so'n Feuerball

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Chapter 1

Setting Up the Model

1.1 Canonical Quantization

1.1.1 Real Scalar Field

Consider a real scalar field ϕ with Lagrangian density ($\eta = \mathrm{diag}(-,+,+,+)$)

$$\mathcal{L} = -\frac{1}{2}(\partial_\mu \phi)(\partial^\mu \phi) - V(\phi) = \frac{1}{2}(\dot{\phi}^2 - \vec{\nabla}^2 \phi) - V(\phi) \tag{1.1}$$

with associated Hamiltonian density

$$\mathcal{H} = \pi\dot{\phi} - \mathcal{L} = \frac{1}{2}(\pi^2 + (\vec{\nabla}\phi)^2) + V(\phi) \tag{1.2}$$

where $\pi = \frac{\partial \mathcal{L}}{\partial \dot{\phi}} = \dot{\phi}$. Choose the free scalar field, $V(\phi) = \frac{1}{2}m^2\phi^2$. The equations of motion arising from this is the Klein-Gordon equation

$$(\partial_\mu \partial^\mu - m^2)\phi(t, \vec{x}) = 0\,. \tag{1.3}$$

The equations of motion (1.3) have the general solution

$$\phi(t,x) = \int \frac{\mathrm{d}^3 p}{(2\pi)^3} \mathcal{N}_{\vec{p}} \left\{ a_{\vec{p}} e^{-i(\omega_{\vec{p}} t - \vec{p}\vec{x})} + b^*_{\vec{p}} e^{i(\omega_{\vec{p}} t - \vec{p}\vec{x})} \right\} \tag{1.4a}$$

$$(\implies) \quad \pi(t,x) = \int \frac{\mathrm{d}^3 p}{(2\pi)^3} \mathcal{N}_{\vec{p}} \left\{ -i\omega_{\vec{p}} a_{\vec{p}} e^{-i(\omega_{\vec{p}} t - \vec{p}\vec{x})} + i\omega_{\vec{p}} b^*_{\vec{p}} e^{i(\omega_{\vec{p}} t - \vec{p}\vec{x})} \right\}\,. \tag{1.4b}$$

only subject to the condition $\omega_{\vec{p}} = \sqrt{m^2 + p^2}$. $a_{\vec{p}}$ and $b^*_{\vec{p}}$ are complex Fourier coefficients. Reality of $\phi(t,x)$ further implies $a_{\vec{p}} = b_{\vec{p}}$. The normalization is typically chosen as $\mathcal{N}^2_{\vec{p}}\omega_{\vec{p}} = \frac{1}{2}$ for reasons that will become clear in a moment. If one uses

Definition 1.1|1: Poisson Brackets on Field Space

$$\{A, B\} = \int \mathrm{d}^3 x \left[\frac{\delta A}{\delta \phi}\frac{\delta B}{\delta \pi} - \frac{\delta A}{\delta \pi}\frac{\delta B}{\delta \phi} \right] \tag{1.5}$$

the field and momentum fields satisfy

$$\{\phi(t,x), \phi(t,y)\} = \{\pi(t,x), \pi(t,y)\} = 0\,, \quad \{\phi(t,x), \pi(t,y)\} = \delta^{(d)}(x-y)\,. \tag{1.6}$$

Quantization is achieved by the replacement

$$i\{\cdot,\cdot\} \to [\cdot,\cdot]\,, \tag{1.7}$$

lifting fields to operators, $\phi \to \hat{\phi}$ and $\pi \to \hat{\pi}$, and therefore also $a_{\vec{p}} \to \hat{a}_{\vec{p}}$ and $a^\dagger_{\vec{p}} \to \hat{a}^\dagger_{\vec{p}}$ (though the $\hat{}$ will be omitted). The fundamental commutator $[\phi(t,x), \pi(t,y)] = i\delta^{(d)}(x-y)$ then implies

Important 1.1|2: Commutators of $a_{\vec{p}}$, $a^\dagger_{\vec{q}}$

$$[a_{\vec{p}}, a_{\vec{q}}] = [a^\dagger_{\vec{p}}, a^\dagger_{\vec{q}}] = 0\,, \quad [a_{\vec{p}}, a^\dagger_{\vec{q}}] = (2\pi)^3\delta^{(3)}(\vec{p}-\vec{q})\,. \tag{1.8}$$

Calculation 1.1|3: Commutators of $a_{\vec{p}}$, $a^\dagger_{\vec{q}}$

Notice the relations

$$a_{\vec{p}} = \frac{1}{2\mathcal{N}_{\vec{p}}}\int \mathrm{d}^3x \left\{\phi(t,x) + \frac{i}{\omega_{\vec{p}}}\pi(t,x)\right\} e^{i(\omega_{\vec{p}}t - \vec{p}\vec{x})} \tag{1.9a}$$

$$a^\dagger_{\vec{p}} = \frac{1}{2\mathcal{N}_{\vec{p}}}\int \mathrm{d}^3x \left\{\phi(t,x) - \frac{i}{\omega_{\vec{p}}}\pi(t,x)\right\} e^{-i(\omega_{\vec{p}}t - \vec{p}\vec{x})} \tag{1.9b}$$

The non-vanishing commutator is derived as follows:

$$\begin{aligned}
[a_{\vec{p}}, a^\dagger_{\vec{q}}] &= \frac{1}{4\mathcal{N}_{\vec{p}}\mathcal{N}_{\vec{q}}}\int \mathrm{d}^3x\mathrm{d}^3y \left\{-\frac{i}{\omega_{\vec{q}}}[\phi(t,x),\pi(t,y)]e^{i\left((\omega_{\vec{p}}-\omega_{\vec{q}})t-(\vec{p}\vec{x}-\vec{q}\vec{y})\right)}\right. \\
&\qquad \left. +\frac{i}{\omega_{\vec{p}}}[\pi(t,x),\phi(t,y)]e^{-i\left((\omega_{\vec{p}}-\omega_{\vec{q}})t-(\vec{p}\vec{x}-\vec{q}\vec{y})\right)}\right\} \\
&= \frac{1}{4\mathcal{N}_{\vec{p}}\mathcal{N}_{\vec{q}}}\int \mathrm{d}^3x \left\{\frac{1}{\omega_{\vec{q}}}e^{i\left((\omega_{\vec{p}}-\omega_{\vec{q}})t-(\vec{p}-\vec{q})x\right)}\right. \\
&\qquad \left. +\frac{1}{\omega_{\vec{p}}}e^{-i\left((\omega_{\vec{p}}-\omega_{\vec{q}})t-(\vec{p}-\vec{q})x\right)}\right\} \\
&= \frac{(2\pi)^3}{2\mathcal{N}^2_{\vec{p}}\omega_{\vec{p}}}\delta^{(3)}(\vec{p}-\vec{q})
\end{aligned}$$

whereas the vanishing commutators are calculated as

$$\begin{aligned}
[a_{\vec{p}}, a_{\vec{q}}] &= \frac{1}{4\mathcal{N}_{\vec{p}}\mathcal{N}_{\vec{q}}}\int \mathrm{d}^3x\mathrm{d}^3y \left\{\frac{i}{\omega_{\vec{q}}}[\phi(t,x),\pi(t,y)]e^{i\left((\omega_{\vec{p}}+\omega_{\vec{q}})t-(\vec{p}\vec{x}+\vec{q}\vec{y})\right)}\right. \\
&\qquad \left. +\frac{i}{\omega_{\vec{p}}}[\pi(t,x),\phi(t,y)]e^{i\left((\omega_{\vec{p}}+\omega_{\vec{q}})t-(\vec{p}\vec{x}+\vec{q}\vec{y})\right)}\right\} \\
&= 0
\end{aligned}$$

and simmilarly for $[a^\dagger_{\vec{p}}, a^\dagger_{\vec{q}}] = 0$.

After this quantization, the fields are written as

$$\phi(t,x) = \int \frac{\mathrm{d}^3p}{(2\pi)^3}\frac{1}{\sqrt{2\omega_{\vec{p}}}}\left\{a_{\vec{p}}e^{-i(\omega_{\vec{p}}t-\vec{p}\vec{x})} + a^\dagger_{\vec{p}}e^{i(\omega_{\vec{p}}t-\vec{p}\vec{x})}\right\} \tag{1.10a}$$

$$\pi(t,x) = \int \frac{\mathrm{d}^3p}{(2\pi)^3}\frac{i\omega_{\vec{p}}}{\sqrt{2\omega_{\vec{p}}}}\left\{-a_{\vec{p}}e^{-i(\omega_{\vec{p}}t-\vec{p}\vec{x})} + a^\dagger_{\vec{p}}e^{i(\omega_{\vec{p}}t-\vec{p}\vec{x})}\right\}\,. \tag{1.10b}$$

To express the Hamiltonian in terms of $a_{\vec{p}}$ and $a^\dagger_{\vec{p}}$ rewrite

$$\phi(t,x) = \int \frac{\mathrm{d}^3p}{(2\pi)^3}\frac{1}{\sqrt{2\omega_{\vec{p}}}}\left\{a_{\vec{p}}e^{-i\omega_{\vec{p}}t} + a^\dagger_{-\vec{p}}e^{i\omega_{\vec{p}}t}\right\}e^{i\vec{p}\vec{x}} \tag{1.11a}$$

$$\pi(t,x) = \int \frac{\mathrm{d}^3p}{(2\pi)^3}\frac{i\omega_{\vec{p}}}{\sqrt{2\omega_{\vec{p}}}}\left\{-a_{\vec{p}}e^{-i\omega_{\vec{p}}t} + a^\dagger_{-\vec{p}}e^{i\omega_{\vec{p}}t}\right\}e^{i\vec{p}\vec{x}}\,. \tag{1.11b}$$

Omit the time dependence for the next calculation, for example by choosing $t = 0$. The Hamiltonian is now easily computed to be

$$
\begin{aligned}
H &= \frac{1}{2}\int \mathrm{d}^3x \frac{\mathrm{d}^3p}{(2\pi)^3}\frac{\mathrm{d}^3q}{(2\pi)^3}\frac{1}{\sqrt{4\omega_{\vec p}\omega_{\vec q}}}e^{i(\vec p+\vec q)\vec x}\Big[-\omega_{\vec p}\omega_{\vec q}(-a_{\vec p}+a^\dagger_{-\vec p})(-a_{\vec q}+a^\dagger_{-\vec q})+ \\
&\qquad + (-\vec p\vec q + m^2)(a_{\vec p}+a^\dagger_{-\vec p})(a_{\vec q}+a^\dagger_{-\vec q})\Big] && (1.12a) \\
&= \frac{1}{2}\int \frac{\mathrm{d}^3p}{(2\pi)^3}\frac{1}{2\omega_{\vec p}}\Big[-\omega_{\vec p}^2(\cancel{a_{\vec p}a_{-\vec p}} - a^\dagger_{-\vec p}a_{-\vec p} - a_{\vec p}a^\dagger_{\vec p} + \cancel{a^\dagger_{-\vec p}a^\dagger_{\vec p}})+ \\
&\qquad + (\vec p^2 + m^2)(\cancel{a_{\vec p}a_{-\vec p}} + a^\dagger_{-\vec p}a_{-\vec p} + a_{\vec p}a^\dagger_{\vec p} + \cancel{a^\dagger_{-\vec p}a^\dagger_{\vec p}})\Big] && (1.12b) \\
&= \int \frac{\mathrm{d}^3p}{(2\pi)^3}\frac{\omega_{\vec p}}{2}(a^\dagger_{\vec p}a_{\vec p} + a_{\vec p}a^\dagger_{\vec p}) && (1.12c) \\
&= \int \frac{\mathrm{d}^3p}{(2\pi)^3}\omega_{\vec p}(a^\dagger_{\vec p}a_{\vec p} + \frac{1}{2}[a_{\vec p}, a^\dagger_{\vec p}]) && (1.12d)
\end{aligned}
$$

Since only the combination $a_{\vec p}a^\dagger_{\vec p}$ shows up, the explicit time dependence would have dropped out anyways. The commutation relation between H, $a_{\vec p}$ and $a^\dagger_{\vec p}$ are given by

$$[H, a_{\vec p}] = -\omega_{\vec p}a_{\vec p}\,, \qquad [H, a^\dagger_{\vec p}] = \omega_{\vec p}a^\dagger_{\vec p} \tag{1.13}$$

From quantum mechanics it is now clear that H every momentum mode $\vec p$ has a discrete spectrum of excitations or energy eigenstates, such that

$$H|n_{\vec p}\rangle = (\omega_{\vec p} + E_0)|n_{\vec p}\rangle \tag{1.14}$$

and the operator $a_{\vec p}$ ($a^\dagger_{\vec p}$) annihilates (creates) excitations,

$$a_{\vec p}|n_{\vec p}\rangle = \sqrt{n_{\vec p}}|(n-1)_{\vec p}\rangle\,, \qquad a^\dagger_{\vec p}|n_{\vec p}\rangle = \sqrt{(n+1)_{\vec p}}|(n+1)_{\vec p}\rangle \tag{1.15}$$

E_0 is the (IR) divergent vacuum energy. The vacuum is defined by $a_{\vec p}|0\rangle = 0\ \forall \vec p$. The operator

$$N_{\vec p} = a^\dagger_{\vec p}a_{\vec p} \tag{1.16}$$

is the number operator for a given momentum mode and its expectation value $n(\vec p) = \langle N_{\vec p}\rangle$ has the interpretation of the momentum space number density,

$$N = \int \frac{\mathrm{d}^3p}{(2\pi)^3}n(\vec p)\,, \qquad n(\vec p) = (2\pi)^3\frac{\mathrm{d}N}{\mathrm{d}^3p} \tag{1.17}$$

1.1.2 Complex Scalar Field

Consider a complex scalar field $\phi = \frac{1}{\sqrt{2}}(\phi_1 + i\phi_2)$ with $\phi_k, k \in \{1, 2\}$ two real scalar fields. The Lagrangian of ϕ can be written as the sum of the Lagrangians $\mathcal{L}_k$ of ϕ_k. Similarly for the Hamiltonian

$$\mathcal{L} = -(\partial_\mu\phi^*)(\partial^\mu\phi) - m^2\phi\phi^* = \sum_k\Big\{-\frac{1}{2}\Big((\partial_\mu\phi_k)(\partial^\mu\phi_k) + m^2\phi_k^2\Big)\Big\} = \sum_k \mathcal{L}_k \tag{1.18}$$

With the conjugate momenta $\pi_k = \dot\phi_k$ the conjugate momentum of ϕ turns out to be

$$\pi = \frac{\partial\mathcal{L}}{\partial\dot\phi} = \dot\phi^* = \frac{\pi_1 - i\pi_2}{\sqrt{2}} \tag{1.19}$$

and the Hamiltonian is therefore

$$\mathcal{H} = \pi\dot\phi + \pi^*\dot\phi^* - \mathcal{L} = \dot\phi\dot\phi^* + (\vec\nabla\phi)(\vec\nabla\phi^*) + m^2\phi\phi^* = \sum_k\frac{1}{2}(\pi_k^2 + (\vec\nabla\phi_k)^2 + m^2\phi_k^2) = \sum_k\mathcal{H}_k \tag{1.20}$$

Quantization rules are imposed as before on the real scalar field ϕ_k. From this, it is immediately clear that $\phi(t,\vec{x})$ takes the form

$$\phi(t,\vec{x}) = \int \frac{\mathrm{d}^3p}{(2\pi)^3}\frac{1}{\sqrt{2\omega_{\vec{p}}}}\Big\{\frac{a_{\vec{p},(1)}+ia_{\vec{p},(2)}}{\sqrt{2}}e^{-i(\omega_{\vec{p}}t-\vec{p}\vec{x})}+\frac{a^\dagger_{\vec{p},(1)}+ia^\dagger_{\vec{p},(2)}}{\sqrt{2}}e^{i(\omega_{\vec{p}}t-\vec{p}\vec{x})}\Big\} \tag{1.21a}$$

$$\pi(t,\vec{x}) = \int \frac{\mathrm{d}^3p}{(2\pi)^3}\frac{i\omega_{\vec{p}}}{\sqrt{2\omega_{\vec{p}}}}\Big\{-\frac{a_{\vec{p},(1)}-ia_{\vec{p},(2)}}{\sqrt{2}}e^{-i(\omega_{\vec{p}}t-\vec{p}\vec{x})}+\frac{a^\dagger_{\vec{p},(1)}-ia^\dagger_{\vec{p},(2)}}{\sqrt{2}}e^{i(\omega_{\vec{p}}t-\vec{p}\vec{x})}\Big\} \tag{1.21b}$$

It is intuitive to define

$$a_{\vec{p}} = \frac{a_{\vec{p},(1)}+ia_{\vec{p},(2)}}{\sqrt{2}}\,, \qquad b^\dagger_{\vec{p}} = \frac{a^\dagger_{\vec{p},(1)}+ia^\dagger_{\vec{p},(2)}}{\sqrt{2}} \tag{1.22}$$

recovering the looking familiar expression

$$\phi(t,\vec{x}) = \int \frac{\mathrm{d}^3p}{(2\pi)^3}\frac{1}{\sqrt{2\omega_{\vec{p}}}}\Big\{a_{\vec{p}}e^{-i(\omega_{\vec{p}}t-\vec{p}\vec{x})}+b^\dagger_{\vec{p}}e^{i(\omega_{\vec{p}}t-\vec{p}\vec{x})}\Big\} \tag{1.23a}$$

$$\pi(t,\vec{x}) = \int \frac{\mathrm{d}^3p}{(2\pi)^3}\frac{i\omega_{\vec{p}}}{\sqrt{2\omega_{\vec{p}}}}\Big\{-b_{\vec{p}}e^{-i(\omega_{\vec{p}}t-\vec{p}\vec{x})}+a^\dagger_{\vec{p}}e^{i(\omega_{\vec{p}}t-\vec{p}\vec{x})}\Big\} \tag{1.23b}$$

where now, unlike before, explicitly $a_{\vec{p}} \neq b_{\vec{p}}$ is found.

The commutation relations of $a_{\vec{p},(k)}$, $a^\dagger_{\vec{p},(k)}$ are trivially given by

$$[a_{\vec{p},(j)}, a^\dagger_{\vec{q},(k)}] = (2\pi)^3\delta^{(3)}(\vec{p}-\vec{q})\delta_{jk}\,, \qquad [a^{(\dagger)}_{\vec{p},(j)}, a^{(\dagger)}_{\vec{p},(k)}] = 0 \tag{1.24}$$

and lead to

$$[a_{\vec{p}}, b_{\vec{q}}] = \frac{1}{2}[a_{\vec{p},(1)}+ia_{\vec{p},(2)}, a_{\vec{q},(1)}-ia_{\vec{q},(2)}] = 0 \tag{1.25a}$$

$$[a^{(\dagger)}_{\vec{p}}, a^{(\dagger)}_{\vec{q}}] = 0 \tag{1.25b}$$

$$[a_{\vec{p}}, b^\dagger_{\vec{q}}] = \frac{1}{2}[a_{\vec{p},(1)}+ia_{\vec{p},(2)}, a^\dagger_{\vec{q},(1)}+ia^\dagger_{\vec{q},(2)}] = \frac{1}{2}\big([a_{\vec{p},(1)}, a^\dagger_{\vec{q},(1)}]-[a_{\vec{p},(2)}, a^\dagger_{\vec{q},(2)}]\big) = 0 \tag{1.25c}$$

$$[a_{\vec{p}}, a^\dagger_{\vec{q}}] = \frac{1}{2}[a_{\vec{p},(1)}+ia_{\vec{p},(2)}, a^\dagger_{\vec{q},(1)}-ia^\dagger_{\vec{q},(2)}] = \frac{1}{2}\big([a_{\vec{p},(1)}, a^\dagger_{\vec{q},(1)}]+[a_{\vec{p},(2)}, a^\dagger_{\vec{q},(2)}]\big) = (2\pi)^3\delta^{(3)}(\vec{p}-\vec{q}) \tag{1.25d}$$

$$[b_{\vec{p}}, b^\dagger_{\vec{q}}] = \frac{1}{2}[a_{\vec{p},(1)}-ia_{\vec{p},(2)}, a^\dagger_{\vec{q},(1)}+ia^\dagger_{\vec{q},(2)}] = \frac{1}{2}\big([a_{\vec{p},(1)}, a^\dagger_{\vec{q},(1)}]+[a_{\vec{p},(2)}, a^\dagger_{\vec{q},(2)}]\big) = (2\pi)^3\delta^{(3)}(\vec{p}-\vec{q}) \tag{1.25e}$$

From this, again the Hamiltonian is derived to be

$$H = \int \frac{\mathrm{d}^3p}{(2\pi)^3}\omega_{\vec{p}}\Big(a^\dagger_{\vec{p},(1)}a_{\vec{p},(1)}+a^\dagger_{\vec{p},(2)}a_{\vec{p},(2)}+\frac{1}{2}\big([a_{\vec{p},(1)}, a^\dagger_{\vec{p},(1)}]+[a_{\vec{p},(2)}, a^\dagger_{\vec{p},(2)}]\big)\Big) \tag{1.26a}$$

$$= \int \frac{\mathrm{d}^3p}{(2\pi)^3}\omega_{\vec{p}}\Big(a^\dagger_{\vec{p}}a_{\vec{p}}+b^\dagger_{\vec{p}}b_{\vec{p}}+\frac{1}{2}\big([a_{\vec{p}}, a^\dagger_{\vec{p}}]+[b_{\vec{p}}, b^\dagger_{\vec{p}}]\big)\Big) \tag{1.26b}$$

using $a^\dagger_{\vec{p}}a_{\vec{p}} = \frac{1}{2}(a^\dagger_{\vec{p},(1)}-ia^\dagger_{\vec{p},(2)})(a_{\vec{p},(1)}+ia_{\vec{p},(2)})$ and $b^\dagger_{\vec{p}}b_{\vec{p}} = \frac{1}{2}(a^\dagger_{\vec{p},(1)}+ia^\dagger_{\vec{p},(2)})(a_{\vec{p},(1)}-ia_{\vec{p},(2)})$. Whereas $n_{\vec{p}} = \langle a^\dagger_{\vec{p}}a_{\vec{p}}\rangle$ has the interpretation of a particle number density, $\overline{n}_{\vec{p},J} = \langle b^\dagger_{\vec{p}}b_{\vec{p}}\rangle$ is understood as the antiparticle density.

1.2 Particle Spectra from Classical Sources

1.2.1 Real Scalar Field

Follow **CITE: REINHARD: FIELD QUANTIZATION** around eq. 4.140. Define the Pauli-Jordan function $\Delta(x)$ via

$$\begin{aligned}
i\Delta(x-y) &= \int \frac{\mathrm{d}^3p}{(2\pi)^3}\frac{1}{2\omega_{\vec{p}}}\Big(e^{-i(\omega_{\vec{p}}(x^0-y^0)-\vec{p}(\vec{x}-\vec{y}))} - e^{i(\omega_{\vec{p}}(x^0-y^0)-\vec{p}(\vec{x}-\vec{y}))}\Big) && (1.27)\\
&= \int \frac{\mathrm{d}^3p}{(2\pi)^3}\frac{1}{2\omega_{\vec{p}}}\Big(e^{ip(x-y)} - e^{-ip(x-y)}\Big) && (1.28)\\
&= \int \frac{\mathrm{d}^3p}{(2\pi)^3}\frac{1}{2\omega_{\vec{p}}}\Big(e^{-i\omega_{\vec{p}}(x^0-y^0)} - e^{i\omega_{\vec{p}}(x^0-y^0)}\Big)e^{i\vec{p}(\vec{x}-\vec{y})} && (1.29)\\
&= 2\pi\int \frac{\mathrm{d}^4p}{(2\pi)^4}\epsilon(p^0)\delta(p^2+m^2)e^{-i(p^0(x^0-y^0)-\vec{p}(\vec{x}-\vec{y}))} && (1.30)\\
&= 2\pi\int \frac{\mathrm{d}^4p}{(2\pi)^4}\epsilon(p^0)\delta(p^2+m^2)e^{ip(x-y)} && (1.31)
\end{aligned}$$

which satisfies (...) $(\partial_\mu\partial^\mu - m^2)\Delta = 0$. $\epsilon(x)$ is the sign function. The retarded and advanced propagators $\Delta_{R,A}(x)$ are given by

$$\Delta_R(x) = \Theta(x^0)\Delta(x)\,, \qquad \Delta_A(x) = \Theta(x^0)\Delta(x)\,. \tag{1.32}$$

which immediately implies $\Delta(x) = \Delta_R(x) - \Delta_A(x)$. These function satisfy

$$(\partial_\mu\partial^\mu - m^2)\Delta_{R,A}(x) = \delta^{(4)}(x) \tag{1.33}$$

Calculation 1.2|2: Greens Functions

Solve $(\partial_\mu\partial^\mu - m^2)D(x) = \delta^{(4)}(x)$. Using

$$D(x) = \int\frac{\mathrm{d}^4p}{(2\pi)^4}\tilde{D}(p)e^{ipx}$$

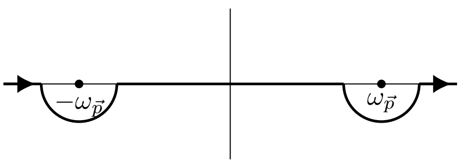
one finds

$$\int\frac{\mathrm{d}^4p}{(2\pi)^4}(-p^2-m^2)\tilde{D}(p) = \delta^{(4)}(x)\tilde{D}(p) = -\frac{1}{p^2+m^2} \tag{1.34}$$

and thus

$$\begin{aligned}
D(x) &= \int\frac{\mathrm{d}^4p}{(2\pi)^4}\frac{-1}{p^2+m^2}e^{-ipx} && (1.35)\\
&= -\int\frac{\mathrm{d}^3p}{(2\pi)^3}\frac{\mathrm{d}p^0}{2\pi}\frac{1}{(p^0+\omega_{\vec{p}})(-p^0+\omega_{\vec{p}})}e^{i(p^0t-\vec{p}\vec{x})} && (1.36)
\end{aligned}$$

Definition 1.2|2: Retarded Propagator, Contour



If $t > 0$, close the integration contour in the upper imaginary half plane.

$$\int \frac{\mathrm{d}p^0}{2\pi} \frac{1}{(p^0+\omega_{\vec{p}})(-p^0+\omega_{\vec{p}})} e^{ip^0 t} = 2\pi i \Big(\lim_{p^0 \to \omega_{\vec{p}}} \frac{1}{2\pi}(p^0-\omega_{\vec{p}}) \frac{e^{ip^0 t}}{(p^0+\omega_{\vec{p}})(-p^0+\omega_{\vec{p}})} + \lim_{p^0 \to -\omega_{\vec{p}}} \frac{1}{2\pi}(p^0+\omega_{\vec{p}}) \frac{e^{ip^0 t}}{(p^0+\omega_{\vec{p}})(-p^0+\omega_{\vec{p}})} \tag{1.37}$$

$$= i \Big(\frac{e^{-i\omega_{\vec{p}} t} - e^{i\omega_{\vec{p}} t}}{2\omega_{\vec{p}}} \Big) \tag{1.38}$$

For $t < 0$, close the integration in the lower half plane, such that there is no residue within the integration contour. This leads to

$$D_R(x) = \frac{1}{i}\Theta(t) \int \frac{\mathrm{d}^3p}{(2\pi)^3} \frac{1}{2\omega_{\vec{p}}} (e^{-i(\omega_{\vec{p}}t - \vec{p}\vec{x})} - e^{i(\omega_{\vec{p}}t-\vec{p}\vec{x})}) = -i\Theta(x^0) \int \frac{\mathrm{d}^3p}{(2\pi)^3} \frac{1}{2\omega_{\vec{p}}} (e^{ipx} - e^{-ipx}) = \Theta(x^0)\Delta(x) \equiv \Delta_R(x) \tag{1.39}$$

Consider now a real scalar field that evolves according to the inhomogeneous Klein-Gordon equation

$$(\partial_\mu \partial^\mu - m^2)\phi = -J \tag{1.40}$$

The solution is constructed by superposition of homogeneous solutions and a particular inhomogeneous solutions. Requesting $\phi \equiv 0$ for vanishing source, one finds

$$\phi_J(x) = -\int \mathrm{d}^4y \Delta_R(x-y) J(y) \tag{1.41a}$$

$$= i \int \mathrm{d}^4y \Theta(x^0 - y^0) \int \frac{\mathrm{d}^3p}{(2\pi)^3} \frac{1}{2\omega_{\vec{p}}} (e^{ip(x-y)} - e^{-ip(x-y)}) J(y) \tag{1.41b}$$

$$\overset{x^0 \gtrless y^0}{=} i \int \frac{\mathrm{d}^3p}{(2\pi)^3} \frac{1}{2\omega_{\vec{p}}} (J(p)e^{ipx} - J(-p)e^{-ipx}) \tag{1.41c}$$

$$= i \int \frac{\mathrm{d}^3p}{(2\pi)^3} \frac{1}{2\omega_{\vec{p}}} (J(p) e^{-i(\omega_{\vec{p}}t - \vec{p}\vec{x})} - J(-p) e^{i(\omega_{\vec{p}}t-\vec{p}\vec{x})}) \tag{1.41d}$$

where $J(p) = \int \mathrm{d}^4y J(x) e^{-ipy}$ was used.

Taking the homogeneous solution ϕ_0 as given by (1.10a) into account, the field after the source has vanished is given by

$$\phi(t,x) = \phi_0(t,\vec{x}) + \phi_J(t,\vec{x}) = \int \frac{\mathrm{d}^3p}{(2\pi)^3} \frac{1}{\sqrt{2\omega_{\vec{p}}}} \Big\{ (a_{\vec{p}} + \frac{iJ(p)}{\sqrt{2\omega_{\vec{p}}}}) e^{-i(\omega_{\vec{p}}t-\vec{p}\vec{x})} + (a^\dagger_{\vec{p}} - \frac{iJ(-p)}{\sqrt{2\omega_{\vec{p}}}}) e^{i(\omega_{\vec{p}}t - \vec{p}\vec{x})} \Big\} \tag{1.42}$$

This is described by effectively replacing annihilation and creation operators via

$$a_{\vec{p}} \mapsto a_{\vec{p}} + \frac{iJ(p)}{\sqrt{2\omega_{\vec{p}}}}, \qquad a^\dagger_{\vec{p}} \mapsto a^\dagger_{\vec{p}} - \frac{iJ(-p)}{\sqrt{2\omega_{\vec{p}}}} \tag{1.43}$$

These replacements are of course compatible, considering that for a real source $J(p) = J^*(-p)$. Since $J(p)$ is just a $\mathbb{C}$-number, it does not alter the commutation relations from which the Hamiltonian and number operator are derived. The number density after the source has vanished, starting from the initial vacuum state $|0\rangle$ without any particles, is given by

$$n_{\vec{p},J} = \langle 0 | \Big(a_{\vec{p}} + \frac{iJ(p)}{\sqrt{2\omega_{\vec{p}}}} \Big) \Big(a^\dagger_{\vec{p}} - \frac{iJ^*(p)}{\sqrt{2\omega_{\vec{p}}}} \Big) |0\rangle = \frac{1}{2\omega_{\vec{p}}} |J(p)|^2 \tag{1.44}$$

1.2.2 Complex Scalar Field

The derivation for the free scalar field is completely analogous. The identity $J(p) = J^*(-p)$ may not be used anymore. Instead, $J(p)$ contributes to the spectrum of particles, whereas $J(-p)$ contributes to the spectrum of antiparticles, in the following way:

$$a_{\vec{p}} \mapsto a_{\vec{p}} + \frac{iJ(p)}{\sqrt{2\omega_{\vec{p}}}}, \qquad b^\dagger_{\vec{p}} \mapsto b^\dagger_{\vec{p}} - \frac{iJ(-p)}{\sqrt{2\omega_{\vec{p}}}} \tag{1.45}$$

The particle and antiparticle momentum space number densities, or spectra, induces by the source J are now

$$n_{\vec{p},J} = \frac{1}{2\omega_{\vec{p}}}|J(p)|^2\,, \qquad \overline{n}_{\vec{p},J} = \frac{1}{2\omega_{\vec{p}}}|J(-p)|^2 \tag{1.46}$$

1.2.3 Extracting the Source from the Late Time Field

Equation (1.41d) allows use to extract

$$J(p)e^{-i\omega_{\vec{p}}t} = \int \mathrm{d}^3x\Big(- i\omega_{\vec{p}}\phi_J(x) + \big(\partial_t\phi_J(x)\big)\Big)e^{-i\vec{p}\vec{x}} \tag{1.47a}$$

$$J(p) = \int \mathrm{d}^3x\Big(\phi_J(x)\overleftarrow{\partial_t}e^{i(\omega_{\vec{p}}t-\vec{p}\vec{x})} - \phi_J(x)\overrightarrow{\partial_t}e^{i(\omega_{\vec{p}}t-\vec{p}\vec{x})}\Big) \tag{1.47b}$$

$$J(-p)e^{i\omega_{\vec{p}}t} = -\int \mathrm{d}^3x\Big(- i\omega_{\vec{p}}\phi_J(x) - \big(\partial_t\phi_J(x)\big)\Big)e^{i\vec{p}\vec{x}} \tag{1.47c}$$

$$J(-p) = \int \mathrm{d}^3x\Big(\phi_J(x)\overleftarrow{\partial_t}e^{-i(\omega_{\vec{p}}t-\vec{p}\vec{x})} - \phi_J(x)\overrightarrow{\partial_t}e^{-i(\omega_{\vec{p}}t-\vec{p}\vec{x})}\Big) \tag{1.47d}$$

1.3 Matching Hydrodynamics with Field Theory

1.3.1 Expanding around Minimum of Linear σ-model

The Lagrangian density

$$\mathscr{L} = \mathcal{L}_{\text{kin}} - V(\sigma,\vec{\pi}) = -\frac{1}{2}(\partial_\mu\sigma)(\partial^\mu\sigma) - \frac{1}{2}(\partial_\mu\vec{\pi})(\partial^\mu\vec{\pi}) + \frac{1}{2}\mu^2(\sigma^2+\vec{\pi}^2) + \frac{\lambda}{4}(\sigma^2+\vec{\pi}^2)^2 + h\sigma \tag{1.48}$$

can be expanded around the minimum at $\sigma_0 = f_\pi + h\cdot\frac{1}{2\mu^2} + \mathcal{O}(h^2)$ where $f_\pi = \frac{\mu}{\sqrt{\lambda}}$. Performing the substitution $\sigma \mapsto v + \sigma$ and neglecting terms of order $\mathcal{O}(h^2, \sigma^3, \sigma\vec{\pi}^2, (\vec{\pi}^2)^2)$ and higher the potential reads

$$V(\sigma,\vec{\pi}) = -\frac{\mu^4}{4\lambda} + \frac{1}{2}m_\sigma\sigma^2 + \frac{1}{2}m_\pi^2\vec{\pi}^2 \tag{1.49}$$

with pion mass $m_\pi^2 = \frac{h}{f_\pi}$ and sigma mass $m_\sigma^2 = 2\mu^2 + \mathcal{O}(h)$. Defining $\pi^\pm = (1/\sqrt{2})(\pi^1 \mp i\pi^2)$ one gets

$$(\pi^1)^2 + (\pi^2)^2 = |\pi^+|^2 + |\pi^-|^2 = 2\pi^+\pi^- \equiv 2\pi^+\overline{\pi^+} \tag{1.50}$$

The expansion of the Lagrangian around σ_0 breaks the $SO(4)$-symmetry associated to the vector $(\sigma,\vec{\pi})$ and chooses explicitly a minimum within the $SO(4)$-symmetric mexican hat potential. The residual symmetry is $SU(3)$. It features the $SO(2)$ subgroup of symmetry transformations

$$\begin{pmatrix}\pi^1\\ \pi^2\end{pmatrix} \mapsto \begin{pmatrix}\cos\alpha & \sin\alpha\\ -\sin\alpha & \cos\alpha\end{pmatrix}\begin{pmatrix}\pi^1\\ \pi^2\end{pmatrix} \qquad \Longleftrightarrow \qquad \pi^\pm \mapsto e^{\pm i\alpha}\pi^\pm \tag{1.51}$$

Note $\pi^- = \overline{\pi^+}$.

The Lagrangians and energy-momentum tensors $T^{\mu\nu} = 2(\partial\mathscr{L}/\partial g_{\mu\nu}) + g^{\mu\nu}\mathscr{L}$ **CITE: BLAU NOTES** for the separate fields read

$$\mathscr{L}_{\pi^\pm} = -(\partial_\mu\pi^+)(\overline{\partial_\mu\pi^+}) - m_\pi^2\pi^+\overline{\pi^+} \qquad T^{\mu\nu}_{\pi^\pm} = 2(\partial^\mu\pi^+)(\overline{\partial^\nu\pi^+}) + g^{\mu\nu}\big(- (\partial_\alpha\pi^+)(\overline{\partial_\alpha\pi^+}) - m_\pi^2\pi^+\overline{\pi^+}\big) \tag{1.52a}$$

$$\mathscr{L}_{\pi^0} = -\frac{1}{2}(\partial_\mu\pi^0)(\partial^\mu\pi^0) - \frac{1}{2}m_\pi^2(\pi^0)^2 \qquad T^{\mu\nu}_{\pi^0} = (\partial^\mu\pi^0)(\partial^\nu\pi^0) + g^{\mu\nu}\big(- \frac{1}{2}(\partial_\alpha\pi^0)(\partial^\alpha\pi^0) - \frac{1}{2}m_\pi^2(\pi^0)^2\big) \tag{1.52b}$$

$$\mathscr{L}_{\sigma} = -\frac{1}{2}(\partial_\mu\sigma)(\partial^\mu\sigma) - \frac{1}{2}m_\sigma^2\sigma^2 \qquad T^{\mu\nu}_{\sigma} = (\partial^\mu\sigma)(\partial^\nu\sigma) + g^{\mu\nu}\big(- \frac{1}{2}(\partial_\alpha\sigma)(\partial^\alpha\sigma) - \frac{1}{2}m_\pi^2(\pi^0)^2\big) \tag{1.52c}$$

Following **CITE WEINBERG COSMOLOGY** the energy momentum tensor of a real scalar field φ is

$$T^{\mu\nu}_{\varphi} = -g^{\mu\nu}\big[\frac{1}{2}g^{\rho\sigma}(\partial_\rho\varphi)(\partial_\sigma\varphi) + V(\varphi)\big] + g^{\mu\rho}g^{\nu\sigma}(\partial_\rho\varphi)(\partial_\sigma\varphi) \tag{1.53a}$$

$$\epsilon = \frac{1}{2}g^{\rho\sigma}(\partial_\rho\varphi)(\partial_\sigma\varphi) + V(\varphi) \tag{1.53b}$$

$$p = \frac{1}{2}g^{\rho\sigma}(\partial_\rho\varphi)(\partial_\sigma\varphi) - V(\varphi) \tag{1.53c}$$

$$u^\mu = -\big[-g^{\rho\sigma}(\partial_\rho\varphi)(\partial_\sigma\varphi)\big]^{-1/2} g^{\mu\nu}\partial_\nu\varphi \tag{1.53d}$$

Matching to Fluid Variables for the Real Field π^0

The only availabe 4-vector in the fluid theory is u_μ. It is thus intuitive to try to identify the real-valued 4vector $\partial_\mu\pi^0 \sim u_\mu$. Taking the normalization $u_\mu u^\mu = -1$ into account, one finds

$$u_\mu = \frac{\partial_\mu\pi^0}{\chi}\,, \qquad 0 < \chi^2 := -(\partial_\mu\pi^0)(\partial^\mu\pi^0) \tag{1.54}$$

From the fluid theory, we try to match the energy density of the hypothetical superfluid

$$\epsilon_{s,\pi^0} = u_\mu u_\nu T^{\mu\nu}_{\pi^0} = \frac{(\partial_\nu\pi^0)(\partial_\mu\pi^0)}{\chi^2}\Big((\partial^\mu\pi^0)(\partial^\nu\pi^0) + g^{\mu\nu}\big(-\frac{1}{2}(\partial_\alpha\pi^0)(\partial^\alpha\pi^0) - \frac{1}{2}m_\pi^2(\pi^0)^2\big)\Big) \tag{1.55a}$$

$$= \chi^2 - \big(\frac{1}{2}\chi^2 - \frac{1}{2}m_\pi^2(\pi^0)^2\big) \tag{1.55b}$$

$$= \frac{m_\pi^2(\pi^0)^2 + \chi^2}{2} \tag{1.55c}$$

Imposing $\epsilon = \text{const.}$ on the freezout surface, which is parametrized by some angle α, yields

$$0 = \mathrm{d}\epsilon = m_\pi^2\pi^0\mathrm{d}\pi^0 + \chi\mathrm{d}\chi \tag{1.56a}$$

$$= m_\pi^2\pi^0(\partial_\mu\pi^0)\mathrm{d}^\mu s + \chi\mathrm{d}\chi \tag{1.56b}$$

$$= m_\pi^2\pi^0\chi u_\mu\mathrm{d}^\mu s + \chi\mathrm{d}\chi \tag{1.56c}$$

The solution π^0 on the freezout surface thus needs to fulfill the ODE

$$\mathrm{d}\pi^0 = \chi u_\mu \mathrm{d}^\mu s\,, \quad \mathrm{d}\chi = -m_\pi^2\pi^0 u_\mu\mathrm{d}^\mu s \tag{1.57}$$

with $\mathrm{d}^\mu s = (\partial x^\mu)/(\partial\alpha)\mathrm{d}\alpha$ the displacement vector on the freezeout surface. Initial conditions leave 1 degree of freedom, namely the ratio of kinetic energy $\epsilon_{\text{kin}} = (1/2)\chi^2$ and $\epsilon_{\text{pot}} = (1/2)m_\pi^2(\pi^0)^2$ at $\alpha = 0$. To be precise, choose $r \in [0,1]$ and set $\epsilon_{\text{pot}}\big|_{\alpha=0} = r\epsilon$ and $\epsilon_{\text{kin}}\big|_{\alpha=0} = (1-r)\epsilon$.

The equations of motion for the real Klein-Gordon field are

$$(-\Box + m_\pi^2)\pi^0 = 0 \tag{1.58}$$

Matching to Fluid Variables for the Complex Fields $\pi^\pm$

The $U(1)$-symmetry $\pi^\pm \mapsto e^{\pm i\alpha}\pi^\pm$ with infinitesimal transformation $\pi^\pm \mapsto (1 \pm i\delta\alpha)\pi^\pm$ generates a conserved Noether current

$$j^\mu \sim \frac{\delta\mathscr{L}}{\delta(\partial_\mu\pi^+)}\delta\pi^+ + \frac{\delta\mathscr{L}}{\delta(\partial_\mu\pi^-)}\delta\pi^- \tag{1.59a}$$

$$\sim \pi^-(\partial^\mu\pi^+) - \pi^+(\partial_\mu\pi^-) \tag{1.59b}$$

$$= \sqrt{n}\big((\partial_\mu\sqrt{n}) + i\sqrt{n}(\partial_\mu\theta)\big) - \sqrt{n}\big((\partial_\mu\sqrt{n}) - i\sqrt{n}(\partial_\mu\theta)\big) \tag{1.59c}$$

$$= n(\partial_\mu\theta) \tag{1.59d}$$

with the parametrization $\pi^\pm = \sqrt{n}e^{\pm i\theta}$. The most intuitive matching is now

$$u^\mu = \frac{\partial_\mu\theta}{\chi_\theta}\,, \qquad 0 < \chi_\theta^2 := -(\partial_\mu\theta)(\partial^\mu\theta) \tag{1.60}$$

leading to the energy density

$$\epsilon_{s,\pi^\pm} = u_\mu u_\nu T^{\mu\nu}_{\pi^\pm} = \frac{(\partial_\mu\theta)(\partial_\nu\theta)}{\chi_\theta^2}\Big(2(\partial^\mu\pi^+)(\partial^\nu\pi^-) + g^{\mu\nu}\big(-(\partial_\alpha\pi^+)(\partial^\alpha\pi^-) - m_\pi^2\pi^+\pi^-\big)\Big) \tag{1.61a}$$

$$= 2\frac{[(\partial_\mu\sqrt{n})(\partial^\mu\theta)]^2}{\chi_\theta^2} + 2n\chi_\theta^2 - \big(n\chi_\theta^2 - (\partial_\mu\sqrt{n})(\partial^\mu\sqrt{n}) - m_\pi^2 n\big) \tag{1.61b}$$

$$= n\chi_\theta^2 + 2\frac{[(\partial_\mu\sqrt{n})(\partial^\mu\theta)]^2}{\chi_\theta^2} + (\partial_\mu\sqrt{n})(\partial^\mu\sqrt{n}) + m_\pi^2 n \tag{1.61c}$$

where the intermediate calculation

$$(\partial^\mu\pi^+)(\partial^\nu\pi^-) = \big((\partial^\mu\sqrt{n}) + i\sqrt{n}(\partial^\mu\theta)\big)\big((\partial^\nu\sqrt{n}) - i\sqrt{n}(\partial^\nu\theta)\big) \tag{1.62a}$$

$$= (\partial^\mu\sqrt{n})(\partial^\nu\sqrt{n}) + n(\partial^\mu\theta)(\partial^\nu\theta) + i\big(\sqrt{n}(\partial^\mu\theta)(\partial^\nu\sqrt{n}) - \sqrt{n}(\partial^\nu\theta)(\partial^\mu\sqrt{n})\big) \tag{1.62b}$$

$$(\partial_\mu\pi^+)(\partial^\mu\pi^-) = -\chi_n^2 - n\chi_\theta^2 \tag{1.62c}$$

is useful. Note how the imaginary part of this tensor is antisymmetric and thus does not contribute upon contraction with a symmetric tensor. Assume further $\partial_\mu\sqrt{n} = u_\mu\chi_n$ then

$$\epsilon_{s,\pi^\pm} = n\chi_\theta^2 + \chi_n^2 + m_\pi^2 n \tag{1.63}$$

Expressing the Lagrangian in terms of (n,θ)

$$\mathscr{L}_{\pi^\pm} = -(\partial_\mu\sqrt{n})(\partial^\mu\sqrt{n}) - n(\partial_\mu\theta)(\partial^\mu\theta) - nm_\pi^2 \tag{1.64}$$

yields as the corresponding equations of motion

$$\partial_\mu\left(\frac{\partial\mathscr{L}_{\pi^\pm}}{\partial(\partial_\mu\sqrt{n})}\right) = \frac{\partial\mathscr{L}_{\pi^\pm}}{\partial\sqrt{n}}: \qquad -2\Box\sqrt{n} = -2\sqrt{n}\big((\partial_\mu\theta)(\partial^\mu\theta) + m_\pi^2\big) \tag{1.65a}$$

$$\partial_\mu\left(\frac{\partial\mathscr{L}_{\pi^\pm}}{\partial(\partial_\mu\theta)}\right) = \frac{\partial\mathscr{L}_{\pi^\pm}}{\partial\theta}: \qquad \partial_\mu\big(-2n(\partial^\mu\theta)\big) = 0 \tag{1.65b}$$

the second of which encodes the conservation law for the $U(1)$-Noether current.

The easiest (and most naive) solution is again $n = \text{const.}$ and $\partial_\mu\theta = p_\mu$ with $p_\mu p^\mu = -m_\pi^2$. This would be a solution with only a single momentum mode. This solution implies $u^\mu = \text{const.}$ which is generally not satisfied by the given data. Assume therefore the existence of a small perturbation $\partial_\mu\theta = p_\mu + \delta q_\mu$ with $q_\mu p^\mu = 0$. From this, $\chi^2 = -(\partial_\mu\theta)(\partial^\mu\theta) = -p_\mu p^\mu - q_\mu q^\mu = m_\pi^2 - \delta^2 q^2$. To linear order in δ

$$\partial_\mu\theta = \chi u_\mu = m_\pi u_\mu \tag{1.66}$$

holds true. To expand the equations of motion and allow for non-constant amplitude, assume $n = n_{(0)} + \delta n_{(1)}(x)$ with $n_{(0)} = \text{const.}$ ($\sqrt{n} \approx \sqrt{n_{(0)}} + \delta\cdot n_{(1)}/(2\sqrt{n_{(0)}})$).

$$\Box\sqrt{n_{(1)}} = 0 + \mathcal{O}(\delta^2) \tag{1.67a}$$

$$0 = n_{(0)}\partial_\mu q^\mu + q^\mu\partial_\mu n_{(1)} \tag{1.67b}$$

1.4 Evaluation on the Freezeout Surface

1.4.1 Invariance of Fourier Transform w.r.t. Deformations of the Hypersurface

Let ϕ_1, ϕ_2 be fields of equal mass evolving according to the KG equation. Then the current

$$J_\mu[\phi_1,\phi_2] = -i(\phi_1\partial_\mu\phi_2^* - (\partial_\mu\phi_1)\phi_2^*) =: -i\phi_1\overleftrightarrow{\partial}_\mu\phi_2^* \tag{1.68}$$

with the antisymmetrized two-sided derivative $\overleftrightarrow{\partial}_\mu = \overrightarrow{\partial}_\mu - \overleftarrow{\partial}_\mu$ is conserved. Recall Gauß law

$$\int_\Omega \mathrm{d}\Omega\,\nabla^\mu J_\mu = \int_{\partial\Omega}\mathrm{d}\sigma^\mu\,J_\mu \tag{1.69}$$

with $\mathrm{d}\sigma_\mu$ the outwards oriented surface normal of the spacetime volume Ω. The bilinear form

$$(\phi_1, \phi_2)_\Sigma = \int_\Sigma \mathrm{d}\Sigma^\mu \, J_\mu[\phi_1, \phi_2] = -i \int_\Sigma \mathrm{d}\Sigma^\mu \, \phi_1 \overleftrightarrow{\partial}_\mu \phi_2^* \tag{1.70}$$

is therefore independent of the choice of (Cauchy) hypersurface Σ (if $\partial\Sigma$ is changed, one must carefully check for further contributions in Gauß law).

Let

$$u_{\vec{p}}(t, \vec{x}) = \exp(-i(\omega_{\vec{p}} t - \vec{p}\vec{x})) \,, \qquad u^*_{\vec{p}}(t, \vec{x}) = \exp(i(\omega_{\vec{p}} t - \vec{p}\vec{x})) \tag{1.71}$$

be the positive and negative frequency eigensolutions to the free Klein-Gordon equation. They form an orthogonal system with respect to the inner product defined above,

$$(u_{\vec{p}}, u_{\vec{q}})_{\Sigma_t} = (2\omega_{\vec{p}})(2\pi)^3 \delta^{(3)}(\vec{p} - \vec{q}) \,, \qquad (u^*_{\vec{p}}, u^*_{\vec{q}})_{\Sigma_t} = -(2\omega_{\vec{p}})(2\pi)^3 \delta^{(3)}(\vec{p} - \vec{q}) \,, \qquad (u_{\vec{p}}, u^*_{\vec{q}})_{\Sigma_t} = 0 \tag{1.72}$$

with the relations stated here on a a hypersurface Σ_t where $t = \text{const.}$. This means that the Fourier coefficients, or equivalently annihilation and creation operators after quantization, for example in equation (1.10a), can be extraced via

$$\sqrt{2\omega_{\vec{p}}} a_{\vec{p}} = (\phi, u_{\vec{p}})_{\Sigma_t} \,, \qquad \sqrt{2\omega_{\vec{p}}} a^\dagger_{\vec{p}} = -(\phi, u^*_{\vec{p}})_{\Sigma_t} \tag{1.73}$$

This leads of course to the same statement as in equations (1.9).

The equations in (1.47) are also precisely of this form, namely

$$J(p) = -\int_{\Sigma_t} \mathrm{d}\Sigma^\mu \, \phi_J \overleftrightarrow{\partial}_\mu u^*_{\vec{p}} = (\phi_J, u_{\vec{p}})_{\Sigma_t} \tag{1.74a}$$

$$J(-p) = -\int_{\Sigma_t} \mathrm{d}\Sigma^\mu \, \phi_J \overleftrightarrow{\partial}_\mu u_{\vec{p}} = (\phi_J, u^*_{\vec{p}})_{\Sigma_t} \tag{1.74b}$$

We finally wish to transform the hypersurface to evaluate the inner product on, and evaluate instead on the Freezeout surface. Assuming that the condensate has no contributions at large rapidities $\eta \to \pm\infty$, one can deform the hypersurface Σ_t at large lab time $t = \text{const.}$ into a hypersurface of large Bjorken time $\Sigma_{\tau \gg \tau_L}$ at a τ much larger then the lifetime τ_L of the fireball. Finally, transform $\Sigma_{\tau \gg \tau_L}$ to the freezeout surface.

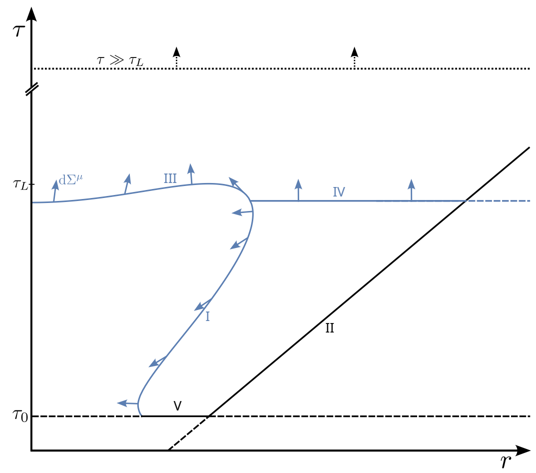


Figure 1.1: Freezeout surface in τ-r-plane[KGF23].

Consider the freezeout on the hypersurface depicted in 1.1. Assume that the condensate contribution as a function in phase space $f_{\text{cond}}(x^\mu, \vec{p})$ vanishes on Σ_{II} and Σ_{V}, i.e. is contained within the union of all light cones starting on the freeze out surface $\Sigma_{FO} \equiv \Sigma_{\text{I}} \cap \Sigma_{\text{III}}$. **To do (1) By causality this seems reasonable, but from Fourier decomposition of a classical field this is not at all clear.** Following the reasoning from [KGF23], we wish to apply Gauß law. Consider separately the contribution on the τ-axis

$$\int_{\Sigma_{r=0}} \mathrm{d}\Sigma^\mu \, J_\mu \qquad \text{or} \qquad \lim_{r \to 0} \int_{\Sigma_r} \mathrm{d}\Sigma^\mu \, J_\mu$$

The surface vector on this hypersurface is $\mathrm{d}\Sigma_\mu = r\tau \mathrm{d}\tau \mathrm{d}\eta \mathrm{d}\varphi (0, 1, 0, 0)$ and thus vanishes at $r = 0$ (the hypersurface $\Sigma_{r=0}$ has zero 3-volume). Since the derivative of a rotationally symmetric integrand introduces no divergencies, the contribution of $\Sigma_{r=0}$ to Gauß law is zero.

$$(\phi_J, u^{(*)}_{\vec{p}})_{\Sigma_t} = (\phi, u^{(*)}_{\vec{p}})_{\Sigma_{\tau \gg \tau_L}} = (\phi, u^{(*)}_{\vec{p}})_{\Sigma_{FO}} \tag{1.75}$$

1.4.2 Coordinates on the Freezeout Surface

The freezeout hypersurface is parametrized as $\Sigma_{\mathrm{FO}} = \{x^\mu \in \mathbb{R}^{(1,3)} | (\tau, r) = (\tau(\alpha), r(\alpha))\}$ with τ, r defined by the coordinate transformation

$$\begin{cases} t = \tau \cosh \eta \\ z = \tau \sinh \eta \\ x = r \cos \varphi \\ y = r \sin \varphi \end{cases} \quad \Longleftrightarrow \quad \begin{cases} \tau = \sqrt{t^2 - z^2} \\ \eta = \operatorname{artanh}(z/t) \\ r = \sqrt{x^2 + y^2} \\ \varphi = \arctan(y/x) \end{cases} \tag{1.76}$$

Calculation 1.4|1: Metric on Hypersurface

Recall the metric $g_{\mu\nu} = \mathrm{diag}(-1, 1, \tau^2, r^2)$ in coordinates (τ, r, η, φ). Orthonormal tangent vectors to the freeze out hypersurface are $(\hat{\partial}_\varphi)^\mu = (0, 0, 0, r^{-1}) = r^{-1}(\partial_\varphi)^\mu$, $(\hat{\partial}_\eta)^\mu = (0, 0, \tau^{-1}, 0) = \tau^{-1}(\partial_\eta)^\mu$ and $(\hat{\partial}_\alpha)^\mu = \sqrt{r'^2(\alpha) - \tau'^2(\alpha)}^{\,-1}(\tau'(\alpha), r'(\alpha), 0, 0) = D(\alpha)(\partial_\alpha)^\mu$ with $D(\alpha) = \sqrt{r'^2(\alpha) - \tau'^2(\alpha)}^{\,-1}$. The projector on the hypersurface is

$$\gamma_{\mu\nu} = (\hat{\partial}_\varphi)_\mu(\hat{\partial}_\varphi)_\nu + (\hat{\partial}_\eta)_\mu(\hat{\partial}_\eta)_\nu + (\hat{\partial}_\alpha)_\mu(\hat{\partial}_\alpha)_\nu = \begin{pmatrix} D^2(\alpha)\tau'^2(\alpha) & -D^2(\alpha)\tau'(\alpha)r'(\alpha) & 0 & 0 \\ -D^2(\alpha)\tau'(\alpha)r'(\alpha) & D^2(\alpha)r'^2(\alpha) & 0 & 0 \\ 0 & 0 & \tau^2 & 0 \\ 0 & 0 & 0 & r^2 \end{pmatrix} \tag{1.77}$$

The normal of the hypersurface is $n^\mu \equiv (\hat{\partial}_\alpha^\perp)^\mu = D(\alpha)(r'(\alpha), \tau'(\alpha), 0, 0)$ and is timelike where D is real. Naturally $\gamma_{\mu\nu}n^\nu = 0$. In the basis $(\partial_\alpha, \partial_\eta, \partial_\varphi, n)$ using (in short form)

$$(\partial_\alpha)^\nu \gamma_{\mu\nu} (\partial_\alpha)^\mu = \begin{pmatrix} \tau' \\ r' \end{pmatrix}^T \begin{pmatrix} -\tau' \\ r' \end{pmatrix} = D^{-2} \tag{1.78}$$

the hypersurface metric in coordinates $x^i = (\alpha, \eta, \varphi)$ reads

$$\gamma_{ij} = \mathrm{diag}(D^{-2}(\alpha), \tau^2(\alpha), r^2(\alpha)) \tag{1.79}$$

and the volume element is given by $\mathrm{d}\Sigma = r(\alpha)\tau(\alpha)D^{-1}(\alpha)\mathrm{d}\alpha\mathrm{d}\eta\mathrm{d}\varphi$. The oriented surface element is

$$\mathrm{d}\Sigma^\mu = n^\mu \mathrm{d}\Sigma = r(\alpha)\tau(\alpha)(r'(\alpha), \tau'(\alpha), 0, 0)\mathrm{d}\alpha\mathrm{d}\eta\mathrm{d}\varphi \tag{1.80}$$

It is also useful to evaluate $p_\mu x^\mu$ in the Bjorken coordinate system. Therefore introduce an analogous coordinate change in momentum space

$$\begin{cases} p_t = m_\perp \cosh \eta_p \\ p_z = m_\perp \sinh \eta_p \\ p_x = p_\perp \cos \varphi_p \\ p_y = p_\perp \sin \varphi_p \end{cases} \tag{1.81}$$

to rewrite the scalar product as

$$p_\mu x^\mu \equiv -\tau(p_t \cosh \eta - p_z \sinh \eta) + r(p_x \cos \varphi + p_y \sin \varphi) = -\tau m_\perp \cosh(\eta - \eta_p) + r p_\perp \cos(\varphi - \varphi_p) \tag{1.82}$$

We used the identities

$$\cosh(a - b) = \cosh a \cosh b - \sinh a \sinh b\,, \qquad \cos(a - b) = \cos a \cos b + \sin a \sin b \tag{1.83}$$

The integral measure changes according to $\mathrm{d}^4 p_{\mathrm{cart}} = \mathrm{d}m_\perp \mathrm{d}p_\perp \mathrm{d}\eta_p \mathrm{d}\varphi_p \cdot m_\perp p_\perp$. The momentum shell condition $p^2 + m^2 = 0$ is equivalently parametrized by $m_\perp^2 = p_\perp^2 + m^2 =: \omega_\perp^2$.

1.4.3 Computing the Inner Product

The projection of the derivative onto the surface normal of Σ_{FO} is (omiiting the α-dependence)

$$\mathrm{d}\Sigma^\mu \partial_\mu = (r'\partial_\tau + \tau'\partial_r) \cdot r\tau\, \mathrm{d}\alpha\mathrm{d}\eta\mathrm{d}\varphi \tag{1.84}$$

It is useful to compute the following integrals related to Bessel functions: `https://dlmf.nist.gov/10.9`

$$\int_0^{2\pi} \mathrm{d}\varphi e^{\pm ia\cos\varphi} = \int_0^{2\pi} \big(\cos(a\cos\varphi) \pm i\sin(a\cos\varphi)\big) = 2\int_0^{\pi} \cos(a\cos\varphi) \tag{1.85a}$$

$$= 2\pi J_0(a) \tag{1.85b}$$

$$\int_{-\infty}^{\infty} \mathrm{d}\eta e^{\pm ia\cosh\eta} = 2\int_0^{\infty} \mathrm{d}\eta\big(\cos(a\cosh\eta) \pm i\sin(a\cosh\eta)\big) \tag{1.85c}$$

$$= \pi\big(-Y_0(a) \pm iJ_0(a)\big) \tag{1.85d}$$

$$= \pm\pi i(J_0(a) \pm iY_0(a)) = \begin{cases} +\pi i H_0^{(1)}(a) & \text{for "+"} \\ -\pi i H_0^{(2)}(a) & \text{for "-"} \end{cases} \tag{1.85e}$$

Additionally to the integral representations, the following computation makes use of `https://dlmf.nist.gov/10.4`

$$J_0'(x) = -J_1(x)\,, \qquad Y_0'(x) = -Y_1(x) \tag{1.85f}$$

$$J(\pm p) = -\int_{-\infty}^{\infty}\mathrm{d}\eta\int_0^{2\pi}\mathrm{d}\varphi\int_0^{\pi}\mathrm{d}\alpha\tau r\Big[\phi(\tau,r)\big(r'\overleftrightarrow{\partial}_\tau + \tau'\overleftrightarrow{\partial}_r\big)e^{\pm i(\tau\omega_\perp\cosh(\eta-\eta_p) - rp_\perp\cos(\varphi-\varphi_p))}\Big] \tag{1.86a}$$

$$= -\int_{-\infty}^{\infty}\mathrm{d}\eta\int_0^{2\pi}\mathrm{d}\varphi\int_0^{\pi}\mathrm{d}\alpha\tau r\Big[\phi(\tau,r)\big(r'\overleftrightarrow{\partial}_\tau + \tau'\overleftrightarrow{\partial}_r\big)e^{\pm i(\tau\omega_\perp\cosh\eta - rp_\perp\cos\varphi)}\Big] \tag{1.86b}$$

$$= -2\pi^2\int_0^{\pi}\mathrm{d}\alpha\tau r\Big[\phi(\tau,r)(r'\overleftrightarrow{\partial}_\tau + \tau'\overleftrightarrow{\partial}_r)\Big[J_0(rp_\perp)\times\big(-Y_0(\tau\omega_\perp) \pm iJ_0(\tau\omega_\perp)\big)\Big]\Big] \tag{1.86c}$$

$$\begin{aligned} &= 2\pi^2\int_0^{\pi}\mathrm{d}\alpha\tau r\Big[(r'\partial_\tau + \tau'\partial_r)\phi(\tau,r)\Big[J_0(rp_\perp)\times\big(-Y_0(\tau\omega_\perp) \pm iJ_0(\tau\omega_\perp)\big)\Big] + \\ &\quad + \phi(\tau,r)\Big[\tau'\times p_\perp J_1(rp_\perp)\times\big(-Y_0(\tau\omega_\perp) \pm iJ_0(\tau\omega_\perp)\big) + \\ &\qquad + r'\times J_0(rp_\perp)\times\omega_\perp\big(-Y_1(\tau\omega_\perp) \pm iJ_1(\tau\omega_\perp)\big)\Big]\Big] \end{aligned} \tag{1.86d}$$

$$J(\overset{(-)}{+}p) = -\int_{-\infty}^{\infty}\mathrm{d}\eta\int_0^{2\pi}\mathrm{d}\varphi\int_0^{\pi}\mathrm{d}\alpha\tau r\Big[\phi(\tau,r)\big(r'\overleftrightarrow{\partial}_\tau + \tau'\overleftrightarrow{\partial}_r\big)e^{\overset{(-)}{+}i(\tau\omega_\perp\cosh(\eta-\eta_p) - rp_\perp\cos(\varphi-\varphi_p))}\Big] \tag{1.87a}$$

$$= -\int_{-\infty}^{\infty}\mathrm{d}\eta\int_0^{2\pi}\mathrm{d}\varphi\int_0^{\pi}\mathrm{d}\alpha\tau r\Big[\phi(\tau,r)\big(r'\overleftrightarrow{\partial}_\tau + \tau'\overleftrightarrow{\partial}_r\big)e^{\overset{(-)}{+}i(\tau\omega_\perp\cosh\eta - rp_\perp\cos\varphi)}\Big] \tag{1.87b}$$

$$= \overset{(+)}{-}2\pi^2 i\int_0^{\pi}\mathrm{d}\alpha\tau r\Big[\phi(\tau,r)(r'\overleftrightarrow{\partial}_\tau + \tau'\overleftrightarrow{\partial}_r)\Big[J_0(rp_\perp)\times H_0^{\overset{(2)}{(1)}}(\tau\omega_\perp)\Big]\Big] \tag{1.87c}$$

$$\begin{aligned} &= \overset{(-)}{+}2\pi^2 i\int_0^{\pi}\mathrm{d}\alpha\tau r\Big[(r'\partial_\tau + \tau'\partial_r)\phi(\tau,r)\Big[J_0(rp_\perp)\times H_0^{\overset{(2)}{(1)}}(\tau\omega_\perp)\Big] + \\ &\quad + \phi(\tau,r)\Big[\tau'\times p_\perp J_1(rp_\perp)\times H_0^{\overset{(2)}{(1)}}(\tau\omega_\perp) + r'\times J_0(rp_\perp)\times\omega_\perp H_1^{\overset{(2)}{(1)}}(\tau\omega_\perp)\Big]\Big] \end{aligned} \tag{1.87d}$$

To do...

☐ 1 (p. 12): By causality this seems reasonable, but from Fourier decomposition of a classical field this is not at all clear.

Bibliography

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