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THE DIRECT SOLUTION OF THE TRANSPORTATION PROBLEM WITH REDUCED MATRICES*

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A discussion of the importance of a direct method in obtaining all the solutions of a transportation problem, and in obtaining solutions of more general problems, is followed by a discussion of methods of reduced matrices in which the transportation matrix is reduced, by a series of subtractions from rows and columns, to a transformed matrix to which the orthogonality condition is applicable. The direct method proceeds in a series of simple steps to the determination of zero terms having associated x_{ij} values which eventually satisfy the row and column equations. Formal and informal versions are presented and application is made to several general problems.

Introduction

We seek methods which produce all the solutions of a transportation problem and which are applicable to more complex transportation problems. It is argued that direct methods should be considered. While the purest direct method is not practical, a modification substitutes some simple techniques for the minimization condition. With the methods of reduced matrices, we make subtractions from the rows and columns of the transportation matrix to produce a transformed matrix with all elements non-negative such that the non-negative integral x_{ij} can be assigned to the zero terms so as to satisfy the specifications for origins, i, and destinations, j.

This work is a revised and extended version of an unpublished paper of 1955 [6]. It is related to published work on (and machines programs for) more general problems published in 1956 [7] and 1957 [9] and [17].

The Problem

The transportation matrix, $C = \| c_{ij} \|$, is an mxn matrix with associated integral $x_{ij} \geqq 0$ to be determined according to the specifications

$$(1) \qquad \textstyle\sum_{i=1}^{m} x_{ij} = b_j \; ; \qquad \sum_{j=1}^{n} x_{ij} = a_i \; ; \qquad \sum_{i=1}^{m} a_i = \sum_{j=1}^{n} b_j = N$$

such that

$$(2) \qquad T = \textstyle\sum_{ij} x_{ij} c_{ij} \text{ is as small as possible.}$$

General Solutions and Solutions of More General Problems

For most purposes it would seem that an important property of a proposed method for solving a problem is that it produces all the solutions, and not just one of them. For some purposes it may even be desirable that the solution may

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which satisfy $0 \leqq z \leqq b_3$, $0 \leqq y \leqq b_4 + b_5 - a_1$ and $b_1 + b_2 - a_3 \leqq y + z \leqq a_3 - b_2$.

Some matrices which are not completely reduced may also be used to give solutions to more general problems with $x_{ij} \geqq 0$. Thus the matrix $C^{(1)}$ of Table 5 with $a_i > 0$ and $b_j > 0$ is a completely reduced matrix for any problem with $b_2 \geqq a_2 + a_3$ and $a_4 \geqq b_1 + b_2$ while the $C^{(2)}$ matrix is completely reduced for any problem with $b_2 \geqq a_2 + a_3$ and $b_3 \leqq a_4 \leqq b_1 + b_3$.

It is useful to note that any negative solution of the problem with fixed specifications corresponds to a non-negative solution of a problem with a_i and b_j increased by at least the amount of the negative solution. Thus the matrix $C^{(3)}$ of Table 5 is a completely reduced matrix for all problems with specifications $a_i = 1 + \beta, 1, 1, 1$; $b_j = 1, 1 + \beta, 1, 1$ for all positive integral values of β and for all matrices which reduce to $C^{(3)}$.

2. *Problems with One a_i and One b_j General*

Not quite so general but still useful is the case in which one a_i becomes $a_i \pm \beta$ and some b_j becomes $b_j \pm \beta$. The method proceeds as before. It is only necessary to avoid deficiencies and negative values of x_{ij}. As an illustration, a problem of Charnes-Kirby [3] is used in which $a_2 = 20$ is replaced by $20 - \beta$ and $b_1 = 11$ is replaced by $11 - \beta$. The solution is given in Table 6 using the informal version. The steps are routine and remarks interpret the results. There is a deficiency in row 2 of the matrix $C^{(2)}$ when $\beta < 4$ but the resulting transformation yields the matrix $C^{(3)}$ which is completely reduced for $0 \leqq \beta \leqq 4$. The solution for integral non-negative β requires no more steps than does the solution for $\beta = 0$.

As a more practical and explicit illustration consider the problem of Table 3. Suppose facilities at Los Angeles are increased by 20 units and deliveries at San Francisco are increased by the same amount. Then the solution is immediately obtained from the completely reduced matrix of Table 3 with x_{46} increased from 43 to 63. If instead the deliveries at St. Louis are increased by 20 units, the matrix $C^{(3)}$ again provides the solution with $x_{46} = 63$, $x_{36} = 16$, $x_{33} = 51$ and the other x_{ij} not changed.

3. *Problems with Inadmissible Squares*

Problems with inadmissible squares are very easy to handle with the direct method using reduced matrices for solutions with $x_{ij} \geqq 0$. It is only necessary to fix $x_{ij} = 0$ for every inadmissible square. This can be accomplished by making the c_{ij} for each inadmissible square so large that the subtractions of the reduction process never reduce it to zero. Operationally this can be managed by placing an X in each inadmissible square and using an X in the next matrix when a constant is subtracted. Of course there may be so many inadmissible squares that no solution is possible but this method of reduced matrices then reveals the fundamental inconsistency since eventually the reduction process leads to some $\Delta = X$ which can not be used in a transformation.

4. Bounded Variable Transportation Problems

Completely reduced matrices are useful in solving bounded variable transportation problems since the relations involving the bounded variables are usually expressed as conditions additional to (1). Solutions are available for types of problems which the x_{ij} are bounded [5, 378] and in which partial sums of the x_{ij} are bounded [15]. It is planned to discuss these results in a separate paper.

5. Constant Fixed Charge Transportation Problems

The general solutions are useful in solving fixed charge transportation problems with constant fixed charges. Once a general parametric solution is available, as it is with the direct method using reduced matrices, it is only necessary to determine the parameters so as to wipe out as many x_{ij} terms as possible. Thus in Table 1, the optimal constant fixed charge solution is given by $z = 0$, $y = 4$ since x_{24}, x_{31}, and x_{34} are wiped out and no other admissible values of y and z can wipe out so many x's. A paper on this subject has been accepted by *The Naval Research Logistics Quarterly* for the September 1966 issue [11].

6. The General Transportation Problem

The k-dimensional transportation problem, with specified capacities at intermediate points, etc. can be solved with the formal version of the direct method, though in this case fractional solutions as well as negative solutions may be encountered. Fractional solution transformations are similar to negative solution transformations. This problem is the mathematical equivalent of the group assembly problem [7]. Earlier versions of solutions are available in the literature and B. A. Galler has prepared a machine program for problems with as many dimensions as 20 [9] and [17]. The direct generality of the formal method is seen from the fact the program in the k dimensions is also designed for the $k = 2$ dimensional problem under consideration in this paper.

Conclusion

The direct method using reduced matrices is applicable to many direct generalizations of the basic problem. However an important claim for its use in the Hitchcock problem is its simplicity and ease and the speed, as compared with available alternative methods, particularly for the informal version, with which a general solution can be reached.

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