Chapter 3

Calculation of Condensed Field

3.1 Relating Fluid and Pion Fields

The signature is $(-,+,+,+)$.

Start with the Lagrangian of the linear σ-model for the real-valued $O(4)$-vector $\varphi_a = (\sigma, \boldsymbol{\pi})$.

$$\mathscr{L} = -\frac{1}{2}\partial_\mu\sigma\partial^\mu\sigma - \frac{1}{2}\partial_\mu\boldsymbol{\pi}\partial^\mu\boldsymbol{\pi} - \frac{1}{2}m^2(\sigma^2+\boldsymbol{\pi}^2) - \frac{\lambda}{4}(\sigma^2+\boldsymbol{\pi}^2)^2 - \epsilon\sigma \tag{3.1}$$

and from 37 we find, that if $m^2 = -\mu^2 < 0$ SSB occurs (plus, $\epsilon \neq 0$ explicitly breaks the symmetry) with the VEV v and masses of the σ and $\boldsymbol{\pi}$ excitation in $\varphi_a = (v+\delta\sigma, \boldsymbol{\pi})$ are given by (note the difference in conventions $\lambda \to 6\lambda$ w.r.t. 37)

$$\begin{cases} v = \dfrac{\mu}{\sqrt{\lambda}} + \epsilon\dfrac{1}{2\mu^2} + \mathcal{O}(\epsilon^2) \\ m_\sigma^2 = 2\mu^2 + \mathcal{O}(\epsilon) \\ m_\pi^2 = \epsilon\dfrac{\sqrt{\lambda}}{\mu} + \mathcal{O}(\epsilon^2) \end{cases} \quad \overset{\text{to lowest order}}{\Longleftrightarrow} \quad \begin{cases} \mu = \dfrac{m_\sigma}{\sqrt{2}} \\ \lambda = \dfrac{m_\sigma^2}{2v} \\ \epsilon = v m_\pi^2 \end{cases} \tag{3.2}$$

Choose now a fixed alignment of the condensate $\boldsymbol{\pi} = \pi\boldsymbol{e}$ with $\boldsymbol{e}\cdot\boldsymbol{e} = 1$ determining the orientation in isospin space. This choice breaks $O(4)$ to $O(2)$ (we restrict ourselves to $SO(2)$) and use to isomorphism $SO(2) \cong U(1)$ to write the linear σ-model as a theory of a complex scalar field $\phi = \frac{1}{\sqrt{2}}(\sigma + i\pi) = \rho e^{i\vartheta}$.

$$\text{in terms of}\,(\phi,\phi^*) \qquad \mathscr{L} = -(\partial_\mu\phi)(\partial^\mu\phi^*) + \mu^2\phi\phi^* - \frac{\lambda}{2}(\phi\phi^*)^2 + \frac{\epsilon}{\sqrt{2}}(\phi+\phi^*) \tag{3.3a}$$

$$\text{in terms of}\,(\rho,\vartheta^*) \qquad \mathscr{L} = -(\partial_\mu\rho)(\partial^\mu\rho) - \rho^2(\partial_\mu\vartheta)(\partial^\mu\vartheta) + \mu^2\rho^2 - \frac{\lambda}{2}\rho^4 + \sqrt{2}\epsilon\rho\cos\vartheta \tag{3.3b}$$

The classical equations of motion arising from this are

$$\text{in terms of}\,(\phi,\phi^*) \qquad -\Box\phi = (\partial_t^2 - \boldsymbol{\nabla}^2)\phi = \big[\mu^2 - \lambda(\phi^*\phi)\big]\phi + \frac{\epsilon}{\sqrt{2}} \tag{3.4a}$$

$$\text{in terms of}\,(\rho,\vartheta^*) \qquad -\Box\rho = \big[-(\partial_\mu\vartheta)(\partial^\mu\vartheta) + \mu^2 - \lambda\rho^2\big]\rho + \frac{\epsilon}{\sqrt{2}}\cos\vartheta \tag{3.4b}$$

$$-\partial_\mu(\rho^2\partial^\mu\vartheta) = -\frac{\epsilon}{\sqrt{2}}\rho\sin\vartheta \tag{3.4c}$$

The conserved Noether current of thse symmetry $\theta \to \theta + \alpha$ (in the limit $\epsilon \to 0$) and energy-momentum tensor are

$$\alpha j^\mu = -\frac{\partial\mathscr{L}}{\partial(\partial_\mu\vartheta)}\delta\vartheta = 2\alpha\rho^2(\partial^\mu\vartheta) \tag{3.5a}$$

$$T^{\mu\nu} = \frac{2}{\sqrt{-g}}\frac{\delta(\sqrt{-g}\mathscr{L})}{\delta g_{\mu\nu}} = 2\frac{2\partial\mathscr{L}}{\partial g_{\mu\nu}} + g^{\mu\nu}\mathscr{L} = 2\big[(\partial^\mu\rho)(\partial^\nu\rho) + \rho^2(\partial^\mu\vartheta)(\partial^\nu\vartheta)\big] + g^{\mu\nu}\mathscr{L} \tag{3.5b}$$

where $\delta\sqrt{-g} = \frac{-1}{2\sqrt{-g}}\delta g = \frac{-1}{\sqrt{-g}}gg^{\mu\nu}\delta g_{\mu\nu} = \frac{\sqrt{-g}}{2}g^{\mu\nu}\delta g_{\mu\nu}$ was used.

In a tree-level approximation one only needs to solve the classical equations of motion. In the limit $\epsilon \to 0$ a valid solution is

$$\partial_\mu \vartheta = \text{const.}\,, \qquad \rho = \sqrt{\frac{\chi^2 + \mu^2}{\lambda}} + \mathcal{O}(\epsilon) \qquad (\chi^2 := -\,(\partial_\mu \vartheta)(\partial^\mu \vartheta)) \tag{3.6}$$

We might generalize the solution to the case $\partial_\mu \vartheta \approx$ const. which should be valid in the limit $\chi^2 \ll \mu^2 = \frac{m_\sigma^2}{2}$. On these solutions one finds

$$\mathscr{L}\big|_{\text{EOM}} = \rho^2(\chi^2 + \mu^2 - \frac{\lambda}{2}\rho^2) = \rho^2 \frac{\chi^2 + \mu^2}{2} \tag{3.7a}$$

$$T^{\mu\nu}\big|_{\text{EOM}} = 2\rho^2(\partial^\mu \vartheta)(\partial^\nu \vartheta) + g^{\mu\nu}\rho^2 \frac{\chi^2 + \mu^2}{2} \tag{3.7b}$$

Assume the dynamics of the field could be described by ideal hydrodynamics, i.e. a conserved current and energy-momentum tensor of the form **To do (39) I really don't know about the signature**

$$j^\mu = n_s \nu^\mu \qquad (v^\mu v_\mu = -1) \tag{3.8a}$$

$$T^{\mu\nu} = (\epsilon_s + P_s) v^\mu v^\nu + g^{\mu\nu} P_s \tag{3.8b}$$

from which the prefactors can be extracted by the contractions

$$n_s = \sqrt{-j^\mu j_\mu}\,, \qquad \epsilon_s = v_\mu v_\nu T^{\mu\nu}\,, \qquad P_s = \frac{1}{3}(g_{\mu\nu} + v_\mu v_\nu) T^{\mu\nu} \tag{3.9}$$

Identifying the field theoretic with the hydrodynamic viewpoint it immediately follows that

$$n_s = 2\rho^2 \chi\,, \qquad v^\mu = \chi^{-1}(\partial^\mu \vartheta) \qquad (\iff \chi = -v^\mu(\partial_\mu \vartheta)) \tag{3.10a}$$

$$\epsilon_s = 2\rho^2\chi^2 - \rho^2 \frac{\chi^2 + \mu^2}{2} = \rho^2 \frac{4\chi^2 - (\chi^2 + \mu^2)}{2} = \frac{(\chi^2 + \mu^2)(3\chi^2 - \mu^2)}{2\lambda} \tag{3.10b}$$

How to apply this?

The freeze-out surface is invariant under rotations (= independent of polar angle φ) around the collision axis and longitudinal boosts (= independent of rapidity η_s) and hence parametrized by a one-dimensional curve in the r-τ-plane. The curve itself may be parametrized by some real parameter α, following some mapping $\alpha \mapsto (r(\alpha), \tau(\alpha))$. From the hydro simulation we wish to identify the gradient $\partial_\mu \vartheta \sim u_\mu$ of the complex phase of the condensate field with the fluid 4-velocity u_μ, hence in order to find the phase of the field, an integration of $\partial_\mu \vartheta$ over the hypersurface is needed and an integration constant ϑ_0 can be chosen freely. Choose $\alpha = \arctan(\tau/r)$ to be the polar angle of the point $(r(\alpha), \tau(\alpha))$ in the r-τ-plane. Since $r, \tau > 0$ α is restricted to the range $[0, \pi]$ and $\vartheta(\alpha)$ on the hypersurface can be calculated via

$$\vartheta(\alpha) = \vartheta_0 + \int_0^\alpha \mathrm{d}s \frac{\mathrm{d}\vartheta}{\mathrm{d}s} = \vartheta_0 + \int_0^\alpha \mathrm{d}s \frac{\partial x^\mu(s)}{\partial s} \partial_\mu \vartheta \tag{3.11}$$

$\partial x^\mu(s)/\partial s$ represents the tangent vector of the freeze-out surface.

The energy density ϵ and 4-velocity u^μ of the fluid is related to the condensate phase and density via

$$-(\partial_\mu \vartheta)(\partial^\mu \vartheta) = \chi^2 = \frac{-\mu^2 + \sqrt{6\epsilon\lambda + 2\mu^4}}{3} \tag{3.12a}$$

$$\partial^\mu \vartheta = \chi u^\mu\,, \qquad \rho^2 = \sqrt{\frac{\chi^2 + \mu^2}{\lambda}} \tag{3.12b}$$

One may use the relations (3.2) to rewrite the above equation in terms of the particle masses and the pion decay constant $f_\pi \equiv v$.

$$\chi^2 = \frac{-m_\sigma^2/2 + \sqrt{3\epsilon m_\sigma^2/f_\pi + m_\sigma^4/2}}{3} = \frac{-m_\sigma^2 + m_\sigma\sqrt{12\epsilon/f_\pi + 2m_\sigma^2}}{6}\,, \qquad \rho = \sqrt{f_\pi \frac{2\chi^2 + m_\sigma^2}{m_\sigma^2}} \tag{3.13}$$

In Milne coordinates $(x^\mu) = (\tau, r, \varphi, \eta_s)$ the 4-velocity has components $(u^\mu) = (\gamma, \gamma v, 0, 0)$, where $v = v(\alpha)$ is a function on the freeze-out surface. **To do (40) Are these upstairs or downstairs indexed coordinates?**

3.2 Finding the Spectrum at the Detector Surface

In general the particle number N and particle number density $n(\vec{x})$ in position space and $n(\vec{p})$ in momentum space associated to the condensate $\phi(\vec{x})$ of a complex scalar field are given by the relations

$$n(\vec{x}) = \phi(\vec{x})\phi^*(\vec{x})\,, \qquad n(\vec{p}) = \phi(\vec{p})\phi^*(\vec{p}) \tag{3.14a}$$

$$N = \int \mathrm{d}^3x n(\vec{x}) = \int \frac{\mathrm{d}^3p}{(2\pi)^3} n(\vec{p}) \tag{3.14b}$$

with the convention $\phi(\vec{x}) = \int \mathrm{d}^3p/(2\pi)^3 \phi(\vec{p}) e^{-i\vec{p}\vec{x}}$ for the Fourier transform.

3.2.1 Treating the Freeze Out Field as Source Term???

Let's evaluate the Fourier transform of the condensate field. The calculation uses the result from linear response theory for the deviation of the expectation value $\overline{\phi}$ induced by a source term j, which we for the moment we assume to be specified by the hydro variables on the freezeout surface.

$$\overline{\phi}(x) \equiv \langle \phi(x) \rangle = \int \mathrm{d}^4 G_{\mathrm{ret}}(x,y) j(y) \tag{3.15a}$$

The retarded Greens function is given by

$$G_{\mathrm{ret}}(x,y) = \Theta(x^0 - y^0)\big[D(x-y) - D(y-x)\big] \tag{3.15b}$$

with the propagator

$$D(x-y) = \int \frac{\mathrm{d}^3q}{(2\pi)^3} \frac{e^{-i[\omega_{\vec{q}}(x^0-y^0) - \vec{q}(\vec{x}-\vec{y})]}}{2\omega_{\vec{q}}} \tag{3.15c}$$

and the relations

$$\int \mathrm{d}^3x e^{i\vec{x}(\vec{p}-\vec{q})} = (2\pi)^3 \delta^{(3)}(\vec{p}-\vec{q})\,, \qquad \phi(t,\vec{p}) = \int \mathrm{d}^3x e^{i\vec{p}\vec{x}} \phi(t,\vec{x}) \tag{3.15d}$$

Putting things together

$$\begin{aligned}
\overline{\phi}(x^0 \equiv t, \vec{p}) &= \int \mathrm{d}^3x e^{i\vec{p}\vec{x}} \overline{\phi}(x^0 \equiv t, \vec{x}) && (3.16a)\\
&= \int \mathrm{d}^3x e^{i\vec{p}\vec{x}} \int \mathrm{d}^4y G_{\mathrm{ret}}(x,y) j(y) && (3.16b)\\
&= \int \mathrm{d}^3x e^{i\vec{p}\vec{x}} \int \mathrm{d}^4y \Theta(x^0-y^0)\big[D(x-y) - D(y-x)\big] j(y) && (3.16c)\\
&= \int \mathrm{d}^3x e^{i\vec{p}\vec{x}} \int \mathrm{d}^4y \int \frac{\mathrm{d}^3q}{(2\pi)^3} \frac{1}{2\omega_{\vec{q}}} \Theta(x^0-y^0) \times\\
&\qquad \times \big[e^{-i[w_{\vec{q}}(x^0-y^0) - \vec{q}(\vec{x}-\vec{y})]} - e^{i[w_{\vec{q}}(x^0-y^0) - \vec{q}(\vec{x}-\vec{y})]}\big] j(y) && (3.16d)\\
&= \int \mathrm{d}^3x \int \mathrm{d}^4y \int \frac{\mathrm{d}^3q}{(2\pi)^3} \frac{1}{2\omega_{\vec{q}}} \Theta(x^0-y^0) \times\\
&\qquad \times \big[e^{-i[w_{\vec{q}}(x^0-y^0) + \vec{q}\vec{y}]} e^{i(\vec{p}+\vec{q})\vec{x}} - e^{i[w_{\vec{q}}(x^0-y^0) + \vec{q}\vec{y}]} e^{i(\vec{p}-\vec{q})\vec{x}}\big] j(y) && (3.16e)\\
&= \int \mathrm{d}^4y \int \mathrm{d}^3q \frac{1}{2\omega_{\vec{q}}} \Theta(x^0-y^0) \times\\
&\qquad \times \big[e^{-i[w_{\vec{q}}(x^0-y^0) + \vec{q}\vec{y}]} \delta^{(3)}(\vec{p}+\vec{q}) - e^{i[w_{\vec{q}}(x^0-y^0) + \vec{q}\vec{y}]} \delta^{(3)}(\vec{p}-\vec{q})\big] j(y) && (3.16f)\\
&= \int \mathrm{d}^4y \frac{1}{2\omega_{\vec{p}}} \Theta(x^0-y^0) \big[e^{-i[w_{\vec{p}}(x^0-y^0) - \vec{p}\vec{y}]} - e^{i[w_{\vec{p}}(x^0-y^0) + \vec{p}\vec{y}]}\big] j(y) && (3.16g)\\
&= \int \mathrm{d}^4y \frac{1}{2\omega_{\vec{p}}} \Theta(x^0-y^0) e^{i\vec{p}\vec{y}} \big[e^{-i\omega_{\vec{p}}(x^0-y^0)} - e^{i\omega_{\vec{p}}(x^0-y^0)}\big] j(y) && (3.16h)\\
&= \int \mathrm{d}^4y \frac{1}{i\omega_{\vec{p}}} \Theta(x^0-y^0) e^{i\vec{p}\vec{y}} \sin(\omega_{\vec{p}}(x^0-y^0)) j(y) && (3.16i)
\end{aligned}$$

Specify this for the relevant case supp$j = \Sigma_{\text{freeze-out}}$ and parametrize the integral over the hypersurface Σ via (see (3.2|1)) $\int_\Sigma \mathrm{d}^3\Sigma = \int \mathrm{d}\varphi \mathrm{d}\eta \mathrm{d}\alpha r(\alpha)\tau(\alpha)\sqrt{r'^2(\alpha) - \tau'^2(\alpha)}$

$$\begin{aligned}\overline{\phi}(x^0 \equiv t, \vec{p}) = \int_0^{2\pi} \mathrm{d}\varphi \int_{-\infty}^{\infty} \mathrm{d}\eta \int_0^{\pi} \mathrm{d}\alpha r(\alpha)\tau(\alpha)\sqrt{r'^2(\alpha) - \tau'^2(\alpha)} \frac{1}{i\omega_{\vec{p}}} \Theta(x^0 - \tau(\alpha)\cosh\eta)\times \\ \times \exp\left(i[\vec{p}_\perp \vec{x}_\perp + p_z \tau(\alpha)\sinh\eta]\right) \sin(\omega_{\vec{p}}(x^0 - \tau(\alpha)\cosh\eta)) j(\tau(\alpha), r(\alpha))\end{aligned} \tag{3.16j}$$

Performing the φ-integration, with φ appearing in the integrand in $\vec{x}_\perp = (r\cos\varphi, r\sin\varphi)$, we choose to align $\varphi = 0$ with the p_x direction. Then $\vec{p}_\perp \vec{x}_\perp = p^\perp r \cos\varphi$. Use the Bessel function J_0 of first kind to simplify

$$\begin{aligned}\overline{\phi}(x^0 \equiv t, \vec{p}) = \int_{-\infty}^{\infty} \mathrm{d}\eta \int_0^{\pi} \mathrm{d}\alpha r(\alpha)\tau(\alpha)\sqrt{r'^2(\alpha) - \tau'^2(\alpha)} \frac{1}{i\omega_{\vec{p}}} \Theta(x^0 - \tau(\alpha)\cosh\eta)\times \\ \times 2\pi J_0(p^\perp r(\alpha)) \exp\left(i p_z \tau(\alpha)\sinh\eta\right) \sin(\omega_{\vec{p}}(x^0 - \tau(\alpha)\cosh\eta)) j(\tau(\alpha), r(\alpha))\end{aligned} \tag{3.16k}$$

Calculation 3.2|1: Metric on Hypersurface

Recall the metric $g_{\mu\nu} = \mathrm{diag}(-1, 1, \tau^2, r^2)$ in coordinates (τ, r, η, φ). Orthonormal tangent vectors to the freeze out hypersurface are $(\hat{\partial}_\varphi)^\mu = (0,0,0,r^{-1}) = r^{-1}(\partial_\varphi)^\mu$, $(\hat{\partial}_\eta)^\mu = (0,0,\tau^{-1},0) = \tau^{-1}(\partial_\eta)^\mu$ and $(\hat{\partial}_\alpha)^\mu = \sqrt{r'^2(\alpha) - \tau'^2(\alpha)}^{-1}(\tau'(\alpha), r'(\alpha), 0, 0) = D(\alpha)(\partial_\alpha)^\mu$ with $D(\alpha) = \sqrt{r'^2(\alpha) - \tau'^2(\alpha)}^{-1}$. The projector on the hypersurface is

$$\gamma_{\mu\nu} = (\hat{\partial}_\varphi)_\mu(\hat{\partial}_\varphi)_\nu + (\hat{\partial}_\eta)_\mu(\hat{\partial}_\eta)_\nu + (\hat{\partial}_\alpha)_\mu(\hat{\partial}_\alpha)_\nu = \begin{pmatrix} D^2(\alpha)\tau'^2(\alpha) & -D^2(\alpha)\tau'(\alpha)r'(\alpha) & 0 & 0 \\ -D^2(\alpha)\tau'(\alpha)r'(\alpha) & D^2(\alpha)r'^2(\alpha) & 0 & 0 \\ 0 & 0 & \tau^2 & 0 \\ 0 & 0 & 0 & r^2 \end{pmatrix} \tag{3.17}$$

The normal of the hypersurface is $n^\mu \equiv (\hat{\partial}_\alpha^\perp)^\mu = D(\alpha)(r'(\alpha), \tau'(\alpha), 0, 0)$ and is timelike where D is real. Naturally $\gamma_{\mu\nu}n^\nu = 0$. In the basis $(\partial_\alpha, \partial_\eta, \partial_\varphi, n)$ using (in short form)

$$(\partial_\alpha)^\nu \gamma_{\mu\nu} (\partial_\alpha)^\mu = \begin{pmatrix} \tau' \\ r' \end{pmatrix}^T \begin{pmatrix} -\tau' \\ r' \end{pmatrix} = D^{-2} \tag{3.18}$$

the hypersurface metric in coordinates $x^i = (\alpha, \eta, \varphi)$ reads

$$\gamma_{ij} = \mathrm{diag}(D^{-2}(\alpha), \tau^2(\alpha), r^2(\alpha)) \tag{3.19}$$

and the volume element is given by $\mathrm{d}\Sigma = r(\alpha)\tau(\alpha)D^{-1}(\alpha)\mathrm{d}\alpha\mathrm{d}\eta\mathrm{d}\varphi$. The oriented surface element is

$$\mathrm{d}\Sigma^\mu = n^\mu \mathrm{d}\Sigma = r(\alpha)\tau(\alpha)(r'(\alpha), \tau'(\alpha), 0, 0)\mathrm{d}\alpha\mathrm{d}\eta\mathrm{d}\varphi \tag{3.20}$$

3.2.2 Converting Spectra between Coordinate Systems

Consider the coordinate change in momentum space

$$\begin{cases} p_x = p^\perp \cos\varphi_p \\ p_y = p^\perp \sin\varphi_p \\ p_z = m^\perp \sinh\eta_p \\ p_t = m^\perp \cosh\eta_p \end{cases} \iff \begin{cases} p^\perp = \sqrt{p_x^2 + p_y^2} \\ \varphi_p = \arctan(p_y/p_y) \\ m^\perp = \sqrt{p_t^2 - p_z^2} \\ \eta_p = \mathrm{artanh}\,(p_z/p_t) \end{cases} \tag{3.21}$$

with Jacobian

$$\left|\frac{\partial(p^\perp, \varphi_p, m^\perp, \eta_p)}{\partial(p_x, p_y, p_z, p_t)}\right| = \frac{1}{m^\perp p^\perp} \tag{3.22}$$

Let $f(p_\mu)$ be some distribution function and F its momentum space integral evaluated on the momentum shell and future directed momenta.

$$F = \int \frac{\mathrm{d}^4 p_{\text{cart}}}{(2\pi)^4} \delta(p^2 - m^2) \Theta(p_t) f(p_\mu) = \int \frac{\mathrm{d}p_t}{2\pi} \int \frac{\mathrm{d}^3 p_{\text{cart}}}{(2\pi)^3} \frac{1}{2\omega_{\vec{p}}} \big(\delta(p_t - \omega_{\vec{p}}) + \delta(p_t + \omega_{\vec{p}})\big) \Theta(p_t) f(p_\mu) \tag{3.23a}$$

$$= \frac{1}{2\pi} \int \frac{\mathrm{d}^3 p_{\text{cart}}}{(2\pi)^3} \frac{1}{2\omega_{\vec{p}}} f(p_\mu)\big|_{p_t=\omega_{\vec{p}}} \tag{3.23b}$$

On the other hand

$$F = \int \frac{\mathrm{d}^4 p_{\text{cart}}}{(2\pi)^4} \delta(p^2 - m^2) \Theta(p_t) f(p_\mu) = \frac{1}{(2\pi)^4} \int_0^\infty \mathrm{d}p^\perp \int_0^\infty \mathrm{d}m^\perp \int_{-\infty}^\infty \mathrm{d}\eta_p \int_0^{2\pi} \mathrm{d}\varphi_p m^\perp p^\perp \times \\ \times \delta((p^\perp)^2 - (m^\perp)^2) f(p_\mu) \tag{3.23c}$$

assume $f(p^\mu) = f(p^\perp, m^\perp, \eta_p)$ and perform the φ-integraion

$$= \frac{1}{(2\pi)^3} \int_0^\infty \mathrm{d}m^\perp \int_{-\infty}^\infty \mathrm{d}\eta_p \frac{m^\perp}{2} f(p_\mu)\big|_{m^\perp=p^\perp} \tag{3.23d}$$

leding to the important result

$$\frac{1}{2\pi} \int \frac{\mathrm{d}^3 p_{\text{cart}}}{(2\pi)^3} \frac{1}{2\omega_{\vec{p}}} f(p_\mu)\big|_{p_t=\omega_{\vec{p}}} = \frac{1}{(2\pi)^3} \int_0^\infty \mathrm{d}m^\perp \int_{-\infty}^\infty \mathrm{d}\eta_p \frac{m^\perp}{2} f(p_\mu)\big|_{m^\perp=p^\perp} \tag{3.23e}$$

Since the restrictions $p_t = \omega_{\vec{p}}$ and $m^\perp = p^\perp$ are equivalent (considering the parametrization that already satisfies $p_t = p^\perp \cosh \eta_p \geq 0$) we find

$$\omega_{\vec{p}} \frac{\mathrm{d}F}{\mathrm{d}p_x \mathrm{d}p_y \mathrm{d}p_z} = \frac{1}{2\pi m^\perp} \frac{\mathrm{d}F}{\mathrm{d}m^\perp \mathrm{d}\eta_p} \tag{3.24}$$

The result applies to the case

$$f(p_\mu)\big|_{p_t=\omega_{\vec{p}}} = 2\omega_{\vec{p}} \cdot 2\pi \cdot n(\vec{p}) \tag{3.25}$$

and $F = N$.

3.2.3 Fourier Decomposition on Freeze Out Surface???

Generally a Fourier Ddecomposition to solve the Klein-Gordon equation in terms of 3-momentum-modes is given by

$$\phi(x^\mu) = \int \frac{\mathrm{d}^4 p}{(2\pi)^4} \tilde{\phi}(\vec{p}) e^{ip_\mu x^\mu} \delta(p^2 - m^2) \Theta(p_0) + \text{c.c.} = \frac{1}{2\pi} \int \frac{\mathrm{d}^3 p}{(2\pi)^3} \frac{1}{2\omega_{\vec{p}}} \phi(\vec{p}) e^{i(\omega_{\vec{p}} x^0 - \vec{p}\vec{x})} + \text{c.c.} \tag{3.26}$$

For simplicity neglect the +c.c..

Considering a hypersurface Σ and using the field data given only on this hypersurface - that is considering the restriction $\phi|_{x\in\Sigma}$ - can we reconstruct $\tilde{\phi}(\vec{p})$? The map $\tilde{\phi}(\vec{p}) \mapsto \phi(x^\mu \in \Sigma)$ is trivially given by the mode decomposition above. Let $\Sigma = \Sigma_t = \{x^\mu \in \mathbb{R}^{(1,3)} | x^0 = t = \text{const}\}$ be a slice of constant lab time. Then the map $\phi(x^\mu \in \Sigma_t) \equiv \phi_t(\vec{x}) \mapsto \tilde{\phi}(\vec{p})$ is easily found to be

$$\phi(\vec{p}) = (2\pi)(2\omega_{\vec{p}}) \int_{\Sigma_t} \mathrm{d}^3 x \phi_t(\vec{x}) e^{-i(\omega_{\vec{p}} t - \vec{p}\vec{x})} \tag{3.27}$$

which of course uses the orthogonality relation

$$\int \mathrm{d}^n x_{\text{cart}} e^{i(p_\mu - q_\mu) x^\mu} = (2\pi)^n \delta^{(n)}(p - q) \tag{3.28}$$

valid in cartesian coordinates.

We would like to generalize this to arbitrary Σ. Let $(y^i)_{i=1}^3$ be the coordinates of a parametrization $x^\mu(y^i)$ of Σ. The naive attempt would be an integral of the form

$$\tilde{\phi}(\vec{p}) \stackrel{?}{=} (2\pi)(2\omega_{\vec{p}}) \int_\Sigma \mathrm{d}^3 y \sqrt{\gamma} \phi(x^\mu(y^i)) e^{-i(\omega_{\vec{p}} t(y^i) - \vec{p}\vec{x}(y^i))} \tag{3.29}$$

where γ is the induced metric determinant. The relevant example is $\Sigma = \{x^\mu \in \mathbb{R}^{(1,3)} | (\tau, r) = (\tau(\alpha), r(\alpha))\}$ with τ, r defined by the coordinate transformation

$$\begin{cases} t = \tau \cosh \eta \\ z = \tau \sinh \eta \\ x = r \cos \varphi \\ y = r \sin \varphi \end{cases} \qquad \Longleftrightarrow \qquad \begin{cases} \tau = \sqrt{t^2 - z^2} \\ \eta = \operatorname{artanh}(z/t) \\ r = \sqrt{x^2 + y^2} \\ \varphi = \arctan(y/x) \end{cases} \tag{3.30}$$

Fourier Transform in Adapted Coordinates

Let's first evaluate $p_\mu x^\mu$ in the Bjorken coordinate system. Therefore introduce an analogous coordinate change in momentum space

$$\begin{cases} p_t = m_\perp \cosh \eta_p \\ p_z = m_\perp \sinh \eta_p \\ p_x = p_\perp \cos \varphi_p \\ p_y = p_\perp \sin \varphi_p \end{cases} \tag{3.31}$$

to rewrite the scalar product as

$$p_\mu x^\mu \equiv \tau(p_t \cosh \eta - p_z \sinh \eta) - r(p_x \cos \varphi + p_y \sin \varphi) = \tau m_\perp \cosh(\eta - \eta_p) - r p_\perp \cos(\varphi - \varphi_p) \tag{3.32}$$

We used the identities

$$\cosh(a - b) = \cosh a \cosh b - \sinh a \sinh b\,, \qquad \cos(a - b) = \cos a \cos b + \sin a \sin b \tag{3.33}$$

The integral measure changes according to $\mathrm{d}^4 p_{\text{cart}} = \mathrm{d}m_\perp \mathrm{d}p_\perp \mathrm{d}\eta_p \mathrm{d}\varphi_p \cdot m_\perp p_\perp$. The momentum shell condition $p^2 = m^2$ is equivalently parametrized by $m_\perp^2 = p_\perp^2 + m^2 =: \omega_\perp^2$.

These coordinates are adapted to boost symmetry $\eta \to \eta'$ along the beam direction and rotational symmetry $\varphi \to \varphi'$ around the beam axis. Investigate first the implications on the mode decomposition, by requesting that $(\partial/\partial\eta)\phi(x^\mu) = 0 = (\partial/\partial\varphi)\phi(x^\mu)$.

$$\begin{aligned} \phi(x^\mu) = \frac{1}{(2\pi)^4} \int_{-\infty}^{\infty} \mathrm{d}m_\perp \int_0^\infty \mathrm{d}p_\perp \int_{-\infty}^{\infty} \mathrm{d}\eta_p \int_0^{2\pi} \mathrm{d}\varphi_p \cdot m_\perp p_\perp \delta(m_\perp^2 - \omega_\perp^2) \Theta(m_\perp) \times \\ \times \tilde{\phi}(p_x(p_\perp, \varphi_p), p_y(p_\perp, \varphi_p), p_z(m_\perp, \eta_p)) e^{i(\tau m_\perp \cosh(\eta - \eta_p) - r p_\perp \cos(\varphi - \varphi_p))} \end{aligned} \tag{3.34a}$$

...shift $\varphi_p \to \varphi_p + \varphi$ and $\eta_p \to \eta_p + \eta$...

$$\begin{aligned} = \frac{1}{(2\pi)^4} \int_{-\infty}^{\infty} \mathrm{d}m_\perp \int_0^\infty \mathrm{d}p_\perp \int_{-\infty}^{\infty} \mathrm{d}\eta_p \int_0^{2\pi} \mathrm{d}\varphi_p \cdot m_\perp p_\perp \delta(m_\perp^2 - \omega_\perp^2) \Theta(m_\perp) \times \\ \times \tilde{\phi}(p_x(p_\perp, \varphi_p + \varphi), p_y(p_\perp, \varphi_p + \varphi), p_z(m_\perp, \eta_p + \eta)) e^{i(\tau m_\perp \cosh \eta_p - r p_\perp \cos \varphi_p)} \end{aligned} \tag{3.34b}$$

From this it follows that $\tilde{\phi}(p_x, p_y, p_z) = \tilde{\phi}(p_\perp)$ and we can simplify the integral. Let's also evaluate the δ-distribution by using that $\delta(m_\perp^2 - \omega_\perp^2) = (1/2\omega_\perp)(\delta(m_\perp - \omega_\perp) + \delta(m_\perp + \omega_\perp))$

$$\phi(x^\mu) = \frac{1}{(2\pi)^4} \frac{1}{2} \int_0^\infty \mathrm{d}p_\perp \int_{-\infty}^{\infty} \mathrm{d}\eta_p \int_0^{2\pi} \mathrm{d}\varphi_p \cdot p_\perp \tilde{\phi}(p_\perp) e^{i(\tau \omega_\perp \cosh \eta_p - r p_\perp \cos \varphi_p)} \tag{3.34c}$$

$$= \frac{1}{2} \frac{1}{(2\pi)^4} \int_0^\infty \mathrm{d}p_\perp p_\perp \tilde{\phi}(p_\perp) \big(2\pi J_0(r p_\perp)\big) \big(\pi(-Y_0(\tau\omega_\perp) + i J_0(\tau\omega_\perp))\big) \tag{3.34d}$$

where in the last step the following integral representation of Bessel functions of the first kind $J_0(x)$ and of the second kind $Y_0(x)$ where used `https://dlmf.nist.gov/10.9`

$$J_0(x \in \mathbb{R}) = \frac{1}{2\pi} \int_0^{2\pi} \mathrm{d}t \exp(\overset{?}{\pm} ix \cos t)\,, \quad J_0(x > 0) = \frac{1}{\pi} \int_{-\infty}^{\infty} \mathrm{d}t \sin(x \cosh t)\,, \quad Y_0(x > 0) = -\frac{2}{\pi} \int_0^\infty \mathrm{d}t \cos(x \cosh t) \tag{3.35a}$$

$$\int_0^{2\pi} \mathrm{d}\varphi \int_{-\infty}^{\infty} \mathrm{d}\eta e^{i(a \cosh \eta - b \cos \varphi)} = \Big[2\pi J_0(b) \times \big(\pi(-Y_0(a) + i J_0(a))\big)\Big] \tag{3.35b}$$

Other relevant properties are

$$\frac{\mathrm{d}}{\mathrm{d}x}J_0(x) = -J_1(x)\,, \qquad \frac{\mathrm{d}}{\mathrm{d}x}Y_0(x) = -Y_1(x) \tag{3.35c}$$

Naively, there seems to be no useful orthogonality relation. . . to extract $\tilde{\phi}(p_\perp)$ from this.

Invariance of Fourier Transform w.r.t. Deformations of the Hypersurface

Let ϕ_1, ϕ_2 be fields of equal mass evolving according to the KG equation. Then the current

$$J_\mu[\phi_1,\phi_2] = -i(\phi_1\partial_\mu\phi_2^* - (\partial_\mu\phi_1)\phi_2^*) =: -i\phi_1\overset{\leftrightarrow}{\partial}_\mu\phi_2 \tag{3.36}$$

is conserved. Recall Gauß law

$$\int_\Omega \mathrm{d}\Omega\nabla_\mu J^\mu = \int_{\partial\Omega}\mathrm{d}\sigma_\mu J^\mu \tag{3.37}$$

with $\mathrm{d}\sigma_\mu$ the outwards oriented surface normal of the spacetime volume Ω. The bilinear form

$$(\phi_1,\phi_2)_\Sigma = \int_\Sigma \mathrm{d}\Sigma_\mu J^\mu[\phi_1,\phi_2] = -i\int_\Sigma \mathrm{d}\Sigma_\mu\phi_1\overset{\leftrightarrow}{\partial}_\mu\phi_2^* \tag{3.38}$$

is therefore independent of the choice of (Cauchy) hypersurface Σ (if $\partial\Sigma$ is changed, one must carefully check for further contributions in Gauß law). Choose a hypersurface Σ_t where $t = \text{const}$. Consider a solution to the KG equation given by its Fourier decomposition

$$\phi(t,\vec{x}) = \int\frac{\mathrm{d}^3p}{(2\pi)^3}\tilde{\phi}(\vec{p})u_{\vec{p}}(t,\vec{x})\,, \qquad u_{\vec{p}}(t,\vec{x}) = \frac{1}{\sqrt{2\omega_{\vec{p}}}}e^{-i(\omega_{\vec{p}}t-\vec{p}\vec{x})} \tag{3.39}$$

The mode functions $u_{\vec{p}}$ are orthogonal w.r.t. to the bilinear form $(\cdot,\cdot)_{\Sigma_t}$ and normalized according to

$$(u_{\vec{p}},u_{\vec{q}})_{\Sigma_t} = (2\pi)^3\delta^{(3)}(\vec{p}-\vec{q}) \tag{3.40}$$

and the Fourier transform $\tilde{\phi}(\vec{p})$ can be extracted via

$$\tilde{\phi}(\vec{p}) = (\phi,u_{\vec{p}})_{\Sigma_t}$$

and can thus be evaluated on any Cauchy surface.

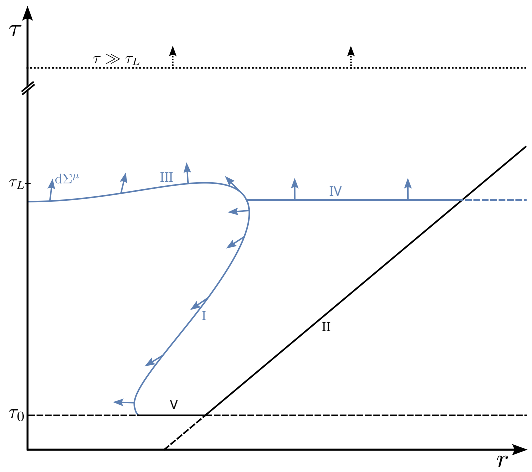

Figure 3.1: Freezeout surface in τ-r-plane[KGF23].

Consider the freezeout on the hypersurface depicted in 3.1. Assume that the condensate contribution as a function in phase space $f_\text{cond}(x^\mu,\vec{p})$ vanishes on Σ_II and Σ_V, i.e. is contained within the union of all light cones starting on the freeze out surface $\Sigma_{FO} \equiv \Sigma_\text{I}\cap\Sigma_\text{III}$. To do (41) **By causality this seems reasonable, but from Fourier decomposition of a classical field this is not at all clear.**

Following the reasoning from [KGF23], we wish to apply Gauß law. Consider separately the contribution on the τ-axis

$$\int_{\Sigma_{r=0}} \mathrm{d}\Sigma_\mu J^\mu \qquad \text{or} \qquad \lim_{r\to 0}\int_{\Sigma_r} \mathrm{d}\Sigma_\mu J^\mu$$

The surface vector on this hypersurface is $\mathrm{d}\Sigma_\mu = r\tau \mathrm{d}\tau \mathrm{d}\eta \mathrm{d}\varphi(0,1,0,0)$ and thus vanishes at $r=0$ (the hypersurface $\Sigma_{r=0}$ has zero 3-volume). Since the derivative in the integrad introduces no divergencies, the contribution of $\Sigma_{r=0}$ to Gauß law is zero.

We can therefore write

$$\tilde{\phi}(\vec{p}) = (\phi, u_{\vec{p}})_{\Sigma_t} = (\phi, u_{\vec{p}})_{\Sigma_{\tau\gg\tau_L}} = (\phi, u_{\vec{p}})_{\Sigma_{FO}} \tag{3.41}$$

The second "=" assumes that $\phi = 0$ for large spacetime rapidities $\eta \to \pm\infty$ and the $\tau = $ const. hypersurface can be deformed to a $t = $ const. hypersurface.

Using the freezeout surface parametrization stated in earlier paragaphs one computes

$$\begin{aligned}
\tilde{\phi}(\vec{p}) = -i\int_{-\infty}^{\infty}\mathrm{d}\eta\int_0^{2\pi}\mathrm{d}\varphi\int_0^{\pi}\mathrm{d}\alpha\tau(\alpha)r(\alpha)\big[r'(\alpha)\phi(\tau,r)\overleftrightarrow{\partial}_\tau e^{i(\tau m_\perp\cosh(\eta-\eta_p)-rp_\perp\cos(\varphi-\varphi_p))}+\\
+\tau'(\alpha)\phi(\tau,r)\overleftrightarrow{\partial}_r e^{i(\tau m_\perp\cosh(\eta-\eta_p)-rp_\perp\cos(\varphi-\varphi_p))}\big] &\qquad (3.42a)\\
= -i\int_{-\infty}^{\infty}\mathrm{d}\eta\int_0^{2\pi}\mathrm{d}\varphi\int_0^{\pi}\mathrm{d}\alpha\tau(\alpha)r(\alpha)\big[r'(\alpha)\phi(\tau,r)\overleftrightarrow{\partial}_\tau e^{i(\tau m_\perp\cosh\eta-rp_\perp\cos\varphi)}+\\
+\tau'(\alpha)\phi(\tau,r)\overleftrightarrow{\partial}_r e^{i(\tau m_\perp\cosh\eta-rp_\perp\cos\varphi)}\big] &\qquad (3.42b)\\
= -i\int_0^{\pi}\mathrm{d}\alpha\tau(\alpha)r(\alpha)\Bigg[r'(\alpha)\phi(\tau,r)\overleftrightarrow{\partial}_\tau\Big[2\pi J_0(rp_\perp)\times\big(\pi(-Y_0(\tau\omega_\perp)+iJ_0(\tau\omega_\perp))\big)\Big]+\\
+\tau'(\alpha)\phi(\tau,r)\overleftrightarrow{\partial}_r\Big[2\pi J_0(rp_\perp)\times\big(\pi(-Y_0(\tau\omega_\perp)+iJ_0(\tau\omega_\perp))\big)\Big]\Bigg] &\qquad (3.42c)\\
= -i\int_0^{\pi}\mathrm{d}\alpha\tau(\alpha)r(\alpha)\Bigg[(r'\partial_\tau+\tau'\partial_r)\phi(\tau,r)[2\pi J_0(rp_\perp)\times\big(\pi(-Y_0(\tau\omega_\perp)+iJ_0(\tau\omega_\perp))\big)\Big]-\\
-\phi(\tau,r)\Big[\tau'\times 2\pi p_\perp J_1(rp_\perp)\times\big(\pi(-Y_0(\tau\omega_\perp)+iJ_0(\tau\omega_\perp))\big)+\\
+r'\times 2\pi J_0(rp_\perp)\times\big(\pi\omega_\perp(Y_1(\tau\omega_\perp)-iJ_0(\tau\omega_\perp))\big)\Big]\Bigg] &\qquad (3.42d)
\end{aligned}$$

where the Bessel function identities (3.35c) were used.

How to treat the projection $r'\partial_\tau + \tau'\partial_r \equiv \partial_\perp \propto n^\mu\partial_\mu$ of ∂_μ onto the normal vector $\propto n^\mu$? One could argue that $\partial_\perp\phi$ not specified by field data on the hypersurface and one needs to find suitable initial conditions. A reasonable choice would be $\pi|_{\Sigma_{FO}} \equiv \dot{\phi}|_{\Sigma_{FO}} = 0$. Since $\partial_\eta\phi = 0$ by symmetry assumption, this implies $\partial_\tau\phi|_{\Sigma_{FO}} = 0$. The derivatives $\partial_\alpha \equiv \tau'\partial_\tau + r'\partial_r$ and $\partial_\perp \equiv r'\partial_\tau + \tau'\partial_r$ therefore can be related by $\partial_\perp\phi|_{\Sigma_{FO}} = \frac{\tau'}{r'}\partial_\alpha\phi|_{\Sigma_{FO}}$.

In the spirit of identifying the pion field with fluid variables, one could argue

$$\partial_\mu\phi = \partial_\mu(\rho\exp(i\vartheta)) = i\phi\partial_\mu\vartheta = i\phi\chi u_\mu \tag{3.43}$$