nu}) + g^{\mu\nu}\mathscr{L}$ **CITE: BLAU NOTES** for the separate fields read

$$\mathscr{L}_{\pi^\pm} = -(\partial_\mu\pi^+)(\overline{\partial_\mu\pi^+}) - m_\pi^2\pi^+\overline{\pi^+} \qquad T^{\mu\nu}_{\pi^\pm} = 2(\partial^\mu\pi^+)(\overline{\partial^\nu\pi^+}) + g^{\mu\nu}\big(- (\partial_\alpha\pi^+)(\overline{\partial_\alpha\pi^+}) - m_\pi^2\pi^+\overline{\pi^+}\big) \tag{1.52a}$$

$$\mathscr{L}_{\pi^0} = -\frac{1}{2}(\partial_\mu\pi^0)(\partial^\mu\pi^0) - \frac{1}{2}m_\pi^2(\pi^0)^2 \qquad T^{\mu\nu}_{\pi^0} = (\partial^\mu\pi^0)(\partial^\nu\pi^0) + g^{\mu\nu}\big(- \frac{1}{2}(\partial_\alpha\pi^0)(\partial^\alpha\pi^0) - \frac{1}{2}m_\pi^2(\pi^0)^2\big) \tag{1.52b}$$

$$\mathscr{L}_{\sigma} = -\frac{1}{2}(\partial_\mu\sigma)(\partial^\mu\sigma) - \frac{1}{2}m_\sigma^2\sigma^2 \qquad T^{\mu\nu}_{\sigma} = (\partial^\mu\sigma)(\partial^\nu\sigma) + g^{\mu\nu}\big(- \frac{1}{2}(\partial_\alpha\sigma)(\partial^\alpha\sigma) - \frac{1}{2}m_\pi^2(\pi^0)^2\big) \tag{1.52c}$$

Following **CITE WEINBERG COSMOLOGY** the energy momentum tensor of a real scalar field φ is

$$T^{\mu\nu}_\varphi = -g^{\mu\nu}\big[\frac{1}{2}g^{\rho\sigma}(\partial_\rho\varphi)(\partial_\sigma\varphi) + V(\varphi)\big] + g^{\mu\rho}g^{\nu\sigma}(\partial_\rho\varphi)(\partial_\sigma\varphi) \tag{1.53a}$$

$$\epsilon = \frac{1}{2}g^{\rho\sigma}(\partial_\rho\varphi)(\partial_\sigma\varphi) + V(\varphi) \tag{1.53b}$$

$$p = \frac{1}{2}g^{\rho\sigma}(\partial_\rho\varphi)(\partial_\sigma\varphi) - V(\varphi) \tag{1.53c}$$

$$u^\mu = -\big[- g^{\rho\sigma}(\partial_\rho\varphi)(\partial_\sigma\varphi)\big]^{-1/2}g^{\mu\nu}\partial_\nu\varphi \tag{1.53d}$$

Matching to Fluid Variables for the Real Field π^0

The only availabe 4-vector in the fluid theory is u_μ. It is thus intuitive to try to identify the real-valued 4vector $\partial_\mu \pi^0 \sim u_\mu$. Taking the normalization $u_\mu u^\mu = -1$ into account, one finds

$$u_\mu = \frac{\partial_\mu \pi^0}{\chi}\,, \qquad 0 < \chi^2 := -(\partial_\mu \pi^0)(\partial^\mu \pi^0) \tag{1.54}$$

From the fluid theory, we try to match the energy density of the hypothetical superfluid

$$\epsilon_{s,\pi^0} = u_\mu u_\nu T^{\mu\nu}_{\pi^0} = \frac{(\partial_\nu \pi^0)(\partial_\mu \pi^0)}{\chi^2}\Big((\partial^\mu \pi^0)(\partial^\nu \pi^0) + g^{\mu\nu}\big(-\frac{1}{2}(\partial_\alpha \pi^0)(\partial^\alpha \pi^0) - \frac{1}{2}m_\pi^2(\pi^0)^2\big)\Big) \tag{1.55a}$$

$$= \chi^2 - \big(\frac{1}{2}\chi^2 - \frac{1}{2}m_\pi^2(\pi^0)^2\big) \tag{1.55b}$$

$$= \frac{m_\pi^2(\pi^0)^2 + \chi^2}{2} \tag{1.55c}$$

Imposing $\epsilon = \text{const.}$ on the freezout surface, which is parametrized by some angle α, yields

$$0 = \mathrm{d}\epsilon = m_\pi^2 \pi^0 \mathrm{d}\pi^0 + \chi \mathrm{d}\chi \tag{1.56a}$$

$$= m_\pi^2 \pi^0 (\partial_\mu \pi^0)\mathrm{d}^\mu s + \chi \mathrm{d}\chi \tag{1.56b}$$

$$= m_\pi^2 \pi^0 \chi u_\mu \mathrm{d}^\mu s + \chi \mathrm{d}\chi \tag{1.56c}$$

The solution π^0 on the freezout surface thus needs to fulfill the ODE

$$\mathrm{d}\pi^0 = \chi u_\mu \mathrm{d}^\mu s\,, \quad \mathrm{d}\chi = -m_\pi^2 \pi^0 u_\mu \mathrm{d}^\mu s \tag{1.57}$$

with $\mathrm{d}^\mu s = (\partial x^\mu)/(\partial\alpha)\mathrm{d}\alpha$ the displacement vector on the freezeout surface. Initial conditions leave 1 degree of freedom, namely the ratio of kinetic energy $\epsilon_{\text{kin}} = (1/2)\chi^2$ and $\epsilon_{\text{pot}} = (1/2)m_\pi^2(\pi^0)^2$ at $\alpha = 0$. To be precise, choose $r \in [0,1]$ and set $\epsilon_{\text{pot}}\big|_{\alpha=0} = r\epsilon$ and $\epsilon_{\text{kin}}\big|_{\alpha=0} = (1-r)\epsilon$.

The equations of motion for the real Klein-Gordon field are

$$(-\Box + m_\pi^2)\pi^0 = 0 \tag{1.58}$$

Matching to Fluid Variables for the Complex Fields $\pi^\pm$

The $U(1)$-symmetry $\pi^\pm \mapsto e^{\pm i\alpha}\pi^\pm$ with infinitesimal transformation $\pi^\pm \mapsto (1 \pm i\delta\alpha)\pi^\pm$ generates a conserved Noether current

$$j^\mu \sim \frac{\delta\mathscr{L}}{\delta(\partial_\mu \pi^+)}\delta\pi^+ + \frac{\delta\mathscr{L}}{\delta(\partial_\mu \pi^-)}\delta\pi^- \tag{1.59a}$$

$$\sim \pi^-(\partial^\mu \pi^+) - \pi^+(\partial_\mu \pi^-) \tag{1.59b}$$

$$= \sqrt{n}\big((\partial_\mu \sqrt{n}) + i\sqrt{n}(\partial_\mu \theta)\big) - \sqrt{n}\big((\partial_\mu \sqrt{n}) - i\sqrt{n}(\partial_\mu \theta)\big) \tag{1.59c}$$

$$= n(\partial_\mu \theta) \tag{1.59d}$$

with the parametrization $\pi^\pm = \sqrt{n}e^{\pm i\theta}$. The most intuitive matching is now

$$u^\mu = \frac{\partial_\mu \theta}{\chi_\theta}\,, \qquad 0 < \chi_\theta^2 := -(\partial_\mu \theta)(\partial^\mu \theta) \tag{1.60}$$

leading to the energy density

$$\epsilon_{s,\pi^\pm} = u_\mu u_\nu T^{\mu\nu}_{\pi^\pm} = \frac{(\partial_\mu \theta)(\partial_\nu \theta)}{\chi_\theta^2}\Big(2(\partial^\mu \pi^+)(\partial^\nu \pi^-) + g^{\mu\nu}\big(-(\partial_\alpha \pi^+)(\partial^\alpha \pi^-) - m_\pi^2 \pi^+ \pi^-\big)\Big) \tag{1.61a}$$

$$= 2\frac{[(\partial_\mu \sqrt{n})(\partial^\mu \theta)]^2}{\chi_\theta^2} + 2n\chi_\theta^2 - \big(n\chi_\theta^2 - (\partial_\mu \sqrt{n})(\partial^\mu \sqrt{n}) - m_\pi^2 n\big) \tag{1.61b}$$

$$= n\chi_\theta^2 + 2\frac{[(\partial_\mu \sqrt{n})(\partial^\mu \theta)]^2}{\chi_\theta^2} + (\partial_\mu \sqrt{n})(\partial^\mu \sqrt{n}) + m_\pi^2 n \tag{1.61c}$$

where the intermediate calculation

$$
\begin{aligned}
(\partial^\mu \pi^+)(\partial^\nu \pi^-) &= \big((\partial^\mu \sqrt{n}) + i\sqrt{n}(\partial^\mu \theta)\big)\big((\partial^\nu \sqrt{n}) - i\sqrt{n}(\partial^\nu \theta)\big) && \text{(1.62a)}\\
&= (\partial^\mu \sqrt{n})(\partial^\nu \sqrt{n}) + n(\partial^\mu\theta)(\partial^\nu\theta) + i\big(\sqrt{n}(\partial^\mu\theta)(\partial^\nu\sqrt{n}) - \sqrt{n}(\partial^\nu\theta)(\partial^\mu\sqrt{n})\big) && \text{(1.62b)}\\
(\partial_\mu \pi^+)(\partial^\mu \pi^-) &= -\chi_n^2 - n\chi_\theta^2 && \text{(1.62c)}
\end{aligned}
$$

is useful. Note how the imaginary part of this tensor is antisymmetric and thus does not contribute upon contraction with a symmetric tensor. Assume further $\partial_\mu \sqrt{n} = u_\mu \chi_n$ then

$$\epsilon_{s,\pi^\pm} = n\chi_\theta^2 + \chi_n^2 + m_\pi^2 n \tag{1.63}$$

Expressing the Lagrangian in terms of (n, θ)

$$\mathscr{L}_{\pi^\pm} = -(\partial_\mu \sqrt{n})(\partial^\mu \sqrt{n}) - n(\partial_\mu\theta)(\partial^\mu\theta) - nm_\pi^2 \tag{1.64}$$

yields as the corresponding equations of motion

$$
\begin{aligned}
\partial_\mu\left(\frac{\partial \mathscr{L}_{\pi^\pm}}{\partial(\partial_\mu\sqrt{n})}\right) &= \frac{\partial \mathscr{L}_{\pi^\pm}}{\partial\sqrt{n}} : & -2\Box\sqrt{n} &= -2\sqrt{n}\big((\partial_\mu\theta)(\partial^\mu\theta) + m_\pi^2\big) && \text{(1.65a)}\\
\partial_\mu\left(\frac{\partial \mathscr{L}_{\pi^\pm}}{\partial(\partial_\mu\theta)}\right) &= \frac{\partial \mathscr{L}_{\pi^\pm}}{\partial\theta} : & \partial_\mu\big(-2n(\partial^\mu\theta)\big) &= 0 && \text{(1.65b)}
\end{aligned}
$$

the second of which encodes the conservation law for the $U(1)$-Noether current.

The easiest (and most naive) solution is again $n = \text{const.}$ and $\partial_\mu\theta = p_\mu$ with $p_\mu p^\mu = -m_\pi^2$. This would be a solution with only a single momentum mode. This solution implies $u^\mu = \text{const.}$ which is generally not satisfied by the given data. Assume therefore the existence of a small perturbation $\partial_\mu\theta = p_\mu + \delta q_\mu$ with $q_\mu p^\mu = 0$. From this, $\chi^2 = -(\partial_\mu\theta)(\partial^\mu\theta) = -p_\mu p^\mu - q_\mu q^\mu = m_\pi^2 - \delta^2 q^2$. To linear order in δ

$$\partial_\mu\theta = \chi u_\mu = m_\pi u_\mu \tag{1.66}$$

holds true. To expand the equations of motion and allow for non-constant amplitude, assume $n = n_{(0)} + \delta n_{(1)}(x)$ with $n_{(0)} = \text{const.}$ ($\sqrt{n} \approx \sqrt{n_{(0)}} + \delta \cdot n_{(1)}/(2\sqrt{n_{(0)}})$).

$$
\begin{aligned}
\Box\sqrt{n_{(1)}} &= 0 + \mathcal{O}(\delta^2) && \text{(1.67a)}\\
0 &= n_{(0)}\partial_\mu q^\mu + q^\mu \partial_\mu n_{(1)} && \text{(1.67b)}
\end{aligned}
$$

Fourier Transform in Adapted Coordinates

The relevant example is $\Sigma = \{x^\mu \in \mathbb{R}^{(1,3)} | (\tau, r) = (\tau(\alpha), r(\alpha))\}$ with τ, r defined by the coordinate transformation

$$
\begin{cases}
t = \tau\cosh\eta\\
z = \tau\sinh\eta\\
x = r\cos\varphi\\
y = r\sin\varphi
\end{cases}
\iff
\begin{cases}
\tau = \sqrt{t^2 - z^2}\\
\eta = \operatorname{artanh}(z/t)\\
r = \sqrt{x^2 + y^2}\\
\varphi = \arctan(y/x)
\end{cases}
\tag{1.68}
$$

Let's first evaluate $p_\mu x^\mu$ in the Bjorken coordinate system. Therefore introduce an analogous coordinate change in momentum space

$$
\begin{cases}
p_t = m_\perp \cosh\eta_p\\
p_z = m_\perp \sinh\eta_p\\
p_x = p_\perp \cos\varphi_p\\
p_y = p_\perp \sin\varphi_p
\end{cases}
\tag{1.69}
$$

to rewrite the scalar product as

$$p_\mu x^\mu \equiv -\tau(p_t\cosh\eta - p_z\sinh\eta) + r(p_x\cos\varphi + p_y\sin\varphi) = -\tau m_\perp\cosh(\eta - \eta_p) + rp_\perp\cos(\varphi - \varphi_p) \tag{1.70}$$

We used the identities

$$\cosh(a-b) = \cosh a\cosh b - \sinh a\sinh b\,, \qquad \cos(a-b) = \cos a\cos b + \sin a\sin b \tag{1.71}$$

The integral measure changes according to $\mathrm{d}^4p_{\text{cart}} = \mathrm{d}m_\perp \mathrm{d}p_\perp \mathrm{d}\eta_p \mathrm{d}\varphi_p \cdot m_\perp p_\perp$. The momentum shell condition $p^2 = m^2$ is equivalently parametrized by $m_\perp^2 = p_\perp^2 + m^2 =: \omega_\perp^2$.

Calculation 1.3|1: Metric on Hypersurface

Recall the metric $g_{\mu\nu} = \text{diag}(-1, 1, \tau^2, r^2)$ in coordinates (τ, r, η, φ). Orthonormal tangent vectors to the freeze out hypersurface are $(\hat{\partial}_\varphi)^\mu = (0, 0, 0, r^{-1}) = r^{-1}(\partial_\varphi)^\mu$, $(\hat{\partial}_\eta)^\mu = (0, 0, \tau^{-1}, 0) = \tau^{-1}(\partial_\eta)^\mu$ and $(\hat{\partial}_\alpha)^\mu = \sqrt{r'^2(\alpha) - \tau'^2(\alpha)}^{-1}(\tau'(\alpha), r'(\alpha), 0, 0) = D(\alpha)(\partial_\alpha)^\mu$ with $D(\alpha) = \sqrt{r'^2(\alpha) - \tau'^2(\alpha)}^{-1}$. The projector on the hypersurface is

$$\gamma_{\mu\nu} = (\hat{\partial}_\varphi)_\mu(\hat{\partial}_\varphi)_\nu + (\hat{\partial}_\eta)_\mu(\hat{\partial}_\eta)_\nu + (\hat{\partial}_\alpha)_\mu(\hat{\partial}_\alpha)_\nu = \begin{pmatrix} D^2(\alpha)\tau'^2(\alpha) & -D^2(\alpha)\tau'(\alpha)r'(\alpha) & 0 & 0 \\ -D^2(\alpha)\tau'(\alpha)r'(\alpha) & D^2(\alpha)r'^2(\alpha) & 0 & 0 \\ 0 & 0 & \tau^2 & 0 \\ 0 & 0 & 0 & r^2 \end{pmatrix} \tag{1.72}$$

The normal of the hypersurface is $n^\mu \equiv (\hat{\partial}_\alpha^\perp)^\mu = D(\alpha)(r'(\alpha), \tau'(\alpha), 0, 0)$ and is timelike where D is real. Naturally $\gamma_{\mu\nu}n^\nu = 0$. In the basis $(\partial_\alpha, \partial_\eta, \partial_\varphi, n)$ using (in short form)

$$(\partial_\alpha)^\nu \gamma_{\mu\nu}(\partial_\alpha)^\mu = \begin{pmatrix} \tau' \\ r' \end{pmatrix}^T \begin{pmatrix} -\tau' \\ r' \end{pmatrix} = D^{-2} \tag{1.73}$$

the hypersurface metric in coordinates $x^i = (\alpha, \eta, \varphi)$ reads

$$\gamma_{ij} = \text{diag}(D^{-2}(\alpha), \tau^2(\alpha), r^2(\alpha)) \tag{1.74}$$

and the volume element is given by $\mathrm{d}\Sigma = r(\alpha)\tau(\alpha)D^{-1}(\alpha)\mathrm{d}\alpha\mathrm{d}\eta\mathrm{d}\varphi$. The oriented surface element is

$$\mathrm{d}\Sigma^\mu = n^\mu \mathrm{d}\Sigma = r(\alpha)\tau(\alpha)(r'(\alpha), \tau'(\alpha), 0, 0)\mathrm{d}\alpha\mathrm{d}\eta\mathrm{d}\varphi \tag{1.75}$$

1.3.2 Properly Considering all Initial Conditions

Adapt the reasoning in [ABL97]. The idea is to model a non-linear, particle producing interaction process with the inhomogeneous KG equation. After all interactions have decayed and the source has vanished, the particle are effectively free. One can use the true e.o.m. to evolve initial conditions and then, after all interactions have decayed, reconstruct the hypothetical source term and derive particle production from this. For a field with the e.o.m.

$$(\Box - m^2)\phi(x) = J(x) \tag{1.76}$$

the particle spectrum after the source has vanished is

$$2\omega_{\vec{p}} \frac{\mathrm{d}N}{\mathrm{d}^3\vec{p}} = \frac{1}{(2\pi)^3}|\tilde{J}(\vec{p})|^2 \tag{1.77}$$

Use following conventions:

Important 1.3|2: Conventions for Fourier Transform

Signature is $(-, +, +, +)$.

$$\tilde{f}(p) = \int \mathrm{d}^4x e^{-ipx} f(x) \tag{1.78a}$$

$$\tilde{f}^{(3)}(t, \vec{p}) = \int \mathrm{d}^3x e^{-i\vec{p}\vec{x}} f(t, \vec{x}) \tag{1.78b}$$

$$\tilde{J}(\vec{p}) = \tilde{J}(p)|_{p^0=\omega_{\vec{p}}} \tag{1.78c}$$

The full solution of a KG-field is specified by 2 initial conditions, e.g. $\phi(t, \vec{x})$ and $\dot{\phi}(t, \vec{x})$ for fixed t, or equivalently

2 spectral functions $a(p)$, $b(p)$ within the following decomposition:

$$\phi(t,\vec{x}) = 2\pi \int \frac{\mathrm{d}^4 p}{(2\pi)^4} \delta(p^2-m^2)\Theta(p^0)\big(a(p)e^{ipx} + b(p)e^{-ipx}\big) \tag{1.79a}$$

$$= \int \frac{\mathrm{d}^3 p}{(2\pi)^3} \frac{1}{2\omega_{\vec{p}}} \big(a(\omega_{\vec{p}},\vec{p})e^{-i(\omega_{\vec{p}}t-\vec{p}\vec{x})} + b(\omega_{\vec{p}},\vec{p})e^{i(\omega_{\vec{p}}t-\vec{p}\vec{x})}\big) \tag{1.79b}$$

$$\dot{\phi}(t,\vec{x}) = \int \frac{\mathrm{d}^3 p}{(2\pi)^3} \frac{1}{2} \big(-ia(\omega_{\vec{p}},\vec{p})e^{-i(\omega_{\vec{p}}t-\vec{p}\vec{x})} + ib(\omega_{\vec{p}},\vec{p})e^{i(\omega_{\vec{p}}t-\vec{p}\vec{x})}\big) \tag{1.79c}$$

From computing the transformations

$$\tilde{\phi}^{(3)}(t,\vec{p}) = \int \mathrm{d}^3 x e^{-i\vec{p}\vec{x}} \phi(t,\vec{x}) = \int \mathrm{d}^3 x e^{-i\vec{p}\vec{x}} \int \frac{\mathrm{d}^3 q}{(2\pi)^3} \frac{1}{2\omega_{\vec{q}}} \big(a(\omega_{\vec{q}},\vec{q})e^{-i(\omega_{\vec{q}}t-\vec{q}\vec{x})} + b(\omega_{\vec{q}},\vec{q})e^{i(\omega_{\vec{q}}t-\vec{q}\vec{x})}\big) \tag{1.79d}$$

$$= \frac{1}{2\omega_{\vec{p}}} \big(a(\omega_{\vec{p}},\vec{p})e^{-i\omega_{\vec{p}}t} + b(\omega_{\vec{p}},-\vec{p})e^{i\omega_{\vec{p}}t}\big) \tag{1.79e}$$

$$\dot{\tilde{\phi}}^{(3)}(t,\vec{p}) = \int \mathrm{d}^3 x e^{-i\vec{p}\vec{x}} \dot{\phi}(t,\vec{x}) = \frac{1}{2}\big(-ia(\omega_{\vec{p}},\vec{p})e^{-i\omega_{\vec{p}}t} + ib(\omega_{\vec{p}},-\vec{p})e^{i\omega_{\vec{p}}t}\big) \tag{1.79f}$$

we can find the inversion of the parametrization

$$a(\omega_{\vec{p}},\vec{p})e^{-i\omega_{\vec{p}}t} = \int \mathrm{d}^3 x \big(\omega_{\vec{p}}\phi(t,\vec{x}) + i\dot{\phi}(t,\vec{x})\big)e^{-i\vec{p}\vec{x}} \tag{1.79g}$$

$$= ie^{-i\omega_{\vec{p}}t} \int \mathrm{d}^3 x \big(\phi(t,\vec{x})\overleftarrow{\partial}_t e^{i(\omega_{\vec{p}}t-\vec{p}\vec{x})} - \phi(t,\vec{x})\overrightarrow{\partial}_t e^{i(\omega_{\vec{p}}t-\vec{p}\vec{x})}\big) \tag{1.79h}$$

$$b(\omega_{\vec{p}},-\vec{p})e^{i\omega_{\vec{p}}t} = \int \mathrm{d}^3 x \big(\omega_{\vec{p}}\phi(t,\vec{x}) - i\dot{\phi}(t,\vec{x})\big)e^{-i\vec{p}\vec{x}} \tag{1.79i}$$

$$= -ie^{i\omega_{\vec{p}}t} \int \mathrm{d}^3 x \big(\phi(t,\vec{x})\overleftarrow{\partial}_t e^{-i(\omega_{\vec{p}}t+\vec{p}\vec{x})} - \phi(t,\vec{x})\overrightarrow{\partial}_t e^{-i(\omega_{\vec{p}}t+\vec{p}\vec{x})}\big) \tag{1.79j}$$

Invariance of Fourier Transform w.r.t. Deformations of the Hypersurface

Let ϕ_1, ϕ_2 be fields of equal mass evolving according to the KG equation. Then the current

$$J_\mu[\phi_1,\phi_2] = -i(\phi_1\partial_\mu\phi_2^* - (\partial_\mu\phi_1)\phi_2^*) =: -i\phi_1\overleftrightarrow{\partial}_\mu\phi_2 \tag{1.80}$$

is conserved. Recall Gauß law

$$\int_\Omega \mathrm{d}\Omega \nabla_\mu J^\mu = \int_{\partial\Omega} \mathrm{d}\sigma_\mu J^\mu \tag{1.81}$$

with $\mathrm{d}\sigma_\mu$ the outwards oriented surface normal of the spacetime volume Ω. The bilinear form

$$(\phi_1,\phi_2)_\Sigma = \int_\Sigma \mathrm{d}\Sigma_\mu J^\mu[\phi_1,\phi_2] = -i\int_\Sigma \mathrm{d}\Sigma_\mu \phi_1 \overleftrightarrow{\partial}^\mu \phi_2^* \tag{1.82}$$

is therefore independent of the choice of (Cauchy) hypersurface Σ (if $\partial\Sigma$ is changed, one must carefully check for further contributions in Gauß law).

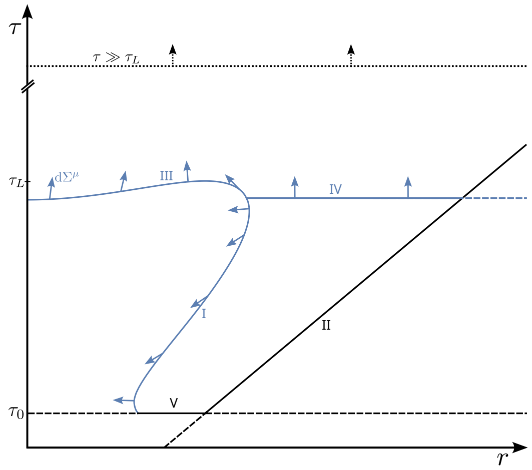


Figure 1.1: Freezeout surface in τ-r-plane[KGF23].

Consider the freezeout on the hypersurface depicted in 1.1. Assume that the condensate contribution as a function in phase space $f_{\text{cond}}(x^\mu, \vec{p})$ vanishes on Σ_{II} and Σ_{V}, i.e. is contained within the union of all light cones starting on the freeze out surface $\Sigma_{FO} \equiv \Sigma_{\text{I}} \cap \Sigma_{\text{III}}$. **To do** (1)**By causality this seems reasonable, but from Fourier decomposition of a classical field this is not at all clear.**

Following the reasoning from [KGF23], we wish to apply Gauß law. Consider separately the contribution on the τ-axis

$$\int_{\Sigma_{r=0}} \mathrm{d}\Sigma_\mu J^\mu \qquad \text{or} \qquad \lim_{r\to 0} \int_{\Sigma_r} \mathrm{d}\Sigma_\mu J^\mu$$

The surface vector on this hypersurface is $\mathrm{d}\Sigma_\mu = r\tau \mathrm{d}\tau \mathrm{d}\eta \mathrm{d}\varphi(0,1,0,0)$ and thus vanishes at $r=0$ (the hypersurface $\Sigma_{r=0}$ has zero 3-volume). Since the derivative in the integrad introduces no divergencies, the contribution of $\Sigma_{r=0}$ to Gauß law is zero.

We can therefore write

$$\tilde{\phi}(\vec{p}) = (\phi, u_{\vec{p}})_{\Sigma_t} = (\phi, u_{\vec{p}})_{\Sigma_{\tau \gg \tau_L}} = (\phi, u_{\vec{p}})_{\Sigma_{FO}} \tag{1.83}$$

The second "=" assumes that $\phi = 0$ for large spacetime rapidities $\eta \to \pm\infty$ and the $\tau = $ const. hypersurface can be deformed to a $t = $ const. hypersurface.

Note how this has the form of the inner product defined in 1.3.2, to be precise

$$a(\omega_{\vec{p}}, \vec{p}) = (\phi, e^{-i(\omega_{\vec{p}} t - \vec{p}\vec{x})})_{\Sigma_t}\,, \qquad b(\omega_{\vec{p}}, -\vec{p}) = -(\phi, e^{i(\omega_{\vec{p}} t - \vec{p}\vec{x})})_{\Sigma_t} \tag{1.84}$$

and can therefore in principle be evaluated on any Cauchy surface.

In [ABL97] it is argued that

$$\phi_{\text{out}}(t, \vec{p}) = \frac{1}{\omega_{\vec{p}}} \Big([i\tilde{J}(\vec{p})] e^{-i\omega_{\vec{p}} t} - [i\tilde{J}(-\vec{p})] e^{i\omega_{\vec{p}} t} \Big) \tag{1.85a}$$

$$\dot{\phi}_{\text{out}}(t, \vec{p}) = \Big(\tilde{J}(\vec{p}) e^{-i\omega_{\vec{p}} t} + \tilde{J}(-\vec{p}) e^{i\omega_{\vec{p}} t} \Big) \tag{1.85b}$$

from which one derives **prefactor of 2?**

$$\tilde{J}(\vec{p}) = -ie^{i\omega_{\vec{p}} t} \big(\omega_{\vec{p}} \phi_{\text{out}}(t, \vec{p}) + i\dot{\phi}_{\text{out}}(t, \vec{p}) \big) \tag{1.86a}$$

$$\tilde{J}(-\vec{p}) = ie^{-i\omega_{\vec{p}} t} (\omega_{\vec{p}} \phi_{\text{out}}(t, \vec{p}) - i\dot{\phi}_{\text{out}}(t, \vec{p})) \tag{1.86b}$$

By substition this gives

$$\tilde{J}(\vec{p}) = -ie^{i\omega_{\vec{p}} t} \int \mathrm{d}^3x \big(\omega_{\vec{p}} \phi(t, \vec{x}) + i\dot{\phi}(t, \vec{x}) \big) e^{-i\vec{p}\vec{x}} \tag{1.86c}$$

$$= \int \mathrm{d}^3x \big(\phi(t, \vec{x}) \overleftarrow{\partial}_t e^{i(\omega_{\vec{p}} t - \vec{p}\vec{x})} - \phi(t, \vec{x}) \overrightarrow{\partial}_t e^{i(\omega_{\vec{p}} t - \vec{p}\vec{x})} \big) \tag{1.86d}$$

and

$$\tilde{J}(-\vec{p}) = ie^{-i\omega_{\vec{p}}t}\int \mathrm{d}^3x\big(\omega_{\vec{p}}\phi(t,\vec{x}) - i\dot{\phi}(t,\vec{x})\big)e^{-i\vec{p}\vec{x}} \tag{1.86e}$$

$$= \int \mathrm{d}^3x\big(\phi(t,\vec{x})\overleftarrow{\partial_t}e^{-i(\omega_{\vec{p}}t+\vec{p}\vec{x})} - \phi(t,\vec{x})\overrightarrow{\partial_t}e^{-i(\omega_{\vec{p}}t+\vec{p}\vec{x})}\big) \tag{1.86f}$$

which, by the same argument as before, is independent of the choice of Cauchy surface. The calculation is the same as before, repeated here for my own clarity (note $\overleftrightarrow{\partial} = \overleftarrow{\partial} - \overrightarrow{\partial}$, unlike before) **and some errors corrected**

$$\tilde{J}(\vec{p}) = \int_{-\infty}^{\infty}\mathrm{d}\eta\int_0^{2\pi}\mathrm{d}\varphi\int_0^{\pi}\mathrm{d}\alpha\tau r\left[\phi(\tau,r)\big(r'\overleftrightarrow{\partial}_\tau + \tau'\overleftrightarrow{\partial}_r\big)e^{i(\tau\omega_\perp\cosh(\eta-\eta_p) - rp_\perp\cos(\varphi-\varphi_p))}\right] \tag{1.87a}$$

$$= \int_{-\infty}^{\infty}\mathrm{d}\eta\int_0^{2\pi}\mathrm{d}\varphi\int_0^{\pi}\mathrm{d}\alpha\tau r\left[\phi(\tau,r)\big(r'\overleftrightarrow{\partial}_\tau + \tau'\overleftrightarrow{\partial}_r\big)e^{i(\tau\omega_\perp\cosh\eta - rp_\perp\cos\varphi)}\right] \tag{1.87b}$$

$$= \int_0^{\pi}\mathrm{d}\alpha\tau r\left[\phi(\tau,r)(r'\overleftrightarrow{\partial}_\tau + \tau'\overleftrightarrow{\partial}_r)\Big[J_0(rp_\perp)\times\big(-Y_0(\tau\omega_\perp) + iJ_0(\tau\omega_\perp)\big)\Big]\right] \tag{1.87c}$$

$$\begin{aligned} &= 2\pi^2\int_0^{\pi}\mathrm{d}\alpha\tau r\Bigg[(r'\partial_\tau + \tau'\partial_r)\phi(\tau,r)\Big[J_0(rp_\perp)\times\big(-Y_0(\tau\omega_\perp) + iJ_0(\tau\omega_\perp)\big)\Big] + \\ &\quad + \phi(\tau,r)\Big[\tau'\times p_\perp J_1(rp_\perp)\times\big(-Y_0(\tau\omega_\perp) + iJ_0(\tau\omega_\perp)\big) + \\ &\qquad + r'\times J_0(rp_\perp)\times\omega_\perp\big(-Y_1(\tau\omega_\perp) + iJ_1(\tau\omega_\perp)\big)\Big]\Bigg] \end{aligned} \tag{1.87d}$$

The above logic holds true for real sources. Real fields in the present approach are (σ,π^a). One should expect that $J_{\pi^\pm} = \frac{1}{\sqrt{2}}(J_{\pi^1} \pm iJ_{\pi^2})$.

$$\pi^\pm = \frac{1}{\sqrt{2}}(\pi^1 \mp i\pi^2) = \sqrt{n}e^{\pm i\theta} \qquad \partial_\mu\pi^\pm = \pm i(\partial_\mu\theta)\pi^\pm = \pm i\chi u_\mu\pi^\pm \tag{1.88a}$$

$$\pi^1 = \sqrt{2}\Re\pi^+ \qquad \partial_\mu\pi^1 = \sqrt{2}\chi u_\mu\Re(i\pi^+) = -\sqrt{2}\chi u_\mu\Im\pi^+ = \chi u_\mu\pi^2 \tag{1.88b}$$

$$\pi^2 = -\sqrt{2}\Im\pi^+ \qquad \partial_\mu\pi^2 = -\sqrt{2}\chi u_\mu\Im(i\pi^+) = -\sqrt{2}\chi u_\mu\Re\pi^+ = -\chi u_\mu\pi^1 \tag{1.88c}$$

If we consider the $U(1)$-symmetry $\pi^\pm \mapsto \pi^\pm e^{\pm i\theta_0}$, taking the above equalities for $\theta_0 = 0$ (i.e. $\pi^\pm = \pi^\pm(0)$ etc.), then

$$\pi^1(\theta_0) = \sqrt{2}\Re(e^{i\theta_0}\pi^+) = \sqrt{2}(\cos\theta_0\Re\pi^+ - \sin\theta_0\Im\pi^+) = \pi^1\cos\theta_0 + \pi^2\sin\theta_0 \tag{1.89a}$$

$$\pi^2(\theta_0) = -\sqrt{2}\Im(e^{i\theta_0}\pi^+) = -\sqrt{2}(\cos\theta_0\Im\pi^+ + \sin\theta_0\Re\pi^+) = \pi^2\cos\theta_0 - \pi^1\sin\theta_0 \tag{1.89b}$$

The final $\pi^\pm$ spectrum will be given by

$$2|J_{\pi^\pm}(\theta_0)|^2 = |J_{\pi^1}(\theta_0) \mp iJ_{\pi^2}(\theta_0)|^2 = |\cos\theta_0 J_{\pi^1} + \sin\theta_0 J_{\pi^2} \mp i(J_\pi^2\cos\theta_0 - J_{\pi^1}\sin\theta_0)|^2 = |(J_{\pi^1} \mp iJ_{\pi^2})e^{\pm i\theta_0}|^2 \tag{1.90}$$

To do...

☐ 1 (p. 14): By causality this seems reasonable, but from Fourier decomposition of a classical field this is not at all clear.

Bibliography

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