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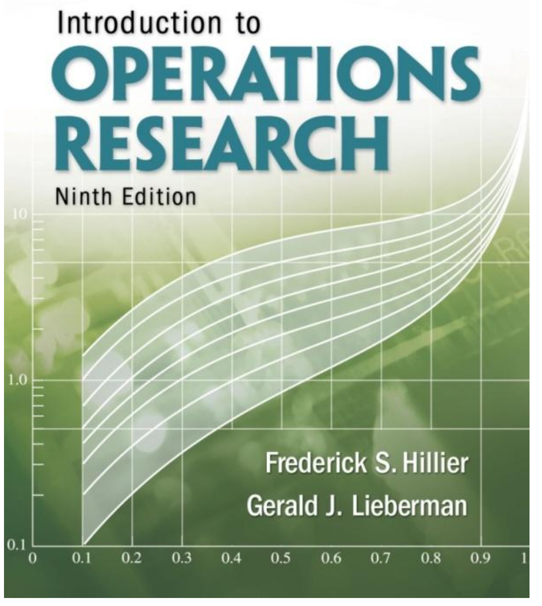

Chapter 8

The Transportation and Assignment Problems

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The Transportation and Assignment Problems

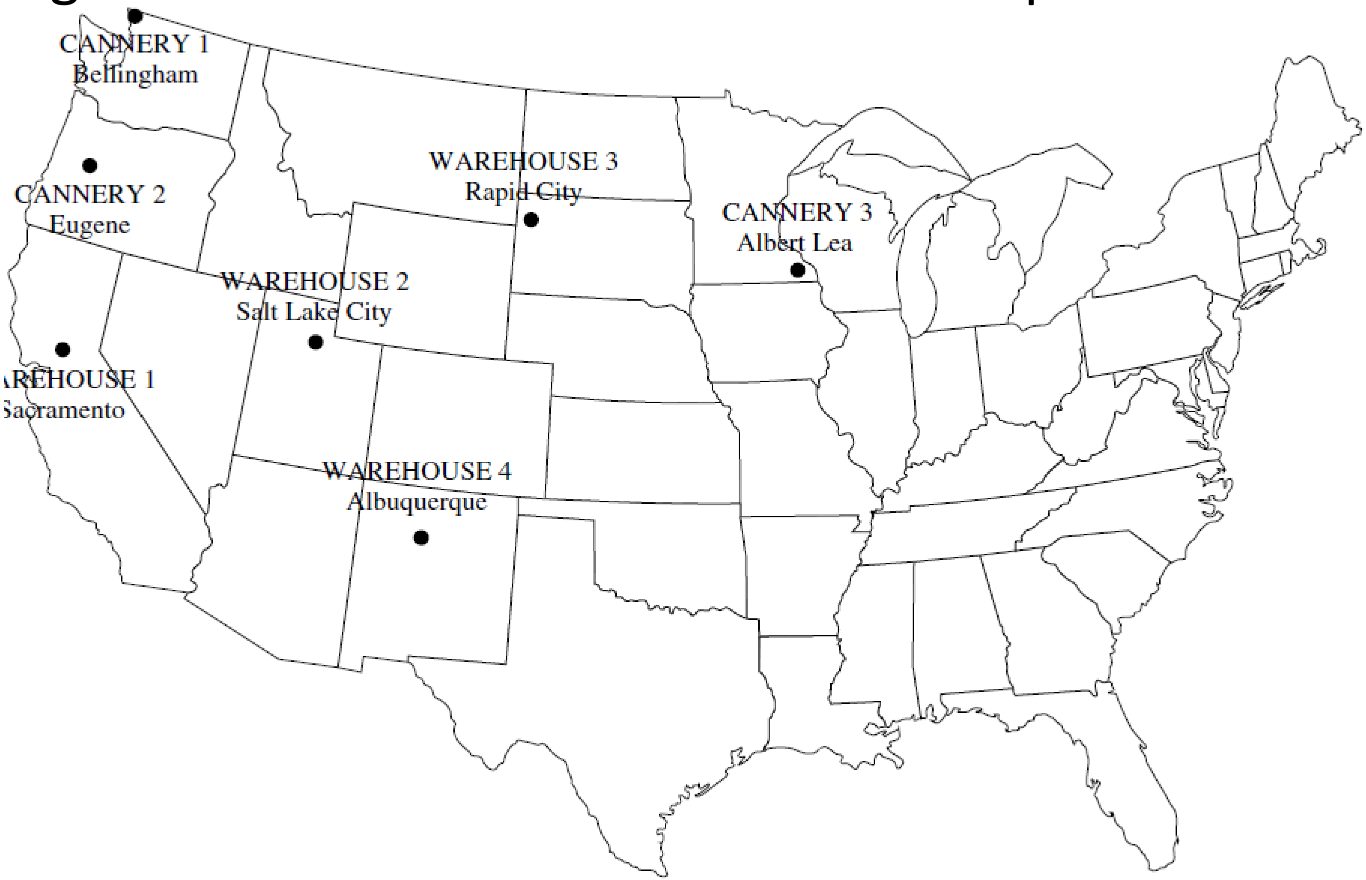
- Transportation Problems: A special class of Linear Programming Problems.
  - Assignment: Special Case of Transportation Problems
- Most of the constraint coefficients (a_{ij}) are zero.
- Thus, it is possible to develop special streamlined algorithms.

TABLE 8.1 Table of constraint coefficients for linear programming

$$\mathbf{A} = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}$$

The Transportation Problem

- P&T Company producing canned peas.
- Peas are prepared at three canneries, then shipped by truck to four distributing warehouses.
- Management wants to minimize total transportation cost.



The Transportation Problem

- Objective: minimum total shipping cost.
- Decisions: Determine the assignments (shipments from canneries to warehouses) , i.e. How much to ship from each cannery to each warehouse.

TABLE 8.2 Shipping data for P & T Co.

		Shipping Cost ($) per Truckload				
		Warehouse				
		1	**2**	**3**	**4**	**Output**
	1	464	513	654	867	75
Cannery	2	352	416	690	791	125
	3	995	682	388	685	100
Allocation		80	65	70	85	

The Transportation Problem

- Network representation
- **Arcs** (arrows, branches) show possible routes for trucks, numbers on arcs are unit cost of shipment on that route.
- Square brackets show **supply** (output) and **demand** (allocation) at each **node** (circle, vertex)

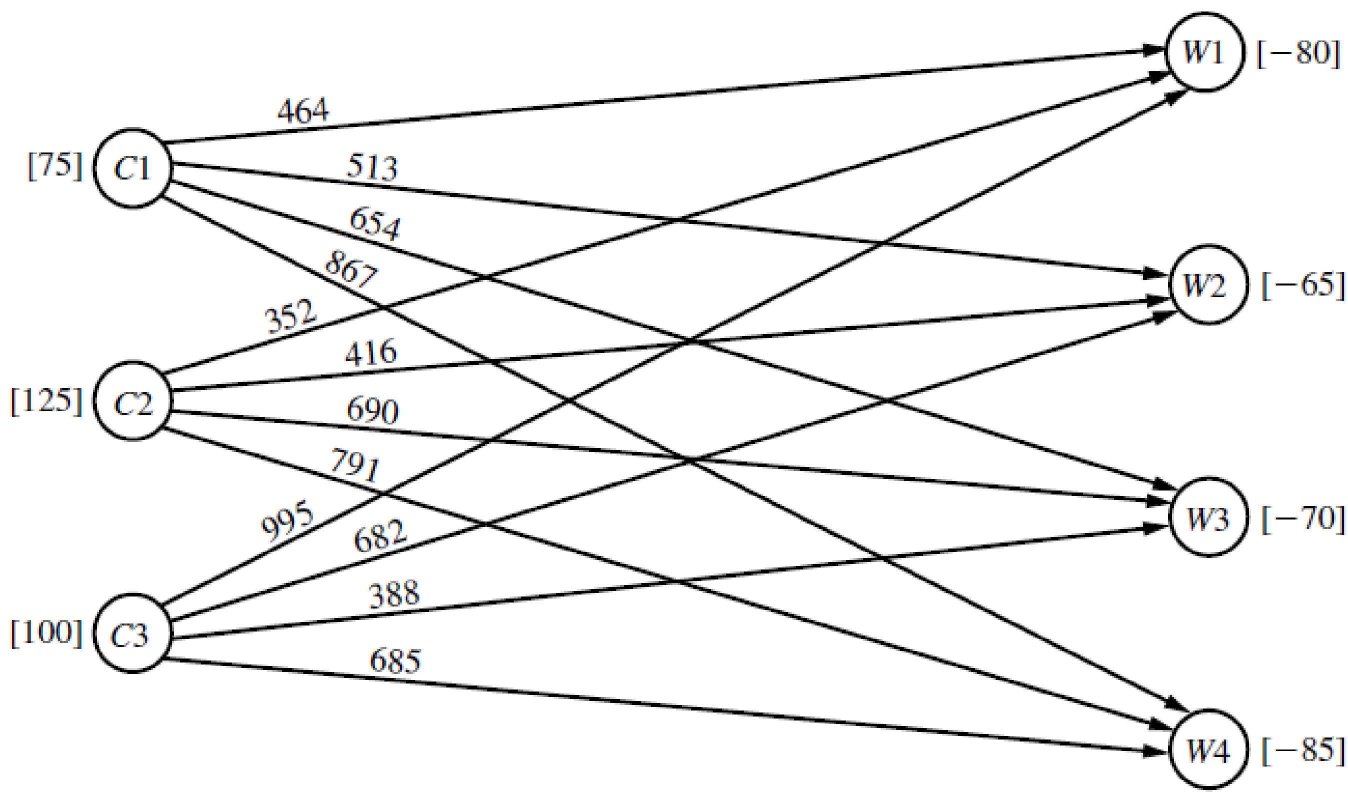


FIGURE 8.2
Network representation of the P & T Co. problem.

The Transportation Problem

- Linear programming problem of the ***transportation problem type***.

let x_{ij} ($i = 1, 2, 3; j = 1, 2, 3, 4$) be the number of truckloads to be shipped from cannery i to warehouse j. Thus, the objective is to choose the values of these 12 decision variables (the x_{ij}) so as to

$$\text{Minimize} \quad Z = 464x_{11} + 513x_{12} + 654x_{13} + 867x_{14} + 352x_{21} + 416x_{22} + 690x_{23} + 791x_{24} + 995x_{31} + 682x_{32} + 388x_{33} + 685x_{34},$$

subject to the constraints

$$\begin{aligned}
x_{11} + x_{12} + x_{13} + x_{14} &= 75\\
x_{21} + x_{22} + x_{23} + x_{24} &= 125\\
x_{31} + x_{32} + x_{33} + x_{34} &= 100\\
x_{11} + x_{21} + x_{31} &= 80\\
x_{12} + x_{22} + x_{32} &= 65\\
x_{13} + x_{23} + x_{33} &= 70\\
x_{14} + x_{24} + x_{34} &= 85
\end{aligned}$$

and

$$x_{ij} \geq 0 \qquad (i = 1, 2, 3; j = 1, 2, 3, 4).$$

The Transportation Problem

- The problem has a special structure of its constraint coefficients.
- This structure distinguishes this problem as a transportation problem.

TABLE 8.3 Constraint coefficients for P & T Co.

	Coefficient of:												
	x_{11}	x_{12}	x_{13}	x_{14}	x_{21}	x_{22}	x_{23}	x_{24}	x_{31}	x_{32}	x_{33}	x_{34}	
	1	1	1	1									Cannery constraints
					1	1	1	1					
									1	1	1	1	
A =	1				1				1				Warehouse constraints
		1				1				1			
			1				1				1		
				1				1				1	

The Transportation Problem

- Generally, a transportation problem is concerned with distributing any commodity from any group of supply centers (**sources**) to any group of receiving centers (**destinations**) in a way to minimize the total transportation cost.

TABLE 8.4 Terminology for the transportation problem

Prototype Example	General Problem
Truckloads of canned peas	Units of a commodity
Three canneries	m sources
Four warehouses	n destinations
Output from cannery i	Supply s_i from source i
Allocation to warehouse j	Demand d_j at destination j
Shipping cost per truckload from cannery i to warehouse j	Cost c_{ij} per unit distributed from source i to destination j

The Transportation Problem

The requirements assumption: Each source has a fixed *supply* of units, where this entire supply must be distributed to the destinations. (We let s_i denote the number of units being supplied by source i, for $i = 1, 2, \dots, m$.) Similarly, each destination has a fixed *demand* for units, where this entire demand must be received from the sources. (We let d_j denote the number of units being received by destination j, for $j = 1, 2, \dots, n$.)

→ There needs to be a balance between the **total supply** and **total demand** in the problem.

The feasible solutions property: A transportation problem will have feasible solutions if and only if

$$\sum_{i=1}^{m} s_i = \sum_{j=1}^{n} d_j.$$

The Transportation Problem

TABLE 8.2 Shipping data for P & T Co.

		Shipping Cost ($) per Truckload				
		Warehouse				
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The feasible solutions property: A transportation problem will have feasible solutions if and only if

$$\sum_{i=1}^{m} s_i = \sum_{j=1}^{n} d_j.$$

The Transportation Problem

- When the problem is unbalanced, we reformulate the problem by defining a **dummy source** or a **dummy destination** for taking up the slack.
- **Cost assumption:** Costs in the model are unit costs.
- ***Any problem*** that can be described as in the below table, and that satisfies the requirements assumption and cost assumption is a transportation problem. The problem may not actually involve transportation of commodities.

TABLE 8.5 Parameter table for the transportation problem

		Cost per Unit Distributed				
		Destination				
		1	**2**	$\cdots$	n	**Supply**
Source	1	c_{11}	c_{12}	$\cdots$	c_{1n}	s_1
	2	c_{21}	c_{22}	$\cdots$	c_{2n}	s_2
	$\vdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\vdots$
	m	c_{m1}	c_{m2}	$\cdots$	c_{mn}	s_m
Demand		d_1	d_2	$\cdots$	d_n	

The Transportation Problem
General Mathematical Model

Letting Z be the total distribution cost and x_{ij} ($i = 1, 2, \ldots, m$; $j = 1, 2, \ldots, n$) be the number of units to be distributed from source i to destination j, the linear programming formulation of this problem is

$$\text{Minimize} \qquad Z = \sum_{i=1}^{m} \sum_{j=1}^{n} c_{ij} x_{ij},$$

subject to

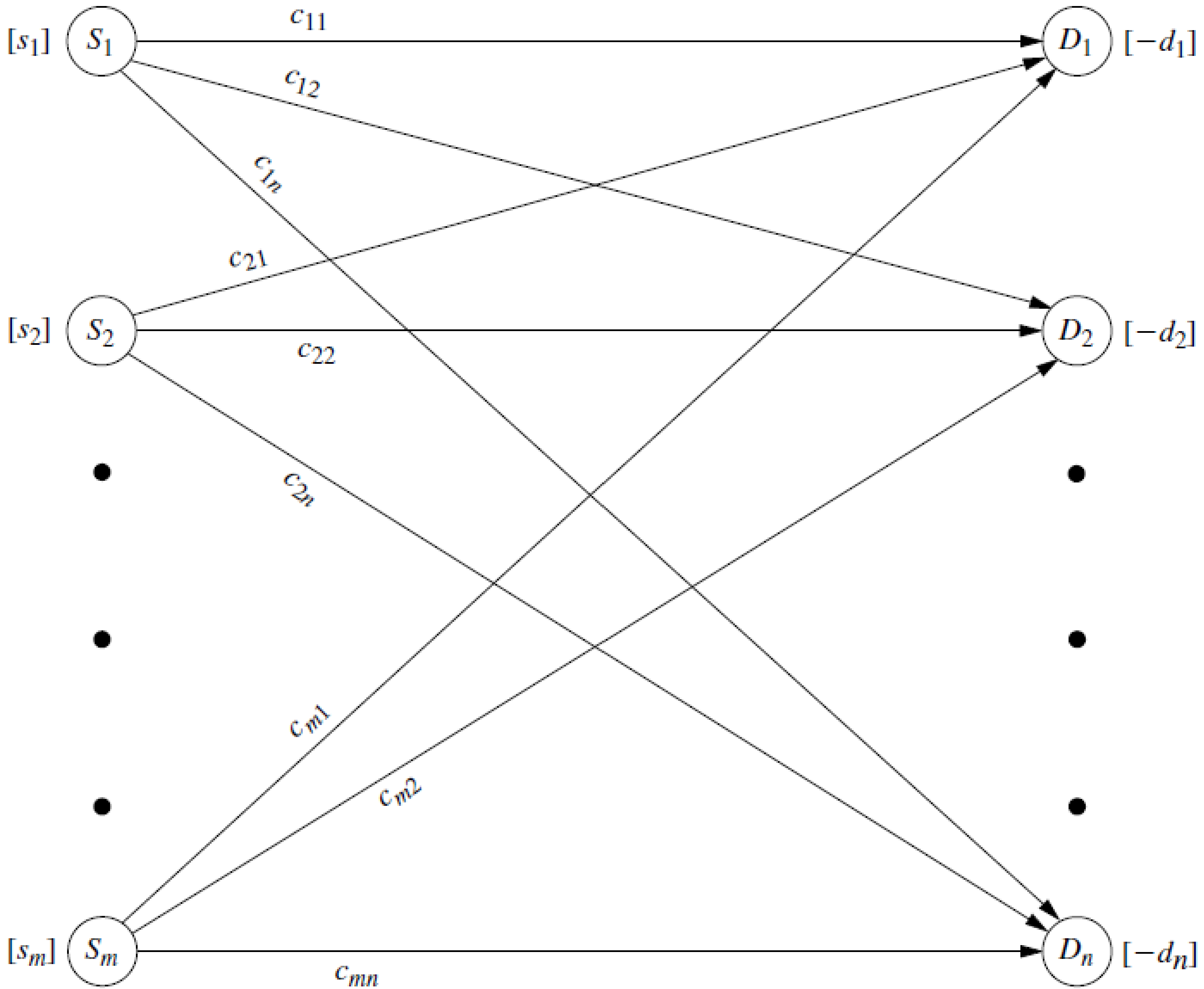
$$\sum_{j=1}^{n} x_{ij} = s_i \qquad \text{for } i = 1, 2, \ldots, m,$$

$$\sum_{i=1}^{m} x_{ij} = d_j \qquad \text{for } j = 1, 2, \ldots, n,$$

and

$$x_{ij} \geq 0, \qquad \text{for all } i \text{ and } j.$$

The Transportation Problem
General Network Representation



The Transportation Problem

Integer solutions property: For transportation problems where every s_i and d_j have an integer value, all the basic variables (allocations) in *every* basic feasible (BF) solution (including an optimal one) also have *integer* values.

The Transportation Problem
Dummy Destination Example

- NORTHERN AIRLINE produces airplanes.
- Last stage in production: Produce jet engines and install them in the completed airplane frame.
- The production of jet engines for planes should be scheduled for the next four months at total minimum cost.

TABLE 8.7 Production scheduling data for Northern Airplane Co.

Month	Scheduled Installations	Maximum Production	Unit Cost* of Production	Unit Cost* of Storage
1	10	25	1.08	0.015
2	15	35	1.11	0.015
3	25	30	1.10	0.015
4	20	10	1.13	

*Cost is expressed in millions of dollars.

The Transportation Problem
Dummy Destination Example

- One way to formulate:
  - Define x_j: jet engines produced in month j, j=1,2,3,4

    →But this will not be a transportation type formulation, therefore will be harder to solve.
- Transportation formulation:

Source i = production of jet engines in month i ($i = 1, 2, 3, 4$)

Destination j = installation of jet engines in month j ($j = 1, 2, 3, 4$)

x_{ij} = number of engines produced in month i for installation in month j

$$c_{ij} = \text{cost associated with each unit of } x_{ij}$$

$$= \begin{cases} \text{cost per unit for production and any storage} & \text{if } i \le j \\ ? & \text{if } i > j \end{cases}$$

$$s_i = ?$$

d_j = number of scheduled installations in month j.

The Transportation Problem
Dummy Destination Example

TABLE 8.8 Incomplete parameter table for Northern Airplane Co.

		Cost per Unit Distributed				
		Destination				
		1	2	3	4	Supply
Source	1	1.080	1.095	1.110	1.125	?
	2	?	1.110	1.125	1.140	?
	3	?	?	1.100	1.115	?
	4	?	?	?	1.130	?
Demand		10	15	25	20	

The Transportation Problem
Dummy Destination Example

- We don't know the exact supplies (exact production quantities for each month), but we know what can be upper bounds on the amount of supplies for each month.

$$x_{11} + x_{12} + x_{13} + x_{14} \leq 25,$$
$$x_{21} + x_{22} + x_{23} + x_{24} \leq 35,$$
$$x_{31} + x_{32} + x_{33} + x_{34} \leq 30,$$
$$x_{41} + x_{42} + x_{43} + x_{44} \leq 10.$$

- To convert these equalities into equalities, we use ***slack variables***. These are allocations to a single **dummy destination** that represent ***total unused production capacity***. The demand of this destination:

$$(25 + 35 + 30 + 10) - (10 + 15 + 25 + 20) = 30.$$

The Transportation Problem

Dummy Destination Example

TABLE 8.9 Complete parameter table for Northern Airplane Co.

		Cost per Unit Distributed					
		Destination					
		1	**2**	**3**	**4**	**5(*D*)**	**Supply**
Source	1	1.080	1.095	1.110	1.125	0	25
	2	*M*	1.110	1.125	1.140	0	35
	3	*M*	*M*	1.100	1.115	0	30
	4	*M*	*M*	*M*	1.130	0	10
Demand		10	15	25	20	30	

The Transportation Problem
Dummy Source Example

- METRO WATER DISTRICT : An agency that handles water distribution in a large geographic region.
- Sources of water: Colombo, Sacron and Calorie rivers.
- Destinations: Berdoo, Los Devils, San Go, Hollyglass.
- No water should be distributed to Hollyglass from Claorie river.
- Allocate all available water **to at least meet minimum needs** at minimum cost.

TABLE 8.10 Water resources data for Metro Water District

	Cost (Tens of Dollars) per Acre Foot				
	Berdoo	**Los Devils**	**San Go**	**Hollyglass**	**Supply**
Colombo River	16	13	22	17	50
Sacron River	14	13	19	15	60
Calorie River	19	20	23	—	50
Minimum needed	30	70	0	10	(in units of 1
Requested	50	70	30	∞	million acre feet)

The Transportation Problem
Dummy Source Example

- After allocating the minimum amounts for other cities, how much extra can we supply for Hollyglass? **(Upper bound)**

$$(50 + 60 + 50) - (30 + 70 + 0) = 60.$$

- Define a **dummy source** to send the extra demand. The supply quantity of this source should be:

$$(50 + 70 + 30 + 60) - (50 + 60 + 50) = 50.$$

TABLE 8.10 Water resources data for Metro Water District

	Cost (Tens of Dollars) per Acre Foot				
	Berdoo	**Los Devils**	**San Go**	**Hollyglass**	**Supply**
Colombo River	16	13	22	17	50
Sacron River	14	13	19	15	60
Calorie River	19	20	23	—	50
Minimum needed	30	70	0	10	(in units of 1
Requested	50	70	30	∞	million acre feet)

The Transportation Problem
Dummy Source Example

TABLE 8.11 Parameter table without minimum needs for Metro Water District

		Cost (Tens of Millions of Dollars) per Unit Distributed				
		Destination				
		Berdoo	**Los Devils**	**San Go**	**Hollyglass**	**Supply**
Source	Colombo River	16	13	22	17	50
	Sacron River	14	13	19	15	60
	Calorie River	19	20	23	*M*	50
	Dummy	0	0	0	0	50
Demand		50	70	30	60	

The Transportation Problem
Dummy Source Example

- How can we take the minimum need of each city into account?
  - Los Devils: **Minimum=70 =Requested=70**. All should be supplied from actual sources. Put M cost in dummy sources row.
  - San Go: **Minimum=0, Requested=30**. No water needs to be supplied from actual sources. Put 0 cost in dummy sources row.
  - Hollyglass: **Minimum=10, Requested (UB)=60**. Since the dummy source's supply is 50, at least 10 will be supplied from actual sources anyway. Put 0 cost in dummy sources row.
  - Berdoo: **Minimum=30, Requested=50**. Split Berdoo into two destinations, one with a demand of 30, one with 20. Put M and 0 in the dummy row for these two destinations.

The Transportation Problem Dummy Source Example

TABLE 8.12 Parameter table for Metro Water District

			Cost (Tens of Millions of Dollars) per Unit Distributed					
			Destination					
			Berdoo (min.) 1	**Berdoo (extra) 2**	**Los Devils 3**	**San Go 4**	**Hollyglass 5**	**Supply**
Source	Colombo River	1	16	16	13	22	17	50
	Sacron River	2	14	14	13	19	15	60
	Calorie River	3	19	19	20	23	*M*	50
	Dummy	4(*D*)	*M*	0	*M*	0	0	50
Demand			30	20	70	30	60	

Transportation Simplex
Recall: Mathematical Model

Letting Z be the total distribution cost and x_{ij} ($i = 1, 2, \ldots, m$; $j = 1, 2, \ldots, n$) be the number of units to be distributed from source i to destination j, the linear programming formulation of this problem is

$$\text{Minimize} \qquad Z = \sum_{i=1}^{m} \sum_{j=1}^{n} c_{ij} x_{ij},$$

subject to

$$\sum_{j=1}^{n} x_{ij} = s_i \qquad \text{for } i = 1, 2, \ldots, m,$$

$$\sum_{i=1}^{m} x_{ij} = d_j \qquad \text{for } j = 1, 2, \ldots, n,$$

and

$$x_{ij} \geq 0, \qquad \text{for all } i \text{ and } j.$$

Transportation Simplex

- Traditional simplex tableau for a transportation problem:
  - Convert the objective function to minimization
  - Introduce artificial variables for = constraints
- Entities not shown are zeros.

TABLE 8.13 Original simplex tableau before simplex method is applied to transportation problem

Basic Variable	Eq.	Coefficient of: Z	$\cdots$	x_{ij}	$\cdots$	z_i	$\cdots$	z_{m+j}	$\cdots$	Right side
Z	(0)	-1		c_{ij}		M		M		0
	(1)									
	$\vdots$									
z_i	(i)	0		1		1				s_i
	$\vdots$									
z_{m+j}	$(m+j)$	0		1				1		d_j
	$\vdots$									
	$(m+n)$									

Transportation Simplex

- At any iteration of traditional simplex:

TABLE 8.14 Row 0 of simplex tableau when simplex method is applied to transportation problem

Basic Variable	Eq.	Coefficient of:								Right Side
		Z	$\cdots$	x_{ij}	$\cdots$	z_i	$\cdots$	z_{m+j}	$\cdots$	
Z	(0)	-1		$c_{ij} - u_i - v_j$		$M - u_i$		$M - v_j$		$-\sum_{i=1}^{m} s_i u_i - \sum_{j=1}^{n} d_j v_j$

- u_i and v_i : dual variables representing how many times the corresponding rows are multiplied and subtracted from row 0.
- If x_{ij} is nonbasic, $(c_{ij}-u_i-v_j)$ is the rate at which Z will change as x_{ij} is increased.

Transportation Simplex

- Regular Simplex:
  - Find an initial BFS by introducing artificial variables
  - Test for optimality
  - Determine the entering variable
  - Determine the leaving variable
  - Find the new BFS
- Transportation Simplex
  - No artificial variables needed to find an initial BFS
  - The current row can be calculated by calculating the current values of u_i and v_j
  - The leaving variable can be identified in a simple way
  - The new BFS can be identified without any algebraic manipulations on the rows of the tableau.

Transportation Simplex

TABLE 8.15 Format of a transportation simplex tableau

		Destination					
		1	**2**	⋯	n	**Supply**	u_i
Source	1	c_{11}	c_{12}	⋯	c_{1n}	s_1	
	2	c_{21}	c_{22}	⋯	c_{2n}	s_2	
	⋮	⋯	⋯	⋯	⋯	⋮	
	m	c_{m1}	c_{m2}	⋯	c_{mn}	s_m	
Demand		d_1	d_2	⋯	d_n	$Z =$	
	v_j						

Additional information to be added to each cell:

If x_{ij} is a basic variable	*If x_{ij} is a nonbasic variable*
c_{ij}	c_{ij}
x_{ij}	$c_{ij} - u_i - v_j$

Transportation Simplex: Initialization

- In an LP with n variables and m constraints :
  - Number of basic variables = $m+n$
- In a transportation problem with n destinations and m sources:
  - Number of basic variables = $m+n-1$
  - This is because one constraint is always redundant in a transportation problem, i.e. if all $m-1$ constraints are satisfied, the remaining constraint will be automatically satisfied.
  - Any BF solution on the transportation tableau will have exactly $m+n-1$ ***circled*** nonnegative allocations (BVs), where the sum of the allocations for each row or column equals its supply or demand.

Transportation Simplex
Initial BFS

1. From the rows and columns still under consideration, select the next BV **according to some criterion**.
2. Allocate as much as possible considering the supply and demand for the related column and row.
3. Eliminate necessary cells from consideration (column or row). If both the supply and demand is consumed fully, arbitrarily select the row. Then, that column will have a degenerate basic variable.
4. If only one row or column remains, complete the allocations by allocating the remaining supply or demand to each cell. Otherwise return to Step 1.

The Transportation Problem
Metro Water District Problem

TABLE 8.12 Parameter table for Metro Water District

			Cost (Tens of Millions of Dollars) per Unit Distributed					
			Destination					
			Berdoo (min.) 1	**Berdoo (extra) 2**	**Los Devils 3**	**San Go 4**	**Hollyglass 5**	**Supply**
Source	Colombo River	1	16	16	13	22	17	50
	Sacron River	2	14	14	13	19	15	60
	Calorie River	3	19	19	20	23	*M*	50
	Dummy	4(*D*)	*M*	0	*M*	0	0	50
Demand			30	20	70	30	60	

Transportation Simplex
Initial BFS – Alternative Criteria

1. **Northwest Corner Rule:** Start by selecting x_{11} (the northwest corner of the tableau). Go on with the one neighboring cell.

TABLE 8.16 Initial BF solution from the Northwest Corner Rule

		Destination						
		1	**2**	**3**	**4**	**5**	**Supply**	u_i
Source	1	16 30	16 20	13	22	17	50	
	2	14	14 0	13 60	19	15	60	
	3	19	19	20 10	23 30	*M* 10	50	
	4(*D*)	*M*	0	*M*	0	0 50	50	
Demand		30	20	70	30	60	$Z = 2{,}470 + 10M$	
	v_j							

Transportation Simplex
Initial BFS – Alternative Criteria

2. Vogel's Approximation Method (VAM):

I. Calculate penalty costs for each row (and column) as the **difference between the minimum unit cost and the next minimum unit cost** in the row (column).

II. Select the row or column with the **largest penalty**. Break ties arbitrarily.

III. Allocate as much as possible to the **smallest-cost-cell** in that row or column. Break ties arbitrarily.

IV. Eliminate necessary cells from consideration. Return to Step I.

Transportation Simplex
Initial BFS – Alternative Criteria - VAM

TABLE 8.17 Initial BF solution from Vogel's approximation method

		Destination						
		1	**2**	**3**	**4**	**5**	**Supply**	**Row Difference**
Source	1	16	16	13	22	17	50	3
	2	14	14	13	19	15	60	1
	3	19	19	20	23	M	50	0
	4(D)	M	0	M	$\boxed{0}$	0	50	0
Demand		30	20	70	30	60	Select $x_{44} = 30$	
Column difference		2	14	0	19	15	Eliminate column 4	

		Destination					
		1	**2**	**3**	**5**	**Supply**	**Row Difference**
Source	1	16	16	13	17	50	3
	2	14	14	13	15	60	1
	3	19	19	20	M	50	0
	4(D)	M	0	M	$\boxed{0}$	20	0
Demand		30	20	70	60	Select $x_{45} = 20$	
Column difference		2	14	0	15	Eliminate row 4(D)	

Transportation Simplex
Initial BFS – Alternative Criteria - VAM

		Destination					
		1	2	3	5	Supply	Row Difference
Source	1	16	16	$\boxed{13}$	17	50	3
	2	14	14	13	15	60	1
	3	19	19	20	M	50	0
Demand		30	20	70	40	Select $x_{13} = 50$	
Column difference		2	2	0	2	Eliminate row 1	

		Destination					
		1	2	3	5	Supply	Row Difference
Source	2	14	14	13	$\boxed{15}$	60	1
	3	19	19	20	M	50	0
Demand		30	20	20	40	Select $x_{25} = 40$	
Column difference		5	5	7	$M - 15$	Eliminate column 5	

Transportation Simplex
Initial BFS – Alternative Criteria - VAM

		Destination 1	2	3	Supply	Row Difference
Source	2	14	14	$\boxed{13}$	20	1
	3	19	19	20	50	0
Demand		30	20	20	Select $x_{23} = 20$	
Column difference		5	5	7	Eliminate row 2	

		Destination 1	2	3	Supply	
Source	3	19	19	20	50	
Demand		30	20	0	Select $x_{31} = 30$ $x_{32} = 20$ $x_{33} = 0$	$Z = 2{,}460$

Transportation Simplex
Initial BFS – Alternative Criteria

3. Russell's Approximation Method (RAM):

I. For each source row still under consideration, determine its $\bar{u}_i$ as the largest unit cost (c_{ij}) remaining in that row.

II. For each destination column still under consideration, determine its $\bar{v}_j$ as the largest unit cost (c_{ij}) remaining in that column.

III. For each variable not previously selected in these rows and columns, calculate:

$$\Delta_{ij} = c_{ij} - \bar{u}_i - \bar{v}_j$$

IV. Select the variable having the most negative delta value. Break ties arbitrarily. Allocate as much as possible. Eliminate necessary cells from consideration. Return to Step I.

The Transportation Problem
RAM – Metro Water District

TABLE 8.12 Parameter table for Metro Water District

			Cost (Tens of Millions of Dollars) per Unit Distributed					
			Destination					
			Berdoo (min.) 1	Berdoo (extra) 2	Los Devils 3	San Go 4	Hollyglass 5	Supply
Source	Colombo River	1	16	16	13	22	17	50
	Sacron River	2	14	14	13	19	15	60
	Calorie River	3	19	19	20	23	M	50
	Dummy	4(D)	M	0	M	0	0	50
Demand			30	20	70	30	60	

TABLE 8.18 Initial BF solution from Russell's approximation method

Iteration	$\bar{u}_1$	$\bar{u}_2$	$\bar{u}_3$	$\bar{u}_4$	$\bar{v}_1$	$\bar{v}_2$	$\bar{v}_3$	$\bar{v}_4$	$\bar{v}_5$	Largest Negative Δ_{ij}	Allocation
1	22	19	M	M	M	19	M	23	M	$\Delta_{45} = -2M$	$x_{45} = 50$
2	22	19	M		19	19	20	23	M	$\Delta_{15} = -5 - M$	$x_{15} = 10$
3	22	19	23		19	19	20	23		$\Delta_{13} = -29$	$x_{13} = 40$
4		19	23		19	19	20	23		$\Delta_{23} = -26$	$x_{23} = 30$
5		19	23		19	19		23		$\Delta_{21} = -24$*	$x_{21} = 30$
6										Irrelevant	$x_{31} = 0$
											$x_{32} = 20$
											$x_{34} = 30$
											$Z = 2{,}570$

*Tie with $\Delta_{22} = -24$ broken arbitrarily.

Transportation Simplex
Initial BFS – Alternative Criteria - RAM

TABLE 8.18 Initial BF solution from Russell's approximation method

Iteration	$\bar{u}_1$	$\bar{u}_2$	$\bar{u}_3$	$\bar{u}_4$	$\bar{v}_1$	$\bar{v}_2$	$\bar{v}_3$	$\bar{v}_4$	$\bar{v}_5$	Largest Negative Δ_{ij}	Allocation
1	22	19	M	M	M	19	M	23	M	$\Delta_{45} = -2M$	$x_{45} = 50$
2	22	19	M		19	19	20	23	M	$\Delta_{15} = -5 - M$	$x_{15} = 10$
3	22	19	23		19	19	20	23		$\Delta_{13} = -29$	$x_{13} = 40$
4		19	23		19	19	20	23		$\Delta_{23} = -26$	$x_{23} = 30$
5		19	23		19	19		23		$\Delta_{21} = -24$*	$x_{21} = 30$
6										Irrelevant	$x_{31} = 0$
											$x_{32} = 20$
											$x_{34} = 30$
											$Z = 2{,}570$

*Tie with $\Delta_{22} = -24$ broken arbitrarily.

Transportation Simplex
Initial BFS – Alternative Criteria - RAM

Iteration 0		Destination 1	2	3	4	5	Supply	u_i
Source	1	16	16	13 (40)	22	17 (10)	50	
	2	14 (30)	14	13 (30)	19	15	60	
	3	19 (0)	19 (20)	20	23 (30)	M	50	
	4(D)	M	0	M	0	0 (50)	50	
Demand		30	20	70	30	60	$Z = 2{,}570$	
	v_j							

Transportation Simplex
Initial BFS – Alternative Criteria

Comparison:

I. Northwest: Quick and easy, usually far from optimal.
II. VAM: Popular, easy to implement by hand, yields nice solutions considering penalties.
III. RAM: Still easy by computer, frequently better than VAM. Which is better on the average is unclear.

→ For this problem VAM gave the optimal solution.

→ For a large problem, use all to find the best initial BFS.

→ Let's use the RAM BFS to start transportation simplex.

Transportation Simplex
Initial BFS

TABLE 8.19 Initial transportation simplex tableau (before we obtain $c_{ij} - u_i - v_j$) from Russell's approximation method

Iteration 0		Destination 1	2	3	4	5	Supply	u_i
Source	1	16	16	13 40	22	17 10	50	
	2	14 30	14	13 30	19	15	60	
	3	19 0	19 20	20	23 30	M	50	
	4(D)	M	0	M	0	0 50	50	
Demand		30	20	70	30	60	$Z = 2{,}570$	
	v_j							

Transportation Simplex

Optimality test: A BF solution is optimal if and only if $c_{ij} - u_i - v_j \geq 0$ for every (i, j) such that x_{ij} is nonbasic.[1]

Since $c_{ij} - u_i - v_j$ is required to be zero if x_{ij} is a basic variable, u_i and v_j satisfy the set of equations

$$c_{ij} = u_i + v_j \qquad \text{for each } (i, j) \text{ such that } x_{ij} \text{ is basic.}$$

There are $m + n - 1$ basic variables, and so there are $m + n - 1$ of these equations. Since the number of unknowns (the u_i and v_j) is $m + n$, one of these variables can be assigned a value arbitrarily without violating the equations. The choice of this one variable and its value does not affect the value of any $c_{ij} - u_i - v_j$, even when x_{ij} is nonbasic, so the only (minor) difference it makes is in the ease of solving these equations. A convenient choice for this purpose is to select the u_i that has the *largest number of allocations in its row* (break any tie arbitrarily) and to assign to it the value zero. Because of the simple structure of these equations, it is then very simple to solve for the remaining variables algebraically.

Transportation Simplex

x_{31}:	$19 = u_3 + v_1.$	Set $u_3 = 0$, so $v_1 = 19$,
x_{32}:	$19 = u_3 + v_2.$	$v_2 = 19$,
x_{34}:	$23 = u_3 + v_4.$	$v_4 = 23$.
x_{21}:	$14 = u_2 + v_1.$	Know $v_1 = 19$, so $u_2 = -5$.
x_{23}:	$13 = u_2 + v_3.$	Know $u_2 = -5$, so $v_3 = 18$.
x_{13}:	$13 = u_1 + v_3.$	Know $v_3 = 18$, so $u_1 = -5$.
x_{15}:	$17 = u_1 + v_5.$	Know $u_1 = -5$, so $v_5 = 22$.
x_{45}:	$0 = u_4 + v_5.$	Know $v_5 = 22$, so $u_4 = -22$.

TABLE 8.19 Initial transportation simplex tableau (before we obtain $c_{ij} - u_i - v_j$) from Russell's approximation method

Iteration 0		Destination 1	2	3	4	5	Supply	u_i
Source	1	16	16	13 40	22	17 10	50	
	2	14 30	14	13 30	19	15	60	
	3	19 0	19 20	20	23 30	M	50	
	4(D)	M	0	M	0	0 50	50	
Demand		30	20	70	30	60	$Z = 2{,}570$	
	v_j							

Transportation Simplex

TABLE 8.20 Completed initial transportation simplex tableau

Iteration 0		Destination 1	Destination 2	Destination 3	Destination 4	Destination 5	Supply	u_i
Source	1	16 +2	16 +2	13 40	22 +4	17 10	50	−5
	2	14 30	14 0	13 30	19 +1	15 $\boxed{-2}$	60	−5
	3	19 0	19 20	20 +2	23 30	M $M-22$	50	0
	4(D)	M $M+3$	0 +3	M $M+4$	0 $\boxed{-1}$	0 50	50	−22
Demand		30	20	70	30	60	$Z = 2{,}570$	
	v_j	19	19	18	23	22		

Compute $c_{ij} - u_i - v_j$

Step 1: Negative, not optimal

Select x_{25} as entering

Transportation Simplex

Step 2. Increasing the entering basic variable from zero sets off a *chain reaction* of compensating changes in other basic variables (allocations), in order to continue satisfying the supply and demand constraints. The first basic variable to be decreased to zero then becomes the leaving basic variable.

TABLE 8.21 Part of initial transportation simplex tableau showing the chain reaction caused by increasing the entering basic variable x_{25}

			Destination			
			3	**4**	**5**	**Supply**
Source	1	...	13 40 $+$	22 $+4$	17 10 $-$	50
	2	...	13 30 $-$	19 $+1$	15 $\boxed{+}$ -2	60
		...	...	...	...	
Demand			70	30	60	

There is always one chain reaction.

Transportation Simplex

TABLE 8.22 Part of second transportation simplex tableau showing the changes in the BF solution

		Destination			
		...	3	4	5	Supply
Source	1	...	13 50	22	17	50
	2	...	13 20	19	15 10	60
		...	...	...	...	
Demand			70	30	60	

We can now highlight a useful interpretation of the $c_{ij} - u_i - v_j$ quantities derived during the optimality test. Because of the shift of 10 allocation units from the donor cells to the recipient cells (shown in Tables 8.21 and 8.22), the total cost changes by

$$\Delta Z = 10(15 - 17 + 13 - 13) = 10(-2) = 10(c_{25} - u_2 - v_5).$$

Transportation Simplex

TABLE 8.23 Complete set of transportation simplex tableaux for the Metro Water District problem

Iteration 0		**Destination 1**	**2**	**3**	**4**	**5**	**Supply**	u_i
Source	1	16 +2	16 +2	13 40 +	22 +4	17 10 −	50	−5
	2	14 30	14 0	13 30 −	19 +1	15 $\boxed{+}$ −2	60	−5
	3	19 0	19 20	20 +2	23 30	M $M - 22$	50	0
	4(*D*)	M $M + 3$	0 +3	M $M + 4$	0 −1	0 50	50	−22
Demand		30	20	70	30	60	$Z = 2,570$	
	v_j	19	19	18	23	22		

Transportation Simplex

TABLE 8.23 (*Continued*)

Iteration 1		Destination 1	2	3	4	5	Supply	u_i
Source	1	16 +2	16 +2	13 50	22 +4	17 +2	50	−5
	2	14 30 −	14 0	13 20	19 +1	15 10 +	60	−5
	3	19 0 +	19 20	20 +2	23 30 −	M $M-20$	50	0
	4(*D*)	M $M+1$	0 +1	M $M+2$	0 $\boxed{+}$ −3	0 50 −	50	−20
Demand		30	20	70	30	60	$Z = 2{,}550$	
	v_j	19	19	18	23	20		

Transportation Simplex

TABLE 8.23 (*Continued*)

Iteration 2		Destination 1	Destination 2	Destination 3	Destination 4	Destination 5	Supply	u_i
Source	1	16 +5	16 +5	13 50	22 +7	17 +2	50	−8
	2	14 +3	14 +3	13 20 −	19 +4	15 40 +	60	−8
	3	19 30	19 20	20 + −1	23 0 −	M $M - 23$	50	0
	4(D)	M $M + 4$	0 +4	M $M + 2$	0 30 +	0 20 −	50	−23
Demand		30	20	70	30	60	$Z = 2,460$	
	v_j	19	19	21	23	23		

Transportation Simplex

TABLE 8.23 (*Continued*)

Iteration 3		Destination 1	2	3	4	5	Supply	u_i
Source	1	16 +4	16 +4	13 (50)	22 +7	17 +2	50	−7
	2	14 +2	14 +2	13 (20)	19 +4	15 (40)	60	−7
	3	19 (30)	19 (20)	20 (0)	23 +1	M $M-22$	50	0
	4(*D*)	M $M+3$	0 +3	M $M+2$	0 (30)	0 (20)	50	−22
Demand		30	20	70	30	60	$Z = 2{,}460$	
	v_j	19	19	20	22	22		

The Assignment Problem

- Special type of LP, in fact a special type of Transportation prob.
- ***Assignees*** (workers, processors, machines, vehicles, plants, time slots) are being assigned to ***tasks*** (jobs, classrooms, people).

1. The number of assignees and the number of tasks are the same. (This number is denoted by n.)
2. Each assignee is to be assigned to exactly *one* task.
3. Each task is to be performed by exactly *one* assignee.
4. There is a cost c_{ij} associated with assignee i ($i = 1, 2, \ldots, n$) performing task j ($j = 1, 2, \ldots, n$).
5. The objective is to determine how all n assignments should be made to minimize the total cost.

The first three assumptions are fairly restrictive. Many potential applications do not quite satisfy these assumptions. However, it often is possible to reformulate the problem to make it fit. For example, *dummy assignees* or *dummy tasks* frequently can be used for this purpose. We illustrate these formulation techniques in the examples.

The Assignment Problem Example

- The JOB SHOP company has 3 new machines.
- There are 4 available locations in the shop where a machine could be installed, but m/c 2 cannot be installed at location 2.
- Objective: Assign machines to the locations to minimize total handling cost.

TABLE 8.24 Materials-handling cost data ($) for Job Shop Co.

		Location			
		1	**2**	**3**	**4**
	1	13	16	12	11
Machine	2	15	—	13	20
	3	5	7	10	6

The Assignment Problem Example

- Introduce a **dummy m/c** for the extra location with zero costs.
- Assign cost M to the assignment of m/c 2 to location 2.

TABLE 8.24 Materials-handling cost data ($) for Job Shop Co.

		Location			
		1	**2**	**3**	**4**
	1	13	16	12	11
Machine	2	15	—	13	20
	3	5	7	10	6

TABLE 8.25 Cost table for the Job Shop Co. assignment problem

		Task (Location)			
		1	**2**	**3**	**4**
	1	13	16	12	11
Assignee	2	15	*M*	13	20
(Machine)	3	5	7	10	6
	4(*D*)	0	0	0	0

The Assignment Problem Mathematical Model (IP)

- Introduce a **binary decision variable.**

$$x_{ij} = \begin{cases} 1 & \text{if assignee } i \text{ performs task } j, \\ 0 & \text{if not,} \end{cases} \quad \text{for } i = 1, 2, \dots, n \text{ and } j = 1, 2, \dots, n.$$

$$\text{Minimize} \qquad Z = \sum_{i=1}^{n} \sum_{j=1}^{n} c_{ij} x_{ij},$$

subject to

$$\sum_{j=1}^{n} x_{ij} = 1 \qquad \text{for } i = 1, 2, \dots, n,$$

$$\sum_{i=1}^{n} x_{ij} = 1 \qquad \text{for } j = 1, 2, \dots, n,$$

and

$$x_{ij} \geq 0, \qquad \text{for all } i \text{ and } j$$
$$(x_{ij} \text{ binary}, \qquad \text{for all } i \text{ and } j).$$

The Assignment Problem
Mathematical Model (IP)

Integer solutions property: For transportation problems where every s_i and d_j have an integer value, all the basic variables (allocations) in *every* basic feasible (BF) solution (including an optimal one) also have *integer* values.

Now focus on the *integer solutions property* in the subsection on the transportation problem model. Because s_i and d_j are integers (= 1) now, this property implies that *every BF solution* (including an optimal one) is an *integer* solution for an assignment problem.

The Assignment Problem Mathematical Model (LP)

- Hence, change **binary decision variables to continuous variables.** Solve the resulting problem as an LP.

$$\text{Minimize} \qquad Z = \sum_{i=1}^{n} \sum_{j=1}^{n} c_{ij} x_{ij},$$

subject to

$$\sum_{j=1}^{n} x_{ij} = 1 \qquad \text{for } i = 1, 2, \ldots, n,$$

$$\sum_{i=1}^{n} x_{ij} = 1 \qquad \text{for } j = 1, 2, \ldots, n,$$

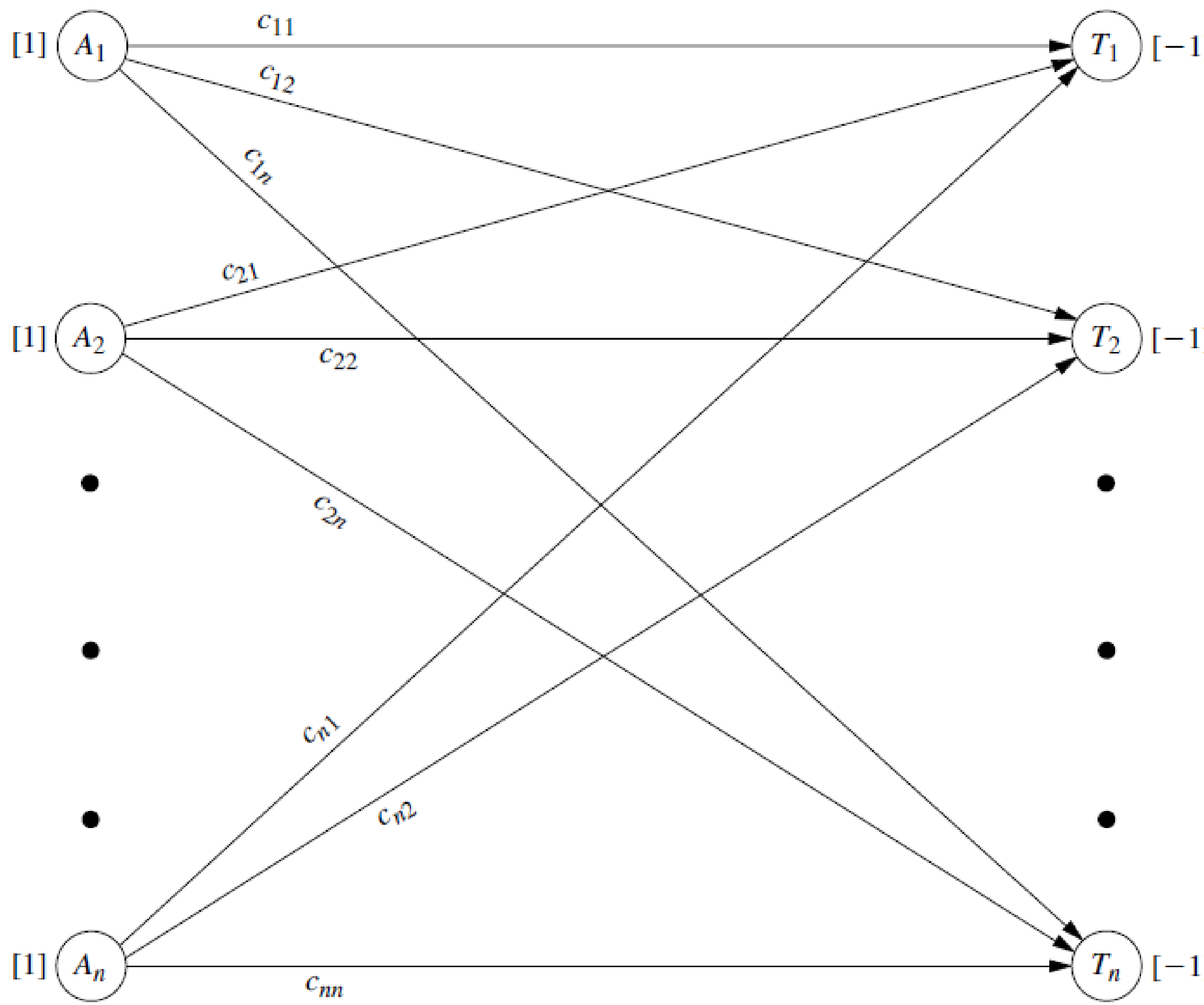
and

$$x_{ij} \le 1, \; x_{ij} \ge 0, \qquad \text{for all } i \text{ and } j$$

~~(x_{ij} binary, for all i and j)~~

The Assignment Problem Network Representation

FIGURE 8.5
Network representation of the assignment problem.



The Assignment Problem

TABLE 8.26 Parameter table for the assignment problem formulated as a transportation problem, illustrated by the Job Shop Co. example

(a) General Case

		Cost per Unit Distributed				
		Destination				
		1	**2**	⋯	***n***	**Supply**
Source	1	c_{11}	c_{12}	⋯	c_{1n}	1
	2	c_{21}	c_{22}	⋯	c_{2n}	1
	⋮	⋯	⋯	⋯	⋯	⋮
	$m = n$	c_{n1}	c_{n2}	⋯	c_{nn}	1
Demand		1	1	⋯	1	

Applying transportation simplex to this tableau, you get:

$$x_{13} = 0, x_{14} = 1, x_{23} = 1,$$
$$x_{31} = 1, x_{41} = 0, x_{42} = 1, x_{43} = 0.$$

(b) Job Shop Co. Example

		Cost per Unit Distributed				
		Destination (Location)				
		1	**2**	**3**	**4**	**Supply**
Source (Machine)	1	13	16	12	11	1
	2	15	*M*	13	20	1
	3	5	7	10	6	1
	4(*D*)	0	0	0	0	1
Demand		1	1	1	1	

The Assignment Problem

The solution with transportation simplex will have too many degenerate variables.

- $m=n$ (number of sources = number of destinations)
- $m+n-1$ basic variables
- only n variables will take nonzero values (assignments)

  → $(2n-1) - n = n-1$ degenerate basic variables.

  → Wasted iterations in transportation simplex.

An even more efficient method for solving an assignment problem:

The Hungarian Algorithm (or **Assignment Algorithm**)

The Assignment Problem: Example

BETTER PRODUCTS Co.

- The company starts the production of 4 new products.
- Production will be in 3 plants, Plant 2 cannot make Product 3.
- Two options:
  1. Permit product splitting, same product can be produced in more than one plant → Transportation problem
  2. Do not permit product splitting → Assignment Problem

TABLE 8.27 Data for the Better Products Co. problem

		Unit Cost ($) for Product				Capacity Available
		1	2	3	4	
Plant	1	41	27	28	24	75
	2	40	29	—	23	75
	3	37	30	27	21	45
Production rate		20	30	30	40	

Example continued

Option 1: Splitting allowed→ Transportation formulation

TABLE 8.28 Parameter table for the transportation problem formulation of Option 1 for the Better Products Co. problem

		Cost per Unit Distributed Destination (Product)					
		1	**2**	**3**	**4**	**5(*D*)**	**Supply**
Source (Plant)	1	41	27	28	24	0	75
	2	40	29	*M*	23	0	75
	3	37	30	27	21	0	45
Demand		20	30	30	40	75	

The optimal solution for this transportation problem has basic variables (allocations) $x_{12} = 30$, $x_{13} = 30$, $x_{15} = 15$, $x_{24} = 15$, $x_{25} = 60$, $x_{31} = 20$, and $x_{34} = 25$, so

Plant 1 produces all of products 2 and 3.
Plant 2 produces 37.5 percent of product 4.
Plant 3 produces 62.5 percent of product 4 and all of product 1.

The total cost is $Z = \$3{,}260$ per day.

Example continued

Option 2: Splitting not allowed→ Assignment formulation

Management has specified that every plant should be assigned at least one of the products. There are more products (four) than plants (three), so one of the plants will need to be assigned two products. Plant 3 has only enough excess capacity to produce one product (see Table 8.27), so *either* Plant 1 or Plant 2 will take the extra product.

TABLE 8.29 Cost table for the assignment problem formulation of Option 2 for the Better Products Co. problem

		Task (Product)				
		1	**2**	**3**	**4**	**5(*D*)**
Assignee (Plant)	1*a*	820	810	840	960	0
	1*b*	820	810	840	960	0
	2*a*	800	870	*M*	920	0
	2*b*	800	870	*M*	920	0
	3	740	900	810	840	*M*

Example continued

Assignment formulation.

Cost of Plant 1 producing one unit of product 1	= \$41
Required (daily) production of product 1	= 20 units
Total (daily) cost of assigning plant 1 to product 1	= 20 (\$41)
	= \$820

TABLE 8.29 Cost table for the assignment problem formulation of Option 2 for the Better Products Co. problem

		Task (Product)				
		1	**2**	**3**	**4**	**5(*D*)**
Assignee (Plant)	1*a*	820	810	840	960	0
	1*b*	820	810	840	960	0
	2*a*	800	870	*M*	920	0
	2*b*	800	870	*M*	920	0
	3	740	900	810	840	*M*

The Assignment Problem
The Hungarian Method

- The algorithm operates on the cost table of the problem.
- Convert costs into opportunity costs:
  1. Subtract the minimum element in each row from all elements of that row. (*Row reduction*)
  2. If the table does not have a zero element in each row and each column, subtract the minimum element in each column from all elements of that column. (*Column reduction*)
  3. The new table will have a zero in each row and column.
  4. If these zero elements provide a complete set of assignments, we are done; the solution is optimal.
- Idea: One can add or subtract any constant from every element of a row or column of the cost table without really changing the problem.

The Assignment Problem
The Hungarian Method

Consider JOB SHOP Co:

TABLE 8.25 Cost table for the Job Shop Co. assignment problem

		Task (Location)			
		1	2	3	4
Assignee (Machine)	1	13	16	12	11
	2	15	*M*	13	20
	3	5	7	10	6
	4(*D*)	0	0	0	0

Row reduction:

	1	2	3	4
1	2	5	1	0
2	2	M	0	7
3	0	2	5	1
4 (D)	0	0	0	0

Column reduction (No need):

	1	2	3	4
1	2	5	1	**0**
2	2	M	**0**	7
3	**0**	2	5	1
4 (D)	0	**0**	0	0

The optimal solution:

Assignment 1-4 → Cost: 11

Assignment 2-3→ Cost: 13

Assignment 3-1→ Cost: 5

Assignment 4-2→ Cost: 0

Total Cost = 29

The Assignment Problem
The Hungarian Method – Iteration 1

Consider BETTER PRODUCT Co:

Every row (but one) already has zero element. Let's start with column reduction:

TABLE 8.29 Cost table for the assignment problem formulation of Option 2 for the Better Products Co. problem

		Task (Product)				
		1	**2**	**3**	**4**	**5(*D*)**
Assignee (Plant)	1*a*	820	810	840	960	0
	1*b*	820	810	840	960	0
	2*a*	800	870	*M*	920	0
	2*b*	800	870	*M*	920	0
	3	740	900	810	840	*M*

	1	**2**	**3**	**4**	**5**
1a	80	0	30	120	0
1b	80	0	30	120	0
2a	60	60	M	80	0
2b	60	60	M	80	0
3	0	90	0	0	M

Every row and column has at least one zero→ No need for row reduction.

But, can we make a complete assignment?

The Assignment Problem
The Hungarian Method – Iteration 1

We CANNOT make a complete assignment, we can only make 3.

To see this, draw the **minimum** number of lines through rows or columns to cover all zeros in the table.

	1	2	3	4	5
1a	80	0	30	120	0
1b	80	0	30	120	0
2a	60	60	M	80	0
2b	60	60	M	80	0
3	0	90	0	0	M

The Assignment Problem
The Hungarian Method – Iteration 1

Three lines mean 3 assignments, but we need to make 5.

The **minimum** number of lines ≠ Number of rows (columns)

	1	2	3	4	5
1a	80	0	30	120	0
1b	80	0	30	120	0
2a	60	60	M	80	0
2b	60	60	M	80	0
3	0	90	0	0	M

The Assignment Problem
The Hungarian Method – Iteration 2

The **minimum** number of lines ≠ Number of rows (columns)

→ Find the **minimum** element not crossed out.

→ Subtract it from the entire table.

→ Add it to each row and column crossed out.

→ Equivalently: Subtract the element from elements not crossed out, and add it to elements crossed out twice.

	1	**2**	**3**	**4**	**5**
1a	80	0	$\boxed{30}$	120	0
1b	80	0	30	120	0
2a	60	60	M	80	0
2b	60	60	M	80	0
3	0	90	0	0	M

The Assignment Problem
The Hungarian Method – Iteration 2

Draw the **minimum** number of lines through rows or columns to cover all zeros in the table.

	1	**2**	**3**	**4**	**5**
1a	50	0	0	90	0
1b	50	0	0	90	0
2a	30	60	M	50	0
2b	30	60	M	50	0
3	0	120	0	0	M

The Assignment Problem
The Hungarian Method – Iteration 2

Four lines mean 4 assignments, we need 5. We should repeat.

	1	2	3	4	5
1a	50	0	0	90	0
1b	50	0	0	90	0
2a	30	60	M	50	0
2b	30	60	M	50	0
3	0	120	0	0	M

The Assignment Problem
The Hungarian Method – Iteration 3

Repeat steps of previous iterations.

	1	2	3	4	5
1a	50	0	0	90	0
1b	50	0	0	90	0
2a	30	60	M	50	0
2b	30	60	M	50	0
3	0	120	0	0	M

The Assignment Problem
The Hungarian Method – Iteration 3

Draw the **minimum** number of lines through rows or columns to cover all zeros in the table.

	1	2	3	4	5
1a	50	0	0	90	30
1b	50	0	0	90	30
2a	0	30	M	20	0
2b	0	30	M	20	0
3	0	120	0	0	M

The Assignment Problem
The Hungarian Method – Iteration 3

Minimum five lines are needed to cover all zeros. We are done!

	1	2	3	4	5
~~1a~~	~~50~~	~~0~~	~~0~~	~~90~~	~~30~~
~~1b~~	~~50~~	~~0~~	~~0~~	~~90~~	~~30~~
~~2a~~	~~0~~	~~30~~	~~M~~	~~20~~	~~0~~
~~2b~~	~~0~~	~~30~~	~~M~~	~~20~~	~~0~~
~~3~~	~~0~~	~~120~~	~~0~~	~~0~~	~~M~~

The Assignment Problem
The Hungarian Method – Iteration 3

Make the assignments. **Red** zeros are one set of assignments, **blue** zeros show an alternative optimal solution. The **purple** zero is common to both solutions.

	1	2	3	4	5
1a	50	**0**	**0**	90	30
1b	50	**0**	**0**	90	30
2a	**0**	30	M	20	**0**
2b	**0**	30	M	20	**0**
3	0	120	0	**0**	M

Z = 810 + 840 + 800 + 0 + 840 = 3290 (from the original cost table) for **red** zeros

Exercise: Check for the **blue** ones!