23.04.

$$\mathcal{L} = \frac{1}{2}(\partial_\mu \varphi_a)(\partial^\mu \varphi_a) - U$$

where $U = -\frac{1}{2}\varphi_a\varphi_a\mu^2 + \frac{\lambda}{4}(\varphi_a\varphi_a)^2 + H_a\varphi_a$ $\quad \frac{\lambda'}{4!} \Rightarrow \lambda' = 6\lambda$ $\quad H_a = (\epsilon, \vec{0})$

$$(\varphi_a\varphi_a)_0 \equiv v^2 = \frac{6\mu^2}{\lambda'} = \frac{\mu^2}{\lambda}$$

(With $\varphi_a = (\underbrace{v + \delta\sigma(x)}_{=\sigma(x)}, \vec{\pi}(x))$ one finds $m_\pi \sim \epsilon$ and $m_{\delta\sigma}^2 = 2\mu^2$)

$$\mathcal{L} = \frac{1}{2}\left((\partial_\mu\sigma)(\partial^\mu\sigma) + (\partial_\mu\vec{\pi})(\partial^\mu\vec{\pi})\right) + \frac{1}{2}\mu^2(\sigma^2 + \vec{\pi}^2) - \frac{\lambda}{4}(\sigma^2 + \vec{\pi}^2)^2 + \sigma\epsilon \;)$$

$$\vec{\pi} = \pi\vec{e} \quad \wedge \quad \phi = \frac{1}{\sqrt{2}}(\sigma + i\pi)$$

$$\boxed{\mathcal{L} = (\partial_\mu\phi)(\partial^\mu\phi^*) + \mu^2\phi\phi^* - \frac{\lambda}{2}(\phi\phi^*)^2 + \frac{\epsilon}{\sqrt{2}}(\phi + \phi^*)}$$

EOM $\qquad \frac{\partial\mathcal{L}}{\partial\phi^*} = \partial_\mu\left(\frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi^*)}\right)$

$$\mu^2\phi - \lambda(\phi\phi^*)\phi + \frac{\epsilon}{\sqrt{2}} = \partial_\mu\partial^\mu\phi$$

parametrize $\phi = \varrho e^{i\vartheta}$ $\quad \Rightarrow \partial_\mu\phi = [(\partial_\mu\varrho) + i(\partial_\mu\vartheta)\varrho]e^{i\vartheta}$, $(\partial_\mu\phi)(\partial^\mu\phi^*) = [(\partial_\mu\varrho)(\partial^\mu\varrho) + \varrho^2(\partial_\mu\vartheta)(\partial^\mu\vartheta)]$

$$\Rightarrow \partial_\mu\partial^\mu\phi = \partial_\mu\left[(\partial_\mu\varrho)e^{i\vartheta} + i(\partial_\mu\vartheta)\varrho e^{i\vartheta}\right]$$

$$= \left(\left[\partial_\mu\partial^\mu\varrho - \underbrace{(\partial_\mu\vartheta)(\partial^\mu\vartheta)}_{=:\eta^2}\varrho\right] + i\left[\underbrace{2(\partial_\mu\varrho)(\partial^\mu\vartheta)}_{=\frac{1}{\varrho}(\partial_\mu\varrho^2)(\partial^\mu\vartheta))} + \varrho(\partial_\mu\partial^\mu\vartheta)\right]\right)e^{i\vartheta}$$

$\mathrm{Re}(\mathrm{EOM}/e^{i\vartheta})$ $\qquad \Box\varrho = (\eta^2 + \mu^2 - \lambda\varrho^2)\varrho + \frac{\epsilon}{\sqrt{2}}$ $\qquad$ (EOM 1)

$\mathrm{Im}(\mathrm{EOM}/e^{i\vartheta})$ $\qquad \frac{1}{\varrho}\partial_\mu(\varrho^2\partial^\mu\vartheta) = 0$ $\qquad$ (EOM 2)

Noether currents:

under $\phi \to \phi e^{i\alpha}$ $(\delta\phi = i\alpha\phi)$, $\phi^* \to \phi^* e^{-i\alpha}$ $(\delta\phi^* = -i\alpha\phi^*)$

$$\alpha j^\mu = \frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)}\delta\phi + \frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi^*)}\delta\phi^* = i\alpha\left[(\partial^\mu\phi^*)\phi - (\partial^\mu\phi)\phi^*\right]$$

$$= i\alpha\left[(\partial_\mu\varrho)\varrho - i(\partial_\mu\vartheta)\varrho^2 - \{(\partial_\mu\varrho)\varrho + i(\partial_\mu\vartheta)\varrho^2\}\right]$$

$$= 2\alpha\varrho^2\partial_\mu\vartheta$$

$$T^{\mu\nu} = \frac{2}{\sqrt{-g}}\frac{\delta(\sqrt{-g}\mathcal{L})}{\delta g_{\mu\nu}} = 2\frac{\partial\mathcal{L}}{\partial g_{\mu\nu}} - g^{\mu\nu}\mathcal{L} = 2\left((\partial^\mu\varrho)(\partial^\nu\varrho) + \varrho^2(\partial^\mu\vartheta)(\partial^\nu\vartheta)\right) - g^{\mu\nu}\mathcal{L}$$

HYDRODYNAMICS:

$$j^\mu = n_s v^\mu \qquad (v^\mu v_\mu = -1)$$

$$T^{\mu\nu} = (\epsilon_s + P_s)v^\mu v^\nu - g^{\mu\nu}P_s$$

$$\Rightarrow n_s = \sqrt{j^\mu j_\mu}$$

$$\epsilon_s = v_\mu v_\nu T^{\mu\nu}$$

$$P_s = -\frac{1}{3}(g_{\mu\nu} - v_\mu v_\nu)T^{\mu\nu}$$

$$\Rightarrow n_s = 2\varrho^2\eta \quad \wedge \quad v^\mu = \frac{\partial^\mu\vartheta}{\eta} \qquad (\rightarrow v^\mu\partial_\mu\vartheta = \eta)$$

$$= 2\eta\frac{\eta^2 + \mu^2}{\lambda}$$

$$\boxed{\begin{aligned} &\eta = \eta(\epsilon_s) = \ldots \quad \wedge \quad \varrho = \sqrt{\frac{\eta^2 + \mu^2}{\lambda}} \\ &\partial_\mu\vartheta = \eta v^\mu \end{aligned}}$$

$$\epsilon_s = 2\varrho^2\eta^2 - \mathcal{L} = \eta n_s - \frac{(\eta^2 + \mu^2)^2}{2\lambda} = \frac{4\eta^2(\eta^2 + \mu^2) - (\eta^4 + 2\eta^2\mu^2 + \mu^4)}{2\lambda}$$

$$= \frac{3\eta^4 + 2\eta^2\mu^2 - \mu^4}{2\lambda} = \frac{(3\eta^2 - \mu^2)(\eta^2 + \mu^2)}{2\lambda}$$

$$\Leftrightarrow \frac{2\lambda}{3}\epsilon_s = \eta^4 + \frac{2\mu^2}{3}\eta^2 - \frac{\mu^4}{3} \Leftrightarrow (\eta^2)_{1,2} = -\frac{\mu^2}{3} \pm \sqrt{\frac{\mu^4}{9} + \frac{\mu^4}{3} + \frac{2\lambda}{3}\epsilon_s}$$

Solutions of EOM assuming $\partial_\mu\partial^\mu\vartheta \approx 0 \Rightarrow \partial_\mu\vartheta \approx p_\mu = \text{const}$

$$\overset{(\mathrm{EOM}\,2)}{\Longrightarrow} \quad \varrho = \text{const}$$

$$\overset{(\mathrm{EOM}\,1)}{\Longrightarrow} \quad \varrho = \sqrt{\frac{\eta^2 + \mu^2}{\lambda}} + \mathcal{O}(\epsilon)$$

$$\mathcal{L}\Big|_{\mathrm{EOM},\,\epsilon\to 0} = \varrho^2\eta^2 + \mu^2\varrho^2 - \frac{\lambda}{2}\varrho^4$$

$$= \left(\frac{\eta^4}{\lambda} + \frac{\eta^2\mu^2}{\lambda} + \frac{\eta^2\mu^2}{\lambda} + \frac{\mu^4}{\lambda}\right) - \frac{\lambda}{2}\left(\frac{\eta^4 + 2\mu^2\eta^2 + \mu^4}{\lambda^2}\right)$$

$$= \frac{\eta^4}{2\lambda} + \frac{2\eta^2\mu^2}{2\lambda} + \frac{\mu^4}{2\lambda} = \frac{(\eta^2 + \mu^2)^2}{2\lambda}$$
