so'n Feuerball

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Contents

Introduction

Have a look at this [KG07]. See [BRS18] for a review.

Chapter 1

Background

1.1 Thermodynamic Basics [Flo16]

Take the variation of entropy $S = S(E, N, V)$ in the microcanonical ensemble

$$\mathrm{d}S = \frac{1}{T}\mathrm{d}E - \frac{\mu}{T}\mathrm{d}N + \frac{p}{T}\mathrm{d}V \tag{1.1}$$

as a definition of temperature, chemical potential and pressure

$$\frac{1}{T} = \frac{\partial S}{\partial E}\Big|_{N,V}, \qquad \mu = -T\frac{\partial S}{\partial N}\Big|_{E,V}, \qquad p = T\frac{\partial S}{\partial V}\Big|_{E,N} \tag{1.2}$$

One also uses inverse temperature $\beta = \frac{1}{T}$ and thermal potential $\alpha = \frac{\mu}{T}$. For homogeneous fluids it is sensible to define densities for energy, particle number and entropy via $E = \epsilon V$, $N = nV$ and $S = sV$. Plugging into the differential (1.1) gives the **Gibbs-Duhem relation**

$$\epsilon + p = Ts + \mu n \tag{1.3}$$

and

$$\mathrm{d}s = \frac{1}{T}\mathrm{d}\epsilon - \frac{\mu}{T}\mathrm{d}n \tag{1.4}$$

(In [Ris22] (1.3) multiplied with V is called **Euler relation**.) Taking the differential of (1.3) and using (1.4) gives the **differential Gibbs-Duhem relation**

$$\mathrm{d}p = s\mathrm{d}T + n\mathrm{d}\mu \tag{1.5}$$

Legendre Transformations

The function $s(\epsilon, n)$ can be inverted to state a function $\epsilon(s, n)$ that follows

$$\mathrm{d}\epsilon = T\mathrm{d}s + \mu\mathrm{d}n \tag{1.6}$$

Performing the Legendre transform of $\epsilon(s, n)$ w.r.t. s gives the free energy density

$$f = f(T, n) = \epsilon - Ts \tag{1.7a}$$
$$\mathrm{d}f = -s\mathrm{d}T + \mu\mathrm{d}n \tag{1.7b}$$

and another Legendre transform w.r.t. n leads to

$$-p = -p(T, \mu) = f - \mu n \tag{1.8a}$$
$$-\mathrm{d}p = -s\mathrm{d}T - n\mathrm{d}\mu \tag{1.8b}$$

hence the pressure is just another thermodynamic potential.

1.2 Fluid Dynamic Approach

The theory of fluid dynamics aims to describe a system of many interacting particles as a fluid, with its state specified by a few spacetime dependent variables, such as temperature $T(x)$ and chemical potential $\mu(x)$, and an equation of state. Dynamics are generated by equations of motion involving the energy-momentum tensor $T^{\mu\nu}(x)$.

$$\begin{aligned} &T^{00} \ldots \text{local energy density} \\ &T^{i0} \ldots \text{local momentum density} \\ &T^{\mu j} \ldots \text{flux of } \mu\text{-component in direction } j \end{aligned} \tag{1.9}$$

To do (1) **Why is $T^{\mu\nu}$ symmetric?** To do (2) **Why does this coincide with the definition $T^{\mu\nu}$ from a Lagrangian?** In a local rest frame the energy-momentum tensor should have the form of static equilibrium. There should be further no flow of particle number or entropy, specifying the form of the particle and entropy current N^μ and S^μ. In a rest frame one finds

$$T^{\mu\nu}_{RF} = \begin{pmatrix} \epsilon & 0 & 0 & 0 \\ 0 & P & 0 & 0 \\ 0 & 0 & P & 0 \\ 0 & 0 & 0 & P \end{pmatrix} \tag{1.10a}$$

$$N^\mu_{RF} = (n, 0, 0, 0)^T \tag{1.10b}$$

$$S^\mu_{RF} = (s, 0, 0, 0)^T \tag{1.10c}$$

with energy density ϵ and pressure P. To do (3) **Check why kinetic and thermodynamic pressure coincide.**

In a general frame these quantitites are obtained by applying a Lorentz boost to (1.10) and read [Ris22]

$$T^{\mu\nu} = -Pg^{\mu\nu} + (P+\epsilon)u^\mu u^\nu = \epsilon u^\mu u^\nu - \Delta^{\mu\nu} P \tag{1.11a}$$

$$N^\mu = nu^\mu \tag{1.11b}$$

$$S^\mu = su^\mu \tag{1.11c}$$

where $u^\mu = (\gamma, \gamma \boldsymbol{v})$ is the local fluid 4-velocity and $\Delta^{\mu\nu} = g^{\mu\nu} - u^\mu u^\nu$ is the projector orthogonal to u^μ. It fulfills

$$\Delta^{\mu\nu} = \Delta^{\nu\mu}\,, \qquad u_\mu \Delta^{\mu\nu} = 0\,, \qquad \Delta^\mu_\lambda \Delta^{\lambda\nu} = \Delta^{\mu\nu} \qquad \Delta^\mu_\mu = 3 = d-1 \tag{1.12}$$

Signature is chosen such that $u_\mu u^\mu = 1$.

Important 1.2|1: Signature Convention

$$\eta^{\mu\nu} = \mathrm{diag}(+,-,-,-)$$

Local energy-momentum conservation and particle number conservation are encoded by the continuity equations

$$\partial_\mu T^{\mu\nu} = 0 \tag{1.13a}$$

$$\partial_\mu N^\mu = 0 \tag{1.13b}$$

which constitutes 5 scalar equations for 6 unkwon functions: $\epsilon(x), P(x), n(x), \boldsymbol{v}(x)$. The equation of state $P = P(\epsilon, n)$ closes the system.

Calculation 1.2|2: Symmetry of $T^{\mu\nu}$

The angular momentum tensor is defined as

$$J^{\lambda,\mu\nu} = x^\mu T^{\lambda\nu} - x^\nu T^{\lambda\mu} \tag{1.14}$$

Angular momentum conservation gives

$$
\begin{aligned}
0 &= \partial_\lambda J^{\lambda,\mu\nu} \\
&= T^{\mu\nu} - T^{\nu\mu} + \underbrace{x^\mu \partial_\lambda T^{\lambda\nu} - x^\nu \partial_\lambda T^{\lambda\mu}}_{=0} \\
&= T^{\mu\nu} - T^{\nu\mu}
\end{aligned}
$$

where conservation of energy-momentum (1.13a) was used.

1.2.1 Covariant Thermodynamics and Entropy Production

The goal is to rewrite the thermodynamic equilibrium equations in a covariant form. Introduce

$$\beta^\mu = \frac{u^\mu}{T} \tag{1.15}$$

and postulate

$$
\begin{aligned}
\mathrm{d}(P\beta^\mu) &= N^\mu \mathrm{d}\alpha - T^{\mu\nu}\mathrm{d}\beta_\nu && \text{(1.16a)} \\
S^\mu &= P\beta^\mu + T^{\mu\nu}\beta_\nu - \alpha N^\mu && \text{(1.16b)}
\end{aligned}
$$

which immediately implies

$$\mathrm{d}S^\mu = \beta_\nu \mathrm{d}T^{\mu\nu} - \alpha \mathrm{d}N^\mu \tag{1.16c}$$

Upon contraction with u_μ equations (1.16) yield again (1.3), (1.4) and (1.5).

Equation (1.16c) furher implies **To do** (4) **Why exactly can we replace the d with a ∂_μ?**

$$\partial_\mu S^\mu = \beta_\nu \partial_\mu T^{\mu\nu} - \alpha \partial_\mu N^\mu \tag{1.17}$$

Assuming conservation of energy-momentum and particle number leads to conservation of entropy,

$$\partial_\mu S^\mu = 0 \tag{1.18}$$

which holds in thermal equilibrium.

It is also possible to postulate conservation of energy-momentum and particle number and entropy in thermal equilibrium and infer the differential form (1.16c).

1.2.2 Derivative Expansion of hydrodynamics [Kov12]

Given some vector $u^\mu(x)$, any spacetime tensor can be decomposed as

$$
\begin{aligned}
T^{\mu\nu} &= \mathcal{E}u^\mu u^\nu + \mathcal{P}\Delta^{\mu\nu} + (q^\mu u^\nu + q^\nu u^\mu) + t^{\mu\nu} && \text{(1.19a)} \\
J^\mu &= \mathcal{N}u^\mu + j^\mu && \text{(1.19b)}
\end{aligned}
$$

with the coefficients given by

$$
\begin{aligned}
\mathcal{E} = u_\mu u_\nu T^{\mu\nu}\,, \quad \mathcal{P} &= \frac{1}{d}\Delta_{\mu\nu}T^{\mu\nu}\,, \quad \mathcal{N} = u_\mu J^\mu && \text{(1.20a)} \\
q_\mu = \Delta_{\mu\alpha}u_\beta T^{\alpha\beta}\,, &\quad j_\mu = \Delta_{\mu\nu}J^\nu && \text{(1.20b)} \\
t_{\mu\nu} = \frac{1}{2}\Big(\Delta_{\mu\alpha}\Delta_{\nu\beta} + \Delta_{\nu\alpha}\Delta_{\mu\beta} &- \frac{2}{d-1}\Delta_{\mu\nu}\Delta_{\alpha\beta}\Big)T^{\alpha\beta} && \text{(1.20c)}
\end{aligned}
$$

Calculation 1.2|3: Decomposition of Rank-1 and Rank-2 Spacetime Tensors

Use $u_\mu u^\mu = 1$ as well as $\Delta^{\mu\nu} + u^\mu u^\nu = g^{\mu\nu}$ (or $\Delta^\mu_\nu + u^\mu u_\nu = \delta^\mu_\nu$) and the projector properties (1.12).

$$\begin{aligned}
J^\mu &= \delta^\mu_\nu J^\nu \\
&= (\Delta^\mu_\nu + u^\mu u_\nu) J^\nu \\
&= j^\mu + \mathcal{N} u^\mu \\
T^{\mu\nu} &= \delta^\mu_\alpha \delta^\nu_\beta T^{\alpha\beta} \\
&= (\Delta^\mu_\alpha + u^\mu u_\alpha)(\Delta^\nu_\beta + u^\nu u_\beta) T^{\alpha\beta} \\
&= (\Delta^\mu_\alpha \Delta^\nu_\beta + \Delta^\nu_\beta u^\mu u_\alpha + \Delta^\mu_\alpha u^\nu u_\beta + u^\mu u^\nu u_\alpha u_\beta) T^{\alpha\beta} \\
&= (\Delta^\mu_\alpha \Delta^\nu_\beta - \frac{1}{d-1}\Delta^{\mu\nu}\Delta_{\alpha\beta} + \frac{1}{d-1}\Delta^{\mu\nu}\Delta_{\alpha\beta} + \Delta^\nu_\beta u^\mu u_\alpha + \Delta^\mu_\alpha u^\nu u_\beta + u^\mu u^\nu u_\alpha u_\beta) T^{\alpha\beta}
\end{aligned}$$

. . . using symmetry of $T^{\alpha\beta}$ . . .

$$\begin{aligned}
&= \frac{1}{2}\Big(\Delta^\mu_\alpha \Delta^\nu_\beta + \Delta^\nu_\alpha \Delta^\mu_\beta - \frac{2}{d-1}\Delta^{\mu\nu}\Delta_{\alpha\beta}\Big) T^{\alpha\beta} + \frac{1}{d-1}\Delta^{\mu\nu}\Delta_{\alpha\beta} T^{\alpha\beta} + (q^\nu u^\mu + q^\mu u^\nu) + \mathcal{E} u^\mu u^\nu \\
&= t^{\mu\nu} + \mathcal{P}\Delta^{\mu\nu} + (q^\nu u^\mu + q^\mu u^\nu) + \mathcal{E} u^\mu u^\nu
\end{aligned}$$

Show $t^{\mu\nu}$ is constructed to be transverse $t^{\mu\nu}u_\nu = 0$ and traceless:

$$\Delta^{\mu\nu} t_{\mu\nu} = \frac{1}{2}\Big(\underbrace{\Delta^\nu_\alpha \Delta_{\nu\beta} + \Delta_{\nu\alpha}\Delta^\nu_\beta}_{=2\Delta_{\alpha\beta}} - \frac{2}{d-1}(d-1)\Delta_{\alpha\beta}\Big) T^{\alpha\beta} = 0$$

The object

$$\Delta^{\mu\nu}_{\alpha\beta} = \frac{1}{2}\Big(\Delta_{\mu\alpha}\Delta_{\nu\beta} + \Delta_{\nu\alpha}\Delta_{\mu\beta} - \frac{2}{d-1}\Delta_{\mu\nu}\Delta_{\alpha\beta}\Big) = \frac{1}{2}\Big(\Delta_{\mu\alpha}\Delta_{\nu\beta} + \Delta_{\nu\alpha}\Delta_{\mu\beta} - \frac{2}{\Delta^\lambda_\lambda}\Delta_{\mu\nu}\Delta_{\alpha\beta}\Big) \tag{1.21}$$

has some properties **To do** (5)**Maybe list the properties here.**

0th Order Hydrodynamics

Ideal hydrodynamics corresponds to the 0th order in a derivative expansion of the (scalar) hydrodynamic variables **To do** (6)**There only scalar hydrodynamic variables, except u^μ, right?** . The tensors q^μ, $t^{\mu\nu}$ can only contain derivatives and hence drop out to lowest order. The 0th order coefficients are identified with the energy density, pressure and particle density of an ideal fluid in thermodynamic equilbrium in a rest frame and promoted to slowly varying fields. One finds

$$T^{\mu\nu}_{(0)} = \epsilon u^\mu u^\nu - p\Delta^{\mu\nu} = (\epsilon + p) u^\mu u^\nu - p g^{\mu\nu} \tag{1.22a}$$

$$N^\mu_{(0)} = n u^\mu \tag{1.22b}$$

Given an equation of state $p(T,\mu)$ entropy and particle number density is found using (1.5) and subsequently energy density via (1.3). Enthalpy density is given by

$$w = \epsilon + p \tag{1.23}$$

The (longitudinal components of) energy-momentum conservation and particle number conservation read

$$0 = u_\nu \partial_\mu T^{\mu\nu} = u^\mu \partial_\mu \epsilon + \epsilon \partial_\mu u^\mu + p \Delta^{\mu\nu}\partial_\mu u_\nu = \partial_\mu(\epsilon u^\mu) + p\partial_\mu u^\mu = \partial_\mu(w u^\mu) - u^\mu \partial_\mu p \tag{1.24a}$$

$$0 = \partial_\mu(n u^\mu) \tag{1.24b}$$

where $0 = u^\nu \partial_\mu u_\nu$ or equivalently $\partial_\mu u_\lambda = \Delta^\nu_\lambda \partial_\mu u_\nu$ as well as $\Delta^{\mu\nu}u_\nu = 0$ which implies $\Delta^{\mu\nu}\partial_\lambda u_\nu = -u_\nu \partial_\lambda \Delta^{\mu\nu}$ was used. Using $w = sT + \mu n$ one can compute

$$\begin{aligned}
\partial_\mu(w u^\mu) &= T\partial_\mu(s u^\mu) + s u^\mu \partial_\mu T + \mu \partial_\mu (n u^\mu) + n u^\mu \partial_\mu \mu \\
&= T\partial_\mu(s u^\mu) + \mu\partial_\mu(n u^\mu) + u^\mu\Big(\frac{\partial p}{\partial T}\partial_\mu T + \frac{\partial p}{\partial \mu}\partial_\mu \mu\Big) \\
&= T\partial_\mu(s u^\mu) + \mu \cdot 0 + u^\mu \partial_\mu p \\
&\overset{!}{=} u^\mu \partial_\mu p \quad \implies \quad \partial_\mu(s u^\mu) = 0
\end{aligned}$$

to show conservation of the entropy current.

1st Order Hydrodynamics

Including first derivatives the coefficients (1.20) take the form

$$\mathcal{E}(x) = \epsilon(x) + f_{\mathcal{E}}(\partial T, \partial\mu, \partial u^\mu) \tag{1.25a}$$
$$\mathcal{P}(x) = -p(x) + f_{\mathcal{P}}(\partial T, \partial\mu, \partial u^\mu) \tag{1.25b}$$
$$\mathcal{N}(x) = n(x) + f_{\mathcal{N}}(\partial T, \partial\mu, \partial u^\mu) \tag{1.25c}$$

Understanding T, μ, u^μ as auxiliary fields in the parametrization of $T^{\mu\nu}$, changing the parametrization To do (7)**Why exactly are the coefficients of $T^{\mu\nu}$ not exactly the energy density and pressure, but only up to derivatives of hydro variables? Does the expansion or the precise form of $f_{\mathcal{E}}\dots$ depend on the spacetime point x? I imagine it like this: In my system I choose a spacetime point x_0 and take the corresponding $T(x_0), p(x_0), u^\mu(x_0)$. If μ, T, u^μ where constant, boosting into the local frame with $u^\mu(x_0)$ enables me to read off ϵ_0, p_0, n_0.**

$$T(x) \to T'(x) = T(x) + \delta T(x) \tag{1.26a}$$
$$\mu(x) \to \mu'(x) = \mu(x) + \delta\mu(x) \tag{1.26b}$$
$$u^\mu(x) \to u^{\mu\prime}(x) = u^\mu(x) + \delta u^\mu(x) \tag{1.26c}$$

where $\delta(u^\mu u_\mu) = 0$ implies $u_\mu \delta u^\mu = 0$ and demanding invariance of $T^{\mu\nu}$ and J^μ changes the coefficients (1.20) to first order byTo do (8)**Do this explicitly, especially for $t^{\mu\nu}$ it is hard to see.**

$$\delta\mathcal{E} = 0\,, \quad \delta\mathcal{P} = 0\,, \quad \delta\mathcal{N} = 0 \tag{1.27a}$$
$$\delta q^\mu = -(\mathcal{E} - \mathcal{P})\delta u^\mu\,, \quad \delta j^\mu = -\mathcal{N}\delta u^\mu \tag{1.27b}$$
$$\delta t^{\mu\nu} = 0 \tag{1.27c}$$

Choosing δu^μ such that $j^\mu = 0$ is called the **Eckart frame**. The choice $q_\mu = 0$ is called **Landau frame**. From (1.27a) it follows that

$$\epsilon(T,\mu) + f_{\mathcal{E}}(\partial T, \partial\mu, \partial u) = \epsilon(T', \mu') + f'_{\mathcal{E}}(\partial T', \partial\mu', \partial u')$$

To do (9)**Is ϵ considered a function of x or of (T,μ)? What is evolved in a hydro evolution, ϵ, T, μ simultaneously?** and analogously for $\mathcal{P}$, $\mathcal{N}$ which implies

$$f'_{\mathcal{E}} = f_{\mathcal{E}} - \left(\frac{\partial\epsilon}{\partial T}\right)_\mu \delta T - \left(\frac{\partial\epsilon}{\partial\mu}\right)_T \delta\mu \tag{1.28a}$$
$$f'_{\mathcal{P}} = f_{\mathcal{P}} + \left(\frac{\partial p}{\partial T}\right)_\mu \delta T + \left(\frac{\partial p}{\partial\mu}\right)_T \delta\mu \tag{1.28b}$$
$$f'_{\mathcal{N}} = f_{\mathcal{N}} - \left(\frac{\partial n}{\partial T}\right)_\mu \delta T - \left(\frac{\partial n}{\partial\mu}\right)_T \delta\mu \tag{1.28c}$$

meaning 2 out of $(f_{\mathcal{E}}, f_{\mathcal{N}}, f_{\mathcal{P}})$ can be set to 0. The combinations

$$l^\mu = j^\mu - \frac{\mathcal{N}}{\mathcal{E} - \mathcal{P}} q^\mu \tag{1.29a}$$
$$f = f_{\mathcal{P}} + \left(\frac{\partial p}{\partial\epsilon}\right)_n f_{\mathcal{E}} + \left(\frac{\partial p}{\partial n}\right)_\epsilon f_{\mathcal{N}} \tag{1.29b}$$

are frame invariant.

Scalar variables that are first order in derivatives are $(\partial_\lambda u^\lambda, u^\lambda\partial_\lambda\mu, u^\lambda\partial_\lambda\mu)$. One can use the 0th order equations (1.24) up to an error of $\mathcal{O}(\partial^2)$ to eliminate 2 out of the 3 scalar variables, conventionally $u^\lambda\partial_\lambda\mu$ and $u^\lambda\partial_\lambda\mu$. The expansion of $\mathcal{P}$ is then

$$\mathcal{P} = -p(x) + \zeta\partial_\lambda u^\lambda + \mathcal{O}(\partial^2) \tag{1.30}$$

with ζ the **bulk viscosity**. Transverse vector valued variables containing 1 derivative are $(\Delta^{\mu\lambda}\partial_\lambda T, \Delta^{\mu\lambda}\partial_\lambda\mu, \Delta^{\mu\nu}\dot{u}_\nu)$ where $\dot{}$ denotes the **comoving derivative**

$$\dot{\#} = u^\lambda\partial_\lambda\# \tag{1.31}$$

The vector valued transverse equation $\Delta_{\lambda\nu}\partial_\mu T^{\mu\nu} = 0$ is used to eliminate $\Delta^{\mu\nu}\dot{u}_\nu$ and the expansion of the transverse vector j^μ becomes

$$j^\mu = -\sigma T\Delta^{\mu\nu}\partial_\nu\left(\frac{\mu}{T}\right) + \chi_T\Delta^{\mu\nu}\partial_\nu T + \mathcal{O}(\partial^2) \tag{1.32}$$

with the **charge conductivity** σ. There is 1 first derivative transverse traceless symmetrc tensor available To do (10) **Why is there only 1 transverse traceless symmetric tensor of order 1 ind derivatives of hydro variables?**

$$\sigma^{\mu\nu} = \Delta^{\mu\alpha}\Delta^{\nu\beta}(\partial_\alpha u_\beta + \partial_\beta u_\alpha - \frac{2}{d-1}\eta_{\alpha\beta}\partial_\mu u^\mu) \tag{1.33}$$

andhence the expansion of $t^{\mu\nu}$ becomes

$$t^{\mu\nu} = -\eta\sigma^{\mu\nu} + \mathcal{O}(\partial^2) \tag{1.34}$$

with the **shear viscosity** η. Hence Lorentz covariance restricts the form of $T^{\mu\nu}$ and J^μ up to four transport coefficients η, ζ, σ and χ_T.

Given a $\mu(x), T(x), u^\mu(x)$ and the equation of state, how can I reconstruct the derivative expansion? Is the expansion unique or does it lead to a specific frame?

1.3 Heavy Ion Collisions

Lorentz contracted nuclei collide and produce many particle that interact on a small volume. For strong enough interactions the system acquires a state of local thermal equilibrium. To do (12) **What distinguishes local and global thermodynamic equilibrium? I would guess homogeneity of hydro-variables, but validity of the differential form of the laws of thermodynamics**

1.3.1 Milne Coordinates [JK15]

$$\begin{cases} \tau = \sqrt{t^2 - z^2} \\ \eta_s = \operatorname{artanh}\frac{z}{t} \\ r = \sqrt{x^2+y^2} \\ \varphi = \operatorname{arctan2}\frac{y}{x} \end{cases} \iff \begin{cases} t = \tau\cosh\eta_s \\ z = \tau\sinh\eta_s \\ x = r\cos\varphi \\ y = r\sin\varphi \end{cases} \tag{1.35}$$

Using

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}x}\arctan x &= \frac{1}{1+x^2} \\ \frac{\mathrm{d}}{\mathrm{d}x}\operatorname{artanh} x &= \frac{1}{1-x^2} \\ \cos x &= \frac{1}{\sqrt{1+\tan^2 x}} & \sin x &= \frac{\tan x}{\sqrt{1+\tan^2 x}} \\ \cosh x &= \frac{1}{\sqrt{1-\tanh^2 x}} & \sinh x &= \frac{\tanh x}{\sqrt{1+\tanh^2 x}} \end{aligned}$$

the Jacobian of this coordinate transformation is

$$J = \frac{\partial(\tau,\eta_s,r,\varphi)}{\partial(t,z,x,y)} = \begin{pmatrix} \frac{t}{\tau} & -\frac{z}{\tau} & 0 & 0 \\ \frac{1}{1-\frac{z^2}{t^2}}\frac{-z}{t^2} & \frac{1}{1-\frac{z^2}{t^2}}\frac{1}{t} & 0 & 0 \\ 0 & 0 & \frac{x}{r} & \frac{y}{r} \\ 0 & 0 & \frac{1}{1+\frac{y^2}{x^2}}\frac{-y}{x} & \frac{1}{1+\frac{y^2}{x^2}}\frac{1}{x} \end{pmatrix} = \begin{pmatrix} \cosh\eta_s & -\sinh\eta_s & 0 & 0 \\ -\frac{\sinh\eta_s}{\tau} & \frac{\cosh\eta_s}{\tau} & 0 & 0 \\ 0 & 0 & \cos\varphi & \sin\varphi \\ 0 & 0 & -\frac{\sin\varphi}{r} & \frac{\cos\varphi}{r} \end{pmatrix} \tag{1.36}$$

$$J^{-1} = \frac{\partial(t,z,x,y)}{\partial(\tau,\eta_s,r,\varphi)} = \begin{pmatrix} \cosh\eta_s & \tau\sinh\eta_s & 0 & 0 \\ \sinh\eta_s & \tau\cosh\eta_s & 0 & 0 \\ 0 & 0 & \cos\varphi & -r\sin\varphi \\ 0 & 0 & \sin\varphi & r\cos\varphi \end{pmatrix} \tag{1.37}$$

Naturally this has block diagonal structure. The metric transforms as

$$\mathrm{d}t^2 - \mathrm{d}x^2 - \mathrm{d}y^2 - \mathrm{d}z^2 = \mathrm{d}\tau^2 - \tau^2\mathrm{d}\eta_s^2 - \mathrm{d}r^2 - r^2\mathrm{d}\varphi^2 \tag{1.38}$$

and hence

$$g_{\mu\nu} = \mathrm{diag}(1, -\tau^2, -1, -r^2) \qquad g^{\mu\nu} = \mathrm{diag}(1, -\frac{1}{\tau^2}, -1, -\frac{1}{r^2}) \tag{1.39}$$

Since for a diagonal metric the Christoffel symbols satisfy for fixed $\lambda \neq \mu \neq \nu \neq \lambda$

$$\Gamma^\lambda_{\mu\nu} = 0 \qquad \Gamma^\lambda_{\mu\mu} = -\frac{1}{2}(g^{-1}_{\lambda\lambda})\partial_\lambda g_{\mu\mu} \tag{1.40a}$$

$$\Gamma^\lambda_{\mu\lambda} = \partial_\mu(\ln\sqrt{|g_{\lambda\lambda}|}) \qquad \Gamma^\lambda_{\lambda\lambda} = \partial_\lambda(\ln\sqrt{|g_{\lambda\lambda}|}) \tag{1.40b}$$

the only non-vanishing Christoffel symbols are

$$\Gamma^\tau_{\eta_s\eta_s} = \tau \qquad \Gamma^r_{\varphi\varphi} = -r \tag{1.41a}$$

$$\Gamma^{\eta_s}_{\tau\eta_s} = \frac{1}{\tau} \qquad \Gamma^\varphi_{r\varphi} = \frac{1}{r} \tag{1.41b}$$

A longitudinal boost in this coordinate system has the form

$$\Lambda(\psi)_{\tau,\eta_s} = J\Lambda(\psi)_{t,z}J^{-1} \tag{1.42a}$$

$$= \begin{pmatrix} \cosh\eta_s & -\sinh\eta_s \\ -\frac{\sinh\eta_s}{\tau} & \frac{\cosh\eta_s}{\tau} \end{pmatrix} \begin{pmatrix} \cosh\psi & -\sinh\psi \\ -\sinh\psi & \cosh\psi \end{pmatrix} \begin{pmatrix} \cosh\eta_s & \tau\sinh\eta_s \\ \sinh\eta_s & \tau\cosh\eta_s \end{pmatrix} \tag{1.42b}$$

$$= \begin{pmatrix} \cosh\eta_s\cosh\psi + \sinh\eta_s\sinh\psi & -\cosh\eta_s\sinh\psi - \sinh\eta_s\cosh\psi \\ -\frac{1}{\tau}(\sinh\eta_s\cosh\psi + \cosh\eta_s\sinh\psi) & \frac{1}{\tau}(\sinh\eta_s\sinh\psi + \cosh\eta_s\cosh\psi) \end{pmatrix} \begin{pmatrix} \cosh\eta_s & \tau\sinh\eta_s \\ \sinh\eta_s & \tau\cosh\eta_s \end{pmatrix} \tag{1.42c}$$

$$= \begin{pmatrix} \cosh\psi & -\tau\sinh\psi \\ -\frac{\sinh\psi}{\tau} & \cosh\psi \end{pmatrix} \tag{1.42d}$$

1.3.2 Bjorken Model [Oll08]

Let the collidition of two incident nuclei start at $z = t = 0$ with the z-axis oriented along the propagation direction of the incident nuclei.

Calculation 1.3|1: Lorentz Boost

We want $\Delta s^2 = c^2t^2 - z^2 = c^2t'^2 - z'^2$ to remain invariant. This is done by

$$\begin{pmatrix} ct' \\ z' \end{pmatrix} = \begin{pmatrix} \cosh\psi & -\sinh\psi \\ -\sinh\psi & \cosh\psi \end{pmatrix} \begin{pmatrix} ct \\ z \end{pmatrix} = \begin{pmatrix} \cosh\psi ct - \sinh\psi z \\ -\sinh\psi ct + \cosh\psi z \end{pmatrix} \tag{1.43}$$

because

$$c^2t'^2 - z'^2 = c^2t^2(\cosh^2\psi - \sinh^2\psi) - z^2(\cosh^2\psi - \sinh^2\psi) = c^2t^2 - z^2$$

using $\cosh^2 x - \sinh^2 x = 1$ for any x. The origin $z' = 0$ of the system Σ' viewed from the unprimed system Σ moves with velocity v given by

$$\beta = \frac{v}{c} = \tanh\psi = \frac{\sinh\psi}{\cosh\psi} \tag{1.44}$$

Define further $\gamma = \frac{1}{\sqrt{1-\beta^2}} = \cosh\psi$. With $|\beta| \in [0,1)$ one finds $\gamma \geq 0$. The Lorentz transformation is then given by

$$\begin{pmatrix} \cosh\psi & -\sinh\psi \\ -\sinh\psi & \cosh\psi \end{pmatrix} = \begin{pmatrix} \gamma & -\gamma\beta \\ -\gamma\beta & \gamma \end{pmatrix} \tag{1.45}$$

The inverse transformation is given by

$$\begin{pmatrix} ct \\ z \end{pmatrix} = \begin{pmatrix} \gamma & \gamma\beta \\ \gamma\beta & \gamma \end{pmatrix} \begin{pmatrix} ct' \\ z' \end{pmatrix} \tag{1.46}$$

as is immediately clear from

$$\begin{pmatrix} \gamma & \gamma\beta \\ \gamma\beta & \gamma \end{pmatrix} \begin{pmatrix} \gamma & -\gamma\beta \\ -\gamma\beta & \gamma \end{pmatrix} = \begin{pmatrix} \gamma^2(1-\beta^2) & 0 \\ 0 & \gamma^2(-\beta^2+1) \end{pmatrix} = \mathbf{1}$$

At $z'=0$ a time interval $\Delta t'$ in Σ' corresponds to a time interval Δt in Σ given by

$$c\Delta t' = c\Delta t\gamma - \gamma\beta\Delta z(z'=0) = c\Delta t\gamma(1-\beta^2) = \frac{c\Delta t}{\gamma} \tag{1.47}$$

meaning $\Delta t' \leq \Delta t$. Similarly, consider 2 point of distance $\Delta z' = l'$ that are at rest in Σ', their distance being defined at $\Delta t' = 0$. Measuring the same 2 points at a single instant in time from the perspective of Σ when $\Delta t = 0$ one finds

$$\begin{aligned} c\Delta t &= \beta\Delta z \\ l' &= \gamma\Delta z \end{aligned}$$

implying that the object appears contracted, $\Delta z < l'$.

Assume all particles are produced in a short interval around $z = t = 0$. Consider the Lorentz boost $\Lambda(\beta)$ in z-direction applied to the 4-velocity $u = \gamma_v(1, \vec{v})$

$$t' = \gamma(t - \beta z) \tag{1.48a}$$
$$z' = \gamma(-\beta t + z) \tag{1.48b}$$
$$u^{t\prime} = \gamma\gamma_v(1 - \beta v_z) \tag{1.48c}$$
$$u^{z\prime} = \gamma\gamma_v(-\beta + v_z) \tag{1.48d}$$

indicating

$$v_z' = \frac{u^{z\prime}}{u^{t\prime}} = \frac{-\beta + v_z}{1 - \beta v_z} \tag{1.49a}$$

whereas the assumption of uniform longitudinal motion $z = v_z t$ implies

$$\frac{z'}{t'} = \frac{-\beta t + z}{t - \beta z} = \frac{-\beta + v_z}{1 - \beta v_z} \tag{1.49b}$$

This shows that

$$v_z = \frac{z}{t} \tag{1.50}$$

is a boost invariant prescription. The assumption that the longitudinal fluid velocity as a function $v_z = v_z(t, z)$ has the form (1.50) is part of the **Bjorken model**.

Define new coordinates

$$\begin{cases} t = \tau\cosh\eta_s \\ z = \tau\sinh\eta_s \\ v_z = \tanh Y \end{cases} \tag{1.51}$$

with the proper time τ, spacetime rapidity η_s and fluid rapidity Y. In these coordinates the Bjorken model dictates $Y = \eta_s$. Under a Lorentz boost of spacetime rapidity ψ

$$\begin{pmatrix} \tau'\cosh\eta_s' \\ \tau'\sinh\eta_s' \end{pmatrix} = \begin{pmatrix} \cosh\psi & -\sinh\psi \\ -\sinh\psi & \cosh\psi \end{pmatrix} \begin{pmatrix} \tau\cosh\eta_s \\ \tau\sinh\eta_s \end{pmatrix} = \begin{pmatrix} \tau(\cosh\eta_s\cosh\psi - \sinh\eta_s\sinh\psi) \\ \tau(-\cosh\eta_s\sinh\psi + \sinh\eta_s\cosh\psi) \end{pmatrix} = \begin{pmatrix} \tau\cosh(\eta_s - \psi) \\ \tau\sinh(\eta_s - \psi) \end{pmatrix} \tag{1.52}$$

where

$$\sinh(a+b) = \sinh a\cosh b + \cosh a\sinh b \tag{1.53a}$$
$$\cosh(a+b) = \cosh a\cosh b - \sinh a\sinh b \tag{1.53b}$$

was used. One finds $\tau' = \tau$, $\eta_s' = \eta_s - \psi$. From (1.49) we also find

$$\tanh Y' = \frac{-\tanh\psi + \tanh Y}{1 - \tanh\psi\tanh Y} = \tanh(Y - \psi) \tag{1.54}$$

and hence $Y' = Y - \psi$. Note that $\eta_s = \operatorname{artanh}\frac{z}{t} = \frac{1}{2}\ln\frac{t+z}{t-z}$.

Let the collision happen around $t = 0$. Let R be the radius of the nuclei. On the axis of collision where $z \approx 0$ the wave eminating from the collision of the nuclei shells has no influence for times as early as $t \lessapprox \frac{R}{c}$.

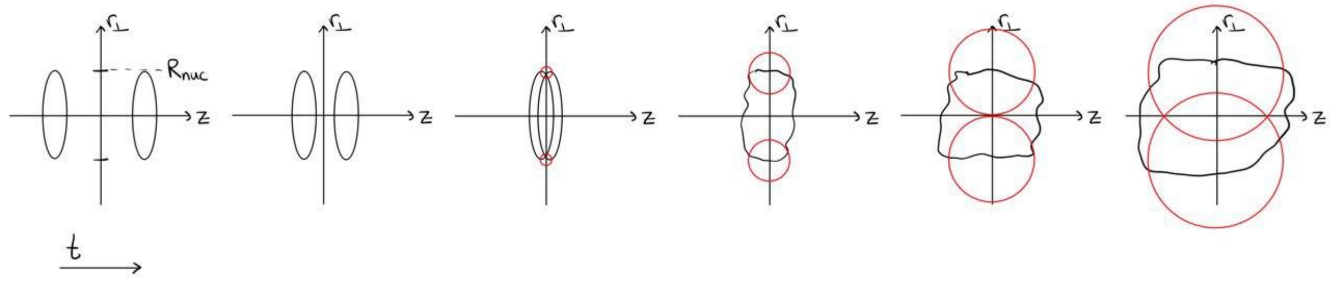


Figure 1.1: At early times the flow at $z \approx 0$ is independent of the extent of the nuclei.

The fluid velocity $u^\mu = (u^\tau, u^{\eta_s}, u^\perp)$, assuming only longitudinal flow, takes the form $u^\mu = (1, 0, 0, 0)$.

Baryon conservation reads

$$0 = \nabla_\mu (n u^\mu) = \partial_\mu (n u^\mu) + \Gamma^\mu_{\mu\nu}(n u^\nu) = \partial_\tau n + \frac{n}{\tau} \tag{1.55}$$

The energy-momentum tensor of ideal hydrodynamics in Milne coordinates has components $T^{\tau\tau} = \epsilon$, $T^{\eta_s\eta_s} = \frac{p}{\tau^2}$, $T^{rr} = p$ and $T^{\varphi\varphi} = \frac{p}{r^2}$. Energy-momentum conservation takes the form

$$0 = \nabla_\mu T^{\mu\nu} = \partial_\mu T^{\mu\nu} + \Gamma^\mu_{\mu\lambda} T^{\lambda\nu} + \Gamma^\nu_{\mu\lambda} \underbrace{T^{\mu\lambda}}_{\text{diagonal} \implies \mu=\lambda} \tag{1.56a}$$

$$= \partial_\mu T^{\mu\nu} + (\Gamma^{\eta_s}_{\eta_s\tau} T^{\tau\nu} + \Gamma^{\varphi}_{\varphi r} T^{r\nu}) + (\Gamma^{\nu}_{\eta_s\eta_s} T^{\eta_s\eta_s} + \Gamma^{\nu}_{\varphi\varphi} T^{\varphi\varphi}) \tag{1.56b}$$

Evaluate different components of this equation

$$\nu = \tau : \quad 0 = \partial_\tau \epsilon + (\frac{\epsilon}{\tau} + 0) + (\tau \frac{p}{\tau^2} + 0) \tag{1.57a}$$

$$\nu = \eta_s : \quad 0 = \partial_{\eta_s}(\frac{p}{\tau^2}) \tag{1.57b}$$

$$\nu = r : \quad 0 = \partial_r p + (0 + \frac{p}{r}) + (0 - r^3 p) \tag{1.57c}$$

$$\nu = \varphi : \quad 0 = \partial_\varphi (r^2 p) \tag{1.57d}$$

This implies that $\epsilon = \epsilon(\tau)$ and $p = p(\tau)$ as well as

$$\partial_\tau \epsilon = -\frac{\epsilon + p}{\tau} \tag{1.58}$$

The equation of state $p(\epsilon)$ closes the system. But from Legendre trafos, p is a function of (T, μ)? How does hydrodynamics and equilibrium thermodynamics come together? After a hydro simulation, e.g. ϵ is given as a function of x, not of some thermo variable. Only 2 variables are evolved (μ, T) all others follow from algebraic relations of thermodynamics

1.3.3 Blast Wave Model

Basic Idea [JK15]

- assume boost-invariant longitudinal flow
- assume functional form of phase space density at kinetic freezout **Find a precise definition of kinetic freezout and also chemical freezout Chemical: Interaction that convert the particle species stop. Kinetic: Interactions that keep energy-pressure balance stop**
- parameters: kinetic temperature, radial flow strength, anisotropy in radial flow, source anisotropy

Work in Milne coordinates $(\tau, \eta_s, r, \varphi)$.

Boost invariance and rotational symmetry imply $u_\varphi = u_{\eta_s} = 0$ **To do** (15) **Why exactly does boost invariance and rotational symmetry imply $u_\varphi = u_{\eta_s} = 0$? My guess: Boost invariance means $v_z = \frac{z}{t}$, a cube in the fluid moves along straight lines of constant η**

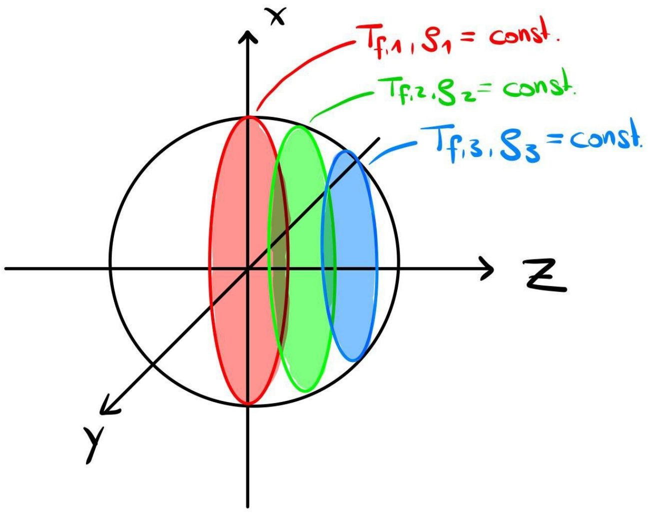


Figure 1.2: Freezout.

To do (16) **Is this image correct? Do the planes freezout at the same proper time? At $z = 0$ this is correct.**

Assumptions of the blast-wave model are that the particle freezout happens at proper time τ_f when the fluid has constant temperature T_f and uniform matter distribution along the transverse plane

$$T = T_f \Theta(R - r) \tag{1.59a}$$
$$u_r = u_0 (\frac{r}{R})^n \Theta(R - r) \tag{1.59b}$$
$$u_\varphi = u_{\eta_s} = 0 \tag{1.59c}$$
$$u_\tau = \sqrt{1 + (u_r)^2} \tag{1.59d}$$

$$\frac{\mathrm{d}r}{\mathrm{d}\tau} = u_0 (\frac{r}{R})^n \qquad \Longrightarrow \qquad \int_{r_0}^{R} \frac{\mathrm{d}r}{r^n} = \int_0^{\tau_f} \mathrm{d}\tau \frac{u_0}{R^n} \tag{1.60}$$

which implies

$$\frac{\tau_f u_0}{R^n} = \begin{cases} n = 1: & \ln \frac{R}{r_0} \\ n > 1: & \frac{1}{n-1} (\frac{1}{R^{n-1}} - \frac{1}{r_0^{n-1}}) \end{cases} \tag{1.61}$$

For $n = 1$ the radius of the fireball at freezout is given by

$$R = r_0 \exp \left(\frac{u_0 \tau_f}{R} \right) \tag{1.62}$$

The freezout time τ_f can be determined under the assumption of Bjorken flow.

$$\tau_f = \tau_0 \left(\frac{\epsilon_i}{\epsilon_{i0}} \right)^{\frac{3}{4}} \tag{1.63}$$

where ϵ is the energy density and the index "0" corresponds to central collision.

1.3.4 Freezout [Dev+20]

The hydrodynamic description of the fireball eventually breaks down as the system cools down. Hydro variables (fluid velocity, temperature) are then replaced by hadronic degrees of freedom, i.e. particle spectra.

The Cooper-Frye procedure relates particle spectra to hydro variables via

$$E_{\boldsymbol{p}} \frac{\mathrm{d}N_a}{\mathrm{d}^3 \boldsymbol{p}} = \frac{\nu_a}{(2\pi)^3} \int_\Sigma \mathrm{d}\Sigma_\mu p^\mu f_a(\overline{E}_{\boldsymbol{p}}, T(x), \mu(x)) \tag{1.64}$$

To do (17) **What about this form of the particle spectrum**

$$\frac{1}{2\pi p_T}\frac{\mathrm{d}^2N}{\mathrm{d}p_T\mathrm{d}Y}\Big|_{Y=0} = \mathcal{N}\int_0^R \mathrm{d}r r m_T I_0\left(\frac{p_T\sinh(\mathbf{artanh}\, u_r)}{T_f}\right)K_1\left(\frac{m_T\cosh(\mathbf{artanh}\, u_r)}{T_f}\right) \tag{1.65}$$

from [Che+21]? parametrize the momenta with Milne coordinates as Flörchinger sketched it and you will find the equivalence. with $\overline{E}_{\boldsymbol{p}} = p^\mu u_\mu$ the energy in the fluid rest-frame (with fluid velocity u^μ) and ν_a the spin degeneracy. f_a is a particle distribution function, which depends on the fluid fields $u^\mu(x)$, $T(x)$, $\mu(x)$ and viscious corrections. The distribution function is modelled by the equilibrium Fermi-Dirac- or Bose-Einstein distribution plus corrections due to higher order hydrodynamics

$$f = f_{\mathrm{eq}} + \delta f_{\mathrm{bulk}} + \delta f_{\mathrm{shear}} \tag{1.66}$$

which take the form

$$f_{\mathrm{eq}} = \frac{1}{\exp(u_\mu p^\mu/T) + s^{(B,F)}} \tag{1.67a}$$

$$\delta f_{\mathrm{bulk}} = \dots \tag{1.67b}$$

$$\delta f_{\mathrm{shear}} = \dots \tag{1.67c}$$

where with $s^{(B)} = -1$ and $s^{(F)} = +1$ for Bosons and Fermions respectively. Unstable resonances (particle types) will decay into long-lived particles. These decays can be accounted for already in the distribution function before integrating. This is done numerically with the help of decay precomputable decay kernels [Maz+19].

1.4 Bose-Einstein Condensation

The partition function in the grand canonical potential is given by

$$Z = \mathrm{Tr} e^{-\frac{\hat{H}-\mu\hat{N}}{T}} \tag{1.68}$$

For a system with excited states of energy E_α which are occupied by n_α particles the partition becomes

$$Z = \sum_{n_\alpha}\exp\left(-\frac{1}{T}\sum_\alpha n_\alpha(E_\alpha-\mu)\right) = \prod_\alpha\left[\sum_{n_\alpha}\exp\left(-\frac{1}{T}n_\alpha(E_\alpha-\mu)\right)\right] \tag{1.69}$$

For bosons $n_\alpha \in \mathbb{N}_0$, for fermions $n_\alpha \in \{0,1\}$ and the sums over n_α can be executed as

$$\text{for bosons:} \qquad \sum_{n_\alpha=0}^{\infty}\exp\left(-\frac{1}{T}n_\alpha(E_\alpha-\mu)\right) = \frac{1}{1-\exp(-\frac{E_\alpha-\mu}{T})} \qquad \text{if } \mu < E_\alpha\ (\forall\alpha) \tag{1.70a}$$

$$\text{for fermions:} \qquad \sum_{n_\alpha=0}^{1}\exp\left(-\frac{1}{T}n_\alpha(E_\alpha-\mu)\right) = 1+\exp(-\frac{E_\alpha-\mu}{T}) \tag{1.70b}$$

Average particle number is then given by

$$N = T\frac{\partial}{\partial_\mu}\ln Z = T\frac{\partial}{\partial_\mu}\left[\pm\ln\left(\prod_\alpha\left[1\mp\exp\left(-\frac{E_\alpha-\mu}{T}\right)\right]\right)\right] = \sum_\alpha\frac{1}{\exp\left(\frac{E_\alpha-\mu}{T}\right)\mp 1} \equiv \sum_\alpha n_\alpha \tag{1.71}$$

where average occupation number of each level is defined by n_α. If the energy levels are independent of the particle spin s and hence have degeneracy $\nu_s = 2s+1$ one replaces the sum $\sum_\alpha = \nu_s\sum_{\alpha'}$ where α' label unique single particle energies. The $\cdot'$ is now omitted. For free particles in $\mathbb{R}^3$ with single particle energies $E = \frac{\boldsymbol{p}^2}{2m}$ the abstract index α becomes continuous. Take instead a finite box and $\boldsymbol{x} \in [0,L]^d$. In this scenario momenta are discretized $\boldsymbol{p} = \frac{2\pi\boldsymbol{k}}{L}$ with $\boldsymbol{k} \in \boldsymbol{Z}^d$. One finds for the discretization $\Delta p_i = \frac{2\pi}{L}\Delta k_i$ with $\Delta k_i = 1$ and hence

$$\sum_\alpha \equiv \sum_{k_1,\dots k_d}\Delta k_1\dots\Delta k_d = \frac{L^d}{(2\pi)^d}\sum_{p_1,\dots p_d}\Delta p_1\dots\Delta p_d \overset{\Delta p_i\to 0}{\to} \int\frac{L^d}{(2\pi)^d}\int \mathrm{d}^d p \tag{1.72}$$

The thermal particle density in the large volume limit is given by

$$n_{\mathrm{therm}}(\mu,T) = \nu_s\int\frac{\mathrm{d}^3p}{(2\pi)^3}\frac{1}{\exp\left(\frac{\boldsymbol{p}^2/(2m)-\mu}{T}\right)-1} = \frac{\nu_s\pi^{\frac{3}{2}}(2mT)^{\frac{3}{2}}}{(2\pi)^3}\mathrm{Li}_{\frac{3}{2}}(\exp(\frac{\mu}{T})) = \nu_s\left(\frac{mT}{2\pi}\right)^{\frac{3}{2}}\mathrm{Li}_{\frac{3}{2}}(\exp(\frac{\mu}{T})) \tag{1.73}$$

Important 1.4|1: Polylogarithms

Define

$$\mathrm{Li}_s(z) = \sum_{n=1}^{\infty} \frac{z^n}{n^s} = \frac{1}{\Gamma(s)} \int_0^\infty \mathrm{d} \frac{t^{s-1}}{e^t z^{-1} - 1} \tag{1.74}$$

which also implies

$$\frac{1}{\Gamma(s)} \int_0^\infty \mathrm{d}t \frac{t^{s-1}}{e^t z^{-1} \mp 1} = \pm \mathrm{Li}_s(\pm z) \tag{1.75}$$

With this

$$\int \mathrm{d}^d p \frac{1}{\exp(a\boldsymbol{p}^2) z^{-1} \mp 1} = \Omega_d \int_0^\infty \mathrm{d}p \frac{(p^2)^{\frac{d-1}{2}}}{\exp(ap^2) z^{-1} \mp 1} \tag{1.76a}$$

$$\overset{\substack{t=ap^2 \\ \mathrm{d}t=2\sqrt{at}\mathrm{d}p}}{=} \Omega_d \int_0^\infty \frac{\mathrm{d}t}{2\sqrt{at}} \frac{a^{\frac{1-d}{2}} t^{\frac{d-1}{2}}}{\exp(t) z^{-1} \mp 1} \tag{1.76b}$$

$$= \frac{\Omega_d}{2a^{\frac{d}{2}}} \int_0^\infty \mathrm{d}t \frac{t^{\frac{d}{2}-1}}{e^t z^{-1} \mp 1} \tag{1.76c}$$

$$= \pm \frac{\Omega_d \Gamma(\frac{d}{2})}{2a^{\frac{d}{2}}} \mathrm{Li}_{\frac{d}{2}}(\pm z) \tag{1.76d}$$

$$\pm = \frac{\pi^{\frac{d}{2}}}{a^{\frac{d}{2}}} \mathrm{Li}_{\frac{d}{2}}(\pm z) \tag{1.76e}$$

where $\Omega_d = \frac{2\pi^{\frac{d}{2}}}{\Gamma(\frac{d}{2})}$ was used.

The chemical potential is restricted to $\mu \leq 0$ and the particle density for fixed temperature is bounded from above by $n_{\text{therm}} < n_{\text{therm}}(\mu \nearrow 0, T)$. However for the ground state of the system $E_\alpha = 0$ and the occupation number n_α diverges to $+\infty$ as $\mu \nearrow 0$, indicating macroscopic occupation which changes the description. The critical density for fixed temperature is

$$n_c = \nu_s \left(\frac{mT}{2\pi}\right)^{\frac{3}{2}} \zeta(\frac{3}{2}) \tag{1.77}$$

or for fixed temperature

$$T_C = \frac{2\pi}{m} \left(\frac{n}{\nu_s \zeta(\frac{3}{2})}\right)^{\frac{2}{3}} \tag{1.78}$$

One can think of the critical values as the limits after which the convergence condition $\mu \leq 0$ is not longer satisfied. Alternatively one can ask the question, whether $\mu = 0$ is reached for $T > 0$ at fixed n. For $T \to 0$ it should definitely approach to 0, since at $T = 0$ no particle should occupy an excited state, $n_{\alpha>0} \overset{T\to 0}{\to} 0$.

Bose-Einstein condensation appears for $T < T_c$ or $n > n_c$. If n_0 is macroscopically large, it is not well described by the integration procedure above. One should rather split the total density into its thermal contribution and the contribution from the ground state,

$$n = n_0 + n_{\text{therm}} \tag{1.79}$$

For fixed n and $T \geq T_C$ $n_0 \approx 0$ can be assumed. n_{therm} is still given by the integral (1.73) and for $T \geq T_C$ equals the total particle density n. Hence the macroscopic density in the ground state for $T < T_C$ is given by

$$n_0 = n - n_{\text{therm}} = n \left[1 - \left(\frac{T}{T_C}\right)^{\frac{3}{2}}\right] \tag{1.80}$$

1.4.1 Relativistic Bose-Einstein Statistics

In $d+1$ spacetime dimensions the particle number density

$$n = \nu_s \int \frac{\mathrm{d}^d p}{(2\pi)^d} \left[\frac{1}{\exp(\frac{E_{\boldsymbol{p}}-\mu}{T}) - 1} - \frac{1}{\exp(\frac{E_{\boldsymbol{p}}+\mu}{T}) - 1}\right] \tag{1.81}$$

is the difference between particles and antiparticles. Note $E_{\boldsymbol{p}} = \sqrt{\boldsymbol{p}^2 + m^2}$. Convergence requires $|\mu| < m$. **This only includes 1-loop order corrections to the effective potential.** **Yes. This assumes a free gas without interaction vertices. For an interacting theory its different.** Note that for $T \ll m$ this reduces to the non-relativistic case, since the second summand/antiparticle contribution becomes negligible.

$$\frac{1}{\exp(\frac{E_{\boldsymbol{p}}-\mu}{T})-1} - \frac{1}{\exp(\frac{E_{\boldsymbol{p}}+\mu}{T})-1} = \frac{\exp(\frac{E_{\boldsymbol{p}}+\mu}{T}) - \exp(\frac{E_{\boldsymbol{p}}-\mu}{T})}{\exp(\frac{2E_{\boldsymbol{p}}}{T}) - \exp(\frac{E_{\boldsymbol{p}}-\mu}{T}) - \exp(\frac{E_{\boldsymbol{p}}+\mu}{T}) + 1} \tag{1.82a}$$

$$= \frac{\sinh(\frac{\mu}{T})}{\cosh(\frac{E_{\boldsymbol{p}}}{T}) - \cosh(\frac{\mu}{T})} \tag{1.82b}$$

$$= \frac{\exp(\frac{E_{\boldsymbol{p}}}{T})(2\frac{\mu}{T} + \mathcal{O}(\frac{\mu}{T})^2)}{(\exp(\frac{E_{\boldsymbol{p}}}{T})+1)^2 + \mathcal{O}(\frac{\mu}{T})^2} \tag{1.82c}$$

and hence

$$n = \nu_s \frac{\Omega_d}{(2\pi)^d} \int_0^\infty \mathrm{d}p p^{d-1} \frac{\sinh(\frac{\mu}{T})}{\cosh(\frac{E_{\boldsymbol{p}}}{T}) - \cosh(\frac{\mu}{T})} = \nu_s \frac{2\pi^{\frac{d}{2}}}{\Gamma(\frac{d}{2})(2\pi)^d} \int_0^\infty \mathrm{d}p p^{d-1} \frac{\sinh(\frac{\mu}{T})}{\cosh(\frac{E_{\boldsymbol{p}}}{T}) - \cosh(\frac{\mu}{T})} \tag{1.83}$$

What is $\lim_{x\to 0} \frac{x^b}{\cosh\sqrt{x^2+a^2}-\cosh a}$? $(b > 0)$

$$\lim_{x\to 0} \frac{x^b}{\cosh\sqrt{x^2+a^2} - \cosh a} = \lim_{x\to 0} \frac{bx^{b-1}}{\sinh(\sqrt{x^2+a^2})\frac{x}{\sqrt{x^2+a^2}}} \tag{1.84a}$$

$$= \lim_{x\to 0} \frac{bx^{b-2}\sqrt{x^2+a^2}}{\sinh\sqrt{x^2+a^2}} \begin{cases} = 0 & \iff \quad b \geq 2 \\ \text{diverges with } x^{b-2} & \iff \quad b < 2 \end{cases} \tag{1.84b}$$

For $a = 0$ one can instead expand $\cosh(x) = 1 + \mathcal{O}(x^2)$ and find the same criterion.

The critical values of density or temperature are now found by setting $|\mu| = m$. Further assume $T \gg m$. Then the integral converges if $d - 1 > 1$ (the integration improves convergence by 1 order) and we find for $d = 3$ **To do (19) Check exactly what the error is for the $p \to 0$ contribution of this integral.**

$$|n| = \nu_s \frac{2\pi^{\frac{3}{2}}}{\Gamma(\frac{3}{2})(2\pi)^3} \frac{2m}{T} \int_0^\infty \mathrm{d}pp^2 \frac{\exp(\frac{p}{T})}{(\exp(\frac{p}{T})+1)^2} = \frac{1}{\frac{\sqrt{\pi}}{2}\pi^{\frac{3}{2}}} \frac{m}{T} T^3 \int_0^\infty \mathrm{d}x x^2 \frac{e^x}{(e^x+1)^2} = \frac{2mT^2}{\pi^2}\frac{\pi^2}{6} = \frac{mT^2}{3} \tag{1.85}$$

implying (if $T \gg m$ or $|n| \gg m^3$)

$$|n_c| = \frac{mT^2}{3} \tag{1.86a}$$

$$T_c = \left(\frac{3|n|}{m}\right)^{\frac{1}{2}} \tag{1.86b}$$

and if $m \ll T < T_c$

$$n_0 = n\left[1 - \left(\frac{T}{T_c}\right)^2\right] \tag{1.86c}$$

The general result in $d+1$ spacetime dimensions in the ultrarelativistic limit is [GdB07]

$$T_c^{UR} = \left[\frac{\Gamma(\frac{d}{2})(2\pi)^d}{4m\pi^{\frac{d}{2}}\Gamma(d)\zeta(d-1)}\right]^{\frac{1}{d-1}} n^{\frac{1}{d-1}} \tag{1.87}$$

The energy of a particle with 4-momentum p^μ measured from an observer with 4-velocity u^μ is $E_{\boldsymbol{p}} = p^\mu u_\mu$ and we get

$$n = \nu_s \int \frac{\mathrm{d}^{d-1}p}{(2\pi)^{d-1}} \left[\frac{1}{\exp(\frac{p^\mu u_\mu - \mu}{T}) - 1} - \frac{1}{\exp(\frac{p^\mu u_\mu + \mu}{T}) - 1}\right] \tag{1.88}$$

1.5 Particle Physics and Group Theory

1.5.1 Isospin

It is found that $m_{\text{proton}} \approx m_{\text{neutron}}$. Take this as a motivation to represent proton and neutron as 2 states of a single entity, just as $|\uparrow\rangle$ and $|\downarrow\rangle$ are basis states for a spin-$\frac{1}{2}$ particle.

$$|p\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix} , \qquad |n\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \tag{1.89}$$

The transformations acting on these states are supposed to be unitary. Neglecting phase factors the corresponding symmetry group is $SU(2)$. The above state form a doublet under **isospin** transformations with total isospin $I = \frac{1}{2}$ and third component $I_3 = \pm\frac{1}{2}$. This idea is extend to up-/down-quarks or up-/down-/strange-quarks since $m_u \approx m_d \approx m_s$.

$$|u\rangle = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} , \qquad |d\rangle = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} , \qquad |s\rangle = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \tag{1.90}$$

The corresponding symmetry neglecting phase factors is $SU(3)$ with 8 generators T_i. **Is isospin the same is flavor symmetry? Isospin is $SU(2)$ betweend u and d or p and n. For u, d, s we would call it flavor symmetry.**

1.5.2 Representations of $SU(3)$

The **3**-representation or fundamental representation is given by $T_i = \frac{1}{2}\lambda_i$ with the Gell-Mann matrices λ_i. They constructed as follows: Choose the first three to reflect ud-symmetry ($\rightarrow SU(2)$)

$$\lambda_1 = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} , \qquad \lambda_2 = \begin{pmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} , \qquad \lambda_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \tag{1.91a}$$

This construction is mirrored to reflect us- and ds-symmetry

$$\lambda_4 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix} , \qquad \lambda_5 = \begin{pmatrix} 0 & 0 & -i \\ 0 & 0 & 0 \\ i & 0 & 0 \end{pmatrix} , \qquad \lambda_8' = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix} \tag{1.91b}$$

$$\lambda_6 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} , \qquad \lambda_7 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -i \\ 0 & i & 0 \end{pmatrix} , \qquad \lambda_8'' = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix} \tag{1.91c}$$

Note that only one of λ_8', λ_8'' can be linearly independent. One instead chooses

$$\lambda_8 = \frac{1}{\sqrt{3}}(\lambda_8' + \lambda_8'') = \frac{1}{\sqrt{3}} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{pmatrix} \tag{1.91d}$$

Alternatively define

$$I_\pm = T_1 \pm iT_2 , \qquad V_\pm = T_4 \pm iT_5 , \qquad U_\pm = T_6 \pm iT_7 \tag{1.92}$$

A basis of $\mathfrak{su}(3)$ is given by $\{T_i\}$ or $\{I_\pm, V_\pm, U_\pm, T_3, T_8\}$. The algebra has two **Casimir operators**

$$C_2 = \sum_{j=1}^{8} T_j^2 , \qquad C_3 = \sum_{j,k,l=1}^{8} T_j^2 \tag{1.93}$$

that commute with all operators and hence all states can be labelled by 2 labels. Since $[T_3, T_8] = 0$ use the eigenstates $|i_3, i_8\rangle$ such that

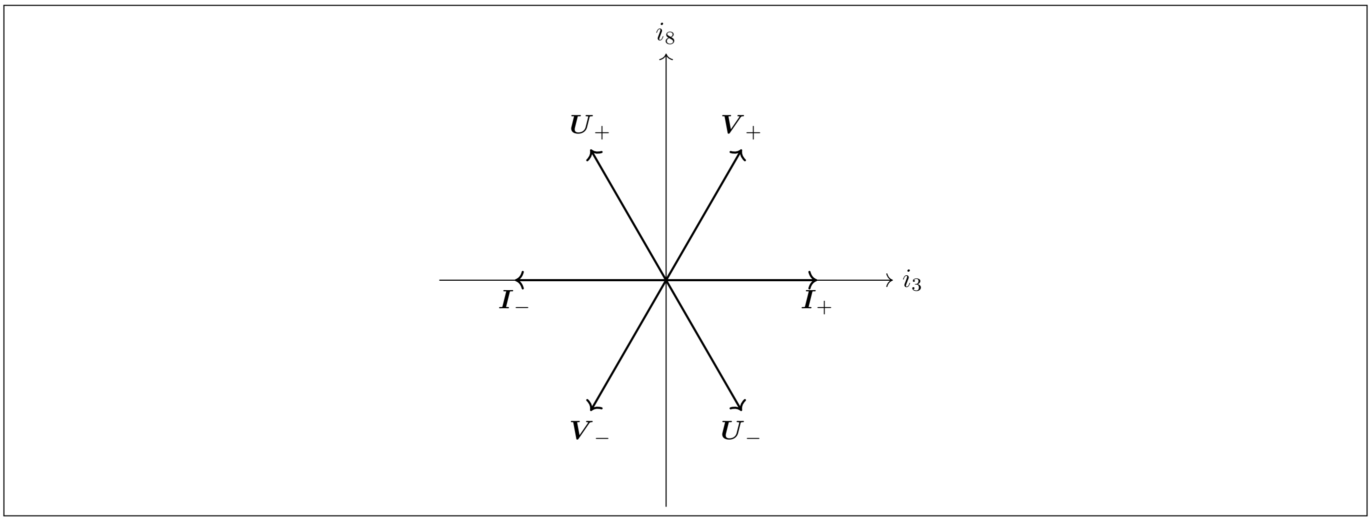
$$T_3|i_3, i_8\rangle = i_3|i_3, i_8\rangle , \qquad T_8|i_3, i_8\rangle = i_8|i_3, i_8\rangle \tag{1.94}$$

The ladder operators act as

$$I_\pm|i_3,i_8\rangle \propto |i_3\pm 1, i_8\rangle \tag{1.95a}$$

$$V_\pm|i_3,i_8\rangle \propto |i_3\pm \frac{1}{2}, i_8 \pm \frac{\sqrt{3}}{2}\rangle \tag{1.95b}$$

$$U_\pm|i_3,i_8\rangle \propto |i_3\mp \frac{1}{2}, i_8 \pm \frac{\sqrt{3}}{2}\rangle \tag{1.95c}$$



One can read off

$$|u\rangle = |\frac{1}{2}, \frac{1}{2\sqrt{3}}\rangle\,, \qquad |d\rangle = |-\frac{1}{2}, \frac{1}{2\sqrt{3}}\rangle\,, \qquad |s\rangle = |0, -\frac{1}{\sqrt{3}}\rangle \tag{1.96}$$

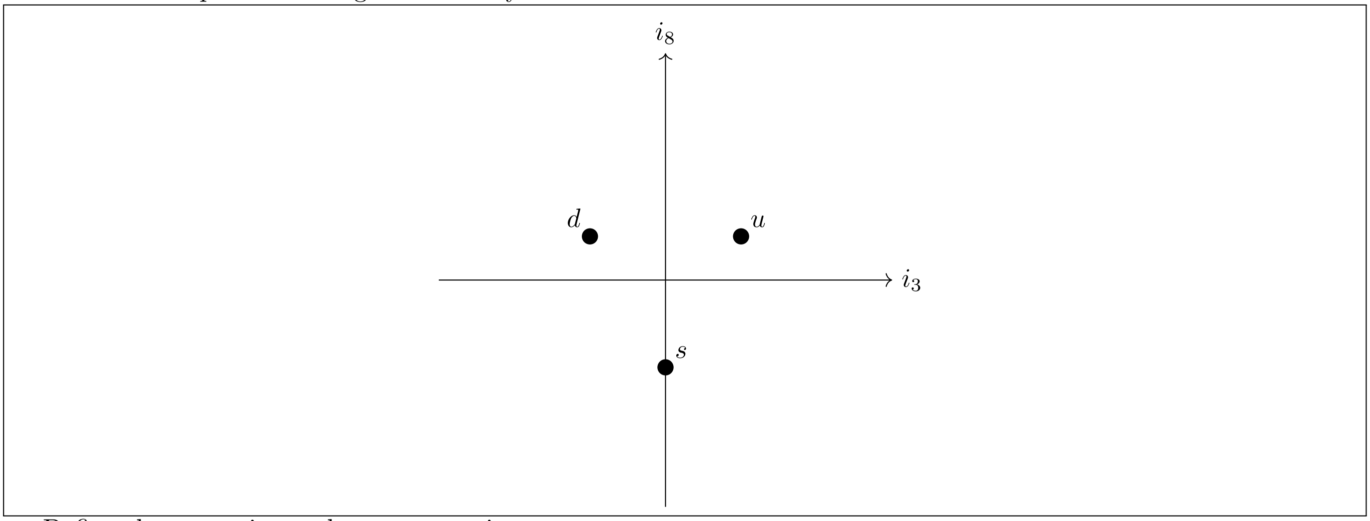
and the ladder operators act as

$$V_-|u\rangle \propto |s\rangle\,, \qquad V_+|s\rangle \propto |u\rangle \tag{1.97a}$$

$$I_-|u\rangle \propto |d\rangle\,, \qquad I_+|d\rangle \propto |u\rangle \tag{1.97b}$$

$$U_-|d\rangle \propto |s\rangle\,, \qquad U_+|s\rangle \propto |d\rangle \tag{1.97c}$$

The states are represented diagrammatically



Define charge conjugated generators via

$$T_j^C := -T_j^* \overset{T_j=T_j^\dagger}{=} -T_j^T \tag{1.98}$$

The sign in this definition ensures

$$[T_j^C, T_k^C] = i f_{jkl} T_l^C \tag{1.99}$$

It is obvious that $T^C_{2,5,7} = T_{2,5,7}$ and $T^C_{1,3,4,6,8} = -T_{1,3,4,6,8}$, hence the charge conjugation operator $\mathcal{C}$ acts on the basis states as

$$\mathcal{C}|i_3, i_8\rangle = |-i_3, -i_8\rangle \tag{1.100}$$

This defines the charge conjugate representation $\mathbf{3}^*$ with the basis

$$|\overline{u}\rangle = |-\frac{1}{2}, -\frac{1}{2\sqrt{3}}\rangle\,, \qquad |\overline{d}\rangle = |\frac{1}{2}, -\frac{1}{2\sqrt{3}}\rangle\,, \qquad |\overline{s}\rangle = |0, -\frac{1}{\sqrt{3}}\rangle \tag{1.101}$$

The **adjoint representation** is defined by linearly mapping the generators to linear maps on the generators via the Lie bracket,

$$\mathrm{adj}(T_j)X = [T_j, X] \tag{1.102}$$

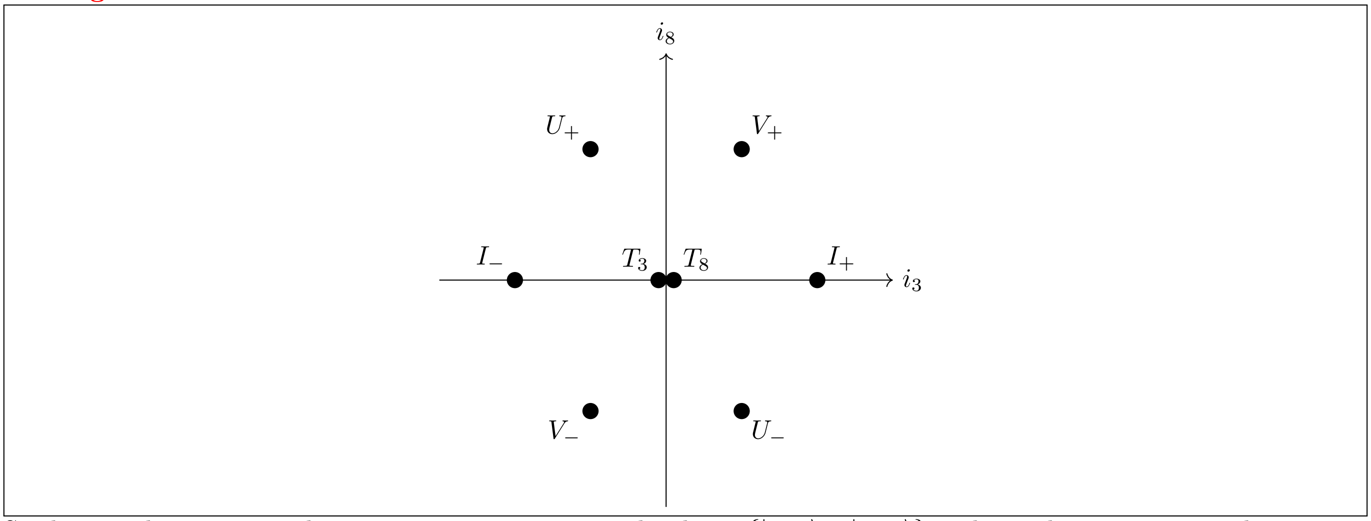
Consider $X = x_k T_k$ to find the matrix elements $[\mathrm{adj}(T_j)]_{ab}$ that fulfill $x'_l = [\mathrm{adj}(T_j)]_{lk} x_k$ if $\mathrm{adj}(T_j)X = X'$.

$$[\mathrm{adj}(T_j)]_{lk} x_k T_l = \mathrm{adj}(T_j)X = x_k[T_j, T_k] = if_{jkl} x_k T_l \qquad \Longleftrightarrow \qquad [\mathrm{adj}(T_j)]_{lk} = if_{jkl} \tag{1.103}$$

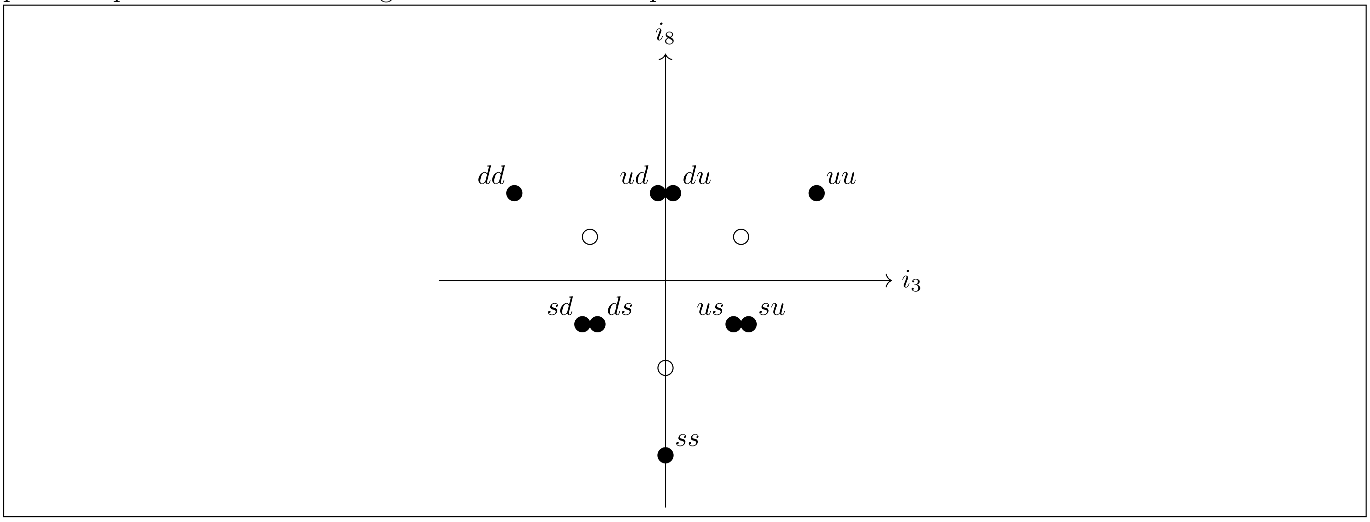
The Jacobi identity of the representation $\{T_j\}$ ensures

$$[\mathrm{adj}(T_j), \mathrm{adj}(T_k)]X = \mathrm{adj}([T_j, T_k])X \equiv if_{jkl}\mathrm{adj}(T_l)X \tag{1.104}$$

such that the adjoint representation is in fact a representation of the same algebra. The corresponding states are now the generators themselves **To do** (21)**It doesn't matter of which representation one thinks about the states, right? There are no different adjoint representations. Still, the eigenvalues depend on the precise choice of basis we did at the beginning** **To do** (22)**Why are exactly the ladder operators the correct eigenstates?**



Study now the tensor product representation $\mathbf{3} \otimes \mathbf{3}$ with a basis $\{|i_3 i_8\rangle \otimes |j_3 j_8\rangle\}$ with i, j the quantum numbers of two copies of $\mathfrak{su}(3)$. Corresponding operators are defined by $T = T^{(1)} + T^{(2)}$ and the eigenvalues of the tensor product space are sums of the eigenvalues in the two copies.



The highest weight state gets annihilated by all of $\{U_+, V_+, I_+\}$. It is therefore $u \otimes u$. To do (23)**So just like addition of angular momenta it has the highest eigenvalues for the Casimirs?** Applying ladder operators to $u \otimes u$ one finds the representation $\mathbf{6}$ of $\mathfrak{su}(3)$ given by

$$u \otimes u = |1, \frac{1}{\sqrt{3}}\rangle\,, \qquad \frac{1}{\sqrt{2}}(u \otimes d + d \otimes u) = |0, \frac{1}{\sqrt{3}}\rangle \tag{1.105a}$$

$$d \otimes d = |-1, \frac{1}{\sqrt{3}}\rangle\,, \qquad \frac{1}{\sqrt{2}}(u \otimes s + s \otimes u) = |\frac{1}{2}, -\frac{1}{2\sqrt{3}}\rangle \tag{1.105b}$$

$$s \otimes s = |0, -\frac{2}{\sqrt{3}}\rangle\,, \qquad \frac{1}{\sqrt{2}}(d \otimes s + s \otimes d) = |-\frac{1}{2}, \frac{1}{2\sqrt{3}}\rangle \tag{1.105c}$$

(1.105d)

The missing states to form a basis are

$$\frac{1}{\sqrt{2}}(d \otimes u - u \otimes d) = |0, \frac{1}{\sqrt{3}}\rangle \tag{1.106a}$$

$$\frac{1}{\sqrt{2}}(s \otimes u - u \otimes s) = |\frac{1}{2}, -\frac{1}{2\sqrt{3}}\rangle \tag{1.106b}$$

$$\frac{1}{\sqrt{2}}(s \otimes d - d \otimes s) = |-\frac{1}{2}, \frac{1}{2\sqrt{3}}\rangle \tag{1.106c}$$

(1.106d)

manifesting the $\mathbf{3}^*$ representation. To do (24)**The bra-ket notation at this point is probably incomplete...there is the quantum number of the Casimir operator missing** In this sense one decomposes $\mathbf{3} \otimes \mathbf{3} = \mathbf{6} \oplus \mathbf{3}^*$. Analogously $\mathbf{3}^* \otimes \mathbf{3}^* = \mathbf{6}^* \oplus \mathbf{3}$. One further finds $\mathbf{3} \otimes \mathbf{3}^* = \mathbf{8} \oplus \mathbf{1}$ and $\mathbf{3} \otimes \mathbf{3} \otimes \mathbf{3} = \mathbf{10} \oplus \mathbf{8} \oplus \mathbf{8} \oplus \mathbf{1}$ where $\mathbf{10}$ has all totally symmetric and $\mathbf{1}$ all totally anti-symmetric states. The states in $\mathbf{3} \otimes \mathbf{3}^*$ are called mesons and those in $\mathbf{3} \otimes \mathbf{3} \otimes \mathbf{3}$ baryons.

Additionally one imposes and additional $SU(3)_C$ color symmetry. It is assumed that all observable states are color neutral and hence singlets under $SU(3)_C$. There are no singlets in $\mathbf{3} \otimes \mathbf{3} \otimes \mathbf{3}^*$ or $\mathbf{3} \otimes \mathbf{3}$ and hence no states corresponding to these representations. **Why is color symmetry not just an additional $\otimes \mathbf{3}_C$**

Quantum numbers are

B :	baryon number
S :	strangeness, # of $\bar{s}$-quarks − # of s-quarks
J :	spin
I_3 :	3-component of isospin
$Y = B + S$:	hypercharge

To do (26)**Why these quantum numbers? I would have expected 1 isospin and 1 z-component for each particle...**

In general, does each particle type have a different field in the QFT? Yes, one could introduce composite fields for example by means of Hubbard Stratonovich. Alternatively one could keep elementary quark fields and look for bound states by investigating correlation functions on the lattice.

1.5.3 From a Field Theory Perspective

Consider N_f flavors of quarks and the Lagrangian

$$\mathscr{L} = \overline{\psi}_a(i\gamma^\mu(\partial_\mu + igA_\mu) - m)\psi^a \tag{1.107}$$

which is invariant under the $SU(N_f)$ transformation

$$\psi \to U\psi\,, \qquad \overline{\psi} \to \overline{\psi}U^\dagger \tag{1.108}$$

where $UU^\dagger = U^\dagger U = \mathbf{1}$ or $U^a{}_b(U^\dagger)^b{}_c = (U^\dagger)^a{}_b U^b{}_c = \delta^a_c$. Unitary transformations are written as $U = \exp(-i\alpha^j T_j)$ with $T_j^\dagger T_j = T_j T_j^\dagger = \mathbf{1}$ and $\mathrm{Tr}(T_j) = 0$. The hermitian traceless matrices T_j are generators of the algebra $\mathfrak{su}(N_f)$. In terms of generators

$$U^a{}_b \approx \delta^a_b - i\alpha_j (T_j)^a{}_b \tag{1.109}$$

such that

$$\delta\psi^a = -i\alpha_j (T_j)^a{}_b \psi^b\,, \qquad \delta\overline{\psi}_a = i\alpha_j \overline{\psi}_b (T_j)^b{}_a \tag{1.110}$$

By virtue of Noethers theorem the current

$$j_\mu^A(x) = -i\frac{\partial \mathscr{L}}{\partial(\partial^\mu \psi^a)}(T^A)^a{}_b\psi^b = \overline{\psi}_a\gamma_\mu (T^A)^a{}_b\psi^b \tag{1.111}$$

is conserved, $\partial^\mu j_\mu^A = 0$. Conserved charges are

$$Q^A = \int \mathrm{d}^3 x j_0^A(x) \tag{1.112}$$

meaning they commute with the generator of time evolution

$$[H, Q^A] = 0 \tag{1.113}$$

and fulfill the $\mathfrak{su}(N_f)$ algebra. To do (28) **Commutation is then related to the commutation of operator valued fields?** To do (29) **A state $|\ldots\rangle$ is some functional of the field. Acting with Q^A on the state means applying the $\hat{\psi}\ldots$ operators to this functional?**

1.6 Chiral Symmetry

Consider N_f flavors of quarks and the Lagrangian

$$\mathscr{L} = \overline{\psi}_a(i\gamma^\mu(\partial_\mu + igA_\mu) - m)\psi^a = \overline{\psi}_a(i\not{D} - m)\psi^a \tag{1.114}$$

where the flavor index $a = 1, \ldots N_f$ is implicitly summed over and $\overline{\psi} = \psi^\dagger\gamma_0$. Define γ_5-matrix via

$$\gamma_5 = i\gamma_0\gamma_1\gamma_2\gamma_3 \tag{1.115}$$

Thus $\{\gamma_5, \gamma_\mu\} = 0$ and $(\gamma_5)^2 = \mathbf{1}_{2\times 2}$.

$$\begin{aligned}(\gamma_5)^2 &= -\gamma_0\gamma_1\gamma_2\gamma_3\gamma_0\gamma_1\gamma_2\gamma_3 \\ &= (\gamma_0)^2\gamma_1\gamma_2\gamma_3\gamma_1\gamma_2\gamma_3 \\ &= (\gamma_0)^2(\gamma_1)^2\gamma_2\gamma_3\gamma_2\gamma_3 \\ &= -\prod(\gamma_\mu)^2 \\ &= \mathbb{1}_{2\times 2}\end{aligned} \tag{1.116}$$

Consider now the transformations

$$U(1)_V: \qquad \psi \to e^{i\alpha}\psi\,, \quad \overline{\psi} \to e^{-i\alpha}\overline{\psi} \tag{1.117a}$$
$$U(1)_A: \qquad \psi \to e^{i\alpha\gamma_5}\psi\,, \quad \overline{\psi} \to \overline{\psi}e^{i\alpha\gamma_5} \tag{1.117b}$$

These are symmetries of the Lagrangian except for the axial symmetry $U(1)_A$ in the case of non-vanishing masses. We wish to apply Noethers theorem

$$\delta\mathscr{L} = \partial_\mu\left(\frac{\partial\mathscr{L}}{\partial(\partial_\mu\phi)}\delta\phi\right) \tag{1.118}$$

The corresponding Noether currents $J^\mu = -\frac{\partial\mathscr{L}}{\partial(\partial_\mu\psi)}\delta\psi$ are

$$U(1)_V: \qquad J^\mu = \overline{\psi}_a\gamma^\mu\psi^a\,, \quad \partial_\mu J^\mu = 0 \tag{1.119a}$$
$$U(1)_A: \qquad J^\mu = \overline{\psi}_a\gamma^\mu\gamma_5\psi^a\,, \quad \partial_\mu J^\mu = 2im\overline{\psi}_a\gamma_5\psi^a \tag{1.119b}$$

The symmetry is explicitly written out if one uses projectors

$$P_\pm = \frac{1\pm\gamma_5}{2} \tag{1.120}$$

that satisfy

$$P_\pm P_\pm = P_\pm \,, \qquad P_\pm P_\mp = 0$$
$$P_+ + P_- = 1 \,, \qquad P_\pm \gamma^\mu = \gamma^\mu P_\mp \tag{1.121}$$

The Lagrangian is then decomposed as

$$\mathscr{L} = \overline{\psi}_a (i\not{D} - m)(P_+^2 + P_-^2)\psi^a = (\overline{\psi}_{a+} i\not{D}\psi_+^a + \overline{\psi}_{a-} i\not{D}\psi_-^a) - m(\overline{\psi}_{a-}\psi_+^a + \overline{\psi}_{a+}\psi_-^a) \tag{1.122}$$

where the projections **To do** (30)**Is γ_5 hermitian?**

$$\psi_\pm = P_\pm \psi \,, \qquad \overline{\psi}_\pm = \psi_\pm^\dagger \gamma_0 = \overline{\psi} P_\mp \tag{1.123}$$

twhere used. Using $\gamma_5 = P_+ - P_-$ The symmetry transformations (1.117) in terms of the projections take the form

$$U(1)_V : \qquad \psi_\pm \to e^{i\alpha}\psi_\pm \,, \quad \overline{\psi}_\pm \to e^{-i\alpha}\overline{\psi}_\pm \tag{1.124a}$$
$$U(1)_A : \qquad \psi_\pm \to e^{\pm i\alpha}\psi_\pm \,, \quad \overline{\psi}_\pm \to e^{\mp i\alpha}\overline{\psi}_\pm \tag{1.124b}$$

Performing simultaneous vector and axial transformations by angles α_V and α_A it follows that $\psi_\pm \to \exp(i[\alpha_V \pm \alpha_A])\psi_\pm \equiv \exp(i\beta_\pm)\psi_\pm$. On this level, it is obvious that

$$U(1)_V \times U(1)_A = U(1)_+ \times U(1)_- \tag{1.125}$$

Note how this symmetry is part of the larger $U(N_f)_+ \times U(N_f)_-$. One can use

$$U(N_f) = (SU(N_f) \times U(1))/\boldsymbol{Z}_{N_f} \tag{1.126}$$

The quotient is due to the fact that all elements $\exp(k\frac{2\pi i}{N_f})$ with $k \in \{0, 1, \dots N_f - 1\}$ viewed as $N_f \times N_f$ matrices are $\in SU(N_f)$, thus the mapping $SU(N_f) \times U(1) \to U(N_f)$ is not injective. The full symmetry group is

$$SU(N_f)_V \times SU(N_f)_A \times U(1)_V \times U(1)_A \tag{1.127}$$

To do (31)**Tong omits the $U(1)_A$, why? I guess $U(1)_A$ is somehow reconstructed from $U(1)_V$ and $SU(N_f)_A$?**
The corresponding transformations take the form

$$SU(N_f)_+ \times U(1)_+ : \qquad \psi_+^a \to e^{i\alpha_+}(U_+)^a{}_b \psi_+^b \,, \qquad \overline{\psi}_{+a} \to e^{-i\alpha_+}\overline{\psi}_{+b}(U_+^\dagger)^b{}_a \tag{1.128a}$$
$$SU(N_f)_- \times U(1)_- : \qquad \psi_-^a \to e^{i\alpha_-}(U_-)^a{}_b \psi_-^b \,, \qquad \overline{\psi}_{-a} \to e^{-i\alpha_-}\overline{\psi}_{-b}(U_-^\dagger)^b{}_a \tag{1.128b}$$

For the vector symmetry $\alpha_+ = \alpha_-$ and $U_+ = U_-$, and for the axial symmetry $\alpha_+ = -\alpha_-$ and $U_+ = U_-^\dagger$. **To do** (32)**How does $U(N > 1)$ vector/axial symmetry really look like in terms of γ_5? Since the generators are hermitian $U_+ = U_-^\dagger$ translates to $\alpha_{+j} = -\alpha_{-j}$**

$$U(N_f)_V : \qquad \psi^a \to (e^{i\alpha_j T_j})^a{}_b \psi^b \,, \qquad \overline{\psi}_b \to \overline{\psi}_b (e^{-i\alpha_j T_j})^b{}_a \tag{1.129a}$$
$$U(N_f)_A : \qquad \psi^a \to (e^{i\gamma_5 \alpha_j T_j})^a{}_b \psi^b \,, \qquad \overline{\psi}_b \to \overline{\psi}_b (e^{i\gamma_5 \alpha_j T_j})^b{}_a \tag{1.129b}$$

The conserved currents are simply

$$U(N_f)_V : \qquad J_j^\mu = \overline{\psi}_a (T_j)^a{}_b \gamma^\mu \psi^a \,, \quad \partial_\mu J^\mu = 0 \tag{1.130a}$$
$$U(N_f)_A : \qquad J_j^\mu = \overline{\psi}_a (T_j)^a{}_b \gamma^\mu \gamma_5 \psi^a \,, \quad \partial_\mu J_j^\mu = 2im\overline{\psi}_a (T_j)^a{}_b \gamma_5 \psi^b \tag{1.130b}$$

1.6.1 Unequal Flavor Masses

If the fermion flavors have different masses $m^{(a)}$ such that

$$m \sum_a (\overline{\psi}_{a-}\psi_+^a + \overline{\psi}_{a+}\psi_-^a) \to \sum_a m^{(a)} (\overline{\psi}_{a-}\psi_+^a + \overline{\psi}_{a+}\psi_-^a) \tag{1.131}$$

then, additionally to the broken axial symmetry, vector symmetry is further broken down

$$SU(N_f)_V \times U(1)_V \to U(1)^{N_f} \tag{1.132}$$

1.6.2 Quark Condensate

Non-Linear σ-Model

A condensate of the form

$$\langle\overline{\psi}_{-a}\psi_+^b\rangle = -\sigma\delta_a^b \tag{1.133}$$

breaks axial chiral symmetry spontaneously, denoted as χSB ("chiral symmetry breaking"). **Why exactly is the manifold of all possible vacua described by $\langle\overline{\psi}_{-a}\psi_+^b\rangle = -\sigma {U_a}^b$ with $U \in SU(N_f)$? Solved, see the following** It transforms according to

$$\langle\overline{\psi}_{-a}\psi_+^b\rangle \to \langle\overline{\psi}_{-c}{(U_-^\dagger)^c}_a{(U_+)^b}_d\psi_+^d\rangle = -\sigma{(U_+U_-^\dagger)^b}_a \tag{1.134}$$

Notice how $U_+U_-^\dagger$ is unitary: $(U_+U_-^\dagger)^\dagger(U_+U_-^\dagger) = U_-U_+^\dagger U_+U_-^\dagger = \mathbb{1}$. Only the vector symmetry $U_+ = U_-$ remains unbroken. The $N_f^2 - 1$ broken generators of the $SU(N_f)_A$ get replaced by Goldstone modes. To do (34) **What about $U(1)_A$? It is already broken by some anomaly, hence no Goldstone mode I guess?** These are low-wavelength excitations of the condensate, parametrized by

$$U(x) = \exp(\frac{2i}{f_\pi}\pi(x)) \qquad \text{with } \pi(x) = \pi_a(x)T_a \tag{1.135}$$

where π_a are the N_f^2-1 pion fields with pion decay constant f_π. Under the action of $U(1)_V \times SU(N_f)_+ \times SU(N_f)_-$ the pion fields transform according to

$$U(x) \to U_-^\dagger U(x)U_+ \tag{1.136}$$

What terms of the pion fields can be included in the Lagrangian that satisfy the symmetry? The only invariant 1-derivative term is $\mathrm{Tr}U^\dagger\partial_\mu U$ which vanishes since $U^\dagger\partial_\mu U$ is $\in \mathfrak{su}(N_f)$ and hence traceless. The 2-derivative terms available are

$$(\mathrm{Tr}U^\dagger\partial_\mu U)^2\,, \quad \mathrm{Tr}(\partial^\mu U^\dagger\partial_\mu U)\,, \quad \mathrm{Tr}(U^\dagger\partial_\mu U)^2 \tag{1.137}$$

however $U^\dagger\partial_\mu U = -(\partial U^\dagger)U$ and hence the unique addition the the Lagrangian is the **chiral Lagrangian** To do (35) **Why do we need to come up with such an action that is not present on the microscopic level? Is this something like an effective theory?**

$$\mathscr{L} = \frac{f_\pi^2}{4}\mathrm{Tr}(\partial^\mu U^\dagger\partial_\mu U) \tag{1.138}$$

The π-fields in this theory are coordinates on some manifold (here $SU(N_f)$) and hence this is not a free theory. This type of theory is called **non-linear sigma models**. We cannot set $U = 0$ and hence the chiral Lagrangian breaks $SU(N_f)_+ \times SU(N_f)_-$ spontaneously. Expand U in terms of π. Note that in general $\partial_\mu\pi$ and π do not commute. Also, upon expanding the exponential in $U^\dagger$ and U respectively, odd powers of π appear with different signs and cancel out after multiplication. The expansion up to $\mathcal{O}(\pi^4)$ takes the following form:

$$U^{(\dagger)} = \exp(\overset{(-)}{+}\frac{2i}{f_\pi}\pi) = \mathbf{1} \overset{(-)}{+} \frac{2i}{1!f_\pi}\pi - \frac{4}{2!f_\pi^2}\pi^2 \overset{(+)}{-} \frac{8i}{3!f_\pi^3}\pi^3 + \mathcal{O}(\pi^4) \tag{1.139a}$$

$$\mathscr{L} = \frac{f_\pi^2}{4}\mathrm{Tr}\left[\frac{4}{f_\pi^2}\partial^\mu\pi\partial_\mu\pi + \frac{16i^2}{1!3!f_\pi^4}(\partial^\mu\pi\partial_\mu(\pi^3) + \partial^\mu(\pi^3)\partial_\mu\pi) + \frac{16}{2!2!f\pi^4}\partial^\mu(\pi^2)\partial_\mu(\pi^2)\right] + \mathcal{O}(\pi^5) \tag{1.139b}$$

$$= \mathrm{Tr}[\partial^\mu\pi\partial_\mu\pi] + \mathrm{Tr}[(-2\cdot 2\cdot\frac{2}{3f_\pi^2} + 2\cdot\frac{1}{f_\pi^2})(\partial^\mu\pi\partial_\mu)\pi\cdot\pi^2 + (-2\cdot\frac{2}{3f_\pi^2} + 2\cdot\frac{1}{f_\pi^2})(\partial^\mu\pi)\pi(\partial_\mu\pi)\pi] + \mathcal{O}(\pi^5) \tag{1.139c}$$

$$= \mathrm{Tr}[\partial^\mu\pi\partial_\mu\pi] + \frac{2}{3f_\pi^2}\mathrm{Tr}[-(\partial^\mu\pi\partial_\mu\pi)\pi^2 + (\partial^\mu\pi)\pi(\partial_\mu\pi)\pi] + \mathcal{O}(\pi^5) \tag{1.139d}$$

With the normalization $T^aT^b = \frac{1}{2}\delta^{ab}$ the kinetic term is canonically normalized. The pion fields originating from $\pi = \pi_jT_j$ in the representation (1.91) are relabelled according to

$$\pi^\pm = \frac{1}{\sqrt{2}}(\pi^1 \mp i\pi^2)\,, \qquad \pi^0 = \pi^3 \tag{1.140a}$$

$$K^\pm = \frac{1}{\sqrt{2}}(\pi^4 \mp i\pi^5)\,, \qquad K^0 = \pi^6 - i\pi^7\,, \qquad \overline{K}^0 = \pi^6 + i\pi^7 \tag{1.140b}$$

$$\eta = \pi^8 \tag{1.140c}$$

To do (36) **In Flörchinger notes, he writes about symmetry breaking patterns and choices of reps to break the symmetry. What does that mean?**

Linear σ-Model

Important 1.6|1: Non-Linear to Linear σ-Model

The condensate is generally of the form

$$\langle\overline{\psi}_{-a}\psi^b_+\rangle = -(\sigma(x)+\delta\sigma(x))(U(x))^b{}_a = (\sigma(x)+\delta\sigma(x))(\delta^b_a + i\pi_j(x)(T_j)^b{}_a + \mathcal{O}(\pi^2)) \tag{1.141}$$

Gauge invariant effective action terms may take the form

$$\mathrm{Tr}(U^\dagger U) = \pi_j\pi_k\mathrm{Tr}(T_jT_k) + \mathcal{O}(\pi^4) \sim \boldsymbol{\pi}^2 + \mathcal{O}(\pi^4) \tag{1.142}$$

if the normalization is chosen such that $\mathrm{Tr}(T_jT_k) \propto \delta_{jk}$.

Take $N_f = 2$ fermion flavors. The $\mathfrak{su}(2)$ algebra $[\lambda_j,\lambda_k] = i\epsilon_{jkl}\lambda_l$ is generated by the matrices

$$\lambda_j = \frac{1}{2}\sigma_j \tag{1.143}$$

where the Pauli matrices are

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad \sigma_1 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \qquad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \tag{1.144}$$

that fulfill $[\sigma_j,\sigma_k] = 2i\epsilon_{jkl}\sigma_l$. The Lagrangian of the linear sigma model reads **To do (37) Why does the σ-field and the π-fields appear with the same coefficients?**

$$\mathscr{L} = \frac{1}{2}\partial_\mu\sigma\partial^\mu\sigma + \frac{1}{2}\partial_\mu\boldsymbol{\pi}\partial^\mu\boldsymbol{\pi} - \frac{1}{2}m^2(\sigma^2+\boldsymbol{\pi}^2) - \frac{\lambda}{24}(\sigma^2+\boldsymbol{\pi}^2)^2 - \epsilon\sigma \tag{1.145}$$

ϵ models the breaking of $SU(2)$ and gives mass to the Goldstone bosons $\boldsymbol{\pi}$. To find the VEV of σ we need to minimize

$$V_\epsilon(\sigma) = \frac{1}{2}m^2\sigma^2 + \frac{\lambda}{24}\sigma^4 - \epsilon\sigma \tag{1.146a}$$

$$\frac{\mathrm{d}}{\mathrm{d}\sigma}V_\epsilon(\sigma) = m^2\sigma + \frac{\lambda}{6}\sigma^3 - \epsilon \overset{!}{=} 0 \tag{1.146b}$$

Take an ansatz $v = \sigma_{\min}(\epsilon) = v_{(0)} + \epsilon v_{(1)} + \dots$. SSB occurs when $-m^2 =: \mu^2 > 0$. Then $v_{(0)}^2 = \frac{6\mu^2}{\lambda}$ and

$$-\mu^2 v_{(1)} + \frac{\lambda}{6}\cdot 3v_{(0)}^2 v_{(1)} + 1 = 0 \tag{1.146c}$$

$$(-\mu^2 + 3\mu^2)v_{(1)} = 1 \tag{1.146d}$$

implying $v_{(1)} = \frac{1}{2\mu^2}$. With the parametrization $\sigma(x) = v + \delta\sigma(x)$ one finds

$$\begin{aligned}\mathscr{L} = {}& \frac{1}{2}\partial_\mu(\delta\sigma)\partial^\mu(\delta\sigma) + \frac{1}{2}\partial_\mu\boldsymbol{\pi}\partial^\mu\boldsymbol{\pi} + \frac{1}{2}\mu^2(v^2 + 2v\delta\sigma + (\delta\sigma)^2 + \boldsymbol{\pi}^2) \\ & - \frac{\lambda}{24}(v^2 + 2v\delta\sigma + (\delta\sigma)^2 + \boldsymbol{\pi}^2)^2 - \epsilon(v+\delta\sigma)\end{aligned} \tag{1.147a}$$

Expand

$$(v^2 + 2v(\delta\sigma) + (\delta\sigma)^2 + \boldsymbol{\pi}^2)^2 = v^4 + 4v^2(\delta\sigma)^2 + (\delta\sigma)^4 + \boldsymbol{\pi}^4 + 4v^3(\delta\sigma) + 2v^2(\delta\sigma)^2 + 2v^2\boldsymbol{\pi}^2 + 4v(\delta\sigma)^3 + 4v(\delta\sigma)\boldsymbol{\pi}^2 + 2(\delta\sigma)^2\boldsymbol{\pi}^2 \tag{1.147b}$$

The mass term of the pions is given by

$$-\frac{1}{2}m_\pi^2 = \frac{1}{2}\mu^2 - \frac{\lambda}{24}2v^2 = \frac{1}{2}(\mu^2 - \frac{\lambda}{6}(v_{(0)} + \epsilon v_{(1)} + \mathcal{O}(\epsilon^2))^2) = -\epsilon\frac{\lambda}{6}v_{(0)}v_{(1)} + \mathcal{O}(\epsilon^2) = -\frac{1}{2}\epsilon\frac{\sqrt{\lambda}}{\sqrt{6}\mu} + \mathcal{O}(\epsilon^2) \tag{1.148}$$

and the $\delta\sigma$-mass by

$$-\frac{1}{2}m_\sigma^2 = \frac{1}{2}\mu^2 - \frac{\lambda}{24}6v^2 = \frac{1}{2}(\mu^2 - \frac{\lambda}{2}v_{(0)}^2 + \mathcal{O}(\epsilon)) = -\frac{1}{2}(2\mu^2) + \mathcal{O}(\epsilon) \tag{1.149}$$

Note that $[\lambda] = [m^2]\cdot[\sigma^{-2}]$ and $[\epsilon] = [m^2]\cdot[\sigma]$.

Remark 1.6|2: Alternative Formulation

One could also write $\phi^a = (\sigma, \boldsymbol{\pi})$ and state the Lagrangian as

$$\mathscr{L} = \frac{1}{2}\partial_\mu\phi_a\partial^\mu\phi_a - \frac{\tilde{\lambda}}{4}(\phi_a\phi_a - v^2)^2 + H_a\phi_a \tag{1.150a}$$
$$= \cdots + \frac{\tilde{\lambda}}{2}v^2\phi_a\phi_a - \frac{\tilde{\lambda}}{4}(\phi_a\phi_a)^2 + H_a\phi_a \tag{1.150b}$$

with the identification

$$\tilde{\lambda} = \frac{\lambda}{3!}, \qquad v^2 = -\frac{m^2}{\tilde{\lambda}} = \frac{6\mu^2}{\lambda}, \qquad H_a = (\epsilon, \mathbf{0}) \tag{1.151}$$

SSB is parametrized best with $\phi_a \to \phi_a + v_a$ (or $(\sigma, \boldsymbol{\pi}) \to (v + \delta\sigma, \boldsymbol{\pi})$) where $v_a = (v, \mathbf{0})$.

$$\mathscr{L} = \frac{1}{2}\partial_\mu\phi_a\partial^\mu\phi_a - \frac{\tilde{\lambda}}{4}(\phi_a\phi_a + 2v_a\phi_a)^2 + H_a\phi_a + H_a v_a \tag{1.152}$$

The equations of motion read

$$\partial_\mu\partial^\mu\phi_a = -\tilde{\lambda}(\phi_b\phi_b + 2v_b\phi_b)(\phi_a + v_a) \tag{1.153}$$

1.7 Linear Response Theory

View the freezout surface of the QGP after a HIC as a source for pioinic fields. Linear response theory investigaes the change of a field expectation value $\langle\chi_a(x)\rangle$ after a perturbation $\Delta S[\phi] = \int \mathrm{d}^d y j_b(y)\chi_b(y)$ was applied to the system. To linear order in j_b the perturbation may be written as

$$\overline{\chi}_a(x) = \langle\chi_a(x)\rangle = \int \mathrm{d}^d y \Delta^R_{ab}(x,y) j_b(y) \tag{1.154}$$

In general, the system at final time t_f (i.e. at the moment of measurement) is given by some density matrix ρ_f and the expectation value of a field becomes

$$\langle\chi_a(x)\rangle_{t_f} = \mathrm{Tr}_{t_f}(\rho_f\chi_a(x)) \tag{1.155}$$

Important 1.7|1: Density Matrix and Unitary Time Evolution in Schwinger-Keldysh formalism

The density matrix is a functional of fields, $\rho_f = \rho_f[\phi_+, \phi_-]$, for example $\rho_t = |\psi_t\rangle\langle\psi_t|$ with $\langle\phi(\boldsymbol{x})|\psi_t\rangle = \psi_t[\phi]$, and hence evolves like (use $U(t_f, t_i) = U(t_i \to t_f)$)

$$\rho_{t_f} = U(t_f, t_i)\rho_{t_i}U^\dagger(t_f, t_i) \tag{1.156a}$$
$$\rho_{t_f}[\phi_+, \phi_-] = \int \mathcal{D}\phi'_+\mathcal{D}\phi'_- U(t_f, t_i)[\phi_+, \phi'_+]\rho_{t_i}[\phi'_+, \phi'_-]U^\dagger(t_f, t_i)[\phi'_-, \phi_-] \tag{1.156b}$$

Generically in a quantum theory, the unitary time evolution operator is given by

$$U(t_f, t_i; q_f, q_i) = \int\limits_{\substack{q(t_i)=q_a \\ q(t_f)=q_f}} \mathcal{D}q(t) \int \mathcal{D}p(t) \exp\left(i\int_{t_i}^{t_f} \mathrm{d}t(p(t)\dot{q}(t) - H(q(t), p(t)))\right) \tag{1.157}$$

where $(q(t), p(t))$ are all possible phase space trajectories subject to the boundary conditions q_i, q_f. Usually the conjugate momenta are integrated out by means of Gaussian integration. In QFT where field configurations parametrize the phase spce, the path integral is generalized to

$$U(t_f, t_i)[\phi_+, \phi_-] = \int\limits_{\substack{\phi(t_i,\boldsymbol{x})=\phi_-(\boldsymbol{x}) \\ \phi(t_f,\boldsymbol{x})=\phi_+(\boldsymbol{x})}} \mathcal{D}\phi \exp\Big(i\underbrace{\int_{t_i}^{t_f} \mathrm{d}^4x\mathcal{L}(\phi, \partial_\mu\phi)}_{=S[\phi]}\Big) \tag{1.158}$$

For thermal states one takes the density matrix

$$\rho = \frac{1}{Z(\beta)} e^{-\beta H}\,, \qquad Z(\beta) = \mathrm{Tr}(e^{-\beta H}) \tag{1.159}$$

This looks analogous to the time evolution operator $e^{-i\Delta t H}$ if $\Delta t = -i\beta$. Hence define the substitution

$$t = -i\tau\,, \qquad \frac{\partial}{\partial \tau} = \frac{\partial t}{\partial \tau}\frac{\partial}{\partial t} = -i\frac{\partial}{\partial t} \tag{1.160}$$

where τ is called Euclidean time, which is now integrated over $[0, \beta)$.

This leads to

$$\langle \chi_a(x)\rangle_{t_f} = \int \mathcal{D}\phi_+ \mathcal{D}\phi_- \rho_{t_f}[\phi_+, \phi_-]\chi_a(x)[\phi_-, \phi_+] \tag{1.161}$$

$$= \int \mathcal{D}\phi_+ \mathcal{D}\phi_- \mathcal{D}\phi'_+ \mathcal{D}\phi'_- U(t_f, t_i)[\phi_+, \phi'_+]\rho_{t_i}[\phi'_+, \phi'_-]U^\dagger(t_f, t_i)[\phi'_-, \phi_-]\chi_a(x)[\phi_-, \phi_+] \tag{1.162}$$

$$\tag{1.163}$$

Now notice for example if ρ_{t_i} represents the thermal state

$$\rho[\phi'_+, \phi'_-] = \frac{1}{Z} \int\limits_{\substack{\phi_0(t_i, \boldsymbol{x}) = \phi'_-(\boldsymbol{x}) \\ \phi_0(t_i - i\beta, \boldsymbol{x}) = \phi'_+(\boldsymbol{x})}} \mathcal{D}\phi_0 e^{-S_E[\phi_0]} \tag{1.164}$$

then we find

$$\int \mathcal{D}\phi'_+(\boldsymbol{x}) U(t_f, t_i)[\phi_+, \phi'_+]\rho_{t_i}[\phi'_+, \phi'_-] = \frac{1}{Z} \int\limits_{\substack{\phi_0(t_i, \boldsymbol{x}) = \phi'_-(\boldsymbol{x}) \\ \phi_0(t_i - i\beta, \boldsymbol{x}) = \phi(t_i, \boldsymbol{x}) \\ \phi(t_f, \boldsymbol{x}) = \phi_+(\boldsymbol{x})}} \mathcal{D}\phi \mathcal{D}\phi_0 e^{iS[\phi]} e^{-S_E[\phi_0]} \tag{1.165}$$

and further (keeping in mind that $U^\dagger(t_f, t_i)[\phi'_-, \phi_-] = U^*(t_f, t_i)[\phi_-, \phi'_-]$)

$$\int \mathcal{D}\phi'_+(\boldsymbol{x}) \mathcal{D}\phi'_-(\boldsymbol{x}) U(t_f, t_i)[\phi_+, \phi'_+]\rho_{t_i}[\phi'_+, \phi'_-]U^\dagger(t_f, t_i)[\phi'_-, \phi_-] = \frac{1}{Z} \int\limits_{\substack{\tilde{\phi}(t_f, \boldsymbol{x}) = \phi_-(\boldsymbol{x}) \\ \tilde{\phi}(t_i, \boldsymbol{x}) = \phi_0(t_i, \boldsymbol{x}) \\ \phi_0(t_i - i\beta, \boldsymbol{x}) = \phi(t_i, \boldsymbol{x}) \\ \phi(t_f, \boldsymbol{x}) = \phi_+(\boldsymbol{x})}} \mathcal{D}\phi \mathcal{D}\phi_0 \mathcal{D}\tilde{\phi} e^{iS[\phi]} e^{-S_E[\phi_0]} e^{-iS^*[\tilde{\phi}]} \tag{1.166}$$

Further using $\chi_a(x)[\phi_-, \phi_+] \propto \delta[\phi_- - \phi_+]$ the functional integrals over $\phi_+(\boldsymbol{x})$ and $\phi_-(\boldsymbol{x})$ become trivial, only adding the boundary condition $\tilde{\phi}(t_i, \boldsymbol{x}) = \phi(t_f, \boldsymbol{x})$. In summary, the expectation value of intereset reads

$$\langle \chi_a(x)\rangle_{t_f} = \frac{1}{Z} \int \mathcal{D}\phi \mathcal{D}\phi_0 \mathcal{D}\tilde{\phi} \chi_a(x)[\phi] e^{iS[\phi]} e^{-S_E[\phi_0]} e^{-iS^*[\tilde{\phi}]} \tag{1.167a}$$

with boundary conditions

$$\tilde{\phi}(t_i, \boldsymbol{x}) = \phi_0(t_i, \boldsymbol{x})\,, \qquad \phi_0(t_i - i\beta, \boldsymbol{x}) = \phi(t_i, \boldsymbol{x})\,, \qquad \phi(t_f, \boldsymbol{x}) = \tilde{\phi}(t_f, \boldsymbol{x}) \tag{1.167b}$$

Expanding the exponentials

$$e^{\overset{(-)}{+} i(S + \Delta S)^{(*)}} \approx e^{\overset{(-)}{+} iS^{(*)}} (1 \overset{(-)}{+} i\Delta S^{(*)}) \tag{1.168}$$

Using cyclicity of the trace and assuming $\langle \chi_a(x)\rangle_{t_f, j\equiv 0} = 0$ we find (assuming j_b and χ_b are real)

$$\langle \chi_a(x)\rangle_{t_f, j} = i \int \mathrm{d}^d y j_b(y) \langle -\chi_b(y)\chi_a(x) + \chi_a(x)\chi_b(y)\rangle_{t_f} = \Theta(y^0 - x^0) i \int \mathrm{d}^d y j_b(y) \langle [\chi_a(x), \chi_b(y)]\rangle \tag{1.169}$$

hence it is clear that Δ^R_{ab} is the retarded propagator.

To simulate the freezout happening around τ_0, one may set $j_b(y)$ to be a narrow distribution peaked around τ_0, depending on hydro-variables like density, fluid velocity…

Chapter 2

Calculation of Condensed Field

2.1 Relating Fluid and Pion Fields

The signature is $(-,+,+,+)$.

Start with the Lagrangian of the linear σ-model for the real-valued $O(4)$-vector $\varphi_a = (\sigma, \boldsymbol{\pi})$.

$$\mathscr{L} = -\frac{1}{2}\partial_\mu\sigma\partial^\mu\sigma - \frac{1}{2}\partial_\mu\boldsymbol{\pi}\partial^\mu\boldsymbol{\pi} - \frac{1}{2}m^2(\sigma^2+\boldsymbol{\pi}^2) - \frac{\lambda}{4}(\sigma^2+\boldsymbol{\pi}^2)^2 - \epsilon\sigma \tag{2.1}$$

and from 36 we find, that if $m^2 = -\mu^2 < 0$ SSB occurs (plus, $\epsilon \neq 0$ explicitly breaks the symmetry) with the VEV v and masses of the σ and $\boldsymbol{\pi}$ excitation in $\varphi_a = (v + \delta\sigma, \boldsymbol{\pi})$ are given by (note the difference in conventions $\lambda \to 6\lambda$ w.r.t. 36)

$$\begin{cases} v_0 = \dfrac{\mu}{\sqrt{\lambda}} + \epsilon\dfrac{1}{2\mu^2} + \mathcal{O}(\epsilon^2) \\ m_\sigma^2 = 2\mu^2 + \mathcal{O}(\epsilon) \\ m_\pi^2 = \epsilon\dfrac{\sqrt{\lambda}}{\mu} + \mathcal{O}(\epsilon^2) \end{cases} \quad \overset{\text{to lowest order}}{\Longleftrightarrow} \quad \begin{cases} \mu = \dfrac{m_\sigma}{\sqrt{2}} \\ \lambda = \dfrac{m_\sigma^2}{2v_0^2} \\ \epsilon = v_0 m_\pi^2 \end{cases} \tag{2.2}$$

Choose now a fixed alignment of the condensate $\boldsymbol{\pi} = \pi\boldsymbol{e}$ with $\boldsymbol{e}\cdot\boldsymbol{e} = 1$ determining the orientation in isospin space. This choice breaks $O(4)$ to $O(2)$ (we restrict ourselves to $SO(2)$) and use to isomorphism $SO(2) \cong U(1)$ to write the linear σ-model as a theory of a complex scalar field $\phi = \frac{1}{\sqrt{2}}(\sigma + i\pi) = \rho e^{i\vartheta}$.

$$\text{in terms of}\,(\phi,\phi^*) \qquad \mathscr{L} = -(\partial_\mu\phi)(\partial^\mu\phi^*) + \mu^2\phi\phi^* - \frac{\lambda}{2}(\phi\phi^*)^2 + \frac{\epsilon}{\sqrt{2}}(\phi+\phi^*) \tag{2.3a}$$

$$\text{in terms of}\,(\rho,\vartheta^*) \qquad \mathscr{L} = -(\partial_\mu\rho)(\partial^\mu\rho) - \rho^2(\partial_\mu\vartheta)(\partial^\mu\vartheta) + \mu^2\rho^2 - \frac{\lambda}{2}\rho^4 + \sqrt{2}\epsilon\rho\cos\vartheta \tag{2.3b}$$

The classical equations of motion arising from this are

$$\text{in terms of}\,(\phi,\phi^*) \qquad -\Box\phi = (\partial_t^2 - \boldsymbol{\nabla}^2)\phi = \big[\mu^2 - \lambda(\phi^*\phi)\big]\phi + \frac{\epsilon}{\sqrt{2}} \tag{2.4a}$$

$$\text{in terms of}\,(\rho,\vartheta^*) \qquad -\Box\rho = \big[-(\partial_\mu\vartheta)(\partial^\mu\vartheta) + \mu^2 - \lambda\rho^2\big]\rho + \frac{\epsilon}{\sqrt{2}}\cos\vartheta \tag{2.4b}$$

$$-\partial_\mu(\rho^2\partial^\mu\vartheta) = -\frac{\epsilon}{\sqrt{2}}\rho\sin\vartheta \tag{2.4c}$$

The conserved Noether current of thse symmetry $\theta \to \theta + \alpha$ (in the limit $\epsilon \to 0$) and energy-momentum tensor are

$$\alpha j^\mu = -\frac{\partial\mathscr{L}}{\partial(\partial_\mu\vartheta)}\delta\vartheta = 2\alpha\rho^2(\partial^\mu\vartheta) \tag{2.5a}$$

$$T^{\mu\nu} = \frac{2}{\sqrt{-g}}\frac{\delta(\sqrt{-g}\mathscr{L})}{\delta g_{\mu\nu}} = 2\frac{\partial\mathscr{L}}{\partial g_{\mu\nu}} + g^{\mu\nu}\mathscr{L} = 2\big[(\partial^\mu\rho)(\partial^\nu\rho) + \rho^2(\partial^\mu\vartheta)(\partial^\nu\vartheta)\big] + g^{\mu\nu}\mathscr{L} \tag{2.5b}$$

where $\delta\sqrt{-g} = \frac{-1}{2\sqrt{-g}}\delta g = \frac{-1}{\sqrt{-g}}gg^{\mu\nu}\delta g_{\mu\nu} = \frac{\sqrt{-g}}{2}g^{\mu\nu}\delta g_{\mu\nu}$ was used.

In a tree-level approximation one only needs to solve the classical equations of motion. In the limit $\epsilon \to 0$ a valid solution is

$$\partial_\mu \vartheta = \text{const.}\,, \qquad \rho = \sqrt{\frac{\chi^2 + \mu^2}{\lambda}} + \mathcal{O}(\epsilon) \qquad (\chi^2 := -\left(\partial_\mu \vartheta\right)(\partial^\mu \vartheta)) \tag{2.6}$$

We might generalize the solution to the case $\partial_\mu \vartheta \approx$ const. which should be valid in the limit $\chi^2 \ll \mu^2 = \frac{m_\sigma^2}{2}$. On these solutions one finds

$$\mathscr{L}\big|_{\text{EOM}} = \rho^2(\chi^2 + \mu^2 - \frac{\lambda}{2}\rho^2) = \rho^2 \frac{\chi^2 + \mu^2}{2} \tag{2.7a}$$

$$T^{\mu\nu}\big|_{\text{EOM}} = 2\rho^2 (\partial^\mu \vartheta)(\partial^\nu \vartheta) + g^{\mu\nu} \rho^2 \frac{\chi^2 + \mu^2}{2} \tag{2.7b}$$

Assume the dynamics of the field could be described by ideal hydrodynamics, i.e. a conserved current and energy-momentum tensor of the form To do (38)**I really don't know about the signature**

$$j^\mu = n_s v^\mu \qquad (v^\mu v_\mu = -1) \tag{2.8a}$$

$$T^{\mu\nu} = (\epsilon_s + P_s) v^\mu v^\nu + g^{\mu\nu} P_s \tag{2.8b}$$

from which the prefactors can be extracted by the contractions

$$n_s = \sqrt{-j^\mu j_\mu}\,, \qquad \epsilon_s = v_\mu v_\nu T^{\mu\nu}\,, \qquad P_s = \frac{1}{3}(g_{\mu\nu} + v_\mu v_\nu) T^{\mu\nu} \tag{2.9}$$

Identifying the field theoretic with the hydrodynamic viewpoint it immediately follows that

$$n_s = 2\rho^2 \chi\,, \qquad v^\mu = \chi^{-1}(\partial^\mu \vartheta) \qquad (\iff \chi = -v^\mu(\partial_\mu \vartheta)) \tag{2.10a}$$

$$\epsilon_s = 2\rho^2\chi^2 - \rho^2 \frac{\chi^2 + \mu^2}{2} = \rho^2 \frac{4\chi^2 - (\chi^2 + \mu^2)}{2} = \frac{(\chi^2 + \mu^2)(3\chi^2 - \mu^2)}{2\lambda} \tag{2.10b}$$

How to apply this?

The freeze-out surface is invariant under rotations (= independent of polar angle φ) around the collision axis and longitudinal boosts (= independent of rapidity η_s) and hence parametrized by a one-dimensional curve in the r-τ-plane. The curve itself may be parametrized by some real parameter α, following some mapping $\alpha \mapsto (r(\alpha), \tau(\alpha))$. From the hydro simulation we wish to identify the gradient $\partial_\mu \vartheta \sim u_\mu$ of the complex phase of the condensate field with the fluid 4-velocity u_μ, hence in order to find the phase of the field, an integration of $\partial_\mu \vartheta$ over the hypersurface is needed and an integration constant ϑ_0 can be chosen freely. Choose $\alpha = \arctan(\tau/r)$ to be the polar angle of the point $(r(\alpha), \tau(\alpha))$ in the r-τ-plane. Since $r, \tau > 0$ α is restricted to the range $[0, \pi]$ and $\vartheta(\alpha)$ on the hypersurface can be calculated via

$$\vartheta(\alpha) = \vartheta_0 + \int_0^\alpha \mathrm{d}s \frac{\mathrm{d}\vartheta}{\mathrm{d}s} = \vartheta_0 + \int_0^\alpha \mathrm{d}s \frac{\partial x^\mu(s)}{\partial s} \partial_\mu \vartheta \tag{2.11}$$

$\partial x^\mu(s)/\partial s$ represents the tangent vector of the freeze-out surface.

The energy density ϵ and 4-velocity u^μ of the fluid is related to the condensate phase and density via

$$-(\partial_\mu \vartheta)(\partial^\mu \vartheta) = \chi^2 = \frac{-\mu^2 + \sqrt{6\epsilon\lambda + 4\mu^4}}{3} \tag{2.12a}$$

$$\partial^\mu \vartheta = \chi u^\mu\,, \qquad \rho^2 = \sqrt{\frac{\chi^2 + \mu^2}{\lambda}} \tag{2.12b}$$

One may use the relations (2.2) to rewrite the above equation in terms of the particle masses and the σ-vev v_0.

$$\chi^2 = \frac{-m_\sigma^2/2 + \sqrt{3\epsilon m_\sigma^2/v_0^2 + m_\sigma^4}}{3} = \frac{-m_\sigma^2 + 2m_\sigma \sqrt{3\epsilon/v_0^2 + m_\sigma^2}}{6}\,, \qquad \rho = \sqrt{v_0 \frac{2\chi^2 + m_\sigma^2}{m_\sigma^2}} \tag{2.13}$$

To do (39)**This is a number of order m_Σ^2. Of what order is u_μ (in eV^{-1}? But then again, isn't u^μ normalized to ± 1?)**

In Milne coordinates $(x^\mu) = (\tau, r, \varphi, \eta_s)$ the 4-velocity has components $(u^\mu) = (\gamma, \gamma v, 0, 0)$, where $v = v(\alpha)$ is a function on the freeze-out surface. To do (40)**Are these upstairs or downstairs indexed coordinates?**

2.1.1 Conserved Currents from Chiral Symmetry

Define

$$\Phi = \sigma\mathbb{1} + i\pi^a\tau^a = \begin{pmatrix} \sigma + i\pi^0 & \sqrt{2}\pi^+ \\ \sqrt{2}\pi^- & \sigma - i\pi^0 \end{pmatrix}, \qquad \pi^\pm = \frac{1}{\sqrt{2}}(\pi^1 \mp i\pi^2) \tag{2.14}$$

where τ^a, $a \in \{0,1,2\}$ are the Pauli matrices (more precisely, $\tau^a = \sigma^{a+1}$ since σ^0 is usually reserved for the identity). Using $(\tau^a)^\dagger = \tau^a$, $\mathrm{Tr}(\tau^a\tau^b) = 2\delta^{ab}$ and $\mathrm{Tr}(\tau^a) = 0$ one immediately finds for example $\mathrm{Tr}(\Phi^\dagger\Phi) \equiv [\Phi^\dagger]_{jk}[\Phi]_{kj} = 2(\sigma^2 + \pi^a\pi^a)$ and the formerly stated linear σ-model Lagrangian can easily be shown to be equivalent to

$$\mathscr{L} = -\frac{1}{2}\mathrm{Tr}\big[(\partial_\mu\Phi^\dagger)(\partial_\mu\Phi)\big] - \frac{1}{2}m^2\mathrm{Tr}\big[\Phi^\dagger\Phi\big] - \frac{\lambda}{4}\Big(\mathrm{Tr}\big[\Phi^\dagger\Phi\big]\Big)^2 - \frac{\epsilon}{2}\mathrm{Tr}\big[\Phi^\dagger + \Phi\big] \tag{2.15}$$

Investigate the transformation behaviour of the field Φ under chiral symmetry transformation, following [Koc97]. These symmetries are

$$U_V: \qquad \psi \mapsto e^{-\frac{i}{2}\alpha_a\tau_a}\psi \tag{2.16a}$$

$$U_A: \qquad \psi \mapsto e^{-\frac{i}{2}\gamma_5\alpha_a\tau_a}\psi \tag{2.16b}$$

Pions and the σ field are certain bound states of quarks, namely $\pi_a = i\overline{\psi}\tau_a\gamma_5\psi$ and $\sigma = \overline{\psi}\psi$. Under the above transformations one finds infinitesimally

$$U_V: \qquad \pi_a \mapsto \pi_a + \epsilon_{abc}\alpha_b\pi_c \qquad \sigma \mapsto \sigma \tag{2.16c}$$

$$U_A: \qquad \pi_a \mapsto \pi_a + \alpha_a\sigma \qquad \sigma \mapsto \sigma - \pi_a\alpha_a \tag{2.16d}$$

and after some calculations

$$U_V: \qquad \Phi^{(\dagger)} \mapsto \Phi^{(\dagger)} - i\frac{\alpha^a}{2}[\tau^a, \Phi^{(\dagger)}] \tag{2.16e}$$

$$U_A: \qquad \Phi^{(\dagger)} \mapsto \Phi^{(\dagger)} \overset{(-)}{+} i\frac{\alpha^a}{2}\{\tau^a, \Phi^{(\dagger)}\} \tag{2.16f}$$

This infintesimal transformation behaviour corresponds to the finite transformations

$$U_V: \qquad \Phi^{(\dagger)} \mapsto U\Phi^{(\dagger)}U^\dagger \tag{2.16g}$$

$$U_A: \qquad \Phi \mapsto U^\dagger\Phi U^\dagger, \quad \Phi^\dagger \mapsto U\Phi^\dagger U \tag{2.16h}$$

with $U = \exp\big(-\frac{i}{2}\alpha^a\tau^a\big)$. Finally, the conserved currents are

$$J_V^\mu = -\frac{\partial\mathscr{L}}{\partial[\partial_\mu\Phi]_{jk}}[\delta\Phi]_{jk} - \frac{\partial\mathscr{L}}{\partial[\partial_\mu\Phi^\dagger]_{jk}}[\delta\Phi^\dagger]_{jk} \tag{2.17a}$$

$$= \frac{-i\alpha^a}{2}\frac{1}{2}\Big([\partial^\mu\Phi^\dagger]_{kj}[\tau^a,\Phi]_{jk} + [\partial^\mu\Phi]_{kj}[\tau^a,\Phi^\dagger]_{jk}\Big) \tag{2.17b}$$

$$= \frac{-i\alpha^a}{2}\frac{1}{2}\mathrm{Tr}\Big((\partial_\mu\Phi^\dagger)[\tau^a,\Phi] + (\partial_\mu\Phi)[\tau^a,\Phi^\dagger]\Big) \tag{2.17c}$$

$$= \frac{-i\alpha^a}{2}\frac{1}{2}\mathrm{Tr}\Big(\big((\partial_\mu\sigma) - i\tau^d(\partial_\mu\pi^d)\big)\cdot 2\epsilon^{abc}\tau^b\pi^c + \big((\partial_\mu\sigma) + i\tau^d(\partial_\mu\pi^d)\big)\cdot(-2)\epsilon^{abc}\tau^b\pi^c\Big) \tag{2.17d}$$

$$= \frac{-i\alpha^a}{2}\frac{1}{2}\cdot\big(-8i\epsilon^{abc}(\partial_\mu\pi^b)\pi^c\big) \tag{2.17e}$$

and

$$J_A^\mu = -\frac{\partial\mathscr{L}}{\partial[\partial_\mu\Phi]_{jk}}[\delta\Phi]_{jk} - \frac{\partial\mathscr{L}}{\partial[\partial_\mu\Phi^\dagger]_{jk}}[\delta\Phi^\dagger]_{jk} \tag{2.17f}$$

$$= \frac{i\alpha^a}{2}\frac{1}{2}\Big([\partial^\mu\Phi^\dagger]_{kj}\{\tau^a,\Phi\}_{jk} - [\partial^\mu\Phi]_{kj}\{\tau^a,\Phi^\dagger\}_{jk}\Big) \tag{2.17g}$$

$$= \frac{i\alpha^a}{2}\frac{1}{2}\mathrm{Tr}\Big((\partial_\mu\Phi^\dagger)\{\tau^a,\Phi\} - (\partial_\mu\Phi)\{\tau^a,\Phi^\dagger\}\Big) \tag{2.17h}$$

$$= \frac{i\alpha^a}{2}\frac{1}{2}\mathrm{Tr}\Big(\big((\partial_\mu\sigma) - i\tau^d(\partial_\mu\pi^d)\big)\cdot 2(\sigma\tau^a + i\pi^a) - \big((\partial_\mu\sigma) + i\tau^d(\partial_\mu\pi^d)\big)\cdot 2(\sigma\tau^a - i\pi^a)\Big) \tag{2.17i}$$

$$= \frac{i\alpha^a}{2}\frac{1}{2}\cdot\big(4i(\partial_\mu\sigma)\pi^a\mathrm{Tr}(\mathbb{1}) - 4i\sigma(\partial_\mu\pi^b)\mathrm{Tr}(\tau^a\tau^b)\big) \tag{2.17j}$$

$$= \frac{i\alpha^a}{2}\frac{1}{2}\cdot 8i\cdot\big((\partial_\mu\sigma)\pi^a - \sigma(\partial_\mu\pi^a)\big) \tag{2.17k}$$

By equivalence of the Lagrangians (2.1) and (2.15) the equations of motion for (σ, π^a), $a \in \{0,1,2\}$ read

$$-\partial_\mu \partial^\mu \sigma = m^2\sigma - \frac{\lambda}{2}(\sigma^2 + \pi^a\pi^a)\sigma - \epsilon \tag{2.18a}$$

$$-\partial_\mu \partial^\mu \pi^a = m^2\pi^a - \frac{\lambda}{2}(\sigma^2 + \pi^b\pi^b)\pi^a \tag{2.18b}$$

2.1.2 Expanding around Minimum of Linear σ-model

The Lagrangian density

$$\mathscr{L} = \mathcal{L}_{\text{kin}} - V(\sigma,\vec{\pi}) = -\frac{1}{2}(\partial_\mu\sigma)(\partial^\mu\sigma) - \frac{1}{2}(\partial_\mu\vec{\pi})(\partial^\mu\vec{\pi}) + \frac{1}{2}\mu^2(\sigma^2+\vec{\pi}^2) + \frac{\lambda}{4}(\sigma^2+\vec{\pi}^2)^2 + h\sigma \tag{2.19}$$

can be expanded around the minimum at $\sigma_0 = f_\pi + h \cdot \frac{1}{2\mu^2} + \mathcal{O}(h^2)$ where $f_\pi = \frac{\mu}{\sqrt{\lambda}}$. Performing the substitution $\sigma \mapsto v + \sigma$ and neglecting terms of order $\mathcal{O}(h^2, \sigma^3, \sigma\vec{\pi}^2, (\vec{\pi}^2)^2)$ and higher the potential reads

$$V(\sigma,\vec{\pi}) = -\frac{\mu^4}{4\lambda} + \frac{1}{2}m_\sigma\sigma^2 + \frac{1}{2}m_\pi^2\vec{\pi}^2 \tag{2.20}$$

with pion mass $m_\pi^2 = \frac{h}{f_\pi}$ and sigma mass $m_\sigma^2 = 2\mu^2 + \mathcal{O}(h)$. Defining $\pi^\pm = (1/\sqrt{2})(\pi^1 \mp i\pi^2)$ one gets

$$(\pi^1)^2 + (\pi^2)^2 = |\pi^+|^2 + |\pi^-|^2 = 2\pi^+\pi^- \equiv 2\pi^+\overline{\pi^+} \tag{2.21}$$

The expansion of the Lagrangian around σ_0 breaks the $SO(4)$-symmetry associated to the vector $(\sigma,\vec{\pi})$ and chooses explicitly a minimum within the $SO(4)$-symmetric mexican hat potential. The residual symmetry is $SU(3)$. It features the $SO(2)$ subgroup of symmetry transformations

$$\begin{pmatrix}\pi^1\\ \pi^2\end{pmatrix} \mapsto \begin{pmatrix}\cos\alpha & \sin\alpha\\ -\sin\alpha & \cos\alpha\end{pmatrix}\begin{pmatrix}\pi^1\\ \pi^2\end{pmatrix} \qquad \Longleftrightarrow \qquad \pi^\pm \mapsto e^{\pm i\alpha}\pi^\pm \tag{2.22}$$

Note $\pi^- = \overline{\pi^+}$.

The Lagrangians and energy-momentum tensors $T^{\mu\nu} = 2(\partial\mathscr{L}/\partial g_{\mu\nu}) + g^{\mu\nu}\mathscr{L}$ **CITE: BLAU NOTES** for the separate fields read

$$\mathscr{L}_{\pi^\pm} = -(\partial_\mu\pi^+)(\overline{\partial_\mu\pi^+}) - m_\pi^2\pi^+\overline{\pi^+} \qquad T^{\mu\nu}_{\pi^\pm} = -2(\partial^\mu\pi^+)(\overline{\partial^\nu\pi^+}) + g^{\mu\nu}\big(-(\partial_\alpha\pi^+)(\overline{\partial_\alpha\pi^+}) - m_\pi^2\pi^+\overline{\pi^+}\big) \tag{2.23a}$$

$$\mathscr{L}_{\pi^0} = -\frac{1}{2}(\partial_\mu\pi^0)(\partial^\mu\pi^0) - \frac{1}{2}m_\pi^2(\pi^0)^2 \qquad T^{\mu\nu}_{\pi^0} = -(\partial^\mu\pi^0)(\partial^\nu\pi^0) + g^{\mu\nu}\big(-\frac{1}{2}(\partial_\alpha\pi^0)(\partial^\alpha\pi^0) - \frac{1}{2}m_\pi^2(\pi^0)^2\big) \tag{2.23b}$$

$$\mathscr{L}_{\sigma} = -\frac{1}{2}(\partial_\mu\sigma)(\partial^\mu\sigma) - \frac{1}{2}m_\sigma^2\sigma^2 \qquad T^{\mu\nu}_{\sigma} = -(\partial^\mu\sigma)(\partial^\nu\sigma) + g^{\mu\nu}\big(-\frac{1}{2}(\partial_\alpha\sigma)(\partial^\alpha\sigma) - \frac{1}{2}m_\pi^2(\pi^0)^2\big) \tag{2.23c}$$

Matching to Fluid Variables for the Real Field π^0

The only availabe 4-vector in the fluid theory is u_μ. It is thus intuitive to try to identify the real-valued 4vector $\partial_\mu\pi^0 \sim u_\mu$. Taking the normalization $u_\mu u^\mu = -1$ into account, one finds

$$u_\mu = \frac{\partial_\mu\pi^0}{\chi}\,, \qquad 0 < \chi^2 := -(\partial_\mu\pi^0)(\partial^\mu\pi^0) \tag{2.24}$$

From the fluid theory, we try to match the energy density of the hypothetical superfluid

$$\epsilon_{s,\pi^0} = u_\mu u_\nu T^{\mu\nu}_{\pi^0} = \frac{(\partial_\nu\pi^0)(\partial_\mu\pi^0)}{\chi^2}\Big(-(\partial^\mu\pi^0)(\partial^\nu\pi^0) + g^{\mu\nu}\big(-\frac{1}{2}(\partial_\alpha\pi^0)(\partial^\alpha\pi^0) - \frac{1}{2}m_\pi^2(\pi^0)^2\big)\Big) \tag{2.25a}$$

$$= -\chi^2 - \big(\frac{1}{2}\chi^2 - \frac{1}{2}m_\pi^2(\pi^0)^2\big) \tag{2.25b}$$

$$= \frac{m_\pi^2(\pi^0)^2 - 3\chi^2}{2} \tag{2.25c}$$

The equations of motion for the real Klein-Gordon field are

$$(-\Box + m_\pi^2)\pi^0 = 0 \tag{2.26}$$

Matching to Fluid Variables for the Complex Fields $\pi^\pm$

The $U(1)$-symmetry $\pi^\pm \mapsto e^{\pm i\alpha}\pi^\pm$ with infinitesimal transformation $\pi^\pm \mapsto (1 \pm i\delta\alpha)\pi^\pm$ generates a conserved Noether current

$$\begin{aligned} j^\mu &\sim \frac{\delta\mathscr{L}}{\delta(\partial_\mu\pi^+)}\delta\pi^+ + \frac{\delta\mathscr{L}}{\delta(\partial_\mu\pi^-)}\delta\pi^- && (2.27a)\\ &\sim \pi^-(\partial^\mu\pi^+) - \pi^+(\partial_\mu\pi^-) && (2.27b)\\ &= \sqrt{n}\big((\partial_\mu\sqrt{n}) + i\sqrt{n}(\partial_\mu\theta)\big) - \sqrt{n}\big((\partial_\mu\sqrt{n}) - i\sqrt{n}(\partial_\mu\theta)\big) && (2.27c)\\ &= n(\partial_\mu\theta) && (2.27d)\end{aligned}$$

with the parametrization $\pi^\pm = \sqrt{n}e^{\pm i\theta}$. The most intuitive matching is now

$$u^\mu = \frac{\partial_\mu\theta}{\chi}, \qquad 0 < \chi^2 := -(\partial_\mu\theta)(\partial^\mu\theta) \tag{2.28}$$

leading to the energy density

$$\begin{aligned} \epsilon_{s,\pi^\pm} = u_\mu u_\nu T^{\mu\nu}_{\pi^\pm} &= \frac{(\partial_\mu\theta)(\partial_\nu\theta)}{\chi^2}\Big(- 2(\partial^\mu\pi^+)(\partial^\nu\pi^-) + g^{\mu\nu}\big(- (\partial_\alpha\pi^+)(\partial^\alpha\pi^-) - m_\pi^2\pi^+\pi^-\big)\Big) && (2.29a)\\ &= -2\frac{[(\partial_\mu\sqrt{n})(\partial^\mu\theta)]^2}{\chi^2} - 2n\chi^2 - \big(n\chi^2 - (\partial_\mu\sqrt{n})(\partial^\mu\sqrt{n}) - m_\pi^2 n\big) && (2.29b)\end{aligned}$$

where the intermediate calculation

$$\begin{aligned} (\partial^\mu\pi^+)(\partial^\mu\pi^-) &= \big((\partial^\mu\sqrt{n}) + i\sqrt{n}(\partial^\mu\theta)\big)\big((\partial^\nu\sqrt{n}) - i\sqrt{n}(\partial^\nu\theta)\big) && (2.30a)\\ &= (\partial^\mu\sqrt{n})(\partial^\nu\sqrt{n}) + n(\partial^\mu\theta)(\partial^\nu\theta) + i\big(\sqrt{n}(\partial^\mu\theta)(\partial^\nu\sqrt{n}) - \sqrt{n}(\partial^\nu\theta)(\partial^\mu\sqrt{n})\big) && (2.30b)\end{aligned}$$

is useful.

Expressing the Lagrangian in terms of (n, θ)

$$\mathscr{L}_{\pi^\pm} = -(\partial_\mu\sqrt{n})(\partial^\mu\sqrt{n}) - n(\partial_\mu\theta)(\partial^\mu\theta) - nm_\pi^2 \tag{2.31}$$

yields as the corresponding equations of motion

$$\begin{aligned} \partial_\mu\left(\frac{\partial\mathscr{L}_{\pi^\pm}}{\partial(\partial_\mu\sqrt{n})}\right) = \frac{\partial\mathscr{L}_{\pi^\pm}}{\partial\sqrt{n}} &: \qquad -2\Box\sqrt{n} = -2\sqrt{n}\big((\partial_\mu\theta)(\partial^\mu\theta) + m_\pi^2\big) && (2.32a)\\ \partial_\mu\left(\frac{\partial\mathscr{L}_{\pi^\pm}}{\partial(\partial_\mu\theta)}\right) = \frac{\partial\mathscr{L}_{\pi^\pm}}{\partial\theta} &: \qquad \partial_\mu\big(- 2n(\partial^\mu\theta)\big) = 0 && (2.32b)\end{aligned}$$

the second of which encodes the conservation law for the $U(1)$-Noether current.

The easiest (and most naive) solution is again $n =$ const. and $\partial_\mu\theta = p_\mu$ with $p_\mu p^\mu = -m_\pi^2$. This would be a solution with only a single momentum mode. This solution implies $u^\mu =$ const. which is generally not satisfied by the given data. Assume therefore the existence of a small perturbation $\partial_\mu\theta = p_\mu + \delta q_\mu$ with $q_\mu p^\mu = 0$. From this, $\chi^2 = -(\partial_\mu\theta)(\partial^\mu\theta) = -p_\mu p^\mu - q_\mu q^\mu = m_\pi^2 - \delta^2 q^2$. To linear order in δ

$$\partial_\mu\theta = \chi u_\mu = m_\pi u_\mu \tag{2.33}$$

holds true. To expand the equations of motion and allow for non-constant amplitude, assume $n = n_{(0)} + \delta n_{(1)}(x)$ with $n_{(0)} =$ const. ($\sqrt{n} \approx \sqrt{n_{(0)}} + \delta \cdot n_{(1)}/(2\sqrt{n_{(0)}})$).

$$\begin{aligned} \Box\sqrt{n_{(1)}} &= 0 + \mathcal{O}(\delta^2) && (2.34a)\\ 0 &= n_{(0)}\partial_\mu q^\mu + q^\mu\partial_\mu n_{(1)} && (2.34b)\end{aligned}$$

2.2 Finding the Spectrum at the Detector Surface

In general the particle number N and particle number density $n(\vec{x})$ in position space and $n(\vec{p})$ in momentum space associated to the condensate $\phi(\vec{x})$ of a complex scalar field are given by the relations

$$n(\vec{x}) = \phi(\vec{x})\phi^*(\vec{x})\,, \qquad n(\vec{p}) = \phi(\vec{p})\phi^*(\vec{p}) \tag{2.35a}$$

$$N = \int \mathrm{d}^3x n(\vec{x}) = \int \frac{\mathrm{d}^3p}{(2\pi)^3} n(\vec{p}) \tag{2.35b}$$

with the convention $\phi(\vec{x}) = \int \mathrm{d}^3p/(2\pi)^3\phi(\vec{p})e^{-i\vec{p}\vec{x}}$ for the Fourier transform.

2.2.1 Treating the Freeze Out Field as Source Term???

Calculation 2.2|1: Metric on Hypersurface

Recall the metric $g_{\mu\nu} = \mathrm{diag}(-1, 1, \tau^2, r^2)$ in coordinates (τ, r, η, φ). Orthonormal tangent vectors to the freeze out hypersurface are $(\hat{\partial}_\varphi)^\mu = (0, 0, 0, r^{-1}) = r^{-1}(\partial_\varphi)^\mu$, $(\hat{\partial}_\eta)^\mu = (0, 0, \tau^{-1}, 0) = \tau^{-1}(\partial_\eta)^\mu$ and $(\hat{\partial}_\alpha)^\mu = \sqrt{r'^2(\alpha) - \tau'^2(\alpha)}^{-1}(\tau'(\alpha), r'(\alpha), 0, 0) = D(\alpha)(\partial_\alpha)^\mu$ with $D(\alpha) = \sqrt{r'^2(\alpha) - \tau'^2(\alpha)}^{-1}$. The projector on the hypersurface is

$$\gamma_{\mu\nu} = (\hat{\partial}_\varphi)_\mu(\hat{\partial}_\varphi)_\nu + (\hat{\partial}_\eta)_\mu(\hat{\partial}_\eta)_\nu + (\hat{\partial}_\alpha)_\mu(\hat{\partial}_\alpha)_\nu = \begin{pmatrix} D^2(\alpha)\tau'^2(\alpha) & -D^2(\alpha)\tau'(\alpha)r'(\alpha) & 0 & 0 \\ -D^2(\alpha)\tau'(\alpha)r'(\alpha) & D^2(\alpha)r'^2(\alpha) & 0 & 0 \\ 0 & 0 & \tau^2 & 0 \\ 0 & 0 & 0 & r^2 \end{pmatrix} \tag{2.36}$$

The normal of the hypersurface is $n^\mu \equiv (\hat{\partial}_\alpha^\perp)^\mu = D(\alpha)(r'(\alpha), \tau'(\alpha), 0, 0)$ and is timelike where D is real. Naturally $\gamma_{\mu\nu}n^\nu = 0$. In the basis $(\partial_\alpha, \partial_\eta, \partial_\varphi, n)$ using (in short form)

$$(\partial_\alpha)^\nu\gamma_{\mu\nu}(\partial_\alpha)^\mu = \begin{pmatrix} \tau' \\ r' \end{pmatrix}^T \begin{pmatrix} -\tau' \\ r' \end{pmatrix} = D^{-2} \tag{2.37}$$

the hypersurface metric in coordinates $x^i = (\alpha, \eta, \varphi)$ reads

$$\gamma_{ij} = \mathrm{diag}(D^{-2}(\alpha), \tau^2(\alpha), r^2(\alpha)) \tag{2.38}$$

and the volume element is given by $\mathrm{d}\Sigma = r(\alpha)\tau(\alpha)D^{-1}(\alpha)\mathrm{d}\alpha\mathrm{d}\eta\mathrm{d}\varphi$. The oriented surface element is

$$\mathrm{d}\Sigma^\mu = n^\mu\mathrm{d}\Sigma = r(\alpha)\tau(\alpha)(r'(\alpha), \tau'(\alpha), 0, 0)\mathrm{d}\alpha\mathrm{d}\eta\mathrm{d}\varphi \tag{2.39}$$

2.2.2 Converting Spectra between Coordinate Systems

Consider the coordinate change in momentum space

$$\begin{cases} p_x = p^\perp\cos\varphi_p \\ p_y = p^\perp\sin\varphi_p \\ p_z = m^\perp\sinh\eta_p \\ p_t = m^\perp\cosh\eta_p \end{cases} \iff \begin{cases} p^\perp = \sqrt{p_x^2 + p_y^2} \\ \varphi_p = \arctan(p_y/p_y) \\ m^\perp = \sqrt{p_t^2 - p_z^2} \\ \eta_p = \mathrm{artanh}\,(p_z/p_t) \end{cases} \tag{2.40}$$

with Jacobian

$$|\frac{\partial(p^\perp, \varphi_p, m^\perp, \eta_p)}{\partial(p_x, p_y, p_z, p_t)}| = \frac{1}{m^\perp p^\perp} \tag{2.41}$$

Let $f(p_\mu)$ be some distribution function and F its momentum space integral evaluated on the momentum shell and future directed momenta.

$$F = \int \frac{\mathrm{d}^4 p_{\text{cart}}}{(2\pi)^4}\delta(p^2-m^2)\Theta(p_t)f(p_\mu) = \int\frac{\mathrm{d}p_t}{2\pi}\int\frac{\mathrm{d}^3 p_{\text{cart}}}{(2\pi)^3}\frac{1}{2\omega_{\vec{p}}}\big(\delta(p_t-\omega_{\vec{p}})+\delta(p_t+\omega_{\vec{p}})\big)\Theta(p_t)f(p_\mu) \tag{2.42a}$$

$$= \frac{1}{2\pi}\int\frac{\mathrm{d}^3 p_{\text{cart}}}{(2\pi)^3}\frac{1}{2\omega_{\vec{p}}}f(p_\mu)\big|_{p_t=\omega_{\vec{p}}} \tag{2.42b}$$

On the other hand

$$F = \int \frac{\mathrm{d}^4 p_{\text{cart}}}{(2\pi)^4}\delta(p^2-m^2)\Theta(p_t)f(p_\mu) = \frac{1}{(2\pi)^4}\int_0^\infty \mathrm{d}p^\perp\int_0^\infty \mathrm{d}m^\perp\int_{-\infty}^\infty \mathrm{d}\eta_p\int_0^{2\pi}\mathrm{d}\varphi_p m^\perp p^\perp\times$$
$$\times\,\delta((p^\perp)^2-(m^\perp)^2)f(p_\mu) \tag{2.42c}$$

assume $f(p^\mu)=f(p^\perp,m^\perp,\eta_p)$ and perform the φ-integraion

$$= \frac{1}{(2\pi)^3}\int_0^\infty \mathrm{d}m^\perp\int_{-\infty}^\infty \mathrm{d}\eta_p\frac{m^\perp}{2}f(p_\mu)\big|_{m^\perp=p^\perp} \tag{2.42d}$$

leding to the important result

$$\frac{1}{2\pi}\int\frac{\mathrm{d}^3 p_{\text{cart}}}{(2\pi)^3}\frac{1}{2\omega_{\vec{p}}}f(p_\mu)\big|_{p_t=\omega_{\vec{p}}} = \frac{1}{(2\pi)^3}\int_0^\infty \mathrm{d}m^\perp\int_{-\infty}^\infty \mathrm{d}\eta_p\frac{m^\perp}{2}f(p_\mu)\big|_{m^\perp=p^\perp} \tag{2.42e}$$

Since the restrictions $p_t=\omega_{\vec{p}}$ and $m^\perp=p^\perp$ are equivalent (considering the parametrization that already satisfies $p_t=p^\perp\cosh\eta_p\geq 0$) we find

$$\omega_{\vec{p}}\frac{\mathrm{d}F}{\mathrm{d}p_x\mathrm{d}p_y\mathrm{d}p_z} = \frac{1}{2\pi m^\perp}\frac{\mathrm{d}F}{\mathrm{d}m^\perp\mathrm{d}\eta_p} \tag{2.43}$$

The result applies to the case

$$f(p_\mu)\big|_{p_t=\omega_{\vec{p}}} = 2\omega_{\vec{p}}\cdot 2\pi\cdot n(\vec{p}) \tag{2.44}$$

and $F=N$.

2.2.3 Fourier Decomposition on Freeze Out Surface???

Generally a Fourier Ddecomposition to solve the Klein-Gordon equation in terms of 3-momentum-modes is given by

$$\phi(x^\mu) = \int\frac{\mathrm{d}^4 p}{(2\pi)^4}\tilde{\phi}(\vec{p})e^{ip_\mu x^\mu}\delta(p^2-m^2)\Theta(p_0)+\text{c.c.} = \frac{1}{2\pi}\int\frac{\mathrm{d}^3 p}{(2\pi)^3}\frac{1}{2\omega_{\vec{p}}}\phi(\vec{p})e^{i(\omega_{\vec{p}}x^0-\vec{p}\vec{x})}+\text{c.c.} \tag{2.45}$$

For simplicity neglect the +c.c..

Considering a hypersurface Σ and using the field data given only on this hypersurface - that is considering the restriction $\phi|_{x\in\Sigma}$ - can we reconstruct $\tilde{\phi}(\vec{p})$? The map $\tilde{\phi}(\vec{p})\mapsto\phi(x^\mu\in\Sigma)$ is trivially given by the mode decomposition above. Let $\Sigma=\Sigma_t=\{x^\mu\in\mathbb{R}^{(1,3)}|x^0=t=\text{const}\}$ be a slice of constant lab time. Then the map $\phi(x^\mu\in\Sigma_t)\equiv\phi_t(\vec{x})\mapsto\tilde{\phi}(\vec{p})$ is easily found to be

$$\phi(\vec{p}) = (2\pi)(2\omega_{\vec{p}})\int_{\Sigma_t}\mathrm{d}^3x\phi_t(\vec{x})e^{-i(\omega_{\vec{p}}t-\vec{p}\vec{x})} \tag{2.46}$$

which of course uses the orthogonality relation

$$\int \mathrm{d}^n x_{\text{cart}}e^{i(p_\mu-q_\mu)x^\mu} = (2\pi)^n\delta^{(n)}(p-q) \tag{2.47}$$

valid in cartesian coordinates.

We would like to generalize this to arbitrary Σ. Let $(y^i)_{i=1}^3$ be the coordinates of a parametrization $x^\mu(y^i)$ of Σ. The naive attempt would be an integral of the form

$$\tilde{\phi}(\vec{p}) \stackrel{?}{=} (2\pi)(2\omega_{\vec{p}})\int_\Sigma \mathrm{d}^3y\sqrt{\gamma}\phi(x^\mu(y^i))e^{-i(\omega_{\vec{p}}t(y^i)-\vec{p}\vec{x}(y^i))} \tag{2.48}$$

where γ is the induced metric determinant. The relevant example is $\Sigma = \{x^\mu \in \mathbb{R}^{(1,3)} | (\tau, r) = (\tau(\alpha), r(\alpha))\}$ with τ, r defined by the coordinate transformation

$$\begin{cases} t = \tau \cosh \eta \\ z = \tau \sinh \eta \\ x = r \cos \varphi \\ y = r \sin \varphi \end{cases} \iff \begin{cases} \tau = \sqrt{t^2 - z^2} \\ \eta = \operatorname{artanh}(z/t) \\ r = \sqrt{x^2 + y^2} \\ \varphi = \arctan(y/x) \end{cases} \tag{2.49}$$

Fourier Transform in Adapted Coordinates

Let's first evaluate $p_\mu x^\mu$ in the Bjorken coordinate system. Therefore introduce an analogous coordinate change in momentum space

$$\begin{cases} p_t = m_\perp \cosh \eta_p \\ p_z = m_\perp \sinh \eta_p \\ p_x = p_\perp \cos \varphi_p \\ p_y = p_\perp \sin \varphi_p \end{cases} \tag{2.50}$$

to rewrite the scalar product as

$$p_\mu x^\mu \equiv \tau(p_t \cosh \eta - p_z \sinh \eta) - r(p_x \cos \varphi + p_y \sin \varphi) = \tau m_\perp \cosh(\eta - \eta_p) - r p_\perp \cos(\varphi - \varphi_p) \tag{2.51}$$

We used the identities

$$\cosh(a-b) = \cosh a \cosh b - \sinh a \sinh b \,, \qquad \cos(a-b) = \cos a \cos b + \sin a \sin b \tag{2.52}$$

The integral measure changes according to $\mathrm{d}^4 p_{\text{cart}} = \mathrm{d}m_\perp \mathrm{d}p_\perp \mathrm{d}\eta_p \mathrm{d}\varphi_p \cdot m_\perp p_\perp$. The momentum shell condition $p^2 = m^2$ is equivalently parametrized by $m_\perp^2 = p_\perp^2 + m^2 =: \omega_\perp^2$.

These coordinates are adapted to boost symmetry $\eta \to \eta'$ along the beam direction and rotational symmetry $\varphi \to \varphi'$ around the beam axis. Investigate first the implications on the mode decomposition, by requesting that $(\partial/\partial\eta)\phi(x^\mu) = 0 = (\partial/\partial\varphi)\phi(x^\mu)$.

$$\begin{aligned} \phi(x^\mu) = \frac{1}{(2\pi)^4} \int_{-\infty}^{\infty} \mathrm{d}m_\perp \int_0^\infty \mathrm{d}p_\perp \int_{-\infty}^{\infty} \mathrm{d}\eta_p \int_0^{2\pi} \mathrm{d}\varphi_p \cdot m_\perp p_\perp \delta(m_\perp^2 - \omega_\perp^2) \Theta(m_\perp) \times \\ \times \tilde{\phi}(p_x(p_\perp, \varphi_p), p_y(p_\perp, \varphi_p), p_z(m_\perp, \eta_p)) e^{i(\tau m_\perp \cosh(\eta - \eta_p) - r p_\perp \cos(\varphi - \varphi_p))} \end{aligned} \tag{2.53a}$$

... shift $\varphi_p \to \varphi_p + \varphi$ and $\eta_p \to \eta_p + \eta$...

$$\begin{aligned} = \frac{1}{(2\pi)^4} \int_{-\infty}^{\infty} \mathrm{d}m_\perp \int_0^\infty \mathrm{d}p_\perp \int_{-\infty}^{\infty} \mathrm{d}\eta_p \int_0^{2\pi} \mathrm{d}\varphi_p \cdot m_\perp p_\perp \delta(m_\perp^2 - \omega_\perp^2) \Theta(m_\perp) \times \\ \times \tilde{\phi}(p_x(p_\perp, \varphi_p + \varphi), p_y(p_\perp, \varphi_p + \varphi), p_z(m_\perp, \eta_p + \eta)) e^{i(\tau m_\perp \cosh \eta_p - r p_\perp \cos \varphi_p)} \end{aligned} \tag{2.53b}$$

From this it follows that $\tilde{\phi}(p_x, p_y, p_z) = \tilde{\phi}(p_\perp)$ and we can simplify the integral. Let's also evaluate the δ-distribution by using that $\delta(m_\perp^2 - \omega_\perp^2) = (1/2\omega_\perp)(\delta(m_\perp - \omega_\perp) + \delta(m_\perp + \omega_\perp))$

$$\phi(x^\mu) = \frac{1}{(2\pi)^4} \frac{1}{2} \int_0^\infty \mathrm{d}p_\perp \int_{-\infty}^{\infty} \mathrm{d}\eta_p \int_0^{2\pi} \mathrm{d}\varphi_p \cdot p_\perp \tilde{\phi}(p_\perp) e^{i(\tau \omega_\perp \cosh \eta_p - r p_\perp \cos \varphi_p)} \tag{2.53c}$$

$$= \frac{1}{2} \frac{1}{(2\pi)^4} \int_0^\infty \mathrm{d}p_\perp p_\perp \tilde{\phi}(p_\perp) \big(2\pi J_0(r p_\perp)\big) \big(\pi(-Y_0(\tau\omega_\perp) + i J_0(\tau\omega_\perp))\big) \tag{2.53d}$$

where in the last step the following integral representation of Bessel functions of the first kind $J_0(x)$ and of the second kind $Y_0(x)$ where used `https://dlmf.nist.gov/10.9`

$$J_0(x \in \mathbb{R}) = \frac{1}{2\pi} \int_0^{2\pi} \mathrm{d}t \exp\left(\overset{?}{\pm} i x \cos t\right), \quad J_0(x > 0) = \frac{1}{\pi} \int_{-\infty}^{\infty} \mathrm{d}t \sin(x \cosh t)\,, \quad Y_0(x > 0) = -\frac{2}{\pi} \int_0^\infty \mathrm{d}t \cos(x \cosh t) \tag{2.54a}$$

$$H_\nu^{(1)}(z) \int_0^{2\pi} \mathrm{d}\varphi \int_{-\infty}^{\infty} \mathrm{d}\eta e^{i(a \cosh \eta - b \cos \varphi)} = \Big[2\pi J_0(b) \times \big(\pi(-Y_0(a) + i J_0(a))\big)\Big] \tag{2.54b}$$

Other relevant properties are

$$\frac{\mathrm{d}}{\mathrm{d}x}J_0(x) = -J_1(x)\,, \qquad \frac{\mathrm{d}}{\mathrm{d}x}Y_0(x) = -Y_1(x) \tag{2.54c}$$

$$H_\nu^{(1)}(z) = J_\nu(z) + iY_\nu(z) \tag{2.54d}$$

Naively, there seems to be no useful orthogonality relation... to extract $\tilde{\phi}(p_\perp)$ from this.

Invariance of Fourier Transform w.r.t. Deformations of the Hypersurface

Let ϕ_1, ϕ_2 be fields of equal mass evolving according to the KG equation. Then the current

$$J_\mu[\phi_1,\phi_2] = -i(\phi_1\partial_\mu\phi_2^* - (\partial_\mu\phi_1)\phi_2^*) =: -i\phi_1\overleftrightarrow{\partial}_\mu\phi_2 \tag{2.55}$$

is conserved. Recall Gauß law

$$\int_\Omega \mathrm{d}\Omega\nabla_\mu J^\mu = \int_{\partial\Omega}\mathrm{d}\sigma_\mu J^\mu \tag{2.56}$$

with $\mathrm{d}\sigma_\mu$ the outwards oriented surface normal of the spacetime volume Ω. The bilinear form

$$(\phi_1,\phi_2)_\Sigma = \int_\Sigma \mathrm{d}\Sigma_\mu J^\mu[\phi_1,\phi_2] = -i\int_\Sigma \mathrm{d}\Sigma_\mu\phi_1\overleftrightarrow{\partial^\mu}\phi_2^* \tag{2.57}$$

is therefore independent of the choice of (Cauchy) hypersurface Σ (if $\partial\Sigma$ is changed, one must carefully check for further contributions in Gauß law). Choose a hypersurface Σ_t where $t = \text{const}$. Consider a solution to the KG equation given by its Fourier decomposition

$$\phi(t,\vec{x}) = \int\frac{\mathrm{d}^3p}{(2\pi)^3}\tilde{\phi}(\vec{p})u_{\vec{p}}(t,\vec{x})\,, \qquad u_{\vec{p}}(t,\vec{x}) = \frac{1}{\sqrt{2\omega_{\vec{p}}}}e^{-i(\omega_{\vec{p}}t-\vec{p}\vec{x})} \tag{2.58}$$

The mode functions $u_{\vec{p}}$ are orthogonal w.r.t. to the bilinear form $(\cdot,\cdot)_{\Sigma_t}$ and normalized according to

$$(u_{\vec{p}},u_{\vec{q}})_{\Sigma_t} = (2\pi)^3\delta^{(3)}(\vec{p}-\vec{q}) \tag{2.59}$$

and the Fourier transform $\tilde{\phi}(\vec{p})$ can be extracted via

$$\tilde{\phi}(\vec{p}) = (\phi, u_{\vec{p}})_{\Sigma_t}$$

and can thus be evaluated on any Cauchy surface.

Remark 2.2|2: Particle Density w.r.t. Convention of Fourier Decomposition

Let $n(t,\vec{x}) = |\phi(t,\vec{x})|^2$ and $N = \int\mathrm{d}^3x n(\vec{x})$. Then

$$N = \int\mathrm{d}^3x\int\frac{\mathrm{d}^3p}{(2\pi)^3}\frac{\mathrm{d}^3q}{(2\pi)^3}\tilde{\phi}(\vec{p})\frac{1}{2\sqrt{\omega_{\vec{p}}\omega_{\vec{q}}}}\tilde{\phi}^*(\vec{q})e^{-i((\omega_{\vec{p}}-\omega_{\vec{q}})t-(\vec{p}-\vec{q})\vec{x})} = \int\frac{\mathrm{d}^3p}{(2\pi)^3}\frac{1}{2\omega_{\vec{p}}}|\tilde{\phi}(\vec{p})|^2 \tag{2.60}$$

leading to $n(\vec{p}) = \frac{1}{2\omega_{\vec{p}}}|\tilde{\phi}(\vec{p})|^2$ which is slightly different convention than before.

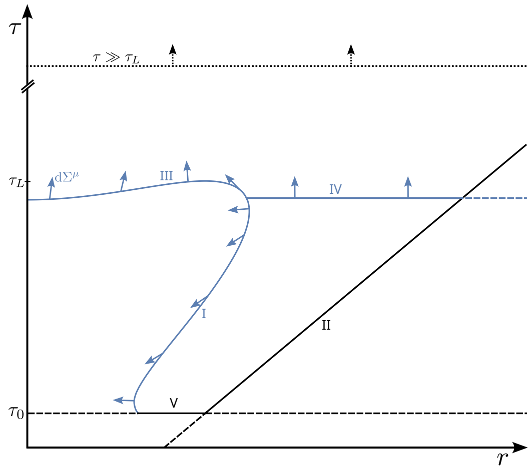


Figure 2.1: Freezeout surface in τ-r-plane[KGF23].

Consider the freezeout on the hypersurface depicted in 2.1. Assume that the condensate contribution as a function in phase space $f_{\text{cond}}(x^\mu, \vec{p})$ vanishes on Σ_{II} and Σ_{V}, i.e. is contained within the union of all light cones starting on the freeze out surface $\Sigma_{FO} \equiv \Sigma_{\text{I}} \cap \Sigma_{\text{III}}$. **To do (41)By causality this seems reasonable, but from Fourier decomposition of a classical field this is not at all clear.**

Following the reasoning from [KGF23], we wish to apply Gauß law. Consider separately the contribution on the τ-axis

$$\int_{\Sigma_{r=0}} \mathrm{d}\Sigma_\mu J^\mu \qquad \text{or} \qquad \lim_{r\to 0} \int_{\Sigma_r} \mathrm{d}\Sigma_\mu J^\mu$$

The surface vector on this hypersurface is $\mathrm{d}\Sigma_\mu = r\tau \mathrm{d}\tau \mathrm{d}\eta \mathrm{d}\varphi(0,1,0,0)$ and thus vanishes at $r = 0$ (the hypersurface $\Sigma_{r=0}$ has zero 3-volume). Since the derivative in the integrad introduces no divergencies, the contribution of $\Sigma_{r=0}$ to Gauß law is zero.

We can therefore write

$$\tilde{\phi}(\vec{p}) = (\phi, u_{\vec{p}})_{\Sigma_t} = (\phi, u_{\vec{p}})_{\Sigma_{\tau \gg \tau_L}} = (\phi, u_{\vec{p}})_{\Sigma_{FO}} \tag{2.61}$$

The second "=" assumes that $\phi = 0$ for large spacetime rapidities $\eta \to \pm\infty$ and the $\tau =$ const. hypersurface can be deformed to a $t =$ const. hypersurface.

2.2.4 Properly Considering all Initial Conditions

Adapt the reasoning in [ABL97]. The idea is to model a non-linear, particle producing interaction process with the inhomogeneous KG equation. After all interactions have decayed and the source has vanished, the particle are effectively free. One can use the true e.o.m. to evolve initial conditions and then, after all interactions have decayed, reconstruct the hypothetical source term and derive particle production from this. For a field with the e.o.m.

$$(\Box - m^2)\phi(x) = J(x) \tag{2.62}$$

the particle spectrum after the source has vanished is

$$2\omega_{\vec{p}} \frac{\mathrm{d}N}{\mathrm{d}^3\vec{p}} = \frac{1}{(2\pi)^3} \left|\tilde{J}(\vec{p})\right|^2 \tag{2.63}$$

Use following conventions:

Important 2.2|3: Conventions for Fourier Transform

Signature is $(-,+,+,+)$.

$$\tilde{f}(p) = \int \mathrm{d}^4 x e^{-ipx} f(x) \tag{2.64a}$$

$$\tilde{f}^{(3)}(t,\vec{p}) = \int \mathrm{d}^3 x e^{-i\vec{p}\vec{x}} f(t,\vec{x}) \tag{2.64b}$$

$$\tilde{J}(\vec{p}) = \tilde{J}(p)|_{p^0=\omega_{\vec{p}}} \tag{2.64c}$$

The full solution of a KG-field is specified by 2 initial conditions, e.g. $\phi(t,\vec{x})$ and $\dot{\phi}(t,\vec{x})$ for fixed t, or equivalently 2 spectral functions $a(p)$, $b(p)$ within the following decomposition:

$$\phi(t,\vec{x}) = 2\pi \int \frac{\mathrm{d}^4 p}{(2\pi)^4} \delta(p^2 - m^2)\Theta(p^0)\big(a(p)e^{ipx} + b(p)e^{-ipx}\big) \tag{2.65a}$$

$$= \int \frac{\mathrm{d}^3 p}{(2\pi)^3} \frac{1}{2\omega_{\vec{p}}} \big(a(\omega_{\vec{p}},\vec{p})e^{-i(\omega_{\vec{p}}t - \vec{p}\vec{x})} + b(\omega_{\vec{p}},\vec{p})e^{i(\omega_{\vec{p}}t - \vec{p}\vec{x})}\big) \tag{2.65b}$$

$$\dot{\phi}(t,\vec{x}) = \int \frac{\mathrm{d}^3 p}{(2\pi)^3} \frac{1}{2} \big(- ia(\omega_{\vec{p}},\vec{p})e^{-i(\omega_{\vec{p}}t - \vec{p}\vec{x})} + ib(\omega_{\vec{p}},\vec{p})e^{i(\omega_{\vec{p}}t - \vec{p}\vec{x})}\big) \tag{2.65c}$$

From computing the transformations

$$\tilde{\phi}^{(3)}(t,\vec{p}) = \int \mathrm{d}^3 x e^{-i\vec{p}\vec{x}} \phi(t,\vec{x}) = \int \mathrm{d}^3 x e^{-i\vec{p}\vec{x}} \int \frac{\mathrm{d}^3 q}{(2\pi)^3} \frac{1}{2\omega_{\vec{q}}} \big(a(\omega_{\vec{q}},\vec{q})e^{-i(\omega_{\vec{q}}t - \vec{q}\vec{x})} + b(\omega_{\vec{q}},\vec{q})e^{i(\omega_{\vec{q}}t - \vec{q}\vec{x})}\big) \tag{2.65d}$$

$$= \frac{1}{2\omega_{\vec{p}}} \big(a(\omega_{\vec{p}},\vec{p})e^{-i\omega_{\vec{p}}t} + b(\omega_{\vec{p}},-\vec{p})e^{i\omega_{\vec{p}}t}\big) \tag{2.65e}$$

$$\dot{\tilde{\phi}}^{(3)}(t,\vec{p}) = \int \mathrm{d}^3 x e^{-i\vec{p}\vec{x}} \dot{\phi}(t,\vec{x}) = \frac{1}{2} \big(- ia(\omega_{\vec{p}},\vec{p})e^{-i\omega_{\vec{p}}t} + ib(\omega_{\vec{p}},-\vec{p})e^{i\omega_{\vec{p}}t}\big) \tag{2.65f}$$

we can find the inversion of the parametrization

$$a(\omega_{\vec{p}},\vec{p})e^{-i\omega_{\vec{p}}t} = \int \mathrm{d}^3 x \big(\omega_{\vec{p}}\phi(t,\vec{x}) + i\dot{\phi}(t,\vec{x})\big)e^{-i\vec{p}\vec{x}} \tag{2.65g}$$

$$= ie^{-i\omega_{\vec{p}}t} \int \mathrm{d}^3 x \big(\phi(t,\vec{x})\overleftarrow{\partial}_t e^{i(\omega_{\vec{p}}t - \vec{p}\vec{x})} - \phi(t,\vec{x})\overrightarrow{\partial}_t e^{i(\omega_{\vec{p}}t - \vec{p}\vec{x})}\big) \tag{2.65h}$$

$$b(\omega_{\vec{p}},-\vec{p})e^{i\omega_{\vec{p}}t} = \int \mathrm{d}^3 x \big(\omega_{\vec{p}}\phi(t,\vec{x}) - i\dot{\phi}(t,\vec{x})\big)e^{-i\vec{p}\vec{x}} \tag{2.65i}$$

$$= -ie^{i\omega_{\vec{p}}t} \int \mathrm{d}^3 x \big(\phi(t,\vec{x})\overleftarrow{\partial}_t e^{-i(\omega_{\vec{p}}t + \vec{p}\vec{x})} - \phi(t,\vec{x})\overrightarrow{\partial}_t e^{-i(\omega_{\vec{p}}t + \vec{p}\vec{x})}\big) \tag{2.65j}$$

Note how this has the form of the inner product defined in 2.2.3, to be precise

$$a(\omega_{\vec{p}},\vec{p}) = (\phi, e^{-i(\omega_{\vec{p}}t - \vec{p}\vec{x})})_{\Sigma_t}\,, \qquad b(\omega_{\vec{p}},-\vec{p}) = -(\phi, e^{i(\omega_{\vec{p}}t - \vec{p}\vec{x})})_{\Sigma_t} \tag{2.66}$$

and can therefore in principle be evaluated on any Cauchy surface.

It is now argued that the Fourier decomposition of a real source can be found by

$$\tilde{J}(\vec{p}) = -ie^{i\omega_{\vec{p}}t}\big(\omega_{\vec{p}}\tilde{\phi}^{(3)}(t,\vec{p}) + i\dot{\tilde{\phi}}^{(3)}(t,\vec{p})\big) \tag{2.67a}$$

from which by substitution we derive the form

$$= -ie^{i\omega_{\vec{p}}t} \int \mathrm{d}^3 x \big(\omega_{\vec{p}}\phi(t,\vec{x}) + i\dot{\phi}(t,\vec{x})\big)e^{-i\vec{p}\vec{x}} \tag{2.67b}$$

$$= \int \mathrm{d}^3 x \big(\phi(t,\vec{x})\overleftarrow{\partial}_t e^{i(\omega_{\vec{p}}t - \vec{p}\vec{x})} - \phi(t,\vec{x})\overrightarrow{\partial}_t e^{i(\omega_{\vec{p}}t - \vec{p}\vec{x})}\big) \tag{2.67c}$$

which, by the same argument as before, is independent of the choice of Cauchy surface. The calculation is the same as before, repeated here for my own clarity (note $\overleftrightarrow{\partial} = \overleftarrow{\partial} - \overrightarrow{\partial}$, unlike before) **and some errors corrected**

$$\tilde{J}(\vec{p}) = \int_{-\infty}^{\infty} \mathrm{d}\eta \int_0^{2\pi} \mathrm{d}\varphi \int_0^{\pi} \mathrm{d}\alpha \tau r \Big[\phi(\tau, r)\big(r' \overleftrightarrow{\partial}_\tau + \tau' \overleftrightarrow{\partial}_r\big) e^{i(\tau\omega_\perp \cosh(\eta-\eta_p) - r p_\perp \cos(\varphi - \varphi_p))}\Big] \tag{2.68a}$$

$$= \int_{-\infty}^{\infty} \mathrm{d}\eta \int_0^{2\pi} \mathrm{d}\varphi \int_0^{\pi} \mathrm{d}\alpha \tau r \Big[\phi(\tau, r)\big(r' \overleftrightarrow{\partial}_\tau + \tau' \overleftrightarrow{\partial}_r\big) e^{i(\tau\omega_\perp \cosh\eta - r p_\perp \cos\varphi)}\Big] \tag{2.68b}$$

$$= \int_0^{\pi} \mathrm{d}\alpha \tau r \Big[\phi(\tau, r)(r' \overleftrightarrow{\partial}_\tau + \tau' \overleftrightarrow{\partial}_r)\Big[J_0(r p_\perp) \times \big(-Y_0(\tau\omega_\perp) + iJ_0(\tau\omega_\perp)\big)\Big]\Big] \tag{2.68c}$$

$$\begin{aligned} = 2\pi^2 \int_0^{\pi} \mathrm{d}\alpha \tau r \Big[&(r'\partial_\tau + \tau'\partial_r)\phi(\tau, r)\Big[J_0(r p_\perp) \times \big(-Y_0(\tau\omega_\perp) + iJ_0(\tau\omega_\perp)\big)\Big] + \\ &+ \phi(\tau, r)\Big[\tau' \times p_\perp J_1(r p_\perp) \times \big(-Y_0(\tau\omega_\perp) + iJ_0(\tau\omega_\perp)\big) + \\ &\qquad + r' \times J_0(r p_\perp) \times \omega_\perp\big(-Y_1(\tau\omega_\perp) + iJ_1(\tau\omega_\perp)\big)\Big]\Big] \end{aligned} \tag{2.68d}$$

Notational Simplification

For numerical ease and clarity, define helper functions

$$H_1 = \Big[J_0(r p_\perp) \times \big(-Y_0(\tau\omega_\perp) + iJ_0(\tau\omega_\perp)\big)\Big] \tag{2.69a}$$

$$\begin{aligned} H_2 = \Big[&\tau' \times p_\perp J_1(r p_\perp) \times \big(-Y_0(\tau\omega_\perp) + iJ_0(\tau\omega_\perp)\big) + \\ &+ r' \times J_0(r p_\perp) \times \omega_\perp\big(-Y_1(\tau\omega_\perp) + iJ_1(\tau\omega_\perp)\big)\Big] \end{aligned} \tag{2.69b}$$

What we are interested in is not the field ϕ, but rather the pionic component $\pi = \sqrt{2}\Im\phi$. We can simply replace all ϕ's with π's since the **homogeneous** KG equation holds separately for imaginary and real part.

In the spirit of identifying the pion field with fluid variables, one could argue

$$\partial_\mu \phi = \partial_\mu(\rho \exp(i\vartheta)) = i\phi \partial_\mu \vartheta = i\phi\chi u_\mu \tag{2.70}$$

Applied separately to real and imaginary part we get

$$\partial_\mu \pi = \sqrt{2}\Im\partial_\mu\phi = \sqrt{2}\chi u_\mu \Re\phi = \sigma\chi u_\mu\,, \qquad \partial_\mu\sigma = \sqrt{2}\Re\partial_\mu\phi = -\sqrt{2}\chi u_\mu \Im\phi = -\pi\chi u_\mu \tag{2.71}$$

Equation (2.11) has an integration constant ϑ_0. Compared to the fields π_0, σ_0 with the choice $\vartheta_0 = 0$, the pion field for any integration constant can be computed by

$$\pi(\vartheta_0) = \Im(\phi e^{i\vartheta_0}) = \sigma_0 \sin\vartheta_0 + \pi_0 \cos\vartheta_0 \tag{2.72}$$

It is thus sensible to compute the following contributions separately

$$C_{1\sigma} = 2\pi^2 \int_0^{\pi} \mathrm{d}\alpha \tau r\ (r'\partial_\tau + \tau'\partial_r)\sigma_0 \times H_1 \tag{2.73a}$$

$$= 2\pi^2 \int_0^{\pi} \mathrm{d}\alpha \tau r\ \cdot (-1) \cdot \pi_0 \chi(r' u_\tau + \tau'\partial_r) \times H_1 \tag{2.73b}$$

$$C_{1\pi} = 2\pi^2 \int_0^{\pi} \mathrm{d}\alpha \tau r\ (r'\partial_\tau + \tau'\partial_r)\pi_0 \times H_1 \tag{2.73c}$$

$$= 2\pi^2 \int_0^{\pi} \mathrm{d}\alpha \tau r\ \sigma_0 \chi(r' u_\tau + \tau'\partial_r) \times H_1 \tag{2.73d}$$

$$C_{2\sigma} = 2\pi^2 \int_0^{\pi} \mathrm{d}\alpha \tau r\ \sigma_0 \times H_2 \tag{2.73e}$$

$$C_{2\pi} = 2\pi^2 \int_0^{\pi} \mathrm{d}\alpha \tau r\ \pi_0 \times H_2 \tag{2.73f}$$

and the final result via

$$\tilde{J}(\vec{p}) = (C_{1\sigma} + C_{2\sigma}) \sin\vartheta_0 + (C_{1\pi} + C_{2\pi}) \cos\vartheta_0 \tag{2.74}$$

To do...

- ☐ 1 (p. 5): Why is $T^{\mu\nu}$ symmetric?
- ☐ 2 (p. 5): Why does this coincide with the definition $T^{\mu\nu}$ from a Lagrangian?
- ☐ 3 (p. 5): Check why kinetic and thermodynamic pressure coincide.
- ☐ 4 (p. 6): Why exactly can we replace the d with a ∂_μ?
- ☐ 5 (p. 7): Maybe list the properties here.
- ☐ 6 (p. 7): There only scalar hydrodynamic variables, except u^μ, right?
- ☐ 7 (p. 8): Why exactly are the coefficients of $T^{\mu\nu}$ not exactly the energy density and pressure, but only up to derivatives of hydro variables? Does the expansion or the precise form of $f_{\mathcal{E}}\ldots$ depend on the spacetime point x?
- ☐ 8 (p. 8): Do this explicitly, especially for $t^{\mu\nu}$ it is hard to see.
- ☐ 9 (p. 8): Is ϵ considered a function of x or of (T,μ)? What is evolved in a hydro evolution, ϵ, T, μ simultaneously?
- ☐ 10 (p. 9): Why is there only 1 transverse traceless symmetric tensor of order 1 ind derivatives of hydro variables?
- ☑ 11 (p. 9): Given a $\mu(x), T(x), u^\mu(x)$ and the equation of state, how can I reconstruct the derivative expansion? Is the expansion unique or does it lead to a specific frame?
- ☐ 12 (p. 9): What distinguishes local and global thermodynamic equilibrium?
- ☑ 13 (p. 12): The equation of state $p(\epsilon)$ closes the system. But from Legendre trafos, p is a function of (T,μ)? How does hydrodynamics and equilibrium thermodynamics come together? After a hydro simulation, e.g. ϵ is given as a function of x, not of some thermo variable.
- ☑ 14 (p. 12): Find a precise definition of kinetic freezout and also chemical freezout
- ☐ 15 (p. 12): Why exactly does boost invariance and rotational symmetry imply $u_\varphi = u_{\eta_s} = 0$?
- ☐ 16 (p. 13): Is this image correct? Do the planes freezout at the same proper time?
- ☐ 17 (p. 14): What about this form of the particle spectrum

$$\frac{1}{2\pi p_T}\frac{\mathrm{d}^2N}{\mathrm{d}p_T\mathrm{d}Y}\bigg|_{Y=0} = \mathcal{N}\int_0^R \mathrm{d}r r m_T I_0\left(\frac{p_T\sinh(\operatorname{artanh} u_r)}{T_f}\right) K_1\left(\frac{m_T\cosh(\operatorname{artanh} u_r)}{T_f}\right) \quad (2.75)$$

  from [Che+21]?
- ☑ 18 (p. 16): This only includes 1-loop order corrections to the effective potential.
- ☐ 19 (p. 16): Check exactly what the error is for the $p \to 0$ contribution of this integral.
- ☑ 20 (p. 17): Is isospin the same is flavor symmetry?
- ☐ 21 (p. 19): It doesn't matter of which representation one thinks about the states, right? There are no different adjoint representations. Still, the eigenvalues depend on the precise choice of basis we did at the beginning
- ☐ 22 (p. 19): Why are exactly the ladder operators the correct eigenstates?

- ☐ 23 (p. 20): So just like addition of angular momenta it has the highest eigenvalues for the Casimirs?
- ☐ 24 (p. 20): The bra-ket notation at this point is probably incomplete...there is the quantum number of the Casimir operator missing
- ☑ 25 It is true that for a symmetry group S that $D_1(S) \otimes D_2(S)$ gives a new valid representation $D_3(S)$, but the representation belong to the same group. On this one cannot simply multiply representations of different groups. (p. 20): Why is color symmetry not just an additional $\otimes \mathbf{3}_C$
- ☐ 26 (p. 20): Why these quantum numbers? I would have expected 1 isospin and 1 z-component for each particle...
- ☑ 27 (p. 20): In general, does each particle type have a different field in the QFT?
- ☐ 28 (p. 21): Commutation is then related to the commutation of operator valued fields?
- ☐ 29 (p. 21): A state $|\dots\rangle$ is some functional of the field. Acting with Q^A on the state means applying the $\hat{\psi}\dots$ operators to this functional?
- ☐ 30 (p. 22): Is γ_5 hermitian?
- ☐ 31 (p. 22): Tong omits the $U(1)_A$, why?
- ☐ 32 (p. 22): How does $U(N > 1)$ vector/axial symmetry really look like in terms of γ_5?
- ☑ 33 (p. 23): Why exactly is the manifold of all possible vacua described by $\langle \overline{\psi}_{-a} \psi_+^b \rangle = -\sigma U_a{}^b$ with $U \in SU(N_f)$?
- ☐ 34 (p. 23): What about $U(1)_A$?
- ☐ 35 (p. 23): Why do we need to come up with such an action that is not present on the microscopic level? Is this something like an effective theory?
- ☐ 36 (p. 23): In Flörchinger notes, he writes about symmetry breaking patterns and choices of reps to break the symmetry. What does that mean?
- ☐ 37 (p. 24): Why does the σ-field and the π-fields appear with the same coefficients?
- ☐ 38 (p. 28): I really don't know about the signature
- ☐ 39 (p. 28): This is a number of order m_Σ^2. Of what order is u_μ (in eV^{-1}? But then again, isn't u^μ normalized to ± 1?)
- ☐ 40 (p. 28): Are these upstairs or downstairs indexed coordinates?
- ☐ 41 (p. 36): By causality this seems reasonable, but from Fourier decomposition of a classical field this is not at all clear.

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